LECTURES ON FRACTALS AND DIMENSION THEORY

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Hausdorff, Cartheadory, Julia, Besicovitch and Mandelbrot

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0. Introduction

For many familiar objects there is a perfectly reasonable intuitive definition of dimension: *A space is d-dimensional if locally it looks like a patch of* $\mathbb{R}^d$. (Of course, "looks" requires some interpretation. For the moment we shall loosely interpret as "diffeomorphic"). This immediately allows us to say: the dimension of a point is zero; the dimension of a line is 1; the dimension of a plane is 2; the dimension of $\mathbb{R}^d$ is d. Moreover, we want the dimension of a circle to be 1; the dimension of a surface to be 2, etc. The difficulty comes with more complicated sets "fractals" for which we might want some notion of dimension which can be any real number.

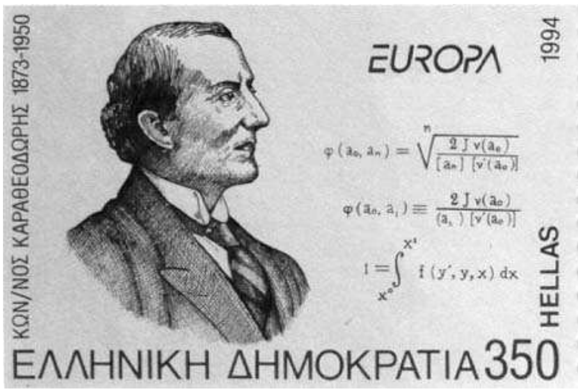
There are several different notions of dimension for more general sets, some more easy to compute and others more convenient in applications. We shall concentrate on *Hausdorff dimension.* Hausdorff introduced his defintion of dimension in 1919 and this was used to study such famous objects such as Koch's snowfalke curve. In fact, his definition was actually based on earlier ideas of Carathéodory. Further contributions and applications, particularly to number theory, were made by Besicovitch.

One could give a provisional mathematical definition of a fractal as a set for which the Hausdorff dimension strictly exceeds the topological dimension, once these terms are defined. However, this is not entirely satisfactory as it excludes sets one would consider fractals. Mandelbrot introduced the term fractal in 1977, based on the latin noun "fractus", derived from the verb "frengere" meaning "to break". The present vogue for fractals is mainly due to Benoit Mandelbrot.

Felix Hausdorff was born on 8th November 1868 in Breslau, Germany (which is now Wroclaw, Poland) into a wealthy family. His Father was a textile merchant. In fact, Felix grew up in Leipzig after his parents moved there when he was a child. He studied Mathematics at Leipzig University, completing his PhD there in 1891. He was subsequently a Privatdozent, and then an Extraordinary Professor in Leipzig. However, Hausdorff really wanted to be a writer and actually published books on philosophy and poetry under a pseudonym. In 1904 he even published a farce which, when eventually produced, turned out to be very successful. Following this literary phase, he concentrated again on mathematics, and during the next dozen years he made major contributions to both topology and set theory. In 1910 he moved to Bonn, and then in 1913 he moved again to take up an ordinary professorship in Greifswalf before finally, in 1921, he returned again to Bonn. In 1919 he introduced the notion of Hausdorff dimension in a seminal paper on analysis. This was essentially a generalisation of an idea introduced earlier by Carathodory, but Hausdorff realised that the construction actually allows a definition of "fractional dimensions". In particular, Hausdorff's paper includes a proof of the famous result that the dimension of the middle-third Cantor set is $\log 2/\log 3$. Unfortunately, the final years of Hausdorff's life were tragic. He had come from a Jewish family, and in 1935 he was forced to retire by the Nazi regime in power in Germany. In 1941 he was scheduled to be sent to an internment camp, but managed to avoid being sent through the intervention of the University. However, this was merely a postponement, and on 26th Januray 1942 Hausdorff, his wife and sister-in-law committed suicide when internment seemed inevitable.

Felix Hausdorff (1868-1942)

Constantin Carathéodory was born on 13th September 1873, in Berlin. He was of Greek extraction, being the son of a secretary in the Greek embassy in Berlin. As a stundent, he studied as a military engineer at the École Militaire de Belgique. Subsequently, he joined the British colonial service and worked on the construction of the Assiut dam in Egypt in 1900. He then went on to study for his PhD in Berlin, and then Gottingen, before becoming a Provatdozent in Bonn in 1908. The following year he married - his own aunt! In the following years Carathéodory went on to hold chairs at Universities in Hanover, Breslau, Gottingen and Bonn. However, in 1919 the Greek Government asked him to help establish a new university in Smyrna. However, this was not a happy experience since the project was thwarted by a turkish attack. Eventually, following this interlude he was appointed to a chair in Munich, which he held until his retirement in 1938. He died there on 2nd February 1950.



Constantin Carathéodory (1873-1950)

Anton Julia was born on 3rd February 1893 in Sidi Bel Abbés, in Algeria. As a soldier in the First World War, he was severely wounded during an attack on the western front. This resulted in a disfiguring injury and he had to wear a leather strap across his face for the rest of his life. In 1918 Julia published "Mémoire sur l'itération des fonctions rationnelles" on the iteration of a rational function f, much of the work done while he was in hospital. In this, Julia gave a precise description of the set of those points whose orbits under the iterates of the map stayed bounded. This received the Grand Prix de l'Académie des Sciences. Julia became a distinguished professor at the École Polytechnique in Paris. He died on 19 March 1978 in Paris. His work was essentially forgotten until B Mandelbrot brought it back to prominence in the 1970s through computer experiments.

Gaston Julia (1893-1978)

Benoit Mandelbrot was born on 20th November 1924, in Warsaw. When his family emigrated to France in 1936 his uncle Szolem Mandelbrojt, who was Professor of Mathematics at the Collége de France, took responsibility for his early education. After studying at Lyon, he studied for his PhD at the École Polytechnique and after a brief spell in the CNRS, accepted an appointment with IBM. In 1945 Mandelbrot's uncle had recommended Julia's 1918 paper. However, is wasn't until the 1970s that he had returned to this problem. By this time rudimentary computer graphics allowed a study of the complicated fractal structure of Julia sets and Mandelbrot sets. This, and subsequent work, has provided and immense impetus to the study of Hausdorff Dimension.

Abram Besicovitch was born on 24th January 1891 in Berdyansk, Russia. His Father used to own a jeweller's shop. He studied mathematics at the University of St Petersburg, taking a chair there in 1991, during the Russian Civil War. Following positions in Copenhagen and Liverpool he moved to Cambridge in 1927, where he worked until his retirement in 1958. His work on sets of non-integer dimension was an early contribution to fractal geometry. Besicovitch extended Hausdorff's work to density properties of sets of finite Hausdorff measure. He died in Cambridge on 2nd November 1970.

Benoit Mandelbrot (1924-)

Abram Besicovitch (1891-1970)

There are an number of excellent mathematical treatments on Hausdorff dimension and its properties. Amongst my particular favorites are *Fractal Geometry* by K.J.Falconer and *Geometry of sets and measures in Euclidean spaces* by P. Matilla. In the context of Dynamical Systems and Dimension Theory an excellent book is *Dimension Theory in Dynamical Systems: Contemporary Views and Applications* by Y. Pesin.

Mark Pollicott

1. Definitions and Examples

1.1 Definitions. To begin at the very beginning: How can we best define the dimension of a closed bounded set X in $\mathbb{R}^n$, say? Ideally, we might want a definition so that:

(i) When X is a manifold then the value of the dimension is an integer which coincides with the usual notion of dimension;
(ii) For more general sets X we can have "fractional" dimensional; and
(iii) Points, and countable unions of points, have zero dimension.

Perhaps the earliest attempt to define the dimension was the following:

First Definition. We can define the *Topological dimension* $\dim_T(X)$ by induction. We say that X has zero dimension if for every point $x \in X$ every sufficiently small ball about x has boundary not intersecting X. We say that X has dimension d if for every point $x \in X$ every sufficiently small ball about x has boundary intersecting X in a set of dimension $d-1$.

This definition satisfies out first requirement, in that it co-incides with the usual notion of dimensions for manifolds. Unfortunately, the topological dimension is always a whole number. (For example, the topological dimension of the Cantor set $\mathcal{C}$ is zero). In particular, this definition fails the second requirement. Thus, let us try another definition.

Second Definition. Given $\epsilon > 0$, let $N(\epsilon)$ be the smallest number of ϵ-balls needed to cover X. We can define the *Box dimension* to be

$$\dim_B(X) = \limsup_{\epsilon \to 0} \frac{\log N(\epsilon)}{\log(1/\epsilon)}$$

Again this co-incides with the usual notion of dimensions for manifolds. Furthermore, the box dimension can be fractional (e.g., the dimension of the Cantor set X is $\log 2/\log 3$). We have used the limit supremum to avoid problems with convergence. Strictly speaking, this is usually called the upper box dimension and the box dimension is usually said to exist when the limit exists (and is thus equal to the limsup). However, we have the following:

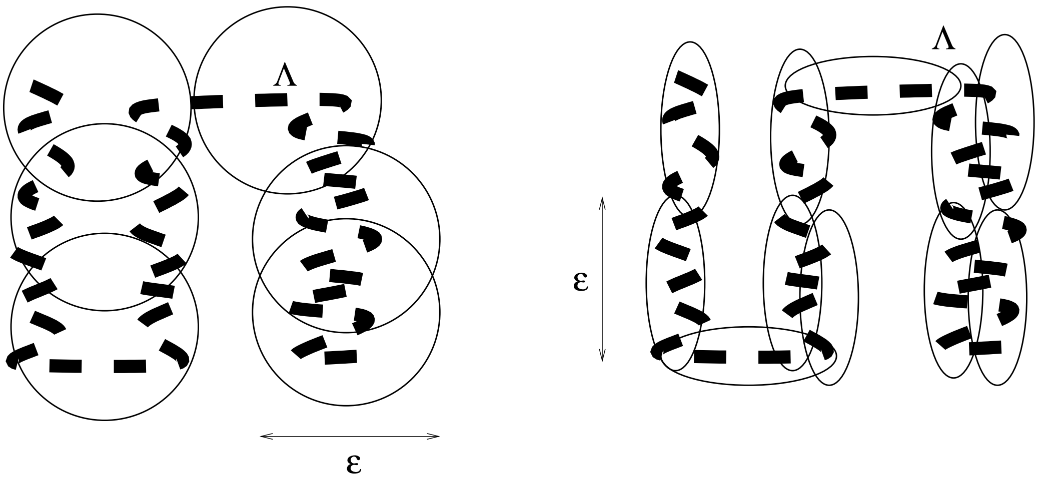
Lemma 1.1.1. *There exist countable sets such that condition (iii) fails for the box dimension.*

Proof. Consider the countable set

$$X = \left\{ \frac{1}{n} : n \geq 1 \right\} \cup \{0\}.$$

Given $0 < \epsilon < 1$, say, we can choose $n = n(\epsilon) \geq 2$ so that $\frac{1}{(n+1)n} \leq \epsilon < \frac{1}{n(n-1)}$. The points $\{1, \frac{1}{2}, \ldots, \frac{1}{n}\}$ are each separated from each other by a minimum distance $\frac{1}{n(n-1)}$. Thus n intervals of length ϵ are needed to cover this portion of the set. The rest of the set X can be covered by another n intervals $[i\epsilon, (i+1)\epsilon)]$ of length $\epsilon > 0$ (for $i = 0, \ldots, n-1$). This gives bounds $n \leq N(\epsilon) \leq 2n$ and we see from the definition that

$$\dim_B(X) = \lim_{\epsilon \to 0} \frac{\log N(\epsilon)}{\log\left(\frac{1}{\epsilon}\right)} = \frac{1}{2},$$



(i) Cover by balls (for $\dim_B(\Lambda)$; (ii) Cover by open sets (for $\dim_H(\Lambda)$)

since $n - 2 < \epsilon^{-1/2} \leq n$

Finally, let us try a third definition,

Third Definition. We can define the *Hausdorff dimension* (or Hausdorff-Besicovitch dimension) as follows. Given X we can consider a cover $\mathcal{U} = \{U_i\}_i$ for X by open sets. For $\delta > 0$ we can define $H_\epsilon^\delta(X) = \inf_{\mathcal{U}}\{\sum_i \operatorname{diam}(U_i)^\delta\}$ where the infimum is taken over all open covers $\mathcal{U} = \{U_i\}$ such that $\operatorname{diam}(U_i) \leq \epsilon$. We define $H^\delta(X) = \lim_{\epsilon\to 0} H_\epsilon^\delta(X)$ and, finally,

$$\dim_H(X) = \inf\{\delta : H^\delta(X) = 0\}.$$

As for the previous two definitions this coincides with the usual notion of dimensions for manifolds. Furthermore, the Hausdorff dimension can be fractional (e.g., the dimension of the Cantor set X is again $\log 2/\log 3$). Finally, for any countable set X property (iii) holds:

Proposition 1.1.2. *For any countable set X we have that $dim_H(X) = 0$.*

Proof. We can enumerate the countable set $X = \{x_n : n \geq 1\}$, say. Given any $\delta > 0$ and $\epsilon > 0$, for each $n \geq 1$, we can choose $\epsilon > \epsilon_n > 0$ sufficiently small that $\sum_{n=1}^\infty \epsilon_n^\delta \leq \epsilon$, say. In particular, we can consider the cover $\mathcal{U}$ for X by balls $B(x_n, \epsilon_n/2)$ centred at x_n and of different diameters ϵ_n. From the definitions, $H_\epsilon^\delta(X) \leq \epsilon$, for any $\epsilon > 0$, and so $H^\delta(X) \leq \lim_{\epsilon\to 0} H_\epsilon^\delta(X) = 0$. Since $\delta > 0$ was arbitrarily, we see from the definition of Hausdorff dimension above that $\dim_H(X) = 0$. □

At first sight, the definition of Hausdorff dimension seems quite elaborate. However, its many useful properties soon become apparent. Conveniently, in many of the examples we will consider later $\dim_H(X) = \dim_B(X)$. In fact, one inequality is true in all cases:

Proposition 1.1.3. *The definitions are related by $dim_H(X) \leq dim_B(X)$.*

Proof. Let $\eta > 0$ and set $\gamma = \dim_B(X) + \eta$ and $\delta = \dim_B(X) + 2\eta$. From the definition of $\dim_B(X)$, for $\epsilon > 0$ sufficiently small we can cover X by $N(\epsilon) \leq \epsilon^{-\gamma}$ ϵ-balls. Taking this as a cover $\mathcal{U}$ we see that $H^\delta_\epsilon(X) \leq \epsilon^{-\gamma}\epsilon^\delta = \epsilon^\eta$ and so $H^\delta(X) = \lim_{\epsilon\to 0} H^\delta_\epsilon(X) = 0$. From the definitions, we see that $\dim_H(X) \leq \delta = \dim_B(X)+\eta$. Finally, since $\eta > 0$ can be chosen arbitrarily small, the result follows. □

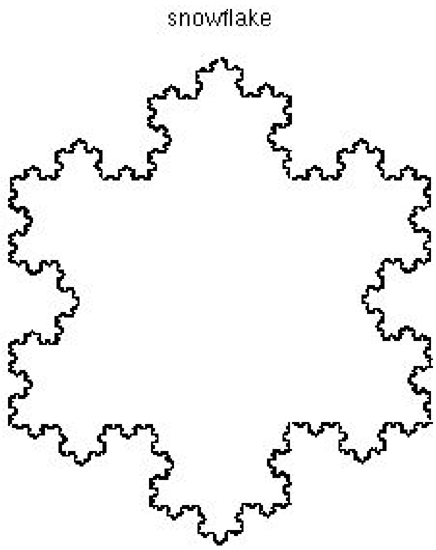
1.2 Examples. To understand the definitions of Box and Hausdorff dimension it is useful to experiment with a few simple examples.

Example 1.2.1: The coastline of countries. Of course, to begin with there is no reason that either the Box dimension or the Hausdorff dimension of a coastline would actually be well defined. However, instead of taking a limit as ϵ tends to zero one could just take ϵ to be "sufficiently small" and see what sort of values one can get. Empirically, we can attempt to estimate what the Box dimension d would be, if it was well defined. More precisely, we can count how many balls are needed to cover the coastline on a range of different scales (e.g., radius 100 miles, 10 miles, 1 mile). This leads to interesting (if not particularly rigorous) results, as was observed by Lewis Fry Richardson. For example:

Germany, d = 1.12;
Great Britain, d = 1.24; and
Portugal, d = 1.12.

Frontiers of different European countries

Example 1.2.2: Snowflake/von Koch curve. The von Koch curve X is a standard fractal construction. Starting from the interval $X_0 = [0,1]$ we associate to each piecewise linear curve X_n in the plane (which is a union of 4^n segments of length 3^{-n}) a new one X_{n+1}. This is done by replacing the middle third of each line segment by the other two sides of an equilateral triangle bases there. Alternatively, one can start from an equilateral triangle and apply this iterative procedure to each of the sides one gets a "snowflake curve".



The top third of this snowflake is the von Koch curve.

Proposition 1.2.1. *For the von Koch curve both the Box dimension and the Hausdorff dimension are* $\frac{\log 4}{\log 3}$.

Proof. When $\epsilon_n = \frac{1}{3^n}$, the set X_n is the union of 4^n intervals of length $\epsilon_n = 3^{-n}$. We can cover X_n by balls of size ϵ_n by associating to each edge a ball of radius $\frac{\epsilon_n}{2}$ centred at the midpoints of the side. It is easy to see that this is also a cover for X. Therefore, we deduce that $N(\epsilon_n) \leq 4^n$.

Moreover, it is easy to see that any ball of diameter ϵ_n intersecting X can intersect at most two intervals from X_n, and thus $N(\epsilon_n) \geq 4^{n-1}$. For any $\epsilon > 0$ we can choose $\epsilon_{n+1} \leq \epsilon < \epsilon_n$ and we know that $N(\epsilon_n) \leq N(\epsilon) \leq N(\epsilon_{n+1})$. Then

$$\frac{n-1}{(n+1)}\frac{\log 4}{\log 3} \leq \frac{\log N(\epsilon_n)}{\log(\frac{1}{\epsilon_{n+1}})} \leq \frac{\log N(\epsilon)}{\log(\frac{1}{\epsilon})} \leq \frac{\log N(\epsilon_{n+1})}{\log(\frac{1}{\epsilon_n})} \leq \frac{(n+1)}{n}\frac{\log 4}{\log 3}.$$

Letting $n \to +\infty$ shows that $\dim_B(X) = \lim_{\epsilon\to 0} \frac{\log N(\epsilon)}{\log(\frac{1}{\epsilon})} = \frac{\log 4}{\log 3}$. We postpone the proof that $\dim_B(X) = \dim_H(X)$ until later, when we shall show a more general result. $\square$

Example 1.2.3. Middle third Cantor set and E_2. Let X denote the middle third Cantor set. This is the set of closed set of points in the unit interval whose triadic expansion does not contain any occurrences of the the digit 1, i.e.,

$$X = \left\{ \sum_{k=1}^{\infty} \frac{i_k}{3^k} \,:\, i_k \in \{0,2\}, k \geq 1 \right\}$$

Proposition 1.2.2. *For the middle third Cantor set both the Box dimension and the Hausdorff dimension are* $\frac{\log 2}{\log 3} = 0.690....$

Proof. When $\epsilon_n = \frac{1}{3^n}$ it is possible to cover the set of X by the union of 2^n intervals

$$X_n = \left\{ \sum_{k=1}^{n} \frac{i_k}{3^k} + \frac{t}{3^n} \,:\, i_k \in \{0,2\}, k \geq 1, \text{ and } 0 \leq t \leq 1 \right\}$$

of length $\frac{1}{3^n}$. Therefore, we deduce that $N(\epsilon_n) \le 2^n$.

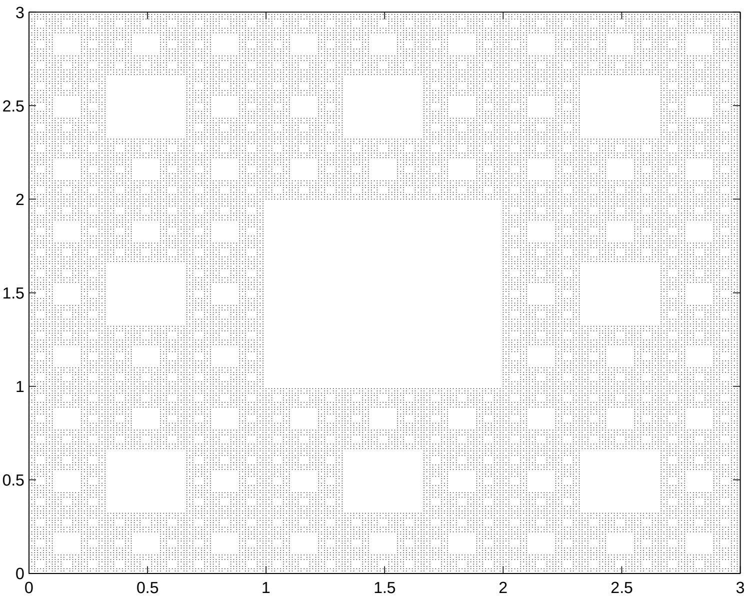
Moreover, it is easy to see that any interval of length ϵ_n intersecting X can intersect at most two intervals from X_n, and thus $N(\epsilon_n) \ge 2^{n-1}$. For any $\epsilon > 0$ we can choose $\epsilon_{n+1} \le \epsilon < \epsilon_n$ and we know that $N(\epsilon_n) \le N(\epsilon) \le N(\epsilon_{n+1})$. Then

$$\frac{n-1}{(n+1)}\frac{\log 2}{\log 3} \le \frac{\log N(\epsilon_n)}{\log(\frac{1}{\epsilon_{n+1}})} \le \frac{\log N(\epsilon)}{\log(\frac{1}{\epsilon})} \le \frac{\log N(\epsilon_{n+1})}{\log(\frac{1}{\epsilon_n})} \le \frac{(n+1)}{n}\frac{\log 2}{\log 3}.$$

Letting $n \to +\infty$ shows that $\dim_B(X) = \lim_{\epsilon\to 0} \frac{\log N(\epsilon)}{\log(\frac{1}{\epsilon})} = \frac{\log 2}{\log 3}$. We again postpone the proof that $\dim_B(X) = \dim_H(X)$ until later, when we shall show a more general result. □

The set E_2 is the set of points whose continued fraction expansion contains only the terms 1 and 2. Unlike the Middle third Cantor set, the dimension of this set is not explicitly known in a closed form and can only be numerically estimated to the desired level of accuracy.

Example 1.2.4 :Sierpinski carpet. Let $X = \left\{\left(\sum_{n=1}^\infty \frac{i_n}{3^n}, \sum_{n=1}^\infty \frac{j_n}{3^n}\right) : (i_n, j_n) \in \mathcal{S}\right\}$ where $\mathcal{S} = \{0,1,2\} \times \{0,1,2\} - \{(1,1)\}$. This is a connected set without interior. We call X a Sierpinski carpet.



THE SIERPINSKI CARPET

Proposition 1.2.3. *For the Sierpinski carpet both the Box dimension and the Hausdorff dimension are equal to* $\frac{\log 8}{\log 3} = 1.892...$

Proof. When $\epsilon_n = \frac{1}{3^n}$ it is possible to cover the set of X by 8^n boxes of size $\frac{1}{3^n}$:

$$X_n = \left\{\left(\sum_{k=1}^n \frac{i_k}{3^k} + \frac{s}{3^n}, \sum_{k=1}^n \frac{j_k}{3^k} + \frac{t}{3^n}\right) : (i_k, j_k) \in \mathcal{S} \text{ and } 0 \le s,t \le 1\right\}$$

Moreover, it is easy to see that there is no cover with less elements. For any $\epsilon > 0$ we can choose $\epsilon_{n+1} \leq \epsilon < \epsilon_n$ and we know that $N(\epsilon_n) \leq N(\epsilon) \leq N(\epsilon_{n+1})$. Then

$$\frac{n}{(n+1)}\frac{\log 8}{\log 3} = \frac{\log N(\epsilon_{n+1})}{\log(\frac{1}{\epsilon_n})} \leq \frac{\log N(\epsilon)}{\log(\frac{1}{\epsilon})} \leq \frac{\log N(\epsilon_n)}{\log(\frac{1}{\epsilon_{n+1}})} = \frac{(n+1)}{n}\frac{\log 8}{\log 3}.$$

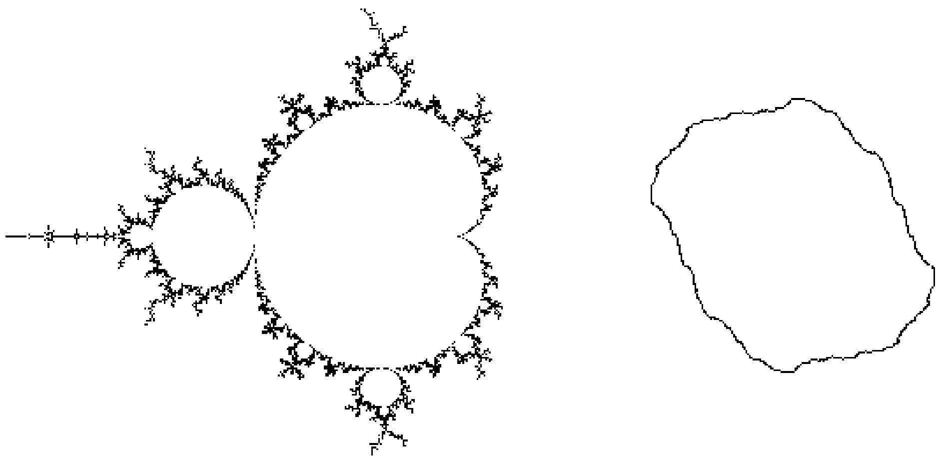
Letting $n \to +\infty$, gives that $\dim_B(X) = \frac{\log 8}{\log 3}$. We postpone the proof that $\dim_B(X) = \dim_H(X)$ until later, when we shall show a more general result. $\square$

1.3 Julia and Mandelbrot sets. The study of Julia sets is one of the areas which has attracted most attention in recent years. We shall begin considering the general setting and specialise later to quadratic maps. Consider a map $T : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ defined by a rational function $T(z) = P(z)/Q(z)$, for non-trivial relatively prime polynomials $P, Q \in \mathbb{C}[z]$. To avoid trivial cases, we always assume that $d := \max(\deg(P), \deg(Q)) \geq 2$.

Definition. We define the *Julia set* J to be the closure of the repelling periodic points i.e.

$$J = \text{closure}\left(\left\{z \in \widehat{\mathbb{C}} : T^n(z) = z, \text{ for some } n \geq 1, \text{ and } |(T^n)'(z)| > 1\right\}\right).$$

The Julia set J is clearly a closed T-invariant set (i.e., $T(J) = J$). There are other alternative definitions, but we shall not require them. By contrast, T has at most finitely many attracting periodic points, which must be disjoint from the Julia set.



We choose the point $c = \frac{i}{4}$ in the parameter space (left picture) and draw the associated Julia set for $T(z) = z^2 + \frac{i}{4}$ (right picture).

Let us now restrict to polynomial maps of degree 2. We can make a change of coordinates to put these maps in a canonical form. For a fixed parameter $c \in \mathbb{C}$ consider the map $T_c : \mathbb{C} \mapsto \mathbb{C}$ defined by $T_c : z \to z^2 + c$. Let J_c be the associated Julia set. To begin with, we see that when $c = 0$ then the Julia set is easily easily calculated.

Example 1.3.1: $c = 0$. For $T_0 z = z^2$, the repelling periodic points of period n are the dense set of points on the unit circle of the form $\xi = \exp(2\pi ik/(2^n - 1))$. The corresponding derivitive is $|(T_0^n)'(\xi)| = 2^n$. In particular, we have $J_0 = \{z \in \mathbb{C} : |z| = 1\}$, i.e., the unit circle. Thus, trivially we have that $\dim(J_0) = 1$.

We next consider the case of values of c of sufficiently small modulus, where the asymptotic behaviour of the limit set is well understood through a result of Ruelle:

Proposition 1.3.1. *For $|c|$ sufficiently small:*

(1) *the Julia set J_c for $T_c(z) = z^2 + c$ is still a Jordan circle, but it has $dim_B(J_c) = dim_H(J_c) > 1$; and*
(2) *the map $c \mapsto dim_H(J_c)$ is real analytic and we have the asymptotic*

$$dim_H(J_c) \sim 1 + \frac{|c|^2}{4\log 2}, \qquad \textit{as } |c| \to 0.$$

In a later section we shall give an outline of the proof of this result using ideas from Dynamical Systems.

At the other extreme, if c has large modulus, the asymptotic behaviour of the limit set is well understood through the following results of Falconer.

Proposition 1.3.2. *For $|c|$ sufficiently large*

(1) *the Julia set for T_c is a Cantor set, with $dim_B(J_c) + dim_H(J_c) > 0$; and*
(2) *the map $c \mapsto dim_H(J_c)$ is real analytic and we have the asymptotic*

$$dim_H(J_c) \sim \frac{2\log 2}{\log |c|} \qquad \textit{as } |c| \to +\infty \quad \text{[Falconer]}.$$

Moreover, there are also a few special cases where the Julia set (and its dimension) are well understood. For example, the case $c = -2$ is particularly simple:

Example 1.3.2. When $c = -2$ then $J_{-2} = [-2, 2]$, i.e., a closed interval and in this case we again trivially have that $\dim(J_{-2}) = 1$. For $c < -2$, the Julia set is contained in the real axis.

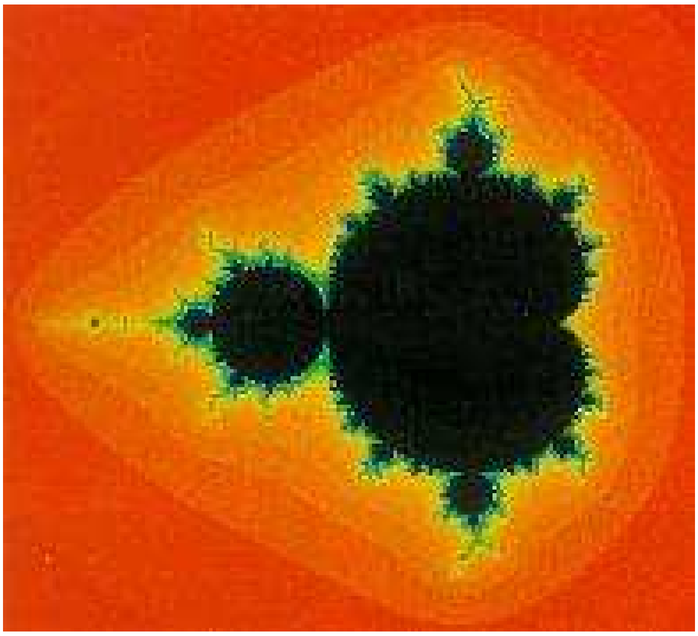
Unfortunately, in general the Hausdorff dimension of the Julia set for most values of c cannot be given explicitly. However, the general nature of the Julia set is characterized by the following famous subset of the parameter space c.

Definition. The *Mandelbrot set* $\mathcal{M} \subset \mathbb{C}$ is defined to be the set of points c in the parameter space such that the orbit $\{T_c^n(0) : n \geq 0\}$ is bounded, i.e.,

$$\mathcal{M} := \{c \in \mathbb{C} : |T_c^n(0)| \not\to +\infty, \text{ as } n \to +\infty\}.$$

In fact, the importance of $z = 0$ in this definition is that it is a critical point for T_c, i.e., $T_c'(0) = 0$. The significance of the Mandelbrot set is that it actually characterizes the type of Julia set J_c one gets for T_c.

Proposition 1.3.3. *If $c \notin \mathcal{M}$ then J_c is a Cantor set. If $c \in \mathcal{M}$ then J_c is a connected set.*



The Mandelbrot set in the parameter space for c

For more specific choices for the parameter c we have to resort to numerical computation if we want to know the Hausdorff dimension of J_c. We shall study this problem in detail in a latter chapter. However, for the moment, we shall illustrate this by examples of each type of behaviour.

Examples 1.3.3.

(i) Let us consider two points in the Mandelbrot set. For $c = i/4$,say, we can estimate $dim_H(J_{i/4}) = 1.02321992890309691....$ For $c = 1/100$, say, we can estimate $dim_H(J_{1/100}) = 1.00003662....$ ..., respectvely.)

(ii) Let us consider two points outside of the Mandelbrot set. For $c = -3/2 + 2i/3$, say, we can estimate $\dim_H(J_{-3/2+2i/3}) = 0.9038745968111....$ For $c = -5$,say, we can estimate $\dim_H(J_{-5}) = 0.4847982944381604305383984 7....$

However, an important ingredient in the method of computation of these values is that the Julia set should satisfy an additional property which is particularly useful in our analysis our analysis. More precisely, we need to assume that T_c is hyperbolic in the following sense.

Definition. We say that the rational map is *hyperbolic* if there exist $\beta > 1$ and $C > 0$ such that for any $z \in \mathbb{C}$ we have $(T^n)'(z)| \geq C\beta^n$, for all $n \geq 1$.

Hyperbolicity, in various guises, is something that underpins a lot of our analysis in different settings. For the particular setting of rational maps, hyperbolicity can be shown to be equivalent to the Julia set J being disjoint from the orbit of the critical points $\mathcal{C} = \{z : T'(z) = 0\}$ (i.e. $J \cap (\cup_{n=0}^{\infty} T^n(\mathcal{C})) = \emptyset$). However, we shall not require this observation in the sequel.

Proposition 1.3.4. *If T_c is hyperbolic then $dim_H(J_c) = dim_B(J_c)$.*

Proof. Actually, in the case of hyperbolic maps we can think of the Julia set as being the limit set of an iterated function scheme with respect to the two inverse branches for T_c. In this case, the result is just a special case of more general results (which we return to in a later chapter). $\square$

As a cautionary tale, we should note that once one takes c outside of the region in the parameter space corresponding to hyperbolic maps, then the situation becomes more complicated. For example, the dimension of the Julia set may no longer be even continuous in c, in contrast to the hyperbolic case where there is actually a real analytic dependence. This is illustrated by the following.

Parabolic Explosions. Of course, as c crosses the boundary of the Mandelbrot set the Julia set J_c (and its Hausdorff dimension) can change more dramatically. Douady studied the case as $c \to \frac{1}{4}$ (along the real axis). As c increases the dimension $\dim(J_c)$ increases monotonically, with derivative tending to infinity. However, as c increases past $\frac{1}{4}$ there is a discontinuity where the dimension suddenly stops.

Let us return to studying the Mandelbrot set. Although the Mandelbrot set is primarily a set in the parameter space for the quadratic maps, it has a particularly interesting structure in its own right. Some of its main features are described in the following proposition.

Proposition 1.3.5.

(1) *The set $\mathcal{M}$ lies within the ball of radius 2 given by $\{c \in \mathbb{C} : |c| \leq 2\}$;*
(2) *The set $\mathcal{M}$ is closed, connected and simply connected;*
(3) *The interior int($\mathcal{M}$) is a union of simply connected components;*
(4) *The largest component of int($\mathcal{M}$) is the main cardioid defined by*

$$\mathcal{M}_1 = \{w \in \mathbb{C} : |1 - \sqrt{1-4w}| < 1\}$$

and for any $c \in \mathcal{M}_1$ the map T_c is hyperbolic;

(5) *For $c \notin \mathcal{M}$, the map T_c is hyperbolic.*

Proof. For part (1), suffices to show that if $|c| \geq 2$ then the sequence $\{T_c^n(0) : n \geq 0\}$ is unbounded. If $|z| > 2$, then $|z^2 + c| \geq |z^2| - |c| > 2|z| - |c|$. If $|z| \geq |c|$, then $2|z| - |c| > |z|$. So, if $|z| > 2$ and $|z| \geq c$, $|z^2 + c| > |z|$, so the sequence is increasing. (It takes a bit more work to prove it is unbounded and diverges.) If $|c| > 2$, the sequence diverges.

The Mandelbrot set is known to be a simply connected set in the plane from a theorem of Douady and Hubbard that there is a conformal isomorphism from the complement of the Mandelbrot set to the complement of the unit disk.

For the other properties we refer the reader to any book on rational maps (e.g., Milnor's). $\square$

Although we don't have a comprehensive knowledge of which parameter values c lead to T_c being hyperbolic, we do have some partial information. For example, it is known that a component H of int($\mathcal{M}$) contains a parameter c for which T_c is hyperbolic if and only if $T_{c'}$ is hyperbolic for *every* $c' \in H$. In particular, any c in the central cartoid $\mathcal{M}_1$ the map T_c has the attracting fixed point $\frac{1}{2}(1 - \sqrt{1-4w})$, and thus is hyperbolic because of another equivalent condition for hyperbolicity is: Either $c \notin \mathcal{M}$ or T_c has an attracting cycle. We call H a *hyperbolic component.*

At first sight, one might imagine that there is little direction between the metric properties of the Mandelbrot set and the associated Julia sets. However, there are are a number of surprising connections. We mention only the following.

Theorem 1.3.6 (Shishikura). *The boundary of $\mathcal{M}$ has Hausdorff dimension 2. For generic points c in the boundary the associated Julia set for T_c has Hausdorff dimension 2.*

Although considerable work has been in recent years done on understanding the structure of the Mandelbrot set, and enormous progress has been made, there remain a number of major outstanding questions. The solution to these would give fundamental insights into the nature of the Mandelbrot set.

Major Open Problems. However, it is a major conjecture that the boundary $\partial\mathcal{M}$ is locally connected (i.e., if every neighbourhood of $\partial\mathcal{M} \cap B(x, \epsilon)$ contains a connected open neighbourhood). Another important question is whether there exist any examples of Julia sets which can have positive measure. Finally, it is apparently unknown whether every component of $\text{int}(\mathcal{M})$ is hyperbolic.

1.4 Fuchsian and Kleinian Limit sets. The Limit sets of Kleinian groups often have similar features to those of Julia sets. Indeed, in the 1970's Sullivan devised a "dictionary" describing many of the corresponding properties.

Let $\mathbb{H}^3 = \{z + jt \in \mathbb{C} \oplus \mathbb{R} : t > 0\}$ be the three dimensional upper half space. We can equip this space with the Poincare metric

$$ds^2 = \frac{dx^2 + dy^2 + dt^2}{t^2}.$$

With this metric the space has curvature $\kappa = -1$. For a detailed description of the space and its geodesics we refer the reader to Bearden's book on *Discrete groups.*

We can identify the isometries for $\mathbb{H}^3$ and this metric with the (orientation preserving) transformations

$$(z, t) \mapsto \left(\frac{az + b}{cz + d}, t + 2 \log |cz + d| \right),$$

where $a, b, c, d \in \mathbb{C}$ with $ad - bc = 1$. In particular, the first component is a linear fractional transformation and we can identify the space of isometries with the matrices $G = SL(2, \mathbb{C})$.

Defintion. A Kleinian group $\Gamma < G$ is a finitely generated discrete group of isometries. Let Γ_0 be the generators of Γ.

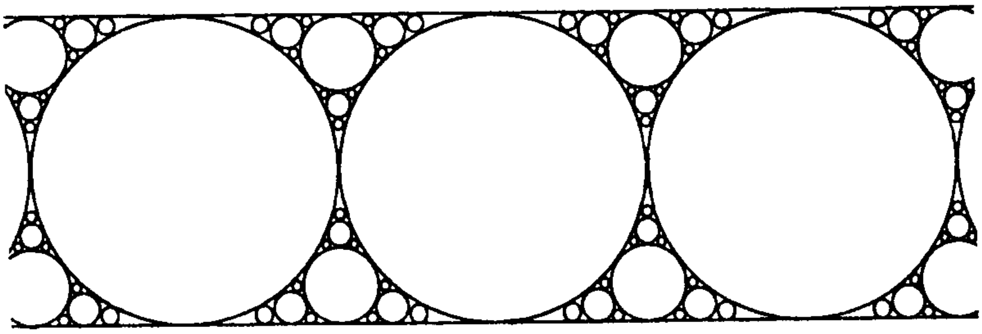
Although the action of $g \in G$ is an isometry on $\mathbb{H}^3$, the action on the boundary is typically not an isometry. In particular, we can associate to each $g \in \Gamma$ its *isometric circle* $C(g) := \{z \in \mathbb{C} : |g'(z)| = 1\}$. This is a Euclidean circle in the complex plane $\mathbb{C}$.

Defintion. We define the limit set $\Lambda = \Lambda_\Gamma \subset \mathbb{C} \cup \{\infty\}$ for Γ to be the set of all limit points (in the Euclidean metric) of the set of points $\{g(j) : g \in \Gamma\}$.

By way of clarification, we should explain that since Γ is a discrete group these limit points must necessarily be in the Euclidean boundary. Moreover, we should really take the limit points using the one point compactification of $\mathbb{C}$ (where the the compactification point is denoted by ∞. Depending on the choice of Γ, the limit set Λ_Γ may have different properties.

These include the possibilities that Λ_Γ is a Cantor set, or all of $\mathbb{C} \cup \{\infty\}$. We begin by considering one of the most famous examples of a Limit set for a Kleinian group - which happens to be neither of these cases.

Example 1.4.1. Apollonian circle packing. Consider three circles C_1, C_2, C_3 in the euclidean plane that are pairwise tangent. Inscribe a fourth circle C_4 which is tangent to all three circles. Within the three triangular region whose sides consist of the new circle and pairs of the other circles inscribe three new circles. Proceed inductively. The limit set is call an Apollonian circle packing.



The Apollonian circle packing

We can associate to each circle $C_i = \{z : |z - z_i| = r_i\}$ (with $z_i \in \mathbb{C}$ and $r_i > 0$) an element $g_i \in G$ associated to the linear fractional transformation

$$g_i : z \mapsto \frac{1}{r_i^2(z - z_i)}.$$

These correspond to generators for a Kleinian group $\Gamma < G$. The limit set is estimated to have dimension $1.305686729\ldots$.

Let us consider some special cases:

Example 1.4.2. Fuchsian Groups:. Let $K = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle in the complex plane $\mathbb{C}$. If each element g preserves K then Γ is a *Fuchsian group.* In this case the isometric circles for each element $g \in \Gamma$ meet K orthogonally.

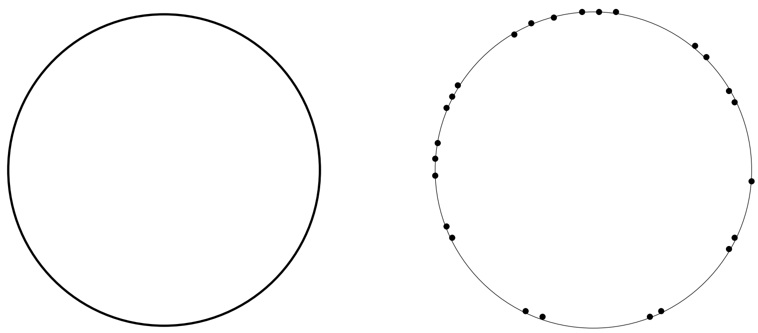
The standard presentation for a (cocompact) Fuchsian group is of the form

$$\Gamma = \langle g_1, \ldots, g_{2d} \in G : \prod_{i=1}^{d} [g_{2i-1}, g_{2i}] = 1 \rangle.$$

where $[g_{2i-1}, g_{2i}] = g_{2i-1} g_{2i} g_{2i-1}^{-1} g_{2i}^{-1}$. We can also consider the limit sets of such groups.

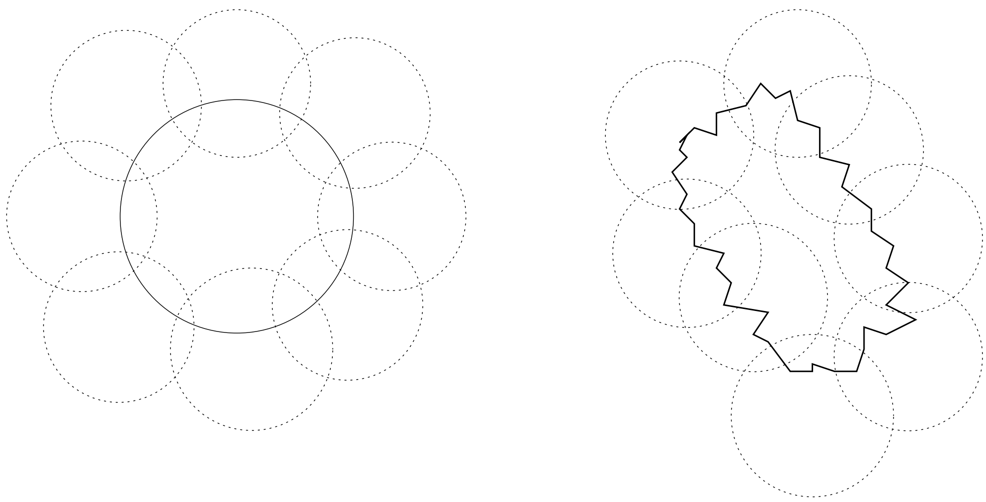
Theorem 1.4.1. *The Limit set of a non-cocompact convex cocompact Fuchsian group is either:*

(1) *a Cantor set lying in the unit circle; or*
(2) *the entire circle.*



For Fuchsian groups (a subclass of Kleianin groups) the limit set could be the entire circle or a Cantor set.

Example 1.4.3. Quasi- Fuchsian Groups:. We can next consider a Kleinian group whose generators (and associated isometric circles) are close to that of a Fuchsian group. Such groups are called *quasi-Fuchsian.* In this case the limit set is still homeomorphic to a closed circle. This is called a *quasi-circle.*



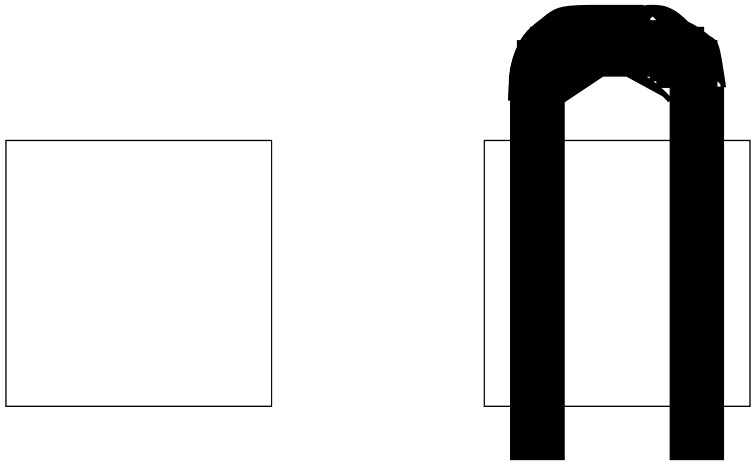
Perturbing the generators of a Fuchsian group changes the limit circle to a quasi-circle. (The dotted circles represent the generators for the Fuchsian group (left) and quasi-Fuchsian group (right).)

However, although the quasi-circle is topologically a circle it can be quite different in terms of geometry.

Theorem 1.4.2. *The Hausdorff dimension of a quasi-circles is greater than or equal to* 1*, with equality only when it is actually a circle.*

This result was originally proved by Bowen, in one of two posthumous papers published after his death in 1978. Quasi-circles whose Hausdorff dimension is strictly bigger than 1 are necessarily non-rectifiable, i.e., they have infinite length.

1.5 Horseshoes. We now recall a famous Cantor set in Dynamical Systems. The "Horseshoe" was introduced by Smale as an example of invariant set for a (hyperbolic) diffeomorphism $f : S^2 \to S^2$ on the two sphere S^2.



f bends the rectangle into a horseshoe. The Cantor set Λ is the set of points that never escape from the rectangle.

In the original construction, f is chosen to expand a given rectangle R (sitting on S^2) vertically; contract it horizontally; and bends it over to a horseshoe shape. The points that remain in the rectangle under all iterates of f (and f^{-1}) are an f-invariant Cantor set, which we shall denote by Λ. The rest of the points on S^2 are arranged to disappear to a fixed point.

In an more general construction, let M be a compact manifold and let $f : M \to M$ be a diffeomorphism. A compact set $\Lambda = \Lambda(f) \subset M$ is called invariant if $f(\Lambda) = \Lambda$. We say that $f : \Lambda \to \Lambda$ is *hyperbolic* if there is a continuous splitting $T_\Lambda M = E^s \oplus E^u$ of the tangent space into Df-invariant bundles and there exists $C > 0$ and $0 < \lambda < 1$ such that

$$||D_x f^n(v)|| \leq C\lambda^n ||v|| \text{ and } v \in E^s$$

$$||D_x f^{-n}(v)|| \leq C\lambda^n ||v|| \text{ and } v \in E^u.$$

We say that Λ is locally maximal if we can choose an open set $U \supset \Lambda$ such that $\Lambda = \cap_{n=-\infty}^{\infty} f^n U$. In general, we can take a horseshoe Λ to be an locally maximal f-invariant hyperbolic Cantor sets a diffeomorphism f on M.

Theorem 1.5.1 (Manning-McClusky). *For Horseshoes $\Lambda(f)$ on surfaces we have that $dim_H(\Lambda(f)) = dim_B(\Lambda(f))$.*

Moreover, Manning and McClusky gave an implicit formula for the Hausdorff dimension, which we shall return to in a later chapter.

Example. Consider the case of the original Smale horseshoe such that $f : R \cap f^{-1}R \to R$ is a linear map which contracts (in the horizontal direction) at a rate α and expands (in the vertical direction) at a rate $1/\beta$. For a linear horseshoe Λ the work of Manning-McClusky gives that:

$$\dim_H(\Lambda) = \dim_B(\Lambda(f)) = \log 2 \left(\frac{1}{\alpha} + \frac{1}{\beta} \right).$$

Let us now consider the dependence of the dimension X on the diffeomeorphism f. Let $\mathcal{D} \subset C^2(M, M)$ be the space of C^2 diffeomorphisms from M to itself. This comes equipped with a standard topology. We can consider a parmeterised family of diffeomorpisms $(-\epsilon, \epsilon) \ni \lambda \mapsto f_\lambda$. The first part of the next result shows smooth dependence of the Hausdorff dimension of horseshoes on surfaces. However, the second part shows this fails dramatically in higher dimensions.

Theorem 1.5.3.

(1) *On surfaces the Hausdorff dimension $dim_H(\Lambda(f_\lambda))$ of the horseshoe varies continuously (even differentiably).*
(2) *There exist examples of horseshoes on three dimensional manifolds for which the Hausdorff dimension does not change continuously.*

Palis and Viana originally showed continuity of the Hausdorff dimension in the case or surfaces, and Mane subsequently showed smoothness. Both results used a study of the "structural stability conjugacy map". Pollicott and Weiss showed the failure in higher dimensions by exploiting number theoretic results of two dimensional expanding maps.

Example. Consider an extension of the original construction of Smale where the rectangle is now replaced by a cube C (sitting on the sphere S^3). We can arrange that f expands the cube in one direction; contracts it in the remaining two directions; and maps it back across C is in the Smale construction. In this case, the dimension depends on the alignment of the intersection of $f(C)$ and C in the two dimensional contracting direction.

1.6 Some useful properies of Hausdorff dimension. A rather simple, but useful, viewpoint is to think of dimension as being a way to distinguish between sets of zero measure.

Proposition 1.6.1. *If $dim_H(X) < d$ then the (d-dimensional) Lebesgue measure of X is zero.*

On surfaces the dimension can be continuous Another useful property is that sets which are the same up to bi-Lipschitz maps have the same dimension (i.e., it is a *invariant* on classifying spaces up to "bi-Lipschitz equivalence"):

Proposition 1.6.2.

(1) *If $L : X_1 \to X_2$ is a surjective Lipschitz map i.e., $\exists C > 0$ such that*
$$|L(x) - L(y)| \leq C|x - y|,$$
then $dim_H(X_1) \leq dim_H(X_2)$.
(2) *If $L : X_1 \to X_2$ is a bijective bi-Lipschitz map i.e., $\exists C > 0$ such that*
$$(1/C)|x - y| \leq |L(x) - L(y)| \leq C|x - y|,$$
then $dim_H(X_1) = dim_H(X_2)$.

Proof. For part 1, consider an open cover $\mathcal{U}$ for X_1 with $\dim(U_i) \leq \epsilon$ for all $U_i \in \mathcal{U}$. Then the images $\mathcal{U}' = \{L(U) : U \in \mathcal{U}\}$ are a cover for X_2 with $\dim(L(U_i)) \leq L\epsilon$ for all $U_i \in \mathcal{U}'$. Thus, from the defintions, $H^\delta_{L\epsilon}(X_2) \geq H^\delta_\epsilon(X_1)$. In particular, letting $\epsilon \to 0$ we see that $H^\delta(X_1) \geq H^\delta(X_2)$. Finally, from the definitions $\dim_H(X_1) \leq \dim_H(X_2)$.

For part 2, we can apply the first part a second time with L replaced by L^{-1}. $\square$

Example. Consider the example of a linear horseshoe. Taking the horizonal and vertical projections we have Cantor sets in the line with smaller Hausdorff dimensions $-\log 2/\log \alpha$ and $-\log 2/\log \beta$.

The next result says that Hausdorff dimension behaves in the way we might have guessed under addition of sets.

Proposition 1.6.3. *Let $\Lambda_1, \Lambda_2 \subset \mathbb{R}$ and let*

$$\Lambda_1 + \Lambda_2 = \{\lambda_1 + \lambda_2 : \lambda_1 \in \Lambda_1, \lambda_2 \in \Lambda_2\}$$

then $dim_H(\Lambda_1 + \Lambda_2) \leq dim_H(\Lambda_1) + dim_H(\Lambda_2)$.

Proof. It is easy to see from the definitions that $\dim_H(\Lambda_1 \times \Lambda_2) = \dim_H(\Lambda_1) + \dim_H(\Lambda_2)$. Since the map $L : \mathbb{R}^2 \to \mathbb{R}$ given by $L(x, y) = x + y$ is Lipshitz, the result follows. $\square$

Example. Consider $\Lambda_1 = \Lambda_2$ to be the middle third Cantor set. We see that $\Lambda_1 + \Lambda_2$ is an interval (by considering the possible triadic expansions) and has dimension 1, whereas the sum of the dimensions is $\log 4/\log 3$.

1.7 Measures and the mass distribution principle. Finally, we complete this lecture with one of the basic techniques for Hausdorff dimension. The usual way to get a lower bound on the Hausdorff dimension is to use probability measures. A measure μ on X is called a probability measure is $\mu(X) = 1$. Assume that μ satisfies the following property.

Hypothesis. Assume that we can find $C > 0$ and $d > 0$ we have that $\mu(B(x, \epsilon)) \leq C\epsilon^d$, for all $x \in X$ and all $\epsilon > 0$.

Proposition 1.7.1. *If we can find a probability measure μ satisfying the above hypothesis then we have $dim_H(X) \geq d$.*

Proof. Let $\epsilon > 0$. Let us consider any arbitrary cover $\mathcal{U}$ for X consisting of balls $B(x_i, \epsilon_i)$, $i = 1, \ldots, N$, say, for which $\max_{1 \leq i \leq N} \epsilon_i \leq \epsilon$. Since μ is a probability measure then we see that

$$1 = \mu\left(\cup_i B(x_i, \epsilon_i)\right) \leq \mu\left(\sum_i \mu(x_i, \epsilon_i)\right) \leq C \sum_i \epsilon_i^d$$

In particular, we see that $H_\epsilon^d(X) \geq 1$. Since $\epsilon > 0$ was chosen arbitrarily we can deduce that $H^d(X) = \lim_{\epsilon \to 0} H_\epsilon^d(X) \geq 1$. It follows from the definitions that $\dim_H \geq d$, as required. $\square$

We can now use this result to show that the Hausdorff dimension and Box dimension are equal in our earlier examples.

Example 7.1: Middle third Cantor set. Let μ be the probability measure which gives weight 2^{-n} to each of the 2^n intervals in X_n, for each $n \geq 1$. For $\frac{1}{3^{n+1}} \leq \epsilon < \frac{1}{3^n}$ we see that for any $x \in X$ the ball $B(x, \epsilon)$ intersects at most two intervals from X_n. In particular, we see that $\mu(B(x, \epsilon)) \leq 2.2^{-n} = 4.2^{-(n+1)}$. Thus, if we let $D = \frac{\log 2}{\log 3}$ then

$$\mu(B(x, \epsilon)) \leq 4.3^{-D(n+1)} \leq 4.\epsilon^D.$$

Applying the Proposition, we deduce that $d \geq D = \frac{\log 2}{\log 3}$. This gives the required equality.

Example 7.2: Sierpinski Gasket. Let μ be the probability measure which gives weight 4^{-n} to each of the 4^n boxes in X_n, for each $n \geq 1$. For $\frac{1}{3^{n+1}} \leq \epsilon < \frac{1}{3^n}$ we see that for any $x \in X$ the ball $B(x, \epsilon)$ intersects at most for boxes from X_n. In particular, we see that $\mu(B(x,\epsilon)) \leq 4.4^{-n} = 16.4^{-(n+1)}$. Thus, if we let $D = \frac{\log 2}{\log 3}$ then

$$\mu(B(x,\epsilon)) \leq 16.4^{-D(n+1)} \leq 16.\epsilon^D.$$

Applying the Proposition, we deduce that $d \geq D = \frac{\log 8}{\log 3}$. This gives the required equality.

2. Iterated function schemes

In this chapter we introduce one of the basic constructions we shall be studying, that of *iterated function schemes* They appear in a surprisingly large number of familiar settings, including several that we have already described in the previous section. Moreover, those sets X for which we stand most chance of computing the dimension are those which exhibit some notion of *self-similarity* (for example, the idea that if you magnify a piece of the set enough then somehow it looks roughly the same). Often, if we have a local distance expanding map on a compact set we can view the natural associated invariant set as the limit set of an iterated function scheme of the inverse branches of this map (e.g., hyperbolic Julia sets, etc.).

In the case of many linear maps, the dimension can be found implicitly in terms of an expression involving only the rates of contraction. In the non-linear case, the corresponding expression involves the so called pressure function.

2.1 Definition and Basic Properties. We begin with a rather familiar notion.

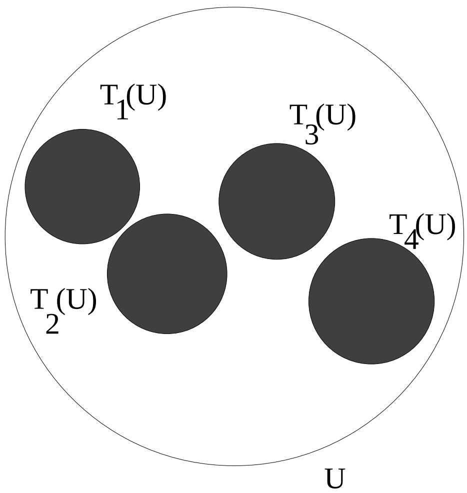
Definition. Let $U \subset \mathbb{R}^d$ be an open set. We say that $S : U \to U$ is a contraction if there exists $0 < \alpha < 1$ such that

$$||S(x) - S(y)|| \leq \alpha ||x - y|| \text{ for all } x, y \in U.$$

(Here $|| \cdot ||$ denotes the usual Euclidean norm on $\mathbb{R}^d$.)

The following definition is fundamental to what follows.

Definition. An *iterated function scheme* on an open set $U \subset \mathbb{R}^d$ consists of a family of contractions $T_1, \dots, T_k : U \to U$.



The images of U under the maps $T_1, \dots, T_4$ in an iterated function scheme.

A modified defintion. In fact, in some examples it is convenient to broaden slightly the definition of an iterated function scheme. More precisely, we might want want to consider contractions $T_i : U_i \to U$ which are only defined on part of the domain U. In this case, we consider only those sequences $(x_n)_{n=0}^{\infty}$ such that $U_{x_n} \supset T_{x_{n-1}}(U_{x_{n-1}})$.

We shall be particularly interested in the associated *limit set* Λ given in the following result.

Proposition 2.1.1. *Let $T_1, \dots, T_k : U \to U$ be a finite family of contractions. There exists a unique closed invariant set $\Lambda = \Lambda(T_1, \cdots, T_k)$ such that $\Lambda = \cup_{i=1}^k T_i \Lambda$.* ∎

Proof. The proof uses a standard application of the contraction mapping principle on sets. We let $\mathcal{X}$ be the set of all compact subsets of X. The set $\mathcal{X}$ can be given the Hausdorff metric defined by

$$d(A, B) = \sup_{x \in A} \left\{ \inf_{y \in B} d(x, y) \right\} + \sup_{x \in B} \left\{ \inf_{y \in A} d(x, y) \right\},$$

where $A, B \subset U$ are compact sets. (Here $d(x, B) := \inf_{y \in B} d(x, y)$ is the distance of x from the set B.) It is an easy exercise to see that this is indeed a metric on $\mathcal{X}$:

(i) $d(A, B) = 0$ if and only if $d(x, B) = 0$ for all $x \in A$ *and* $d(x, A) = 0$ for all $x \in B$. Equivalently, $A \subset B$ and $B \subset A$, i.e., $A = B$.

(ii) $d(A, B) = d(B, A)$ by the symmetry of the definitions, i.e., d is reflexive.

(iii) Given A, B, C and $\epsilon > 0$ we can choose $a, a' \in A$, $b, b' \in B$ and $c, c' \in C$, then by the triangle inequality $d(a, b') \le d(a, c) + d(c, b')$ and $d(a', b) \le d(a', c') + d(c', b)$ and we can write

$$\begin{aligned} d(a, B) + d(A, b) &\le d(a, b') + d(a', b) \\ &\le [d(a, c) + d(a', c')] + [d(c, b') + d(c', b)] \\ &\le [d(c, A) + d(a', C)] + [d(b', C) + d(B, c')] \end{aligned}$$

By suitable choices of a', c, b', c' we can arrange that the last term is bounded by $d(A, C) + d(B, C) + \epsilon$. Takeing the supremum over all a, b completes the proof.

With this metric, $\mathcal{X}$ is a complete metric space. To see this, imagine $B_n \subset X$ are a Cauchy sequence of compact sets. In particular, each $k \ge 1$ we can choose $n(k)$ such that for $n \ge n(k)$ we have that $B_n \subset C_k := \{x \in \mathbb{R}^n : d(x, B_{n(k)}) \le \frac{1}{2^k}\}$, which is again a compact set. But by construction $C_1 \supset C_2 \supset C_3 \supset \cdots$ and by compactness $B := \cap_{k=1}^\infty C_k \in \mathcal{X}$ exists. It is easy to see that $d(B_n, B) \to 0$ as $n \to +\infty$.

We can define a map $T : \mathcal{X} \to \mathcal{X}$ by $TA = \cup_{i=1}^k T_i A$. One can show that T is a contraction on $\mathcal{X}$. In fact,

$$d(TA, TB) \le \max_{1 \le i \le k} d(T_i A, T_i B) \le \alpha d(A, B),$$

where α_i is the contraction constant for T_i and we denote $\alpha = (\sup_{1 \le i \le k} \alpha_i) < 1$. We can apply the contraction mapping theorem to $T : \mathcal{X} \to \mathcal{X}$ to deduce that there is a unique fixed point $T(\Lambda) = \Lambda$. □

An alternative approach to constructing the limit set is as follows.

Definition. Consider a family of contractions $T_1, \dots, T_k : U \to U$. Fix any point $z \in U$ then we define the *limit set* Λ by the set of all limit points of sequences:

$$\Lambda = \left\{ \lim_{n \to +\infty} T_{x_0} \circ T_{x_1} \circ \dots \circ T_{x_n}(z) : x_0, x_1, \dots \in \{1, \dots, k\} \right\}$$

It is easy to see that the individual limits exist. More precisely, given a sequence $(x_n)_{n=0}^\infty$ we can denote $\Lambda_k := T_{x_0} \circ \dots \circ T_{x_k}(\Lambda)$, for each $k \ge 0$. Since this is a nested sequence of compact sets the intersection is non-empty. Moreover, since all of the maps T_i are contracting it is easy to see that the limit consists of a single point.

Lemma 2.1.2. *The limit set Λ agrees with the attractor defined above. In particular, it is independent of the choice of z.*

Proof. The set of limit points defined above is clearly mapped into itself by $T : \mathcal{X} \to \mathcal{X}$. Moreover, it is easy to see that it is fixed by T. Since Λ was the unique fixed point (by the contraction mapping theorem) this suffices to show that the two definitions of limit sets coincide. $\square$

This second point of view has the additional advantage that every point is coded by some infinite sequence. We can define a metric on the space of sequences $\{1, \ldots, k\}^{\mathbb{Z}^+}$ as follows. Given distinct sequences $\underline{x} = (x_n)_{n=0}^{\infty}, \underline{y} = (y_n)_{n=0}^{\infty} \in \{1, \ldots, k\}^{\mathbb{Z}^+}$ we denote

$$n(\underline{x}, \underline{y}) = \min\{n \geq 0 : x_i = y_i \text{ for } 0 \leq i \leq k, \text{ but } x_k \neq y_k\}.$$

We then define the metric by

$$d(\underline{x}, \underline{y}) = \begin{cases} 2^{-n(\underline{x},\underline{y})} & \text{if } \underline{x} \neq \underline{y} \\ 0 & \text{otherwise} \end{cases}$$

It is easy to check that this is a metric. We can define a continuous map $\pi : \{1, \ldots, k\}^{\mathbb{Z}^+} \to \mathbb{R}^d$ by

$$\pi(x) := \lim_{n \to +\infty} T_{x_0} \circ T_{x_1} \circ \ldots \circ T_{x_n}(z)$$

Lemma 2.1.3. *The map π is Hölder continuous (i.e., $\exists C > 0, \beta > 0$ such that $||\pi(\underline{x}) - \pi(\underline{y})|| \leq Cd(\underline{x}, \underline{y}))^{\beta}$ for any $\underline{x}, \underline{y}$.)*

Proof. By definition, if $d(\underline{x}, \underline{y}) = 2^{-n}$, say, then $\pi(\underline{x}), \pi(\underline{y}) \in T_{x_0} \circ \ldots \circ T_{x_n}(\Lambda)$. However,

$$\begin{aligned} ||\pi(\underline{x}) - \pi(\underline{y})|| &\leq \text{diam } (T_{x_0} \circ \ldots \circ T_{x_n}(\Lambda)) \\ &\leq \alpha^n \text{diam } (\Lambda) \\ &\leq (d(\underline{x}, \underline{y}))^{\beta} \text{diam } (\Lambda) \end{aligned}$$

where $\beta = \log \alpha / \log(1/2)$. $\square$

We shall assume for this chapter that $T_1, \ldots, T_k$ are conformal, i.e., the contraction is the same in each direction. Of course, for contractions on the line this is automatically satisfied, and is no restriction. In the one dimensional setting, such iterated function schemes are often called *cookie cutters*.

If we identify $\mathbb{R}^2$ with $\mathbb{C}$ then this naturally leads to simple and familiar examples of conformal maps.

Examples.

(1) Any linear fractional transformation $T : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ on the Riemann sphere $\widehat{\mathbb{C}}$ is conformal. Moreover, if $Tz = (az+b)/(cz+d)$ where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{C})$ then $T'(z) = 1/(cz + d)^2$. (More generally, Mobius tranformations $T : \mathbb{S}^d \to \mathbb{S}^d$ are conformal.)

(2) Any analytic function $T : U \to \mathbb{C}$, where $U \subset C$ is conformal. For example, we could consider T to be a rational map on a neighbourhood of U of the hyperbolic Julia set.

In addition, we shall also want to make the following assumption.

Definition. We say that a family of maps satisfies the *open set condition* if there exists an open set $U \subset \mathbb{R}^d$ such that the sets $T_1(U), \ldots, T_k(U)$ are all contained in U and are disjoint.

The next result shows that for conformal iterated function schemes, the Hausdorff dimension and Box dimension of the limit set actually coincide.

Proposition 2.1.4. *For conformal iterated function schemes satisfying the open set condition* $dim_B(\Lambda) = dim_H(\Lambda)$.

Proof. We need to show that $\dim_B(\Lambda) \leq \dim_H(\Lambda)$. This is down using the Mass Distribution Principle. Let us denote $d = \dim_B(\Lambda)$. In order to employ this method, we want to show that there is a probability measure μ on Λ and constants $C_1, C_2 > 0$ such that

$$\begin{aligned} C_1 \text{diam } (T_{x_0} \circ \ldots \circ T_{x_n}(\Lambda))^d &\leq \mu(T_{x_0} \circ \ldots \circ T_{x_n}(\Lambda)) \\ &\leq C_2 \text{diam } (T_{x_0} \circ \ldots \circ T_{x_n}(\Lambda))^d. \end{aligned}$$

In fact, the existence of such a measure is due to ideas from *Thermodynamic Formalism*, which we shall discuss later. In particular, if $x = \pi((x_n)_{n=0}^{\infty})$ then

$$\lim_{\epsilon \to 0} \frac{\log \mu(B(x,\epsilon))}{\log \epsilon} = \lim_{n \to +\infty} \frac{\log \mu(T_{x_0} \circ \ldots \circ T_{x_n}(\Lambda))}{\log \text{diam } (T_{x_0} \circ \ldots \circ T_{x_n}(\Lambda))} = d.$$

Thus by the Mass distribution principle we have that $\dim_B(\Lambda) \geq d = \dim_H(\Lambda)$. □

In particular, this applies to two of our favorite examples.

Corollary. *For hyperbolic Julia sets and Schottky group limit sets the Hausdorff dimension and the Box dimension coincide.*

We now turn the issue of calculating the dimension of limit sets. We begin with a special case, and then subsequently consider the more general case.

2.2 The case of similarities. We can make even stronger assumptions (and then relaxing them in the next section). We now want to consider a very special class of contractions. We say that $S : \mathbb{R}^d \to \mathbb{R}^d$ is a *similarity* if there exists $\alpha > 0$ such that

$$||S(x) - S(y)|| = \alpha ||x - y|| \text{ for all } x, y \in \mathbb{R}^d$$

This condition is even stronger than asking that the T_i are conformal. These correspond to the case that $T_i : \mathbb{R}^d \to \mathbb{R}^d$ are affine maps (i.e., a linear part $a_i \in GL(d, \mathbb{R})$, satisfying $||a_i|| < 1$, followed by a translation $b_i \in \mathbb{R}^d$ such that $T_i(x) = a_i x + b_i$).

Let us consider a class of iterated function schemes where it is easiest to find an expression for the dimension.

Definition. We say that the limit set $\Lambda = \Lambda(T_1, \ldots, T_k) \subset \mathbb{R}^d$ is *self-similar* if each of the maps T_i, $i = 1, \ldots, k$, are similarities.

Example 2.2.1. The middle third Cantor set is the limit set for $T_1, T_2 : \mathbb{R} \to \mathbb{R}$ defined by $T_1 x = \frac{x}{3}$ and $T_2 x = \frac{x}{3} + \frac{2}{3}$.

Example 2.2.2. The Sierpinski Carpet is the limit set for 8 (conformal) contractions $T_{i,j} : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T_{i,j}(x,y) = (\frac{x}{3}, \frac{y}{3}) + (\frac{i}{3}, \frac{j}{3})$, where $(i,j) \in \mathcal{S}$.

The following basic theorem can be attributed to Moran (1946).

Theorem 2.2.1. *If $T_1, \dots, T_k : \mathbb{R}^d \to \mathbb{R}^d$ are similarities satisfying the open set condition, then the dimension is the unique solution $s = dim_H(\Lambda)$ to the identity*

$$1 = (\alpha_1)^s + \dots + (\alpha_k)^s,$$

where $\alpha_i = ||a_i||$.

Proof. For simplicity, we consider the case of just two maps $T_1, T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ with limit set Λ. It is also convenient to write the two contractions as

$$\begin{cases} \lambda := & |\lambda_1| \\ \lambda^\alpha := & |\lambda_2|, \text{for some } 0 < \alpha < 1, \end{cases}$$

say. We can assume, for simplicity, that the open set in the open set condition is a ball $U = \{x \in \mathbb{R}^2 : ||x|| < r\}$.

Given $k > 1$ we can consider a cover for Λ by all balls of the form

$$T_{i_1} \dots T_{i_m} U \text{ where } M \text{ is chosen with} \frac{\lambda}{k} \le |\lambda_{i_1}| \dots |\lambda_{i_m}| \le \frac{1}{k} \tag{2.1}$$

Let M_k be the total number of such disks, and let $N_k = N(1/k)$.

It is easy to see that there are constants $C_1, C_2 > 0$ with $C_1 N_k \le M_k \le C_2 N_k$.

For example, we are considering

$$\begin{cases} \underbrace{T_1 T_1 \dots T_1}_{\times n} U \\ T_2 \underbrace{T_1 \dots T_1}_{\times (n-1)} U \\ T_1 T_2 \underbrace{T_1 \dots T_1}_{\times (n-2)} U \\ \dots \\ \underbrace{T_2 \dots T_2}_{\times [\alpha n]} \end{cases}$$

(where $[\alpha n]$ is the largest integer smaller than $[\alpha n]$).

If T_1 occurs $[(1-\beta)n]$ times, for some $0 < \beta < 1$, then for (2.1) to be satisfied we require that T_2 occurs approximately $[\beta \alpha n]$ times. Moreover, then number of contributions to the above list depends on their ordering, which is approximately $\binom{[(1-\beta+\alpha\beta)n]}{[\beta\alpha n]}$.

The total number M_k of disks satisfies:

$$\max_\beta \binom{[(1-\beta+\alpha\beta)n]}{[\beta\alpha n]} \le M_k \le n \left(\max_\beta \binom{[(1-\beta+\alpha\beta)n]}{[\beta\alpha n]} \right)$$

and to esimate this we need to maximize $\binom{[(1-\beta+\alpha\beta)n]}{[\beta\alpha n]}$ in β.

By Stirling's formula we know that $\log n! \sim n \log n$, as $n \to +\infty$. Thus

$$\begin{aligned} \log \binom{[(1-\beta+\alpha\beta)n]}{[\beta\alpha n]} &= \log \left(\frac{[(1-\beta+\alpha\beta)]!}{[\beta\alpha n]![(1-\beta]!} \right) \\ &\sim n\left((x+y)\log(x+y) - x\log x - y\log y\right) \end{aligned}$$

where $x = \alpha\beta$ and $y = (1-\beta)$. Writing $f(x,y) = (x+y)\log(x+y) - x\log x - y\log y$, we have a problem of maximizing this function subject to the condition $g(x,y) = x + \alpha y = \alpha$. Using a Lagrange multiplier γ this reduces to solving

$$\begin{aligned}\nabla f &= (\log(x+y) - \log x, \log(x+y) - \log y)\\ &= \gamma\nabla g = \gamma(1,\alpha)\end{aligned}$$

In particular, we get $\left(\frac{x}{x+y}\right)^{\alpha} = \left(\frac{x}{x+y}\right)$ and so setting $\lambda^d := \frac{x}{x+y}$ solves $\lambda^d + (\lambda^\alpha)^d = 1$. Thus

$$d = \lim_{k\to+\infty} \frac{\log N_k}{\log k} = \lim_{k\to+\infty} \frac{\log\left(\frac{1}{\lambda^d}\right)^k}{\log\left(\lambda^k\right)}$$

as required. □

Warning. Without the open set condition, things can go hideously wrong! (as we shall see later). Consider, as an example, the maps $T_i x = \lambda x + i$, for $i = 0, 1, 3$ and let Λ_λ be the limit set

(1) For almost all $1/4 < \lambda < 1/3$ we have that $\dim_H(\Lambda_\lambda) = \frac{\log 3}{\log(1/\lambda)}$ (as expected); However,
(2) For a dense set of values λ we have that $\dim_H(\Lambda_\lambda) < \frac{\log 3}{\log(1/\lambda)}$.

In particular, the dimension of the set Λ_λ is not continuous in λ. We shall return to this example later.

For the present, let us just see how Moran's theorem allows us to deduce the dimensions of the limits sets in three familiar simple examples.

Example 2.2.3. Consider the middle third Cantor set. We have $\alpha_1 = \alpha_2 = \frac{1}{3}$ and observe that

$$1 = \left(\frac{1}{3}\right)^{\frac{\log 2}{\log 3}} + \left(\frac{1}{3}\right)^{\frac{\log 2}{\log 3}}$$

thus $\dim_H(X) = \frac{\log 2}{\log 3}$.

Example 2.2.4. Consider the Sierpinski Carpet. Consider the eight contractions defined by

$$T_{(i,j)}(x,y) = \left(\frac{x+i}{3}, \frac{y+j}{3}\right)$$

where $0 \le i,j \le 2$, and $(i,j) \ne (1,1)$. We can then identify the Sierpinski gasket as the limit set $\Lambda = \Lambda(T_{(0,0)}, \cdots, T_{(2,2)})$. We have $\alpha_{ij} = \frac{1}{3}$ for $(i,j) \in \mathcal{S}$ and observe that

$$1 = \underbrace{\left(\frac{1}{3}\right)^{\frac{\log 8}{\log 3}} + \ldots + \left(\frac{1}{3}\right)^{\frac{\log 8}{\log 3}}}_{\times 8}$$

thus $\dim_H(X_S) = \frac{\log 8}{\log 3}$.

Example 2.2.5. Consider the Koch Curve (1904). We can consider four affine contractions

$$\begin{cases} T_1 : (x,y) \mapsto \left(\frac{x}{3}, \frac{y}{3}\right) \\ T_2 : (x,y) \mapsto \left(\frac{1}{3} + \frac{x}{6}, \frac{y}{2\sqrt{3}}\right) \\ T_3 : (x,y) \mapsto \left(\frac{1}{2} + \frac{x}{6}, \frac{1}{2\sqrt{3}} - \frac{y}{2\sqrt{3}}\right) \\ T_4 : (x,y) \mapsto \left(\frac{2}{3} + \frac{x}{3}, \frac{y}{3}\right). \end{cases}$$

Each branch contracts by $\frac{1}{3}$ the limit figure and observe that

$$1 = \left(\frac{1}{3}\right)^{\frac{\log 4}{\log 3}} + \left(\frac{1}{3}\right)^{\frac{\log 4}{\log 3}} + \left(\frac{1}{3}\right)^{\frac{\log 4}{\log 3}} + \left(\frac{1}{3}\right)^{\frac{\log 4}{\log 3}},$$

thus $\frac{\log 4}{\log 3} = 1.2619....$

The situation becomes interesting when we drop the assumption that the iterated function scheme is made up of similarities. (However, dropping the conformal assumption or the open set condition is, for the moment, something we prefer not even to contemplate!)

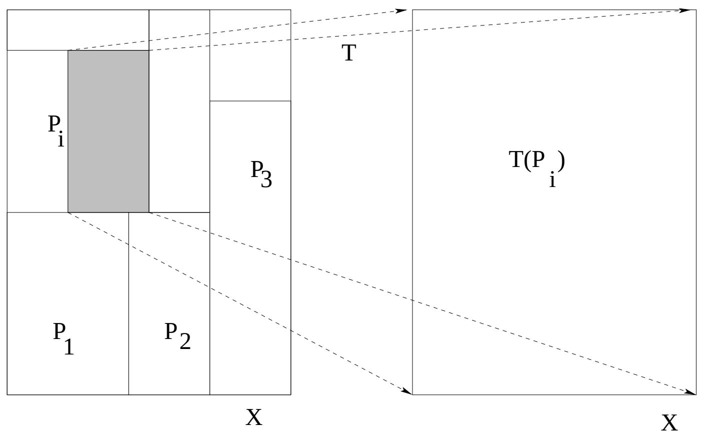
2.3 Expanding maps and conformal iterated function schemes. In many of our examples, the iterated function scheme arises from the inverse branches of an expanding map. Let $T : X \to X$ be a C^1 conformal expanding map (i.e., the derivative is the same in all directions and $|T'(x)| \geq \lambda > 1$) on a compact space.

Example 2.3.1. For the set $E_2 \subset [0,1]$ consisting of numbers whose continued fraction expansions contains only 1s or 2s, we can take $T : E_2 \to E_2$ to be $T(x) = \frac{1}{x} - [\frac{1}{x}]$. We can consider the local inverses $T_1 : [0,1] \to [0,1]$ and $T_2 : [0,1] \to [0,1]$ defined by $T_1(x) = 1/(1+x)$ and $T_1(x) = 1/(2+x)$. We can then view E_2 as the limit set $\Lambda = \Lambda(T_1, T_2)$.

More generally, to associate an iterated function scheme, we want to introduce the idea of a Markov Partition. The contractions in an associated iterated function scheme will then essentially be the inverse branches to the expanding maps. Let $T : X \to X$ be a $C^{1+\alpha}$ locally expanding map on $X \subset \mathbb{R}^d$.

Definition. We call a finite collection of closed subsets $\mathcal{P} = \{P_i\}_{i=1}^k$ a *Markov Partition* if it satisfies the following:

1. Their union is X (i.e., $\cup_{i=1}^k P_i = X$);
2. The sets are proper (i.e., each P_i is the closure of their interiors, relative to X);
3. Each image TP_i, for $i = 1, \dots, k$, is the union of finitely many elements from $\mathcal{P}$ and $T : P_i \to TP_i$ is a local homeomorphism.



The set X is partitioned into pieces $P_1, \dots, P_k$ each of which is mapped under T onto X.

In many examples we consider, each image $TP_i = X$, for $i = 1, \dots, k$, in condition (iii). (Such partitions might more appropriately be called Bernoulli Partitions.)

We shall want to make use of the following standard result.

Proposition 2.3.1. *For $T : X \to X$ a $C^{1+\alpha}$ locally expanding map, there exists a Markov Partition.*

The proof of this result will be outlined in a later Appendix.

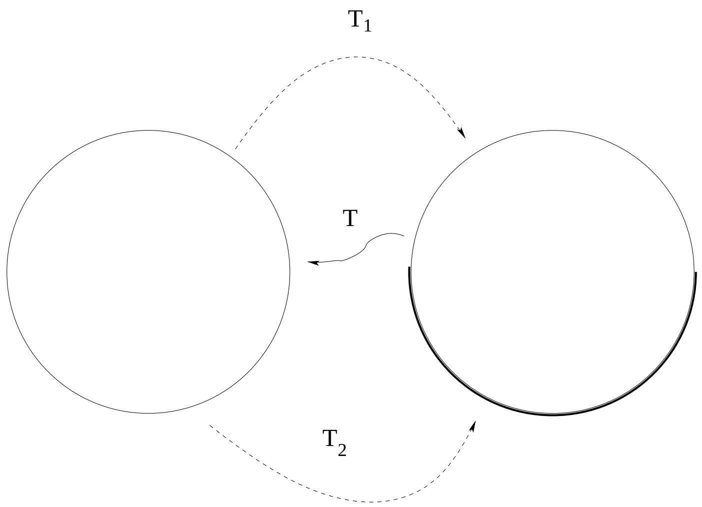
The usefulness of this result is that we can now consider the local inverses $T_i : TP_i \to P_i$, i.e., $T \circ T_i(x) = x$ for $x \in TP_i$, (extended to suitable open neighbourhoods) to be an iterated function scheme for which X is the associated limit set.

Example 2.3.2. Hyperbolic Julia sets. Let $T : J \to J$ be a linear fractional transformation on the Julia set. Assume that the transformation $T : J \to J$ is hyperbolic (i.e., $\exists C > 0, \lambda > 1$ such that $|(T^n)'(x)| \geq C\lambda^n$, for all $x \in J$ and $n \geq 1$). Then Proposition 2.3.1 applies to give a Markov partition.

If we consider the particular case of a quadratic map $Tz = z^2 + c$, with $|c|$ small then we can define the local inverses by

$$T_1(z) = +\sqrt{z-c} \text{ and } T_2(z) = -\sqrt{z-c}$$

Of course, in order for these maps we well defined, we need to define them on domains carefully chosen relative to the cut locus.



The Julia set J_c for $T(z) = z^2 + c$ has two pieces P_1, P_2: The "northern hemisphere" and the "southern hemisphere". The local inverses $T_1 : J_c \to P_1$ and $T_1 : J_c \to P_2$ form an iterated function scheme.

Example 2.3.3. Limit sets for Kleinian groups. We will mainly be concerned with the special case of Schottky groups. In this case, we have $2n$ pairs of disjoint disks $D_i^+, D_i{}^-$, with $0 \leq i \leq n$, whose boundaries are the isometric circles associated to the generators $g_1, \dots, g_n$ (and there inverses). In particular, we can define $T : \Lambda \to \Lambda$ by

$$T(z) = \begin{cases} g_i(z) & \text{if } z \in D_i^+ \\ g_i^{-1}(z) & \text{if } z \in D_i^- \end{cases}$$

If all of the closed disks are disjoint then $T : \Lambda \to \Lambda$ is expanding.

We now want to state the generalization of Moran's theorem to the nonlinear setting . The main ingredient that we require if the following:

Definition. Given any continuous function $f : X \to \mathbb{R}$ we define its *pressure* $P(f)$ (with respect to T) as

$$P(f) := \limsup_{n \to +\infty} \frac{1}{n} \log \underbrace{\left(\sum_{\substack{T^n x = x \\ x \in X}} e^{f(x)+f(Tx)+\ldots+f(T^{n-1}x)} \right)}_{\text{Sum over periodic points}}$$

(As we shall presently see, the limit actually exists and so the "lim sup" can actually be replaced by a "lim".) In practise, we shall mainly be interested in a family of functions $f_t(x) = -t \log |T'(x)|$, $x \in X$ and $0 \le t \le d$, so that the above function reduces to

$$[0, d] \to \mathbb{R}$$

$$t \mapsto P(f_t) = \limsup_{n \to +\infty} \frac{1}{n} \log \left(\sum_{\substack{T^n x = x \\ x \in X}} \frac{1}{|(T^n)'(x)|^t} \right)$$

The following standard result is essentially due Bowen and Ruelle. Bowen showed the result in the context of quasi-circles and Ruelle developed the method for the case of hyperbolic Julia sets.

Theorem 2.3.2 (Bowen-Ruelle). *Let $T : X \to X$ be a $C^{1+\alpha}$ conformal expanding map. There is a unique solution $0 \le s \le d$ to*

$$P(-s \log |T'|) = 0,$$

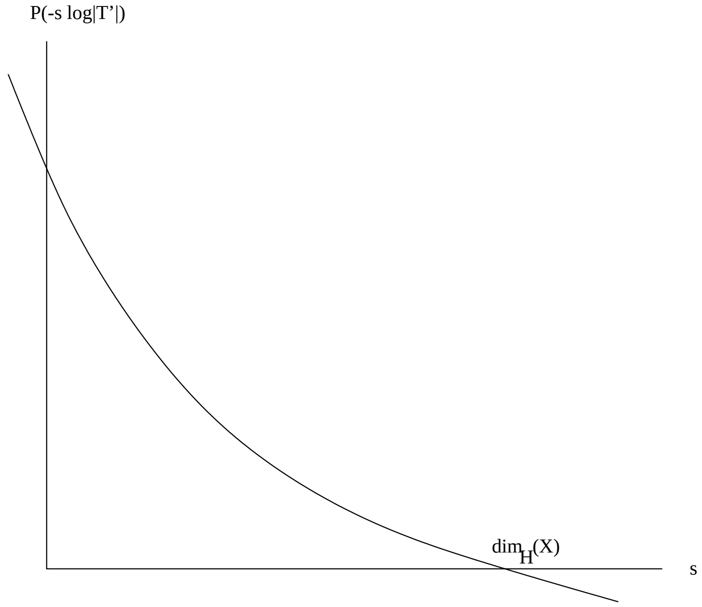
which occurs precisely at $s = dim_H(X) (= dim_B(X))$.

Proof. We shall explain the main ideas in the proof in the next section. □

Reduction to the case of linear contractions. In the case of linear iterated functions schemes this reduces to Moran's theorem. Let us assume that $T_i = a_i x + d_i$ then we can write

$$\begin{aligned} \sum_{\substack{T^n x = x \\ x \in X}} \frac{1}{|(T^n)'(x)|^t} &= \sum_{i_1, \cdots, i_n} \frac{1}{|a_{i_1}|^t \cdots |a_{i_k}|^t} \\ &= \left(\frac{1}{|a_1|^t} + \cdots + \frac{1}{|a_n|^t} \right)^n \end{aligned}$$

In particular, since one readily sees that this expression is monotone decreasing as a function of t we see from the definitions that the value s such that $P(-s \log |T'|) = 0$ is precisely the same as that for which $1 = \frac{1}{|a_1|^s} + \cdots + \frac{1}{|a_k|^s}$, i.e., the value given by Moran's Theorem.



A plot of pressure gives that $\dim_H(X)$ from the graph.

Finally, we observe that the function $t \mapsto P(f_t)$ has the following interesting proprties

(i) $P(0) = \log k$;
(ii) $t \mapsto P(f_t)$ is strictly monotone decreasing;
(iii) $t \mapsto P(f_t)$ is analytic on $[0, d]$.

Property (i) is immediate from the definition. We shall return to the proofs of properties (ii) and (iii) later. For the present, we can interpret analytic to mean having a convergent power series in a sufficiently small neighbourhood of each point.

One particularly nice application of the above theorem and properties of pressure is to showing the analyticity of dimension as we change the associated expanding map. More precisely:

Corollary. *Let T_λ, with $-\epsilon \leq \lambda \leq \epsilon$, be an analytic family of expanding maps. Then $\lambda \mapsto dim_H(\Lambda_\lambda)$ is analytic.*

Proof. The function $f(\lambda, t) = P(-t \log |T'_\lambda|)$ is analytic and satisfies $\frac{\partial f}{\partial \lambda}(\lambda, t) \neq 0$. Using the Implicit Function Theorem, we can often deduce that for an analytic family T_λ the dimension $\lambda \mapsto \dim(\Lambda_\lambda)$ is analytic too. $\square$

This applies, in particular, to the examples of hyperbolic Julia sets and limit sets for Schottky groups.

Example. Quadratic maps. The map $T_c(z) = z^2 + c$ has a hyperbolic Julia set J_c provided $|c|$ is sufficiently small. Ruelle used the above method to show that $c \mapsto \dim(J_c)$ is analytic for $|c|$ sufficiently small. (He also gave the first few terms in the expansion for $\dim(J_c)$, as given in the previous chapter).

In the next section we explain the details of the proof of Theorem 2.3.2.

2.4 Proving the Bowen-Ruelle result. Let $T : X \to X$ be a map on $X \subset \mathbb{R}^d$. By an expanding map we mean one which locally expands distances. In the present context we can assume that there exists $C > 0$ and $\lambda > 1$ such that

$$||D_x T^n(v)|| \geq C\lambda^n ||v||, \text{ for } n \geq 1.$$

The hypothesis that T is $C^{1+\alpha}$ means that the derivative DT is α-Hölder continuous, i.e.,

$$||DT||_\alpha := \sup_{x \neq y} \frac{||D_xT - D_yT||}{||x-y||} < +\infty.$$

Here the norm in the numerator on the Right Hand Side is the norm on linear maps from $\mathbb{R}^d$ to itself (or equivalently, on $d \times d$ matrices).

Let $T : X \to X$ be a $C^{1+\alpha}$ locally expanding map on $X \subset \mathbb{R}^d$. Consider a Markov Partition $\mathcal{P} = \{P_i\}_{i=1}^k$ for T. If we write $T_i : X \to P_i$ for the local inverses then this describes an iterated function scheme. For each $n \geq 1$ we want to consider n-tuples $\underline{i} = (i_1, \ldots, i_n) \in \{1, \ldots, k\}^n$. We shall assume that $TP_{i_r} \supset P_{i_{r-1}}$, for $r = 2, \ldots, n$. It is then an easy observation that

$$P_{\underline{i}} := T_{i_n} \cdots T_{i_2} P_{i_1}$$

is again a non-empty closed subset, and the union of such sets is equal to X.

We would like to estimate the dimension of X by making a cover using the sets $P_{\underline{i}}$, $|\underline{i}| = n$. A slight technical difficulty is that these sets are closed, rather than open. Moreover, if we try to use their interiors we see that they might not cover X. The solution is rather easy: we simply make a cover by choosing open neighbourhoods $U_{\underline{i}} \supset P_{\underline{i}}$ which are slightly larger, and thus do form a cover for X. Let us assume that there is $0 < \theta < 1$ such that

$$\frac{\mathrm{diam}(U_{\underline{i}})}{\mathrm{diam}(P_{\underline{i}})} \leq 1 + O(\theta^n), \text{ for all } \underline{i}.$$

Let us define $T_{\underline{i}} : P_{i_1} \to P_{\underline{i}}$ by $T_{\underline{i}} = T_{i_1} \circ \cdots \circ T_{i_n}$.

We can now obtain the following bounds.

Proposition 2.4.1.

(1) *There exist $B_1, B_2 > 0$ such that for all $\underline{i}$ and all $x, y \in X$:*

$$B_1 \leq \frac{|T'_{\underline{i}}(x)|}{|T'_{\underline{i}}(y)|} \leq B_2$$

(2) *There exist $C_1, C_2 > 0$ such that for all $\underline{i}$ and for all $x \in X$:*

$$C_1 \leq \frac{\mathrm{diam}(P_{\underline{i}})}{|T'_{\underline{i}}(x)|} \leq C_2.$$

In particular, for $t > 0$, there exist $C_1, C_2 > 0$ such that for any x and $n \geq 1$:

$$C_1 \leq \frac{\sum_{|\underline{i}|=n} \mathrm{diam}(U_{\underline{i}})^t}{\sum_{|\underline{i}|=n} |(T_{\underline{i}})'(x)|^t} \leq C_2$$

Proof. Part (1) is sometimes referred to as a telescope lemma. If $D = \sup_i || \log |T'_i|||_\alpha$ and $\theta = \sup_i ||T'_i||_\infty < 1$:

$$\begin{aligned}
|\log |T'_{\underline{i}}(x)| - \log |T'_{\underline{i}}(y)|| &= \sum_{j=1}^n \left| \log |T'_{i_j}(T_{i_{j+1}} \cdots T_{i_n} x)| - \log |T'_{i_j}(T_{i_{j+1}} \cdots T_{i_n} y)| \right| \\
&\leq D \sum_{j=1}^n d(T_{i_{j+1}} \cdots T_{i_n} x, T_{i_{j+1}} \cdots T_{i_n} y)^\alpha \\
&\leq D \sum_{j=1}^n \theta^{n\alpha} d(x,y)^\alpha \leq \left(\frac{D}{1-\theta^\alpha} \right) d(x,y)^\alpha
\end{aligned}$$

This uses the Chain Rule and Holder continuity. In particular, setting $C = \frac{D}{1-\theta^\alpha} > 0$ we have that for and $x, y \in X$ and all $n \geq 1$ and $|\underline{i}| = n$ with $i_1 = i$:

$$\Big| \log |T_{\underline{i}}'(x)| - \log |T_{\underline{i}}'(y)| \Big| \leq C d(x,y)^\alpha.$$

In particular, part (1) follows since:

$$\underbrace{e^{-C\mathrm{diam}(X)^\alpha}}_{=:B_1} \leq \frac{|T_{\underline{i}}'(x)|}{|T_{\underline{i}}'(y)|} = \exp\Big(\log |T_{\underline{i}}'(x)| - \log |T_{\underline{i}}'(y)|\Big) \leq \underbrace{e^{C\mathrm{diam}(X)^\alpha}}_{=:B_2}.$$

Since the contractions are conformal we can estimate

$$B_1 |T_{\underline{i}}'(x)| \leq \text{diam } (P_{\underline{i}}) \leq B_2 |T_{\underline{i}}'(x)|.$$

This suffices to deduce Part (2). $\square$

It is not surprising that the part of the approach to proving the Bowen-Ruelle result involves understanding the asymptotics of the expression $\sum_{|\underline{i}|=n} \mathrm{diam}(U_{\underline{i}})^d$ as $n \to \infty$, since this is intimately related to definition involving covers of the Hausdorff dimension of X. Moreover, the last Propostion tells us that it is an equivalent problem to understand the behaviour of $\sum_{|\underline{i}|=n} |(T_{\underline{i}})'(x)|$. Perhaps, at first sight, this doesn't seem to be an improvement. However, the key idea is to introduce a transfer operator.

Definition. Let $C^\alpha(\mathcal{P})$ be the space of Hölder continuous functions on the disjoint union of the sets in $\mathcal{P}$. This is a Banach space with the norm $||f|| = ||f||_\infty + ||f||_\alpha$ where

$$||f||_\infty = \sup_{x \in X} |f(x)| \text{ and } ||f||_\alpha = \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x,y)^\alpha}.$$

For each $t > 0$ we define a bounded linear operator $\mathcal{L}_t : C^\alpha(\mathcal{P}) \to C^\alpha(\mathcal{P})$ by

$$\mathcal{L}_t w(x) = \sum_i |T_i'(x)|^t w(T_i x).$$

To understand the role played by the transfer operator, we need only observe that iterates of the operator applied to the constant function 1 take the required form: for $x \in X$

$$\mathcal{L}_t^n 1(x) = \sum_{|\underline{i}|=n} |(T_{\underline{i}})'(x)|^t,$$

i.e., the numerator in the last line of Proposition 2.4.1 (2). In particular, to understand what happens as n tends to infinity is now reduced to the behaviour of the operator $\mathcal{L}_t$.

Proposition 2.4.2 (Ruelle Operator Theorem).

(1) *The operator $\mathcal{L}_t$ has a simple maximal positive eigenvalue λ_t. Moreover the rest of the spectrum is contained in a disk of strictly smaller radius, i.e., we can choose $0 < \theta < 1$ and $C > 0$ such that $|\mathcal{L}_t^n 1 - \lambda_t^n| \leq C \lambda_t^n \theta^n$, for $n \geq 1$.*

(2) *There exists a probability measure* μ *and* $D_1, D_2 > 0$ *such that for any* $n \geq 1$ *and* $|\underline{i}| = n$ *and* $x \in X$*:*

$$D_1 \lambda_t^n \leq \frac{\mu(P_{\underline{i}})}{|T_{\underline{i}}'(x)|^t} \leq D_2 \lambda_t^n.$$

(3) *The map* $\lambda : \mathbb{R} \to \mathbb{R}$ *given by* $\lambda(t) = \lambda_t$ *is real analytic and* $\lambda'(t) < 0$ *for all* $t \in \mathbb{R}$*.*

We shall return to the proof of this result later. However, for the present we have an immediate corollary.

Corollary. *We can write* $P(-t \log |T'|) = \log \lambda_t$*.*

Proof. For each $|\underline{i}| = n$ we can choose a periodic point $T^n x = x$ such that By Proposition 2.4.1 (1), if we let $C_1 = B_1^t, C_2 = B_2^t > 0$ then for any $x_0 \in X$ we have $C_1 |(T^n)'(x_0)|^{-t} \leq |(T^n)'(x)|^{-t} \leq C_2 |(T^n)'(x_0)|^{-t}$. Summing over all possible $|\underline{i}| = n$ we have that:

$$C_1(\mathcal{L}_t^n 1)(x_0) \leq \sum_{T^n x = x} |(T^n)'(x)|^{-t} \leq C_2(\mathcal{L}_t^n 1)(x_0). \tag{2.2}$$

The result then follows from the definition of pressure and part (2) of Proposition 2.4.2. $\square$

In particular, properties (ii) and (iii) follow from this corollary.

By Part (2) of Proposition 2.4.1 and (2.2) we see that for some $D_1, D_2 > 0$ and $0 \leq t \leq n$:

$$D_1 \lambda_t^n \leq \sum_{|\underline{i}|=n} \mathrm{diam}(U_{\underline{i}})^t \leq D_2 \lambda_t^n, \text{ for } n \geq 1.$$

Recalling the definition of Hausdorff dimension we can bound

$$H_\epsilon^t(X) = \inf_{\mathcal{U}} \left\{ \sum_{U_i \in \mathcal{U}} \mathrm{diam}\ (U_i)^t \right\} \leq \sum_{|\underline{i}|=n} \mathrm{diam}(U_{\underline{i}})^t \leq D_2 \lambda_t^n,$$

where the infimum is over open covers $\mathcal{U}$ whose elements have diameter at most $\epsilon > 0$, say, and n is chosen such that $\epsilon = \max_{|\underline{i}|=n}\{\mathrm{diam}(U_{\underline{i}})\}$. We can therefore deduce that if $t > d$ then $\lambda_t < 1$ and thus $\lim_{\epsilon \to 0} H_\epsilon^t(X) = 0$. In particular, from the definition of Hausdorff dimension we see that $\mathrm{diam}_H(X) \leq d$.

To obtain the lower bound for $\dim_H(X)$ we can use the mass distribution principle with the measure μ. In particular, for any $|\underline{i}| = n$ and $x \in X$ we can estimate

$$\begin{aligned} \mu(P_{\underline{i}}) = \int (\mathcal{L}_t^n \chi_{P_{\underline{i}}}) d\mu &\leq D_2 \lambda_d^n |T_{\underline{i}}'(x)|^d \\ &\leq D_2 C_1^{-1} \lambda_d^n (\mathrm{diam}(P_{\underline{i}}))^d \end{aligned}$$

Given any $x \in X$ and any $\epsilon > 0$ we can choose n so that we can cover the ball $B(x, \epsilon)$ by a uniformly bounded number of sets $P_{\underline{i}}$ with $|\underline{i}| = n$.

In particular, since $\lambda_d = 1$ we can deduce that there exists $C > 0$ such that $\mu(B(x, \epsilon)) \leq C\epsilon^d$ for $\epsilon > 0$. Thus, by the mass distribution we dedude that $\dim_H(X) \geq d$.

This completes the proof of the Bowen-Ruelle Theorem (except for the proof of Proposition 2.4.2). It remains to prove Proposition 2.4.2

Proof of Proposition 2.4.2. Fix $C > 0$. We can consider the cone of functions

$$\mathcal{C} = \{f : C \to \mathbb{R} : 0 \leq f(x) \leq 1 \text{ and } f(x) \leq f(x)e^{C||x-y||^\alpha}, \forall x, y \in X\}.$$

It is easy to see that $\mathcal{C}$ is convex and closed with respect to the norm $||\cdot||_\infty$.

If $g \in \mathcal{C}$ then for $x \neq y$ we have that

$$\begin{aligned}|g(x) - g(y)| &\leq |g(y)| \left(\exp\left(C||f||_\alpha ||x-y||^\alpha\right) - 1\right)\\ &\leq ||g||_\infty C||f||_\alpha \exp\left(C||f||_\alpha\right) ||x-y||^\alpha,\end{aligned}$$

from which we deduce that $\mathcal{C}$ is uniformly continuous in the $||\cdot||_\infty$ norm, and thus compact by the Arzela-Ascoli theorem.

Given $n \geq 1$ we can define $L_n(g) = \mathcal{L}(g+1/n)/||\mathcal{L}(g+1/n)||$. Since the operator $\mathcal{L}$ is positive, the numerator is non-zero and thus the operator L_n is well defined. Moreover, providing C is sufficiently large we have that

$$L_n f(x) \leq L_n f(x) e^{C||x-y||^\alpha}$$

from which we can easily deduce that $L_n(\mathcal{C}) \subset \mathcal{C}$. Using the Schauder-Tychanoff Theorem there is a fixed point $\mathcal{L}_n g_n = g_n \in \mathcal{C}$, i.e.,

$$\mathcal{L}(g_n + 1/n) = ||\mathcal{L}(g_n + 1/n)||(g_n + 1/n). \tag{2.3}$$

Finally, we can again use that $\mathcal{C}$ is compact in the $||\cdot||_\infty$ norm to choose a limit point $h \in \mathcal{C}$ of $\{h\}_{n=1}^\infty$. Taking limits in (2.3) we get $\mathcal{L}_t h = \lambda_t h$, where $\lambda_t = ||\mathcal{L}_t h||_\infty$.

Next observe that $\mathcal{L}_t(h+1/n)(x) \geq \inf\{(h_n(x) + 1/n)e^{-||f||_\infty}\}$ and so $||\mathcal{L}_t(h + 1/n)||_\infty \geq e^{-||f||_\infty}$. Taking the limit we see that $\lambda_t \geq e^{-||f||_\infty} > 0$. To show that $h > 0$, assume for a contradiction that $h(x_0) = 0$. Then since $\mathcal{L}_t^n h(x_0) = \sum_{|\underline{i}|=n} \lambda_t^n |T_{\underline{i}}'(x_0)| h(T_{\underline{i}} x_0)$ we conclude that $h(T_{\underline{i}} x_0) = 0$ for all $|\underline{i}| = n$ and all $n \geq 1$. In particular, $h(x)$ is zero on a dense set, but then it must be identically zero contradicting $\lambda_t = ||\mathcal{L}_t h||_\infty > 0$. To see that λ_t is a simple eigenvalue, observe that if we have a second eigenvector g with $\mathcal{L}_t g = \lambda_t g$ and we let $t = \inf\{g(x)/h(x)\} = g(x_0)/h(x_0)$ then $g(x) - th(x) \geq 0$, but with $g(x_0) - th(x_0) = 0$. Since $g - th$ is again a positive eigenvector for $\mathcal{L}_t$, the preceding argument shows that $g - th = 0$, i.e., g is a multiple of h.

Let us define a new operator $\mathcal{M}_t w(x) = \lambda_t^{-1} w(x)^{-1} \sum_i |T_i'(x)|^t h_t(T_i x) w(T_i)$. By defintion, we have that $\mathcal{M}_t 1 = 1$, i.e., $\mathcal{M}_t$ preserves the constants. Let $\mathcal{M}$ be the space of probability measures on X. The space $\mathcal{M}$ is convex and compact in the weak star topology, by Alaoglu's theorem. Since $\mathcal{M}_t : \mathcal{M} \to \mathcal{M}$ we see by the Schauder-Tychanof theorem that $\mathcal{M}_t \mu = \mu$, or equivalently, $\mathcal{L}_t \nu = \lambda_t \nu$, where $\nu = h\mu$, i.e.,

$$\int (\mathcal{L}_t w)(x) d\nu(x) = \lambda_t \int w(x) d\mu(x) \tag{2.4}$$

for all $w \in C(X)$. We can consider the characteristic function $\chi_{P_{\underline{i}}}$ and then

$$\mu(P_{\underline{i}}) = \int \chi_{P_{\underline{i}}} d\mu_t = \lambda_t^{-n} \int \mathcal{L}_t^n \chi_{P_{\underline{i}}} d\mu_t = \lambda_t^{-n} \int |(T_{\underline{i}})'(y)| d\mu_t(y)$$

However, by Proposition 2.4.1 (1) we can bound

$$B_1 B_2^{-1} |(T_{\underline{i}})'(x)| \leq \int |(T_{\underline{i}})'(y)| d\mu_t(y) \leq B_2 B_1^{-1} |(T_{\underline{i}})'(x)|$$

for all $x \in X$. Thus Part (2) of Proposition 2.4.2 follows.

It is a simple calculation to show that there exists $C > 0$ such that

$$||\mathcal{M}_t^n h||_\alpha \leq C||h||_\infty + \alpha^n ||h||_\alpha, \text{ for } n \geq 1. \tag{2.5}$$

We first claim that $\mathcal{M}_t^n h \to \int g d\mu$ in the $||\cdot||_\infty$ topology. To see this we first observe from (2.5) that the family $\{\mathcal{M}_t^n h\}_{n=1}^\infty$ is equicontinuous. We can then choose a limit point $\overline{h}$. In particular, since $\mathcal{M}_t 1 = 1$ we see that $\sup \overline{h} \geq \sup \mathcal{M}_t \overline{h} \geq \cdots \geq \sup \mathcal{M}_t^n \overline{h} \to \sup \overline{h}$, from which we deduce $\sup \mathcal{M}_t^n \overline{h} = \sup \overline{h} = \overline{h}(x)$, say, for all $n \geq 1$. In particular, $\overline{h}(T_{\underline{i}} x) = \overline{h}(x)$ for all $|\underline{i}| = n$ and $n \geq 1$ and so $\overline{h}$ is a constant function. We can denote by $\mathbb{C}^\perp$ the functions $h \in C^\alpha(X)$ which satisfy $\int h d\mu = 0$. To show that the rest of the spectrum is in a disk of smaller radius we shall apply the spectral radius theorem to $\mathcal{M}_t : \mathbb{C}^\perp \to \mathbb{C}^\perp$ to show that its spectrum is strictly within the unit disk. (The spectra of $\mathcal{M}_t$ and $\mathcal{L}_t$ agree up to scaling by β_t). For $h \in \mathbb{C}^\perp$ the convergence result becomes $||\mathcal{M}_t^n h||_\infty \to 0$. By applying (2.5) twice we can estimate:

$$\begin{aligned} ||\mathcal{M}_t^{2n} h||_\alpha &\leq C||\mathcal{M}_t^n h||_\infty + \alpha^n ||\mathcal{M}_t^n h||_\alpha \\ &\leq C||\mathcal{M}_t^n h||_\infty + \alpha^n (C||h||_\infty + ||h||_\alpha) \\ &\to 0 \text{ as } n \to +\infty. \end{aligned}$$

In particular, for n sufficiently large we see that $||\mathcal{M}_t^{2n} h||_\alpha < 1$ and so the result on the spectrum follows.

For the final part, we observe that since λ_t is a simple isolated eigenvalue it follows by perturbation theory that it has an analytic dependence on t (as does its associated eigenfunction h_t, say). To show that λ_t is monotone decreasing we consider its derivative. Differentiating $\mathcal{L}_t h_t = \lambda_t h_t$ we can write

$$\lambda_t' h_t + \lambda_t h_t' = \mathcal{L}_t h_t' + \mathcal{L}_t (\log |T'| h_t)$$

Integrating with respect to μ_t and applying (2.4) we can cancel two of the terms to get $\lambda_t' \int h_t d\mu_t = \int \log |T_t'| h_t d\mu_t$ $\square$

3. Computing Hausdorff dimension

We now come to one of the main themes we want to discuss: *How can one compute the Hausdorff Dimension of a set?*

3.1 Algorithms. In some of the simpler examples, particularly those constructed by affine maps, it was possible to give explicit formulae for the Hausdorff dimension. In this chapter we shall consider more general cases. Typically, it is not possible to give a simple closed form for the dimension and it is necessary to resort to algorithms to compute the dimension as efficiently as we can. The original definition of Haudorff Dimension isn't particularly convenient for computation in the type of examples we have been discussing. However, the use of pressure for interated function schemes provides a much more promising approach.

We shall describe a couple of different variations on this idea. The main hypotheses on the compact X is that there exists a transformation $T : X \to X$ such that:

(1) *Markov dynamics:* There is a Markov partition (to help describe the local inverses as an interted function scheme);
(2) *Hyperbolicity:* There exists some $\lambda > 1$ such that $|T'(x)| \geq \lambda$ for all $x \in X$;
(3) *Conformality:* T is a conformal map;
(4) *Local maximality:* For any sufficiently small open neighbourhood U of the invariant set X we have $X = \cap_{n=0}^{\infty} T^{-n}U$ (such an X is sometimes called a *repeller*).

Our two main examples are the following:

Example 3.1.1. Consider a hyperbolic rational map $T : \widehat{C} \to \widehat{C}$ of degree $d \geq 2$ and let J be the Julia set. This satisfies the hypotheses (1)-(4). We let U be a sufficiently small neighbourhood of J.

Using the Markov partitions can write $J = \cup_{i=1}^{k} J_i$ and inverse branches $T_i : J \to J_i$ such that $T \circ T_i(z)$ and $i = 1, \ldots, k$ for all $z \in J_i$. J is the limit set for this iterated function schemes.

Example 3.1.2. Consider a Schottky group $\Gamma = \langle g_1, \cdots, g_n, g_{n+1} = g_1^{-1}, \cdots, g_{2n} = g_n^{-1} \rangle$ and let Λ be the limit set. We let $U = \cup_{i=1}^{2n} U_i$ be the union of the disjoint open sets $U_i = \{z \in \ : |g_i'(z)| > 1\}$ of isometric circles. We define $T : \Lambda \to \Lambda$ by $T(z) = g_i(z)$, for $z \in U_i \cap \Lambda$ and $i = 1, \ldots, 2n$. This satisfies the hypotheses (1)-(4).

We can define inverse branches $T_i : g_i(U_i \cap \Lambda) \to U_i \cap \Lambda$ such that $T \circ T_i(z)$ and $i = 1, \ldots, n$ for all $z \in U_i \cap \Lambda$. The limit set Λ is the same as that given by the iterated function scheme.

We now describe three different approaches to estimating Hausdorff dimension.

A first approach: Using the definition of pressure. The most direct approach is to try to estimate the pressure directly from its definition, and thus the dimension from the last chapter.

Lemma 3.1. *For each $n \geq 1$ we can choose s_n to be the unique solution to*

$$\frac{1}{n} \log \left(\sum_{T^n x = x} |(T^n)'(x)|^{-s_n} \right) = 1.$$

Then $s_n = \dim_H(X) + O\left(\frac{1}{n}\right)$.

Proof. Fix a point x_0. There exists $C > 0$, we can associate to each preimage $y \in T^{-n}x_0$ a periodic point $T^n x = x$ with $|(T^n)'(y)|/|(T^n)'(x)| \leq C$ (in the last chapter). We can estimate

$$e^{-Cs} \sum_{T^n y = x_0} |(T^n)'(y)|^{-s} \leq \sum_{T^n x = x} |(T^n)'(x)|^{-s} \leq e^{Cs} \sum_{T^n y = x_0} |(T^n)'(y)|^{-s}$$

We can identify

$$\mathcal{L}_s 1^n(x) = \sum_{T^n y = x} |(T^n)'(y)|^{-s}. \tag{3.1}$$

Recall that the Ruelle operator theorem allows us to write that $\mathcal{L}_s^n 1(x) = \lambda_s^n (1 + o(1))$, where $s > 0$, and thus

$$\log \lambda_s = \frac{1}{n} \log \left(\sum_{T^n x = x} |(T^n)'(x)|^{-s} \right) + O\left(\frac{1}{n}\right).$$

We can deduce the result from the the Bowen-Ruelle Theorem (since the derivative of $\log \lambda_s$ is non-zero). $\square$

In particular, in order to get an estimate with error of size $\epsilon > 0$, say, one expects to need the information on periodic points of period approximately $1/\epsilon$. This does not suggest itself as a very promising approach for very accurate approximations, since the number of periodic points we need to consider grows exponentially quickly with $n \asymp \frac{1}{\epsilon}$.

A second approach: Using the transfer operator. McMullen observed that working with the transfer operator one can quite effectively compute the pressure and the dimension. In practise, the numerical competition uses the approximation of the operator by matrices. Some of the flavour is given by the following statement.

Proposition 3.2. *Given $x \in X$, and then for each $n \geq 1$ we can choose s_n to be the unique solution to $\sum_{T^n y = x} |(T^n)'(y)|^{-s_n} = 1$. Then $s_n = \dim_H(X) + O(\theta^n)$, for some $0 < \theta < 1$.*

Proof. We begin from the identity (3.1). The stronger form of the Ruelle operator theorem means we can write that $\mathcal{L}_s^n 1(x) = \lambda_s^n (1 + O(\alpha^n))$ where $0 < \alpha < 1$. The derivative $\frac{1}{\lambda_s}\frac{\partial \lambda_s}{\partial s}$ of $\log \lambda_s$ can be seen to be non-zero, and so we can deduce the result from the Bowen-Ruelle Theorem. $\square$

For many practical purposes, this gives a pretty accurate approximation to the Hausdorff dimension of X. However, we now turn to the main method we want to discuss.

A third approach: Using determinants. Finally, we want to consider an approach based on determinants of transfer operators. The advantage of this approach is that it gives very fast, super-exponential, convergence to the Hausdorff dimension of the compact set X. This is based on the map $T : X \to X$ satisfying the additional assumption:

(5) *Analyticity:* T is real-analytic.

We need to introduce some notation.

Definition. Let us define a sequence of real numbers

$$a_n = \frac{1}{n} \sum_{|\underline{i}|=n} \frac{|T_{\underline{i}}(z_{\underline{i}})|^{-s}}{\det\left(I - \left[T_{\underline{i}}(z_{\underline{i}})\right]^{-1}\right)}, \text{ for } n \geq 1,$$

where the summation is over all n-strings of contractions, $T'_{\underline{i}}(z_{\underline{i}})$ denotes the derivative of $T_{\underline{i}}$ at the fixed point $z_{\underline{i}} = T_{\underline{i}}(z_{\underline{i}})$, and $|T'_{\underline{i}}(z_{\underline{i}})|$ denotes the modulus of the derivative. Next we define a sequence of functions by

$$\Delta_N(s) = 1 + \sum_{n=1}^{N} \sum_{\substack{(n_1,\dots,n_m) \\ n_1+\dots+n_m=n}} \frac{(-1)^m}{m!} a_{n_1} \dots a_{n_m},$$

where the second summation is over all ordered m-tuples of positive integers whose sum is n.

The main result relating these functions to the Hausdorff dimension of X is the following.

Theorem 3.3. *Let $X \subset \mathbb{R}^d$ and assume that $T : X \to X$ satisfies conditions (1)-(5). We can find $C > 0$ and $0 < \theta < 1$ such that if s_N is the largest real zero of Δ_N then*

$$|dim(X) - s_N| \leq C\theta^{N^{\left(1+\frac{1}{d}\right)}} \text{ for each } N \geq 1.$$

In the case of Cantor sets in an interval then we would take $d = 1$. In the case of Julia sets and Kleinian group limit sets we would take $d = 2$.

Practical points.

(1) In practise, we can get estimates for $C > 0$ and $0 < \theta < 1$ in terms of T. For example, θ is typically smaller for systems which are more hyperbolic.
(2) To implement this on a desktop computer, the main issue is amount memory required. In most examples it is difficult to get N larger than 18, say.

3.2 Examples.

Example 1: E_2. We can consider the non-linear Cantor set

$$E_2 = \left\{ \frac{1}{i_1 + \frac{1}{i_2 + \frac{1}{i_3 + \dots}}} : i_n \in \{1, 2\} \right\}.$$

For $X = E_2$, we can define $Tx = \frac{1}{x} \pmod 1$. This forms a Cantor set in the line, contained in the interval $[\frac{1}{2}(\sqrt{3}-1), \sqrt{3}-1]$, of zero Lebesgue measure.[1]

A number of authors have considered the problem of estimating the Hausdorff dimension $\dim_H(E_2)$ of the set E_2. In 1941, Good showed that $0.5194 \leq \dim_H(E_2) \leq$

[1] It represents sets of numbers with certain diophantine approximatibility conditions and its Hausdorff dimension has other number theoretic significance in terms of the Markloff spectrum in diophantine approximation, as we shall see in the next chapter.

0.5433. In 1982, Bumby improved these bounds to $0.5312 \leq \dim_H(E_2) \leq 0.5314$. In 1989 Hensley showed that $0.53128049 \leq \dim_H(E_2) \leq 0.53128051$. In 1996, he improved this estimate to 0.5312805062772051416.

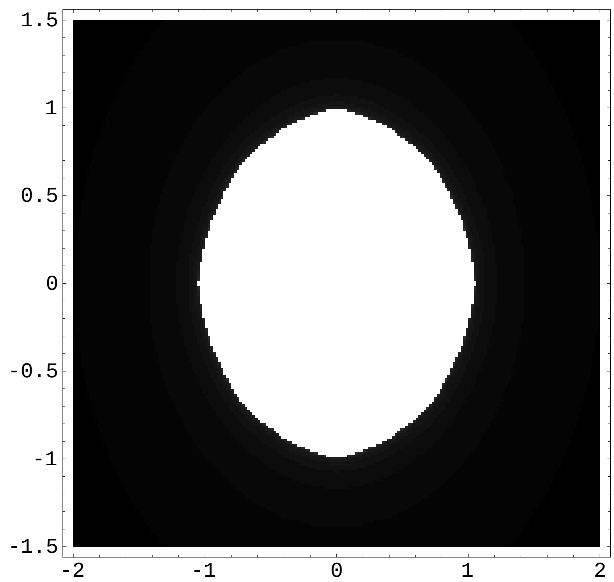
We can apply Theorem 3.3 to estimating $\dim_H(E_2)$. In practice we can choose $N = 16$, say, and if we solve for $\Delta_{16}(s_{16}) = 0$ then we derive the approximation

$$\dim_H(E_2) = 0.5312805062772051416244686\ldots$$

which is correct to the 25 decimal places given.

Example 2: Julia sets. We can consider Julia sets for quadratic polynomials $f_c(z) = z^2 + c$ with different values of c.

Example 2(a). Inside the main cardioid of the Mandelbrot set. Let $c = -0.06$, which is in the main cardioid of the Mandelbrot set. Thus the quadratic map T_c is hyperbolic and its Julia set is a quasi-circle (which looks quite "close" to a circle).



The Julia set for $z^2 - 0.06$ is the boundary between the white and black regions. (The white points are those which do not escape to infinity)

Bodart & Zinsmeister estimated the Hausdorff dimension of the Julia set to be $dim_H(J_c) = 1.001141$, whereas McMullen gave an estimate of $dim_H(J_c) = 1.0012$. Using Theorem 3.3 we can recover and improve on these estimates. Working with $N = 8$ we obtain the approximation

$$dim_H(J_c) = 1.0012136624817464642\ldots$$

Example 2(b). Outside the Mandelbrot set. Let $c = -20$, which is outside the Mandelbrot set. Thus the quadratic map T_c is hyperbolic and its Julia set is a Cantor set. With $N = 12$ this gives the approximation

$$dim_H(J_c) = 0.3185080957\ldots$$

which is correct to ten decimal places. This improves on an earlier estimate of Bodart & Zinsmeister.

3.3 Proof of Theorem 3.3 (outline). The proof of this Theorem is based on the study of the transfer operator on Hilbert spaces of real analytic functions. To explain the ideas, we shall first outline the main steps in the general case (without proofs) and then restrict to a special case (where more proofs will be provided). The difficulties in extending from the particular case to the general case are more notational than technical.

(i) Real Analytic Functions. We have a natural identification

$$\mathbb{R}^d = \mathbb{R}^d \times \{0\} \subset \mathbb{R}^d \times i\mathbb{R}^d = \mathbb{C}^d.$$

A function $f : U \to \mathbb{R}^k$ on an neighbourhood $U \subset \mathbb{R}^d$ is real analytic if about every point $x \in U$ there is a convergent power series expansion. Equivalently, it has a complex analytic extension to a function $f : D \to \mathbb{C}^k$, where $U \subset D \subset \mathbb{C}^d$ is an open set in $\mathbb{C}^d$.

(ii) Expanding maps and Markov Partitions. We start from an expanding map $T : X \to X$ with a Markov Partition $\mathcal{P} = \{X_j\}$, say. For each $1 \leq j \leq k$, let us assume that U_j is an open neighbourhood of a element X_j of the Markov Partition. We may assume that for each (i, j), the local inverse $T_{ji} : X_j \to X_i$ for $T : X_i \cap T^{-1}X_j \to X_j$ are contracting maps in an interated function scheme. Using analyticity (and choosing a smaller Markov partition $\mathcal{P}$, if necessary) we can assume that $U_j \times \{0\} \subset D_j$ where $\mathrm{D}_j = D_j^{(1)} \times \ldots \times D_j^{(d)} \subset \mathbb{C}^d$ is chosen is an open polydisc, i.e., a product of open discs $D_j^{(l)}$ in $\mathbb{C}$. Thus, we can assume that these extend holomorphically to maps $T_{ji} : D_i \to D_j$, and $|DT_{ji}(\cdot)| : D_i \to \mathbb{C}$ too, such that both

$$\overline{T_{ji}(D_i)} \subset D_j \qquad \text{and} \qquad \sup_{z \in D_i} |DT_{ji}(z)| < 1, \tag{3.1}$$

i.e., the discs are mapped are mapped so that their closures are contained inside the interior of the range disk, and the derivative is smaller than 1.

(iii) A Hilbert space and a linear operator. For any open set $U \subset \mathbb{C}^d$, let $\mathcal{A}_2(U)$ denote the Hilbert space of square integrable holomorphic functions on U equipped with the norm

$$||f||_{\mathcal{A}_2(U)} = \sqrt{\int_U |f|^2 d(\mathrm{vol})}.$$

For any $s \in \mathbb{R}$, and any admissible pair (i, j), define the analytic *weight function* $w_{s,(j,i)} \in \mathcal{H}(D_i)$ by $w_{s,(j,i)}(z) = |DT_{ji}(z)|^s$.[2] We then define the bounded linear operator $\mathcal{L}_{s,(j,i)} : \mathcal{H}(D_j) \to \mathcal{H}(D_i)$ by

$$\mathcal{L}_{s,(j,i)}g(z) = g(T_{ji}z)w_{s,(j,i)}(z).$$

For a fixed i we sum over all (admissible) composition-type operators $\mathcal{L}_{s,(j,i)}$ to form the *transfer operator* $\mathcal{L}_{s,i}$, i.e.,

$$\mathcal{L}_{s,i}h(z) = \sum_{j:A(i,j)=1} h(T_{ji}z)w_{s,(j,i)}(z). \tag{3.2}$$

[2] It is here that we need to consider real analyticity, because of the need for the modulus $|\cdot|$.

Finally, let $D = \coprod_i D_i$ be the disjoint union of the disks, then we define the *transfer operator* $\mathcal{L}_s : \mathcal{A}_2(D) \to \mathcal{A}_2(D)$ by setting

$$\mathcal{L}_s h|_{D_i} = \mathcal{L}_{s,i} h$$

for each $h \in \mathcal{A}_2(D)$ and each $i \in \{1, \ldots, k\}$.

The strategy we shall follow is the following. The operators $\mathcal{L}_s$ are defined on analytic functions on the disjoint union of the disks D_i. This in turn allows us to define their Fredholm determinants $det(I - z\mathcal{L}_s)$. These are entire function of z which, in particular, have as a zero the value $z = 1/\lambda_s$. In this context we can get very good approximations to $det(I - z\mathcal{L}_s)$ using polynomials whose coefficients involve the traces $tr(\mathcal{L}_s^n)$. Finally, these expressions can be evaluated in terms of fixed points of the iterated function scheme, leading to the functions $\Delta_N(s)$ introduced above.

(iv) Nuclear operators and approximation numbers. Given a bounded linear operator $L : H \to H$ on a Hilbert space H, its i^{th} *approximation number* $s_i(L)$ is defined as

$$s_i(L) = \inf\{||L - K|| : \text{rank}(K) \leq i - 1\},$$

where K is a bounded linear operator on H.

Definition. A linear operator $L : H \to H$ on a Hilbert space H is called *nuclear* if there exist $u_n \in H$, $l_n \in H^*$ (with $||u_n|| = 1$ and $||l_n|| = 1$) and $\sum_{n=0}^{\infty} |\rho_n| < +\infty$ such that

$$L(v) = \sum_{n=0}^{\infty} \rho_n l_n(v) u_n, \qquad \text{for all } v \in H. \tag{3.4}$$

The following theorem is due to Ruelle.

Proposition 3.4. *The transfer operator $\mathcal{L} : \mathcal{A}_2(D) \to \mathcal{A}_2(D)$ is nuclear.*

(iv) Determinants. We now associate to the transfer operators a function of a two complex variables.

Definition. For $s \in \mathbb{C}$ and $z \in \mathbb{C}$ we define the *Fredholm determinant* $\det(I - z\mathcal{L}_s)$ of the transfer operator $\mathcal{L}_s$ by

$$\det(I - z\mathcal{L}_s) = \exp\left(-\sum_{n=1}^{\infty} \frac{z^n}{n} \text{tr}(\mathcal{L}_s^n)\right) \tag{3.5}$$

This is similar to the way in which one associates to a matrix the determinant.

We can compute the traces explicitly.

The key to our method is the following explicit formula for the traces of the powers $\mathcal{L}_s^n$ in terms of the fixed points of our iterated function scheme.

Proposition 3.5. *If $\mathcal{L}_s : \mathcal{A}_\infty(D) \to \mathcal{A}_\infty(D)$ is the transfer operator associated to a conformal iterated function scheme then*

$$tr(\mathcal{L}_s^n) = \sum_{|\underline{i}|=n} \frac{|T'_{\underline{i}}(z_{\underline{i}})|^s}{det(I - T'_{\underline{i}}(z_{\underline{i}}))},$$

where $T'_{\underline{i}}(\cdot)$ is the (conformal) derivative of the map $T_{\underline{i}}$.

This allows us to compute the determinant:

$$\det(I - z\mathcal{L}_s) = \exp\left(-\sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{\underline{i} \in \mathrm{Fix}(n)} \frac{|DT_{\underline{i}}(z_{\underline{i}})|^s}{\det(I - DT_{\underline{i}}(z_{\underline{i}}))}\right).$$

(iv) Pressure, Hausdorff Dimension and Determinants. We can now make the final connection with the Hausdorff dimension.

Proposition 3.6. *For any $s \in \mathbb{C}$, let $\lambda_r(s)$, $r = 1, 2, \ldots$ be an enumeration of the non-zero eigenvalues of $\mathcal{L}_s$, counted with algebraic multiplicities. Then*

$$\det(I - z\mathcal{L}_s) = \prod_{r=1}^{\infty}(1 - z\lambda_r(s)).$$

In particular, the set of zeros z of the Fredholm determinant $\det(I - z\mathcal{L}_s)$, counted with algebraic multiplicities, is equal to the set of reciprocals of non-zero eigenvalues of $\mathcal{L}_s$, counted with algebraic multiplicities.

This brings us to the connection we want.

Proposition 3.7. *Given an iterated function scheme, the Hausdorff dimension $\dim(\Lambda)$ of its limit set Λ is the largest real zero of the function $s \mapsto \det(I - \mathcal{L}_s)$.*

Proof. If s is real then by the previous section the operator $\mathcal{L}_s$ has simple maximal eigenvalue λ_s, which equals 1 if and only if $s = \dim(\Lambda)$. But Proposition 3.7 tells us that 1 is an eigenvalue of $\mathcal{L}_s$ if and only if s is a zero of $\det(I - \mathcal{L}_s)$.

To see that $\dim_H(\Lambda)$ is actually the *largest* real zero of $\det(I - \mathcal{L}_s)$, observe that if $s > \dim(\Lambda)$ then the spectral radius of $\mathcal{L}_s$ is less than 1, so that 1 cannot be an eigenvalue of $\mathcal{L}_s$, and hence cannot be a zero of $\det(I - \mathcal{L}_s)$.

The reason that $\det(I - z\mathcal{L}_s)$ is particularly useful for estimating λ_s is because of the following result.

Proposition 3.8. *The function $\det(I - z\mathcal{L}_s)$ is entire as a function of $z \in \mathbb{C}$ (i.e., it has an analytic extension to the entire complex plane). In particular, we can expand*

$$\det(I - z\mathcal{L}_s) = 1 + \sum_{n=1}^{\infty} b_n(s) z^n$$

where $|b_n(s)| \le C\theta^{n^{1+1/d}}$, for some $C > 0$ and $0 < \theta < 1$.

We can rewrite $\det(I - \mathcal{L}_s)$ by applying the series expansion for $e^{-x} = 1 + \sum_{m=1}^{\infty}(-1)^m \frac{x^m}{m!}$ to the trace formula representation of $\det(I - z\mathcal{L}_s)$, and then regrouping powers of z. More precisely, we can expand the presentation

$$\begin{aligned}\det(I - z\mathcal{L}_s) &= \exp\left(-\sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{|\underline{i}|=n} \frac{|T_{\underline{i}}(\underline{z}_{\underline{i}}^*)|^{-s}}{\det(I - T_{\underline{i}}(\underline{z}_{\underline{i}}^*))}\right) \\ &= 1 + \sum_{n=1}^{\infty} b_n(s) z^n\end{aligned} \tag{3.6}$$

using the Taylor series $e^{-x} = 1 + \sum_{m=1}^{\infty}(-1)^m \frac{x^m}{m!}$. Collecting together the coefficients of z^N we have the following:

Proposition 3.9. *Let* $det(I - z\mathcal{L}_s) = 1 + \sum_{N=1}^{\infty} d_N(s) z^N$ *be the power series expansion of the Fredholm determinant of the transfer operator* $\mathcal{L}_s$*. Then*

$$b_N(s) = \sum_{\substack{(n_1,\dots,n_m)\\ n_1+\dots+n_m=N}} \frac{(-1)^m}{m!} \prod_{l=1}^{m} \frac{1}{n_l} \sum_{|\underline{i}|=n_l} \frac{|DT_{\underline{i}}(z_{\underline{i}})|^s}{det(I - DT_{\underline{i}}(z_{\underline{i}}))}, \tag{3.7}$$

where the summation is over all ordered m*-tuples of positive integers whose sum is* N*.*

In conclusion, (3.7) allows an *explicit* calculation of any coefficient $d_N(s)$, in terms of fixed points of compositions of at most N contractions.

3.4 Proof of Theorem 3.3 (special case). We shall try to illustrate the basic ideas of the proof, by proving these results with in the simplest setting: $d = 1$. Let $\Delta_r = \{z \in \mathbb{C} : |z| < r\}$ denote the open disk of radius r centered at the origin in the complex plane. Assume that X is contained in the unit disk Δ_1 and that $T : X \to X$ has two inverse branches T_1, T_2 which have analytic extensions $T_1 : \Delta_1 \to \Delta_1$ and $T_2 : \Delta_1 \to \Delta_1$ which have analytic extensions to $\Delta_{1+\epsilon}$ satisfying $T_1(\Delta_{1+\epsilon}) \cup T_2(\Delta_{1+\epsilon}) \subset \Delta_1$. Thus T_1 and T_2 are strict contractions of $\Delta_{1+\epsilon}$ into Δ_1 with the radii being reduced by a factor of $\theta = 1/(1+\epsilon) < 1$.

Let $\mathcal{A}_2(\Delta_r)$ denote the Hilbert space of analytic functions on Δ_r with inner product $\langle f, g\rangle := \int_{\Delta_r} f(z)\overline{g(z)}\, dx\, dy$.

Let us assume that $|T_1'(z)|$ and $|T_2'(z)|$ have analytic extensions from X to $\Delta_{1+\epsilon}$. We define the transfer operator $\mathcal{L}_s : \mathcal{A}_2(\Delta_1) \to \mathcal{A}_2(\Delta_1)$ by

$$\mathcal{L}_s h(z) = |T_1'(z)|^s h(T_1 z) + |T_2'(z)|^s h(T_2 z), \text{ for } z \in \Delta_{1+\epsilon}.$$

Observe that $\mathcal{L}_s(\mathcal{A}_2(\Delta_1)) \subset \mathcal{A}_2(\Delta_{1+\epsilon})$ and then

$$\begin{aligned}
\mathcal{L}_s h(z) &= \int_{|\xi=1+\epsilon|} \frac{\mathcal{L}_s h(\xi)}{z-\xi} d\xi \\
&= \frac{1}{2\pi i} \int_{|\xi|=1+\epsilon} \mathcal{L}_s h(\xi) \left(\frac{1}{\xi} \sum_{n=0}^{\infty} \left(\frac{z}{\xi}\right)^n \right) d\xi \\
&= \sum_{n=0}^{\infty} z^n \frac{1}{2\pi i} \int_{|\xi|=1+\epsilon} \frac{\mathcal{L}_s h(\xi)}{\xi^{n+1}} d\xi,
\end{aligned}$$

where $u_n(z) = z^n \in \mathcal{A}_2(\Delta_{1+\epsilon})$ and $l_n(h) = \frac{1}{2\pi i} \int_{|\xi|=1+2\epsilon} \frac{\mathcal{L}_s h(\xi)}{\xi^{n+1}} \in \mathcal{A}_2(\Delta_{1+\epsilon})^*$ is a linear functional. We can deduce that $\mathcal{L}_s$ is a nuclear operator, the uniform convergence of the series coming from $|z/\xi| = \theta < 1$.

Aside on Operator Theory. A bounded linear operator $T : H \to H$ on a Hilbert space H is called *compact* if the image $T(B) \subset H$ of the unit ball $\{x \in H : ||x|| \leq 1\}$ has a compact closure. In particular, a nuclear operator is automatically compact.

We denote the norm of the operator by $||T||_H = \sup_{||f||=1} ||T(f)||$.

We recall the following classical result.

Weyl's Lemma. *Let $A : H \to H$ be a compact operator with eigenvalues $(\lambda_n)_{n=1}^{\infty}$. We can bound $|\lambda_1 \lambda_2 \cdots \lambda_n| \le s_1 s_2 \cdots s_n$*

Proof. Given a bounded linear operator $A : H \to H$ on a Hilbert space H we can associate a bounded self-adjoint linear operator $B : H \to H$ by $B = A^* A$. Since B is non-negative (i.e, $\langle Bf, f\rangle = ||Af||^2 \ge 0$ for all $f \in H$) the eigenvalues $\mu_1 \ge \mu_2 \ge \cdots$ for B are described by the minimax identity:

$$\mu_1 = \max_{f \neq 0} \frac{\langle Bf, f\rangle}{||f||^2} \text{ and}$$
$$\mu_{n+1} = \max_{\dim L = n} \max_{f \in L^\perp} \frac{\langle Bf, f\rangle}{||f||^2} \text{ for } n \ge 1,$$

where L denotes an n-dimensional subspace.

Claim 1. $\mu_n \le s_n(A)$

Proof of Claim 1. For any linear operator $K : H \to H$ with n-dimensional image $K(H) \subset H$ we can use the minimax identity to write

$$\mu_n \le \max_{f \in \ker(K)} \frac{\langle Bf, f\rangle}{||f||^2} = \max_{f \in \ker(K)} \frac{\langle (B-K)f, f\rangle}{||f||^2} \le ||B - K||$$

Taking the infimum over all such K proves the claim. □

Claim 2. *Given an orthonormal set $\{\phi_i\}_{i=1}^n \subset H$ we can write*

$$\det(\langle A\phi_i, A\phi_j\rangle)_{i,j=1}^n \le s_1^2 s_2^2 \cdots s_n^2 \det(\langle \phi_i, \phi_j\rangle)_{i,j=1}^n$$

Proof of Claim 2. Let $\{e_n\}_{m=0}^{\infty}$ be a complete orthonormal basis of eigenvectors for B. We can write $\langle A\phi_i, A\phi_j\rangle = \langle B\phi_i, \phi_j\rangle = \sum_{m=0}^{\infty} \mu_m \langle \phi_j, e_m\rangle \langle e_m, \phi_k\rangle$. In particular, we can write the original matrix as a product of two infinite matrices.

$$(\langle A\phi_i, A\phi_j\rangle)_{i,j=1}^n = (\sqrt{\mu_m}\langle \phi_j, e_m\rangle)_{m=1}^{\infty}{}_{j=1}^{n} \times (\sqrt{\mu_m}\overline{\langle e_m, \phi_k\rangle})_{k=1}^{n}{}_{m=1}^{\infty}. \tag{3.7}$$

Considering determinants gives:

$$\det(\langle A\phi_i, A\phi_j\rangle)_{i,j=1}^n = \sum_{C,C'} \det(C)\det(C'),$$

where the sum is over all possible $n \times n$ submatrices C and C' of the two matrices on the rights hand side of (*3.7), respectively. In this latter expression, we can take out a factor of $\sqrt{\mu_1 \mu_2 \cdots \mu_n}$ from each matrix to leave $\det(\langle \phi_i, \phi_j\rangle)_{i,j=1}^n$. Since, by Claim 1, $\mu_1 \mu_2 \cdots \mu_n \le s_1 s_2 \cdots s_n$ this gives the desired result. □

It remains to complete the proof of Weyl's Lemma. Since A is a compact operator we can choose an orthonormal basis $(e)_{n=0}^{\infty}$ for H such that $Ae_n = a_{n1}e_1 + a_{n2}e_2 + \cdots + a_{nn}e_n$, (i.e., the matrix (a_{nm}) is triangular) and $a_{nn} = \lambda_n$ is an eigenvalue. In particular, if $i < j$ than

$$\langle Ae_i, Ae_j\rangle = \sum_{k=1}^{i} \langle A\phi_i, \phi_k\rangle \overline{\langle A\phi_k, A\phi_j\rangle}$$

and thus

$$\begin{aligned}\det\left(\langle Ae_i, Ae_j\rangle\right)_{i,j=1}^n &= \det\left(\langle Ae_i, e_j\rangle\right)_{i,j=1}^n \det\left(\overline{\langle Ae_i, e_j\rangle}\right)_{i,j=1}^n \\ &= \left|\det\left(\langle Ae_i, e_j\rangle\right)_{i,j=1}^n\right|^2 = |\lambda_1 \cdots \lambda_n|^2.\end{aligned}$$

This completes the proof □

We now return to the explicit case of analytic functions.

Lemma 3.10. *The singular values of the transfer operator* $\mathcal{L}_s : A_2(\Delta_1) \to A_2(\Delta_1)$ *satisfy*

$$s_j(\mathcal{L}_s) \le \frac{||\mathcal{L}_s||_{A_2(\Delta_{1+\epsilon})}}{1-\theta}\theta^j, \text{ for all } j \ge 1.$$

Proof. Let $g \in A_2(\Delta_1)$ and write $\mathcal{L}_s g = \sum_{k=0}^\infty l_k(g) p_k$, where $p_k(z) = z^k$. We can easily check $||p_k||_{A_2(\Delta_1)} = \sqrt{\frac{\pi}{k+1}}$ and $||p_k||_{A_2(\Delta_{1+\epsilon})} = \sqrt{\frac{\pi}{k+1}}(1+\epsilon)^{k+1}$. The functions $\{p_k\}_{k=0}^\infty$ form a complete orthogonal family for $\mathcal{A}_2(\Delta_{1+\epsilon})$, and so $\langle \mathcal{L}_s g, p_k\rangle_{\mathcal{A}_2(\Delta_{1+\epsilon})} = l_k(g)||p_k||^2_{\mathcal{A}_2(\Delta_{1+\epsilon})}$. The Cauchy-Schwarz inequality implies that

$$|l_k(g)| \le ||\mathcal{L}_s g||_{\mathcal{A}_2(\Delta_{1+\epsilon})}\, ||p_k||^{-1}_{\mathcal{A}_2(\Delta_{1+\epsilon})}.$$

We denote the rank-j projection operator $L_s^{(j)}$ by $L_s^{(j)}(g) = \sum_{k=0}^{j-1} l_k(g) p_k$. For any $g \in A_2(\Delta_1)$ we can estimate

$$||\left(\mathcal{L}_s - L_s^{(j)}\right)(g)||_{A_2(\Delta_1)} \le ||\mathcal{L}_s g||_{A_2(\Delta_1)} \sum_{k=j}^\infty \theta^{k+1}.$$

It follows that

$$||\mathcal{L}_s - L_s^{(j)}||_{A_2(\Delta_1)} \le \frac{||\mathcal{L}_s||_{A_2(\Delta_1)}}{1-\theta}\theta^{j+1} \text{ and so } s_j(L_s) \le \frac{||\mathcal{L}_s||_{A_2(\Delta_1)}}{1-\theta}\theta^{j+1},$$

and the result follows. □

We now show that the coefficients of the power series of the determinant decay to zero with super-exponential speed.

Lemma 3.11. *If we write* $\prod_{j=1}^\infty (1 + z s_j) = 1 + \sum_{m=1}^\infty c_m z^m$, *then*

$$|c_m| \le B\left(||\mathcal{L}_s||_{\mathcal{A}_2(\Delta_1)}\right)^m \theta^{\frac{m(m+1)}{2}},$$

where $B = \prod_{m=1}^\infty (1-\theta^m)^{-1} < \infty$.

Proof. The coefficients c_n in the power series expansion of the determinant have the form $c_m = \sum_{i_1 < \ldots < i_m} s_{i_1} \cdots s_{i_m}$, the summation is over all m-tuples $(i_1, \ldots, i_m)$ of positive integers satisfying $i_1 < \ldots < i_m$. Thus by Lemma 3.10 we can bound

$$\begin{aligned}|c_m| &\le \left(\frac{||\mathcal{L}_s||_{A_2(\Delta_1)}}{1-\theta}\right)^m \frac{\theta^{m(m+1)/2}}{(1-\theta)(1-\theta^2)\cdots(1-\theta^m)}. \\ &\le B\left(\frac{||\mathcal{L}_s||_{A_2(\Delta_1)}}{1-\theta}\right)^m \theta^{m(m+1)/2}.\end{aligned}$$

For some $B > 0$. $\square$

The coefficients of $\det(I - z\mathcal{L}_s) = 1 + \sum_{n=1}^{\infty} b_n z^n$ are given by Cauchy's Theorem:

$$|b_n| \leq \frac{1}{r^n} \sup_{|z|=r} |\det(I - z\mathcal{L}_s)|, \text{ for any } r > 0. \tag{3.8}$$

We recall the following standard bound of Hardy, Littlewood and Polya: Let $\{a_n\}$, $\{b_n\}$ be not increasing sequences of real numbers such that $\sum_{j=1}^{n} a_j \leq \sum_{j=1}^{n} b_j$ and let $\Phi : \mathbb{R} \to \mathbb{R}$ be a convex function then $\sum_{j=1}^{n} \Phi(a_j) \leq \sum_{j=1}^{n} \Phi(b_j)$. Letting $a_j = \log|\lambda_j|$, $b_j = \log s_j$ and $\Phi(x) = \log(1 + tx)$ (and letting $n \to +\infty$) we deduce that if $|z| = r$ then

$$\begin{aligned} |\det(I - z\mathcal{L}_s)| &\leq \prod_{j=1}^{\infty} (1 + |z|\lambda_j) \leq \prod_{j=1}^{\infty} (1 + |z| s_j) \\ &\leq \left(1 + B \sum_{m=1}^{\infty} (r\alpha)^m \theta^{\frac{m(m+1)}{2}}\right) \end{aligned} \tag{3.9}$$

where $\alpha = ||\mathcal{L}_s||_{\mathcal{A}_2(\Delta_1)}$. If we choose $r = r(n) := \frac{\theta^{-n/2}}{\alpha}$ then we can bound

$$(r\alpha)^m \theta^{m^2/2} \leq \begin{cases} \theta^{n^2/2} & \text{for } m = 1, \ldots, \left[\frac{n}{2}\right] \\ \theta^{((m-n)^2 + nm)/2} \leq (\theta^{n/2})^m & \text{for } m > \left[\frac{n}{2}\right] \end{cases} \tag{3.10}$$

Comparing (3.8), (3.9) and (3.10) we can bound

$$|b_n| \leq \left[\frac{n}{2}\right] \theta^{n^2/2} + \frac{(\theta^{n/2})^{n/2}}{1 - \theta^{n/2}}$$

This proves the super-exponential decay of the coefficients provided we replace θ by a value larger than $\theta^{1/4}$.

Lemma 3.12. *We can compute the traces:*

$$tr(\mathcal{L}_s^n) = \sum_{|\underline{i}|=n} \frac{|T_{\underline{i}}'(x)|^s}{1 - |T_{\underline{i}}'(x)|^{-1}}$$

Proof. For each string $\underline{i} = (i_1, \ldots, i_n) \in \prod_{j=1}^{n} \{0, 1\}$ let us first define operators $\mathcal{L}_{s,\underline{i}} : \mathcal{A}_2(\Delta_1) \to \mathcal{A}_2(\Delta_1)$ by $\mathcal{L}_{s,\underline{i}} g(z) = g(T_{\underline{i}} z) w_{s,\underline{i}}(z)$, where the analytic weight functions $w_{s,\underline{i}}$ are given by $w_{s,\underline{i}}(z) = |DT_{\underline{i}}(z)|^s$. The n^{th} iterate of the transfer operator $\mathcal{L}_s$ is given by

$$\mathcal{L}_s^n = \sum_{|\underline{i}|=n} \mathcal{L}_{s,\underline{i}}.$$

The additivity of the trace means we can write

$$\mathrm{tr}(\mathcal{L}_s^n) = \sum_{|\underline{i}|=n} \mathrm{tr}(\mathcal{L}_{s,\underline{i}}). \tag{3.11}$$

For each $\underline{i}$ there is a unique fixed point $z_{\underline{i}}$ of the contraction $T_{\underline{i}} : \Delta_1 \to \Delta_1$. We can compute the trace of $\mathcal{L}_{s,\underline{i}}$ by evaluating the eigenvalues of this operator and summing. In particular, consider the eigenfunction equation $\mathcal{L}_{s,\underline{i}}h(z) = \lambda h(z)$. We can evaluate this at $z = z_{\underline{i}}$ to deduce that $w_{s,\underline{i}}(z_{\underline{i}})h(z_{\underline{i}}) = \lambda h(z_{\underline{i}})$. If $h(z_{\underline{i}}) \neq 0$ then we see that the only solution corresponds to $\lambda = 1$. If $h(z_{\underline{i}}) = 0$, then we can differentiate the eigenvalue equation to get that

$$w'_{s,\underline{i}}(z)h(z) + w_{s,\underline{i}}(z)h'(z) = \lambda h'(z)$$

Evaluating this at $z = z_{\underline{i}}$ (and recalling that $h(z_{\underline{i}}) = 0$) we get that

$$w_{s,\underline{i}}(z_{\underline{i}})h'(z_{\underline{i}}) = \lambda h'(z_{\underline{i}})$$

If $h'(z_{\underline{i}}) \neq 0$ then we see that the only solution corresponds to $\lambda = w_{s,\underline{i}}(z_{\underline{i}})$. Proceeding inductively, we can see that the only eigenvalues are $\{\lambda_n\}_{n=1}^{\infty} = \{w_{s,\underline{i}}(z_{\underline{i}})^k : k \geq 0\}$. (Moreover, one can see that these eigenvalues are realized). Summing over these eigenvalues gives:

$$\mathrm{tr}(\mathcal{L}_{s,\underline{i}}) = \sum_{n=1}^{\infty} \lambda_n = \frac{w_{s,\underline{i}}(z_{\underline{i}})}{(1 - T'_{\underline{i}}(z_{\underline{i}}))} = \frac{|T'_{\underline{i}}(z_{\underline{i}})|^s}{(1 - T'_{\underline{i}}(z_{\underline{i}}))}. \tag{3.12}$$

Finally, comparing (3.11) and (3.12) completes the proof. □

Remark. Fried actually corrected a minor error in Grothendieck's original paper which was reproduced in Ruelle's paper.

3.5 Julia sets. For practical purposes, our algorithm is effective in computing the dimension $\dim_H(J_c)$ of the Julia set J_c if we choose c either in the main cardioid of the Mandelbrot set $\mathcal{M}$, or c outside of $\mathcal{M}$, say. In the latter case all periodic points are repelling, while in the former case all periodic points are repelling except for a single attractive fixed point. We can give explicitly estimate $\gamma = \gamma_c$ for c close to 0.

For quadratic maps we know that $T'(z) = 2z$ and if $T^n(z) = z$ then by the chain rule

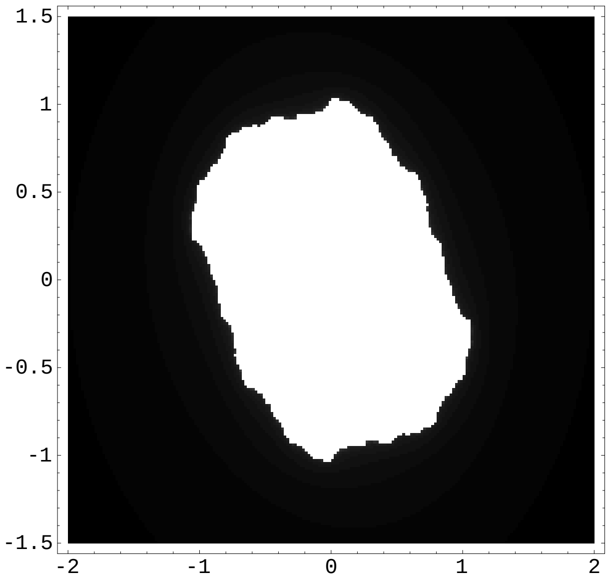
$$(T^n)'(z) = T'(T^{n-1}z) \cdots T'(Tz).T'(z) = 2^n(T^{n-1}z) \cdots (Tz).z$$

and so the coefficients in the expansions take a simpler form.

Example 3.5.1 ($c = i/4$). First we consider the purely imaginary value $c = i/4$, which lies in the main cardioid of the Mandelbrot set. Table 1 illustrates the successive approximations s_N to $\dim_H(J_{i/4})$ arising from our algorithm.

Example 3.5.2 ($c = -\frac{3}{2} + \frac{2}{3}i$). If we take the parameter value $c = -\frac{3}{2} + \frac{2}{3}i$, which lies outside the Mandelbrot set, then the sequence of approximations to the dimension of J_c are given in Table 2.

Example 3.5.3 $c = -5$. For real values of c which are strictly less than -2, the Julia set J_c is a Cantor set completely contained in the real line. For such cases we have,



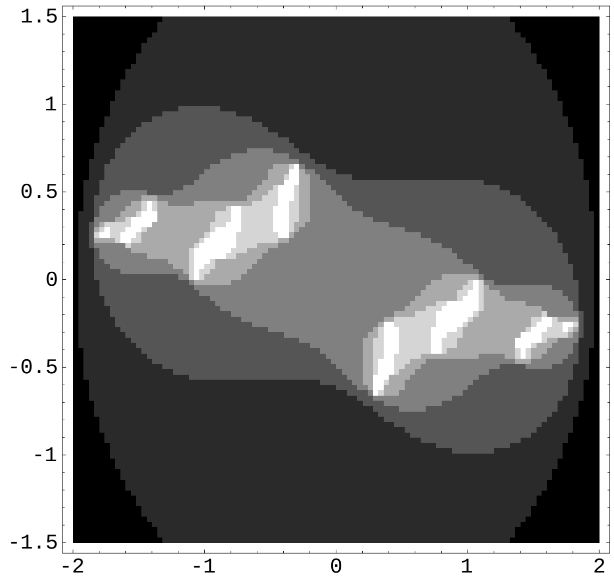
The Julia set for $z^2 + i/4$ is the boundary between the white and black regions. (The white points are those which do not escape to infinity)

N	N^{th} approximation to $\dim(J_{i/4})$
3	1.1677078534172827136
4	0.9974580934808979848
5	1.0169164188641603339
6	1.0218764720532313644
7	1.0230776911089017648
8	1.0232246810534996595
9	1.0232072525392922127
10	1.0231992637099065199
11	1.0231993120941968028
12	1.0231992857944621198
13	1.0231992888227184780
14	1.0231992890455073830
15	1.0231992890300189633
16	1.0231992890307255210
17	1.0231992890309781268
18	1.0231992890309686742
19	1.0231992890309691466
20	1.0231992890309691251

Table 1. Successive approximations to $\dim(J_{i/4})$

by Corollary 3.1, the faster $O(\delta^{N^2})$ convergence rate to $\dim(J_c)$, as illustrated in Table 3 for the case $c = -5$.

Example 3.5.4 ($c = -20$). For larger negative real values of c, the hyperbolicity of $f_c : J_c \to J_c$ is more pronounced, so that the constant $0 < \delta < 1$ in the $O(\delta^{N^2})$



The Julia set for $z^2 - \frac{3}{2} + \frac{2}{3}i$ is a zero measure Cantor set - so invisible to the computer. The lighter regions are points "nearer" the Julia set which take longer to escape.

N	N^{th} approximation to $\dim(J_{-3/2+2i/3})$
1	0.7149355610391974853
2	0.9991996994914223217
3	0.8948837401931045135
4	0.8990693400138277172
5	0.9048525377869365908
6	0.9040847144651654898
7	0.9038472818583009063
8	0.9038738383368002502
9	0.9038748469934538668
10	0.9038745896021979531
11	0.9038745956441220338
12	0.9038745968650866636
13	0.9038745968171929578
14	0.9038745968108846487
15	0.9038745968111623979
16	0.9038745968111848616

Table 2. Successive approximations to $\dim_H(J_{-3/2+2i/3})$

estimate is closer to zero, and the convergence to $\dim_H(J_c)$ consequently faster. Table 4 illustrates this for $c = -20$.

Remark. Of particular interest are those c in the intersection $\mathcal{M} \cap \mathbb{R} = [-2, \frac{1}{4}]$, i.e., the where the real axis intersects the Mandelbrot set. For values $-3/4 < c < 1/4$ (in the main Cartoid) the map T_c is expanding and the dimension $c \mapsto \dim(J_c)$

N	N^{th} approximation to $\dim(J_{-5})$
1	0.45139935847641746096759591012413833349
2	0.48415186841941229924646359003260707715
3	0.48479795874869757786122829089756625571
4	0.48479829435618956997307175635763670900
5	0.48479829443816350575185119434209429957
6	0.48479829443816043053474878912718259909
7	0.48479829443816043053839847657937295512
8	0.48479829443816043053839847817268307747

TABLE 3. Successive approximations to $\dim_H(J_{-5})$

N	N^{th} approximation to $\dim_H(J_{-20})$
1	0.31485651652009699091265279629753355933688857812644665851918
2	0.31850483144363986562810164826944017431378984622904321285835
3	0.31850809576591085725942984004207253452015913804880055477625
4	0.31850809575800523882867786043747732330759968092023152922729
5	0.31850809575800524988789850335472906645586111530021825766595
6	0.31850809575800524988789848098884346788677292871828344714065
7	0.31850809575800524988789848098884348414792438297975066097358
8	0.31850809575800524988789848098884348414792438305840652044425

TABLE 4. Successive approximations to $\dim_H(J_{-20})$

changes analytically. Indeed, about $c = 0$ we have the asymptotic expansion of Ruelle, mentioned before. However, at $c = 0$ the map $T_{c=\frac{1}{4}}$ is *not expanding* (since $T_{c=\frac{1}{4}}$ has a parabolic fixed point of derivative 1 at the point $z = \frac{1}{2}$). Moreover, $c \mapsto \dim(J_c)$ is actually discontinuous at $c = 1/4$. This phenomenon was studied by Douady, Sentenac & Zinsmeister. Havard & Zinsmeister proved that when restricted to the real line, the left derivative of the map $c \mapsto \dim(\mathcal{J}_c)$ at the point $c = 1/4$ is infinite.

One advantage of this method is that it leads to effective estimates on the rate of convergence of the algorithm. This is illustrated by the following result.

Proposition 3.13. *For any $\eta > 1/2$ there exists $\epsilon > 0$ such that if $|c| < \epsilon$ then the expansion coefficient for T_c is less than η.*

The proof is very easy.

3.6 Schottky groups Limit sets.

Example 3.6.1. Fix $2p$ disjoint closed discs $D_1, \ldots, D_{2p}$ in the plane, and Möbius maps $g_1, \ldots, g_p$ such that each g_i maps the interior of D_i to the exterior of D_{p+i}. The corresponding Schottky group is defined as the group generated by $g_1, \ldots, g_p$.

The associated limit set Λ is a Cantor subset of the union of the interiors of the discs $D_1, \dots, D_{2p}$. We define a map T on this union by $T|_{int(D_i)} = g_i$ and $T|_{int(D_{p+i})} = g_i^{-1}$. A *reflection group* is a Schottky group with $D_i = D_{p+i}$ for all $i = 1, \dots, p$.

Example 3.6.2. Quasifuchsian groups. Such groups are isomorphic to the fundamental group of a compact Riemann surface, and are obtained by a quasiconformal deformation of a Fuchsian group (a Kleinian group whose limit set is contained in some circle). The limit set Λ of a quasifuchsian group is a simple closed curve. We can associate an expanding map T with the limit set of any Fuchsian group, and the quasiconformal deformation induces an expanding map on Λ.

We show that the Hausdorff dimension of the limit sets Λ of both Schottky and quasifuchsian groups can be efficiently calculated via a knowledge of the derivatives $(T^n)'(z)$, evaluated at periodic points $T^n z = z$.

Theorem 3.14. (Kleinian groups) *Let Γ be a finitely generated non-elementary convex cocompact Schottky or quasifuchsian group, with associated limit set Λ. The algorithm applies.*

First suppose Γ is a Schottky group. We define a map T on the union $\cup_{j=1}^{2p} D_j$ by $T|_{int(D_j)} = g_j$ and $T|_{int(D_{p+j})} = g_j^{-1}$, for $j = 1, \dots, p$, A Markov partition for this map just consists of the collection of interiors $\{int(D_i)\}_{i=1}^{2p}$.

Suppose Γ is quasifuchsian, with limit set Λ. Now Γ is quasi-conformally conjugate to some Fuchsian group Γ'. Bowen & Series proved there exists an expanding Markov map $S : S^1 \to S^1$ which faithfully models the action of Γ', and the quasiconformal deformation conjugates this to an expanding Markov map $T : \Lambda \to \Lambda$. Conformality and real-analyticity are clearly satisfied.

Example 3.6.3. The following family of reflection groups was considered by McMullen. Consider three circles $C_0, C_1, C_2 \subset \mathbb{C}$ of equal radius, arranged symmetrically around S^1, each intersecting the unit circle S^1 orthogonally, and meeting S^1 in an arc of length θ. We do not want the C_i to intersect each other, so we ask that $0 < \theta < 2\pi/3$. For definiteness let us suppose each C_i has radius $r = r_\theta = \tan\frac{\theta}{2}$, and that the circle centres are at the points $z_0 = a$, $z_1 = ae^{2\pi i/3}$ and $z_2 = ae^{-2\pi i/3}$ (where $a = a_\theta = \sqrt{1+r^2} = \sec\frac{\theta}{2}$). The reflection $\rho_i : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ in the circle C_i takes the explicit form

$$\rho_i(z) = \frac{r^2}{|z - z_i|^2}(z - z_i) + z_i.$$

Let $\Lambda_\theta \subset \mathbb{S}^1$ denote the limit set associated to the group Γ_θ of transformations given by reflection in these circles. For example, with the value $\theta = \pi/6$ we show that the dimension of the limit set $\Lambda_{\pi/6}$ is

$$\dim(\Lambda_{\pi/6}) = 0.18398306124833918694118127344474173288\dots$$

which is empirically accurate to the 38 decimal places given. The approximations are shown in Table 5.

N	Largest zero of Δ_N
2	0.14633481296007741055454748401454596
3	0.18423440272351767688822531747382350
4	0.18399977929621235204864644797773486
5	0.18398305039516509087579859265399133
6	0.18398305988417009403195596234810316
7	0.18398306122261622100816402885866734
8	0.18398306124841998285455137338908131
9	0.18398306124833255797187772764544302
10	0.18398306124833929946685349025674957
11	0.18398306124833918404985469216386875
12	0.18398306124833918700689278881066430
13	0.18398306124833918693967757277042711
14	0.18398306124833918694121655021916395
15	0.18398306124833918694118046846226018
16	0.18398306124833918694118129222351397
17	0.18398306124833918694118127301338345
18	0.18398306124833918694118127345475071
19	0.18398306124833918694118127344451095
20	0.18398306124833918694118127344474707

TABLE 5. Successive approximations to $\dim(\Lambda_{\pi/6})$

4. Some applications: Number Theory and Geometry

4.1 Diophantine approximation. Given any irrational number $\alpha \in \mathbb{R}$, we can approximate it arbitrarily closely by rational numbers, since they are dense in the real numbers. The following is a very classical result.

Dirichlet's Theorem. *Let α be an irrational number. We can find infinitely many distinct $p, q \in \mathbb{Z}$ ($q \neq 0$) such that*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2} \tag{4.1}$$

Proof. The proof just uses the "pigeon-hole principle". Let $N \geq 1$. Consider the $N+1$ fractional parts $\{\alpha\}, \{2\alpha\}, \{3\alpha\}, \cdots, \{(N+1)\alpha\} \in [0,1]$ (where $0 \leq \{j\alpha\} < 1$ is the fractional part of $j\alpha$, i.e., $j\alpha = \{j\alpha\} + [j\alpha]$ with $[j\alpha] \in \mathbb{N}$). If we divide up the unit interval into N-intervals $[0, \frac{1}{N}]$, $[\frac{1}{N}, \frac{2}{N}]$, ..., $[\frac{N-1}{N}, 1]$, each of length $\frac{1}{N}$, then one of the intervals must contain at least two terms $\{i\alpha\}, \{j\alpha\}$, say, for some $1 \leq i < j \leq N+1$. In particular, $0 \leq \{i\alpha\} - \{j\alpha\} \leq \frac{1}{N}$ from which we see that

$$0 \leq \alpha \underbrace{(i-j)}_{=:q} - \underbrace{([\alpha i] - [\alpha j])}_{=:p} = \{i\alpha\} - \{j\alpha\} \leq \frac{1}{N}$$

where $0 \leq q \leq N$. In particular, writing $p = [\alpha i] - [\alpha j]$ and $q = i - j$ we have that $|\alpha - \frac{p}{q}| \leq \frac{1}{q^2}$. Moreover, by successively choosing N sufficiently large we can exclude previous choices of $\frac{p}{q}$ and thus generate an infinite sequence of approximations (4.1) $\square$

In particular, since almost every number is irrational, almost every $0 < \alpha < 1$ satisfies (4.1). We want to consider what happens if we try still stronger approximations.

First version: Replace exponent in the denominator by a larger value. Considers instead the inequality (4.1) with the Right Hand Side decreased from $\frac{1}{q^2}$ to $\frac{1}{q^{2+\eta}}$, say, for some $\eta > 0$. In this case, the set Λ_η of $0 < \alpha < 1$ for which the stronger inequality

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^{2+\eta}} \tag{4.2}$$

has infinitely many solutions is smaller. In fact, the set has Hausdorff Dimension strictly smaller than 1 and so, in particular, has zero measure. This follows from the following classical result.

Janik-Besicovitch Theorem. *For $\eta > 0$, the set of α with infinitely many solutions to (4.2) has zero measure. Moreover this set has Hausdorff dimension, i.e.,*

$$dim_H \underbrace{\left\{\alpha : \left|\alpha - \frac{p}{q}\right| < \frac{1}{q^{2+\eta}} \text{ for infinitely } p \in \mathbb{Z}, q \in \mathbb{Z} - \{0\}\right\}}_{=:\Lambda_\eta} = \frac{2}{2+\eta} < 1.$$

Proof. The upper bound on the dimension is easy to prove. Given $\epsilon > 0$, we can choose $q \geq 2$ such that $\frac{1}{q^{2+\eta}} < \delta \leq \frac{1}{(q-1)^{2+\eta}}$. For each $q \geq 1$, we can choose a cover for this set by the $q(q+1)/2$-intervals

$$\left(\frac{p}{q} - \frac{1}{q^{2+\eta}}, \frac{p}{q} + \frac{1}{q^{2+\eta}}\right), \text{ for } 0 \leq p \leq q.$$

Since these each have diameter $q^{-(2+\eta)} < \epsilon$ we deduce that $H^d_\epsilon \leq q^{2-d(2+\eta)}$. In particular, if $d > \frac{2}{2+\eta}$ then we see that $\lim_{\epsilon \to 0} H^d_\epsilon = 0$. We thus deduce that the Hausdorff dimension is at most $\frac{2}{2+\eta}$. We omit the other inequality, referring to the book of Falconer for the details. $\square$

Second version: replace 1 *in numerator by a different value C.* A natural question to ask is how big a value of $C = C(\alpha) \geq 1$ we can choose such that we can still find infinitely many distinct $p, q \in \mathbb{Z}$ ($q \neq 0$) such that

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{Cq^2}. \tag{4.3}$$

To begin with, we recall that there is a slightly stronger version of Dirichlet's theorem due to Hurewicz.

Hurewicz's Theorem. *Let α be an irrational number. We can find infinitely many distinct $p, q \in \mathbb{Z}$ ($q \neq 0$) such that*

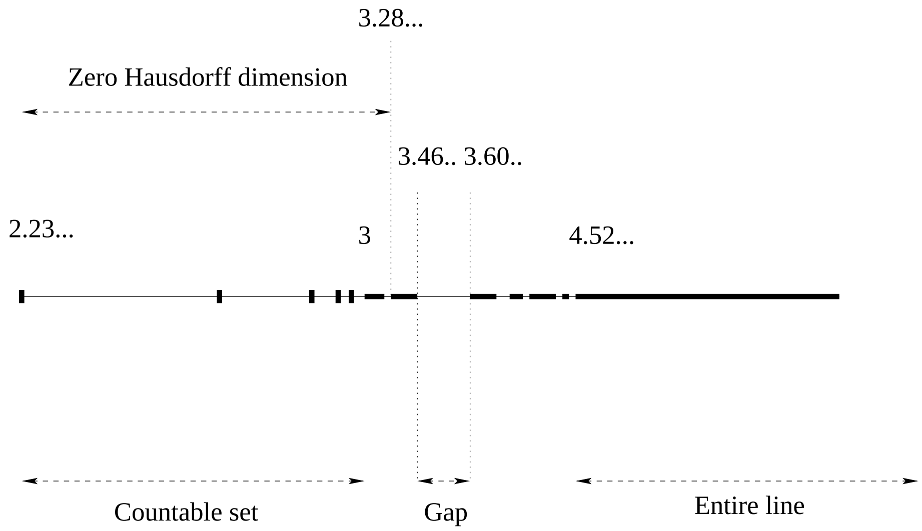
$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{\sqrt{5}q^2}$$

In particular, we can always choose $C \geq \sqrt{5} = 2.23607\ldots$. (The proof, which is not difficult, uses Continued Fractions and can be found in the book of Hardy and Wright).

Notation. For a given irrational number $0 < \alpha < 1$ we define $C(\alpha) \geq \sqrt{5}$ to be the largest C such that $|\alpha - p/q| < 1/(Cq^2)$, for infinitely many p, q, i.e.,

$$C(\alpha) = \liminf_{q \to \infty} \left[\max_{p \in \mathbb{N}} |q^2\alpha - pq|^{-1} \right].$$

We next want to consider the set of all possible values $C(\alpha)$, where α ranges over all irrational numbers between 0 and 1, say. We define the *Lagrange spectrum* to be the set $\mathbb{L} = \{C(\alpha) : \alpha \in (0, 1) - \mathbb{Q}\}$.



THE LAGRANGE SPECTRUM

By Hurewitz's theorem we know that $\mathbb{L} \subset [\sqrt{5}, +\infty)$. Moreover, it is also known that for $\alpha = 1/\sqrt{2}$, say, we have $C(1/\sqrt{2}) = \sqrt{5} \in \mathbb{L}$. In particular, we see that $\sqrt{5}$ is the smallest point in $\mathbb{L}$. In fact, the portion of the spectra below the value 3 is a countable set which is known exactly. For completeness, we quote the following result without proof.

Proposition 4.1. *We can identify*

$$\mathbb{L} \cap [0, 3] = \left\{ \frac{1}{z}\sqrt{9z^2 - 4} : x^2 + y^2 + z^2 = 3xyz, \text{ where } x, y, z \in \mathbb{N} \text{ and } x, y \leq z \right\}$$

In particular, the smallest value in the spectrum is $\sqrt{5}$ and the next smallest values (in ascending order) are: $\sqrt{8} = 2.82843\ldots$, $\sqrt{221}/5 = 2.97321\ldots$, $\sqrt{1517}/13 = 2.99605\ldots$, $\sqrt{7565}/29 = 2.99921\ldots$.

Since this portion $\mathbb{L} \cap [0, 3]$ is countable, we have the following corollary.

Corollary. $dim_H(\mathbb{L} \cap [\sqrt{5}, 3]) = 0$.

At the other extreme, the spectrum is known to contain the whole interval $[\mu, +\infty)$, where $\mu \approx 4.527829566$.

It is an interesting question to ask how large an interval $[\sqrt{5}, t]$ $(t > 3)$ we can choose such that we still have $\dim_H(\mathbb{L} \cap [\sqrt{5}, t]) < 1$ or $\mathbb{L} \cap [\sqrt{5}, t]$ has zero Lebesgue measure. We shall return to this in a moment.

There is an alternative definition of $\mathbb{L}$ which is particularly useful in studying the region $\mathbb{L} \cap [\sqrt{5}, 4.527\ldots]$.

Proposition 4.2. *The set* $\mathbb{L}$ *can also be defined in terms of doubly infinite sequences of positive integers. Given* $a = (a_n)_{n \in \mathbb{Z}}$ *we define*

$$\lambda_i(a) = a_i + [a_{i+1}, a_{i+2} \ldots] + [a_{i-1}, a_{i-2}, \ldots], \quad i \in \mathbb{Z}$$

where, as usual, $[c_0, c_1, \ldots] = 1/(c_0 + (1/c_1 + \ldots))$ *denotes the continued fraction with* $c_0, c_1, \ldots \in \mathbb{N}$. *We then have*

$$\mathbb{L} = \left\{ L(a) = \limsup_{|i| \to \infty} \lambda_i(a) \, : \, a \in \mathbb{N}^{\mathbb{Z}} \right\}.$$

The proof is outside the scope of these notes, and is so omitted.

A little calculation shows:

(1) If $a = (a_n)_{n \in \mathbb{Z}}$ has at least one entry greater than 2 then $L(a) \geq \sqrt{13}$. and indeed $L(a) = \sqrt{13}$ if and only if $a = (\ldots, 3, 3, 3, \ldots)$. However,
(2) if a has entries only 1's and 2's then $L(a) \leq \sqrt{12}$

In particular, we can deduce the following result.

Corollary. *There are gaps in the spectrum (i.e., intervals which don't intersect* $\mathbb{L}$

Proof. This is apparent, since $(\sqrt{12}, \sqrt{13}) \cap \mathbb{L} \neq \emptyset$, as we saw above. $\square$

We can now consider the problem of finding the Lebesgue measure and Hausdorff dimension of various portions of the spectrum. Let us define $\mathbb{L}_t = \mathbb{L} \cap [0, t]$. We have the following result.

Theorem 4.3. *We can estimate*

$$dim_H(\mathbb{L}_{\sqrt{10}}) \approx 0.8121505756228 \text{ and}$$
$$dim_H(\mathbb{L}_{\sqrt{689}/8}) \approx 0.9716519526$$

(where $sqrt10 \approx 3.1622\ldots$ *and* $\sqrt{689}/8 \approx 3.2811\ldots$*).*

Sketch Proof. If we consider $\Lambda_1 \subset E_2$ to be those numbers whose continued fraction expansions do *not* have consecutive triples $(i_k i_{k+1} i_{k+2}) = (121)$ then $\mathbb{L}_{\sqrt{10}} = \mathbb{L} \cap [0, \sqrt{10}] \subset \Lambda_1 + \Lambda_1$ In particular, $\dim_H(\mathbb{L}_{\sqrt{10}}) \leq 2\dim_H(\Lambda_1)$, and we can estimate the numerical value of $\dim_H(\Lambda_1)$ by the method in Chapter 3. Similarly, if we consider $\Lambda_2 \subset E_2$ to be those numbers whose continued fraction expansions do *not* have consecutive quadruples $(i_k i_{k+1} i_{k+2} i_{k+3}) = (1212)$ then $\mathbb{L}_{\sqrt{689}/8} = \mathbb{L} \cap [0, \sqrt{689}/8] \subset \Lambda_2 + \Lambda_2$ and $\dim_H(\mathbb{L}_{\sqrt{10}}) \leq 2\dim_H(\Lambda_2)$. Using degree-16 truncated equations we can estimate $\dim_H(\Lambda_1) \approx 0.4060752878114$ and $\dim_H(\Lambda_2) \approx 0.4858259763$, giving the upper bounds on the dimension in the theorem. On the other hand, a result of Moreira-Yoccoz implies equality. $\square$

In particular the above result implies that:

Corollary. $\mathbb{L}_{\sqrt{689}/8}$ *has zero Lebesgue measure.*

Observe that $\sqrt{689}/8 \approx 3.2811...$ The strongest result in this direction is due to Bumby, who showed that $\mathbb{L}_{3.33437}$ has zero Lebesgue measure.

Remark. The triples (x, y, z) are known as *Markoff triples.* A closely related notion is that of the Markoff spectrum. $\mathbb{M}$. Consider quadratic forms $f(x, y) = ax^2 + bxy + cy^2$ (with $a, b, c \in \mathbb{Z}$) for which $d(f) := b^2 - 4ac > 0$. If we denote $m(f) = \inf|f(x, y)|$, then *Markoff spectrum* $\mathbb{M}$ is defined to be the set of all possible values of $\sqrt{d(f)}/m(f)$. which can be defined in terms of minima of certain indefinite quadratic forms. The Lagrange spectrum $\mathbb{L}$ is a closed subset of $\mathbb{R}$. It is clear from this definition that the Lagrange spectrum is a subset of the Markoff spectrum. It is in the interval $(3, \mu)$ where the Markof and Lagrange spectra differ. The largest known number in $\mathbb{M}$ but not in $\mathbb{L}$ is $\beta \approx 3.293$ (the number is known exactly).

4.2 Eigenvalues of the Laplacian and Kleinian groups. Given any Kleinian group Γ of isometries of n-dimensional hyperbolic space $\mathbb{H}^n$ we can associate the quotient manifold $M = \mathbb{H}^n/\Gamma$. The Laplacian $\Delta_M : C^\infty(M) \to C^\infty(M)$ is a self-adjoint second order linear differential operator. This extends to a self-adjoint linear operator Δ_M on the Hilbert space $L^2(M)$. In particular, the spectrum of $-\Delta_M$ is contained in the interval $[\lambda_0, +\infty)$, where λ_0 is the smallest eigenvalue. If M is compact then the constant functions are an eigenfunction and so $\lambda_0 = 0$. More generally, we can have $\lambda_0 > 0$.

Perhaps surprisingly, λ_0 is related to the Hausdorff dimension $\dim_H(\Lambda)$ of the Limit set by the following result.

Sullivan's Theorem. $\lambda_0 = \min\{d(1-d), 1/4\}$

McMullen's Example. This problem is very closely related to the geometry of an associated surface of constant curvature $\kappa = -1$. Consider the unit disk

$$\mathbb{D}^2 = \{x + iy \in \mathbb{C} : x^2 + y^2 < 1\}$$

with the Poincaré metric

$$ds^2 = 4\frac{dx^2 + dy^2}{(1 - x^2 - y^2)^2}$$

then $(\mathbb{D}^2, ds^2)$ has constant curvature $\kappa = -1$. Let $C_1, C_2, C_3 \subset \mathbb{C}$ be the three similar circles in the complex plane which meet the unit circle orthogonally and enclose an arc of length θ. We can identify the reflections in these circles with isometries $R_1, R_2, R_3 \subset \mathrm{Isom}(\mathbb{D}^2)$ and then consider the Kleinian group Γ_θ they generate. We can then let $M = \mathbb{D}^2/\Gamma$ be the quotient manifold.

The Laplacian $\Delta_M : C^\infty(M) \to C^\infty(M)$ is given by

$$\Delta_M = (1 - x^2 - y^2)^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right).$$

The smallest eigengvalue of $-\Delta_M$ is related to the dimension d of the boundary by Sullivan's Theorem. In particular, we have the following corollary.

Proposition 4.4. *When $\theta = \pi/6$ then we can estimate $\lambda_0 = 0.24922656...$*

Proof. In Chapter 3 we estimated that $\dim_H(\Lambda) = 0.4721891278821...$[3]. By applying Sullivan's Theorem, the result follows. $\square$

On can also study the asymptotic behavior of $\dim_H(\Lambda_\theta)$. McMullen showed the following:

Propositon 4.5. *The asymptotic behaviour of $dim_H(\Lambda_\theta)$ is described by the following result:*

(1)
$$dim_H(\Lambda_\theta) \sim \frac{1}{|\log\theta|} \text{ as } \theta \to 0;$$

(2)
$$dim_H(\Lambda_\theta) \sim 1 - \frac{1}{2}\left(\frac{2\pi}{3} - \theta\right) \text{ as } \theta \to \frac{2\pi}{3}.$$

(Equivalently, the associated smallest eigenvalue $\lambda_0(\theta)$ satisfies $\lambda_0(\theta) \sim \frac{1}{|\log\theta|}$ as $\theta \to 0$ and $\lambda_0(\theta) \sim \frac{1}{2}\left(\frac{2\pi}{3} - \theta\right)$ as $\theta \to \frac{2\pi}{3}$.)

Proof. For small θ, the radii of the circles C_i is well approximated by $\theta/2$. The derivative on C_j $(i \neq j)$ of the hyperbolic reflection in C_i is approximately $(\theta/2)^2/|C_i - C_j| \sim \theta^2/12$. Every periodic orbit $T^n x = x$ satisfies a uniform estimate $|(T^n)'(x)|^{1/n} \sim \theta^2/12$ from which we deduce that $P(-t\log|T'|)\ 2 - t(\theta^2/12)$, since there are 32^{n-1} periodic orbits of period n, for $n \geq 2$. Thus, solving for $2 - t\log(\theta^2/12) = 0$ gives that $t \sim \frac{1}{|\log\theta|}$.

The proof for $\theta \sim \frac{2\pi}{3}$ relies of Sullivan's theorem and asymptotic behaviour of the eigenvalues, as controlled by a minimax principle. In particular, $\dim_H(\Lambda_\theta) \sim 1 - \lambda_0(\theta) \to 1$. However, one can write $\lambda_0(\theta) = \inf_f \int |\nabla f|^2 d\mathrm{vol} / \int |f|^2 d\mathrm{vol} \sim l_\theta$, where l_θ is the length of the boundary curves on the quotient surface. For θ close to $2\pi/3$ on can estimate $l_\theta \sim \sqrt{2\pi/3 - \theta}$. $\square$

4.3 Limit sets for non-conformal maps. Let us now return to the problem of Hausdorff dimension for non-conformal maps, and examples of where number theoretic properties of parameters can lead to complicated behaviour.

Consider a family of affine maps $T_i x = a_i x + b_i$, $i = 1, \dots, k$, on $\mathbb{R}^2$. In particular, a_i is a $d \times d$ matrix and b_i is a vector in $\mathbb{R}^d$. Let Λ denote the limit set of this family of maps, defined precisely as before.

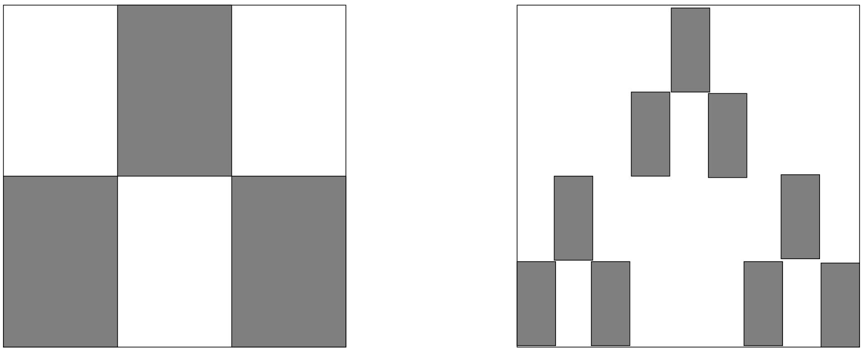
There are simple examples of affine maps where the dimension disagrees. The following is a simple illustration.

Example 1 (Bedford-McMullen). Consider the following three affine maps of $\mathbb{R}^2$:

$$T_i : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} c_i \\ d_i \end{pmatrix}, \quad i = 1, 2, 3,$$

where
$$\begin{pmatrix} c_1 \\ d_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} c_2 \\ d_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} c_3 \\ d_3 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ 0 \end{pmatrix}.$$

[3]McMullen previously estimated $d = \dim_H(X) = 0.47218913...$



THE FIRST TWO STEPS IN THE BEDFORD-MCMULLEN EXAMPLE

The limit set takes the form

$$\Lambda = \left\{ \left(\sum_{n=1}^{\infty} \frac{i_n}{3^n}, \sum_{n=1}^{\infty} \frac{j_n}{2^n} \right) \, : \, (i_n, j_n) \in \{(0,0), (1,1), (2,0)\} \right\},$$

and is closely related to what is called Hironaka's curve. The Box dimension and the Haudorff dimension of the limit set Λ can be explicitly computed in such examples, and be show to be different. More precisely,

$$\begin{aligned} \dim_H(\Lambda) &= \log_2(1 + 2^{\log_3 2}) = 1.34968\ldots \\ < \dim_B(\Lambda) &= 1 + \log_3(\frac{3}{2}) = 1.36907\ldots. \end{aligned}$$

This is part of more general result.

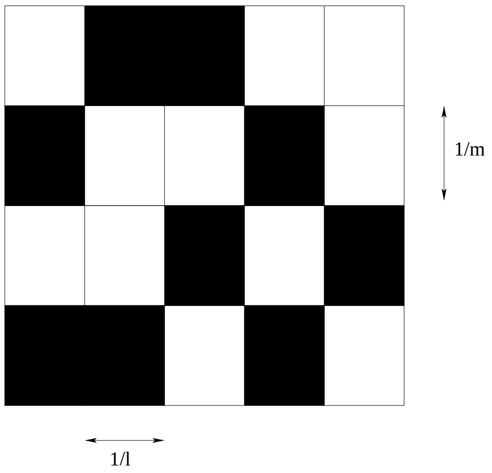
Bedford-McMullen Theorem. *Let $l > m \geq 2$ be integers. Given $\mathcal{S} \subset \{0, 1, \ldots, m-1\} \times \{0, 1, \ldots, l-1\}$ we can associate an affine "Sierpinski carpet":*

$$\Lambda = \left\{ \left(\sum_{n=1}^{\infty} \frac{i_n}{l^n}, \sum_{n=1}^{\infty} \frac{j_n}{m^n} \right) \, : \, (i_n, j_n) \in \mathcal{S} \right\}$$

Assume that every row contains a rectangle. If we denote $t_j = Card\{i : (i,j) \in \mathcal{S}\}$, and $a = Card(\mathcal{S})$ then

$$\begin{aligned} dim_H(\Lambda) &= \log_m \left(\sum_{j=0}^{m-1} t_j^{\log_l m} \right), \ \textit{and} \\ dim_B(\Lambda) &= 1 + \log_l \left(\frac{a}{m} \right) \end{aligned}$$

Proof. At the j the level of the construction we have S^j rectangles of size $l^{-j} \times m^{-j}$. Moreover, we can cover each rectangle by approximately $(l/m)^j$ squares of size m^{-j}. Moreover, because no rows are empty this many are needed.



THE GENERALIZED CONSTRUCTION OF BEDFORD-MCMULLEN

Thus for $\epsilon = l^{-j}$ we have that $N(l^{-j}) = a^j (l/m)^j$. Thus

$$\begin{aligned}\dim_B(\Lambda) &= \lim_{\epsilon\to 0} -\frac{\log N(\epsilon)}{\log \epsilon}\\ &= \lim_{j\to+\infty} \frac{\log(a(l/m)^j)}{\log l^j}\\ &= \frac{\log a}{\log l} + 1 - \frac{\log m}{\log l}\\ &= 1 + \log_l \frac{a}{m},\end{aligned}$$

as required. The calculation of $\dim_H(\Lambda)$ is a little more elaborate (and postponed). □

Example 2. One can consider "genericity" in the linear part of the affine map (rather than the translation). Consider contractions $T_1, T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T_i : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} c_i \\ d_i \end{pmatrix}, \quad i = 1, 2,$$

where $\lambda_1 < \lambda_2$.

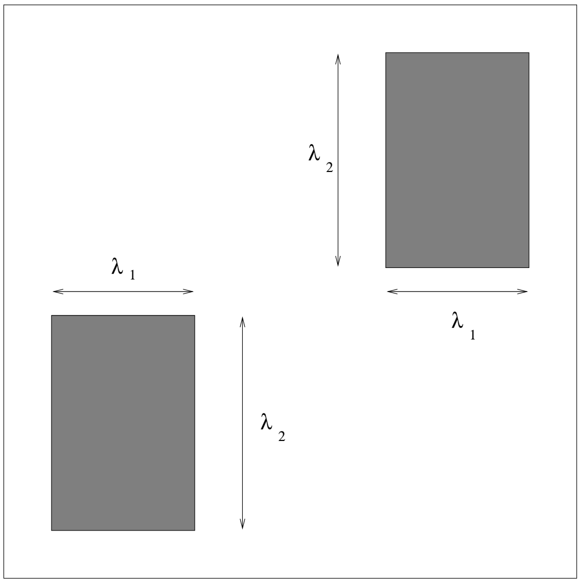
There are the following estimates on the Hausdorff and Box dimensions of the limit set.

Proposition 4.6. *For any choices $c_i, d_i \in \mathbb{R}$ $(i = 1, 2)$ we have:*

(1) *For $0 < \lambda_1 < \lambda_2 < \frac{1}{2}$, $dim_H(\Lambda) = dim_B(\Lambda) = -\frac{\log 2}{\log \lambda_2}$;*

(2) *For $0 < \lambda_1 < \frac{1}{2} < \lambda_2 < 1$, $dim_B(\Lambda) = -\frac{\log\left(\frac{2\lambda_2}{\lambda_1}\right)}{\log \lambda_1}$ and*

$$dim_H(\Lambda) \begin{cases} = -\frac{\log\left(\frac{2\lambda_2}{\lambda_1}\right)}{\log \lambda_1} & \textit{for almost every } \lambda_2, \textit{ but} \\ < -\frac{\log\left(\frac{2\lambda_2}{\lambda_1}\right)}{\log \lambda_1} & \textit{whenever } 1/\lambda_1 \textit{ is a Pisot number.} \end{cases}$$



Two affine contractions

A Pisot number is an algebraic number for which all the other roots of the integer polynomial defining it have modulus less than one. For example, $\frac{\sqrt{5}-1}{2}$ is a Pisot number.

Proof. For part (1), observe that since $\lambda_1 < \lambda_2 < \frac{1}{2}$ the projection onto the vertical axis is a homeomorphic to a Cantor set C in the line generated by two contractions with $\lambda_2 < \frac{1}{2}$. In particular, $\dim_H(\Lambda) \geq \dim_H C \geq \frac{\log 2}{\log \lambda_2}$. On the other hand, when $\lambda_2^{n-1} < \epsilon \leq \lambda_2^n$ we can cover Λ by 2^n ϵ-balls. In particular, $N(\epsilon) \leq 2^n$ and thus

$$\dim_H(\Lambda) \leq \dim_B(\Lambda) = \lim_{\epsilon \to 0} -\frac{\log N(\epsilon)}{\log \epsilon} \leq -\frac{\log 2}{\log \lambda_2}$$

The proof of the second part is postponed. $\square$

In particular, we conclude that

Corollary. *$\dim_B(\Lambda)$ is continuous in λ_1, λ_2, but $\dim_H(\Lambda)$ isn't.*

These examples are easily converted into estimates on limit sets for invertible maps (Smale horsehoes) in three dimensions, by "adding" a one dimensional expanding direction.

Example 3. We can also consider the case of more contractions. Assume that $T_i : \mathbb{R}^2 \to \mathbb{R}^2$, $i = 1, 2, 3, 4$ are defined by

$$T_i : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} c_i \\ d_i \end{pmatrix}, \quad i = 1, 2, 3, 4$$

where $\lambda_1 < \lambda_2 < \frac{1}{4}$. If we let

$$\begin{pmatrix} c_1 \\ d_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} c_2 \\ d_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{4} \end{pmatrix}, \begin{pmatrix} c_3 \\ d_3 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} c_4 \\ d_4 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{3}{4} \end{pmatrix},$$

then the limit set is the product of a point on the x-axis with a Cantor set on the y-axis (with Hausdorff dimension $-\log 4/\log \lambda_2$). In particular, $\dim_H(\Lambda) = -\log 4/\log \lambda_2$. On the other hand, if we let

$$\begin{pmatrix} c_1 \\ d_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} c_2 \\ d_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} c_3 \\ d_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}, \begin{pmatrix} c_4 \\ d_4 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix},$$

then the limit set is the product of a Cantor on the x-axis (of Hausdorff dimension $-\log 2/\log \lambda_1$) with a Cantor set on the y-axis (of Hausdorff dimension $-\log 2/\log \lambda_2$). In particular, $\dim_H(\Lambda) = -\log 2/\log \lambda_1 - \log 2/\log \lambda_2$.

Since $\lambda_1 \neq \lambda_2$, the dimensions of these two different limit sets do not agree, and we conclude that $\dim_H(\Lambda)$ depends not only on the contraction rates but also on the translational part of the affine maps.

5. Measures and Dimension

5.1 Hausdorff dimension of measures. Let μ denote a probability measure on a set X. We can define the Hausdorff dimension μ in terms of the Hausdorff dimension of subsets of Λ.

Definition. For a given probability measure μ we define the Hausdorff dimension of the measure by

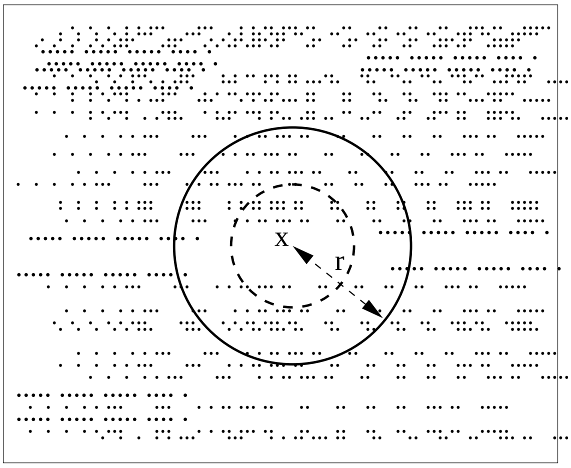
$$\dim_H(\mu) = \inf\{\dim_H(X) : \mu(X) = 1\}.$$

We next want to define a local notion of dimension for a measure μ at a typical point $x \in X$.

Definition. The upper and lower pointwise dimensions of a measure μ are measurable functions $\overline{d}_\mu, \underline{d}_\mu : X \to \mathbb{R} \cup \{\infty\}$ defined by

$$\overline{d}_\mu(x) = \limsup_{r \to 0} \frac{\log \mu(B(x,r))}{\log r} \text{ and } \underline{d}_\mu(x) = \liminf_{r \to 0} \frac{\log \mu(B(x,r))}{\log r}$$

where $B(x,r)$ is a ball of radius $r > 0$ about x.



The pointwise dimensions describe how the measure μ is distributed. We compare the measure of a ball about x to its radius r, as r tends to zero.

There are interesting connections between these different notions of dimension for measures.

Theorem 5.1. *If $\underline{d}_\mu(x) \geq d$ for a.e. (μ) $x \in X$ then $\dim_H(\mu) \geq d$.*

Proof. We can choose a set of full μ measure $X_0 \subset X$ (i.e., $\mu(X_0) = 1$) such that $\underline{d}_\mu(x) \geq d$ for all $x \in X_0$. In particular, for any $\epsilon > 0$ and $x \in X$ we have $\limsup_{r \to 0} \mu(B(x,r))/r^{d-\epsilon} = 0$. Fix $C > 0$ and $\delta > 0$, and let us denote

$$X_\delta = \{x \in X_) : \mu(B(x,r)) \leq C r^{d-\epsilon}, \quad \forall 0 < r \leq \delta\}.$$

Let $\{U_i\}$ be any δ-cover for X. Then if $x \in U_i$, $\mu(U_i) \leq C\mathrm{diam}(U_i)^{d-\epsilon}$. In particular,

$$\mu(X_\delta) \leq \sum_{U_i \cap X_\delta} \mu(U_i) \leq C \sum_i \mathrm{diam}(U_i)^{d-\epsilon}.$$

Thus, taking the infimum over all such cover we have $\mu(X_\delta) \leq CH_\delta^{d-\epsilon}(X_\delta) \leq CH^{d-\epsilon}(X)$. Now letting $\delta \to 0$ we have that $1 = \mu(X_0) \leq CH^{d-\epsilon}(X)$. Since $C > 0$ can be chosen arbitrarily large we deduce that $H^{d-\epsilon}(X) = +\infty$. In particular, $\dim_H(X) \geq d-\epsilon$ for all $\epsilon > 0$. Since $\epsilon > 0$ is arbitrary, we conclude that $\dim_H(X) \geq d$. $\square$

We have the following simple corollary, which is immediate from the definition of $\dim_H(\mu)$.

Corollary. *Given a set $X \subset \mathbb{R}^d$, assume that there is a probability measure μ with $\mu(X) = 1$ and $\underline{d}_\mu(x) \geq d$ for a.e. (μ) $x \in X$. Then $\dim_H(X) \geq d$.*

In the opposite direction we have that a uniform bound on pointwise dimensions leads to an upper bound on the Hausdorff Dimension.

Theorem 5.2. *If $\overline{d}_\mu(x) \leq d$ for a.e. (μ) $x \in X$ then $\dim_H(\mu) \leq d$.*

Moreover, if there is a probability measure μ with $\mu(X) = 1$ and $\overline{d}_\mu(x) \leq d$ for **every** *$x \in X$ then $\dim_H(X) \leq d$.*

Proof. We begin with the second statement. For any $\epsilon > 0$ and $x \in X$ we have $\limsup_{r \to 0} \mu(B(x,r))/r^{d+\epsilon} = \infty$. Fix $C > 0$. Given $\delta > 0$, consider the cover $\mathcal{U}$ for X by the balls

$$\{B(x,r) : 0 < r \leq \delta \text{ and } \mu(B(x,r)) > Cr^{d+\epsilon}\}.$$

We recall the following classical result.

Besicovitch covering lemma. There exists $N = N(d) \geq 1$ such that for any cover by balls we can choose a sub-cover $\{U_i\}$ such that any point x lies in at most N balls.

Thus we can bound

$$H_\delta^{d+\epsilon}(X) \leq \sum_i \mathrm{diam}\ (U_i)^{d+\epsilon} \leq \frac{1}{C} \sum_i \mu(B_i) \leq \frac{N}{C}.$$

Letting $\delta \to 0$ we have that $H^{d+\epsilon}(X) \leq \frac{N}{C}$. Since $C > 0$ can be chosen arbitrarily large we deduce that $H^{d+\epsilon}(X) = 0$. In particular, $\dim_H(X) \leq d + \epsilon$ for all $\epsilon > 0$. Since $\epsilon > 0$ is arbitrary, we deduce that $\dim_H(X) \leq d$.

The proof of the first statement is similar, except that we replace X by a set of full measure for which $\overline{d}_\mu(x) \leq d$. $\square$

Let us consider the particular case of iterated function schemes.

Example (Iterated Function Schemes and Bernoulli measures). For an iterated function scheme $T_1, \cdots, T_k : U \to U$ we can denote as before

$$\Sigma = \{\underline{x} = (x_m)_{m=0}^\infty : x_m \in \{1, \cdots, k\}\}$$

with the Tychonoff product topology. The shift map $\sigma : \Sigma \to \Sigma$ is a local homeomorphism defined by $(\sigma x)_m = x_{m+1}$. The kth level cylinder is defined by,

$$[x_0, \dots , x_{k-1}] = \{(i_m)_{m=0}^{\infty} \in \Sigma : \, i_m = x_m \text{ for } 0 \le m \le k-1\},$$

(i.e., all sequences which begin with $x_0, \dots , x_{k-1}$). We denote by $W_k = \{[x_0, \dots , x_{k-1}]\}$ the set of all kth level cylinders (of which there are precisely k^n).

Notation. For a sequence $\underline{i} \in \Sigma$ and a symbol $r \in \{1, \dots , k\}$ we denote by $k_r(\underline{i}) = \text{card}\{0 \le m \le k-1 : i_m = r\}$ the number of occurrences of r in the first k terms of $\underline{i}$.

Consider a probability vector $\underline{p} = (p_0, \dots , p_{n-1})$ and define the Bernoulli measure of any kth level cylinder to be,

$$\mu([i_0, \dots , i_{k-1}]) = {p_0}^{k_0(\underline{i})} {p_1}^{k_1(\underline{i})} \cdots {p_{n-1}}^{k_{n-1}(\underline{i})}.$$

A probability measure μ on σ is said to be invariant under the shift map if for any Borel set $B \subset X$, $\mu(B) = \mu(\sigma^{-1}(B))$. We say that μ is ergodic if any Borel set $B \subseteq \Sigma$ such that $\sigma^{-1}(X) = X$ satisfies $\mu(X) = 0$ or $\mu(X) = 1$. A Bernoulli measure is both invariant and ergodic.

Definition. For any ergodic and invariant measure μ on Σ the entropy of μ is defined to be the value

$$h_\mu(\sigma) = \lim_{k \to \infty} -\frac{1}{k} \sum_{\omega_k \in W_k} \mu(\omega_k) \log(\mu(\omega_k)).$$

In particular, for a Bernoulli measure μ associated to a probability vector $\underline{p} = (p_0, \dots , p_{n-1})$ the entropy can easily seen to be simply

$$h_\mu(\sigma) = -\sum_{i=0}^{n-1} p_i \log p_i.$$

An important classical result for entropy is the following.

Shannon-McMillan-Brieman Theorem. *Let μ be an ergodic σ-invariant measure on Σ. For μ almost all $\underline{i} \in \Sigma$,*

$$\lim_{k \to \infty} -\frac{1}{k} \log \mu([i_0, \dots , i_{n-1}]) = h_\mu(\sigma).$$

We can define a continuous map $\Pi : \Sigma \to \Lambda$ by $\Pi(\underline{i}) = \lim_{k \to \infty} T_{i_0} \cdots T_{i_k}(0)$. We can associated to a probability measure μ on Σ a measure ν on Λ defined by $\nu = \mu \circ \Pi_\lambda^{-1}$. In particular, when μ is a $\underline{p}$-Bernoulli measure the measure ν satisfies,

$$\nu(A) = \sum_{i=0}^{n-1} p_i \nu(f_i^{-1}(A)).$$

In the case where all the contractions $T_1, \dots , T_k$ are similarities it is possible to use the Shannon-Mcmillan-Brieman Theorem to get an upper bound on the Hausdorff dimension of ν. Let T_i have contraction ratio $|T_i'| = r_i < 1$, say, and let

$$\chi = \sum_{i=0}^{n-1} p_i \log r_i < 0$$

be the Lyapunov exponent of ν.

Proposition 5.3. *Consider a conformal linear iterated function scheme $T_1, \cdots, T_k$ satisfying the open set condition. Let ν be the image of a Bernoulli measure. Then*

$$\dim_H(\nu) = \frac{\sum_{i=0}^{n-1} p_i \log p_i}{\sum_{i=0}^{n-1} p_i \log r_i} \left(= \frac{h_\mu(\sigma)}{|\chi|} \right)$$

Without the open set condition we still get an inequality $\leq$.

Proof. The idea is to apply Theorem 5.1 and Theorem 5.2.

For two distinct sequences $\omega, \tau \in \Sigma$ we denote by $|\omega \wedge \tau| = \min\{k : \omega_k \neq \tau_k\}$ the first term in which the two sequences differ. For two sequences $\omega, \tau \in \Sigma$ we denote by $|\omega \wedge \tau| = \min\{k : \omega_k \neq \tau_k\}$ the first term in which the two sequences differ. Given $\omega, \tau \in \Sigma$ let $m = |\omega \wedge \tau|$, then we define a metric by

$$d(\omega, \tau) = \prod_{i=0}^{k-1} r_i^{m_i(\omega)} \left(= \prod_{i=0}^{k-1} r_i^{m_i(\tau)} \right).$$

A useful property of this metric d is that the diameter of any cylinder in the shift space is the same as the diameter of the projection of the cylinder in $\mathbb{R}^n$. Fix $\tau \in \Sigma_n$ and let $x = \Pi^{-1}\tau$. For $r > 0$ there exists $k(r)$ such that,

$$[i_1, \dots, i_{k(r)}, i_{k(r)+1}] \leq 2r \leq [i_1, \dots, i_{k(r)}]$$

and $k(r) \to \infty$ as $r \to 0$. Hence

$$\lim_{r\to 0} \frac{\log(\nu(B(x,r)))}{\log r} = \lim_{k\to\infty} \frac{\log(\mu([\tau_0, \dots, \tau_{k-1}]))}{\log(\operatorname{diam}([\tau_0, \dots, \tau_{k-1}]))}.$$

(Without the open set condition $\nu(B(x,r))$ can be much bigger than $\mu([\tau_1, \dots, \tau_{k(r)-1}])$.)

By the Shannon-McMillan-Brieman Theorem we have that,

$$\lim_{n\to\infty} \frac{1}{n} \log(\mu([\tau_0, \dots, \tau_{n-1}])) \to \sum_{i=0}^{n-1} p_i \log p_i = h_\mu(\sigma)$$

for μ almost all τ and by the Birkhoff Ergodic theorem we have that

$$\lim_{n\to\infty} \frac{1}{n} \log \operatorname{diam}[\tau_0, \dots, \tau_{n-1}] \to \sum_{i=0}^{n-1} p_i \log r_i = \chi$$

for μ almost all τ. Hence for μ almost all τ where $x = \Pi\tau$ (or equivalently, ν almost all x)

$$\lim_{r\to 0} \frac{\log(\nu(B(x,r)))}{\log r} = \frac{h_\mu(\sigma)}{\chi}.$$

Thus by Theorem 5.1 and Theorem 5.2 the result follows $\square$.

It is follows from the proof that we still get an upper bound $\dim_H(\nu)$ if we replace μ by any other ergodic σ-invariant measure on Σ or if we don't assume the Open Set Condition.

A more general statement is the following:

Proposition 5.4. *Let $T : X \to X$ be a conformal expanding map on a compact metric space. If μ is an ergodic invariant measure then the pointwise dimension $d_\mu(x)$ exists for μ-almost every x. Moreover*

$$d_\mu(x) = \frac{h_\mu(T)}{\int_X \log |T'| \, d\mu}$$

for μ-almost every x.

Proof. The proof follows the same general lines as above. Let $\mathcal{P} = \{P_1, \dots, P_k\}$ be an Markov partition for T and let $C_n(x) = \cap_{i=0}^{n-1} T^{-i} P_{x_i}$ be a cylinder set containing a point x. By the Shannon-McMillan Brieman theorem $-\frac{1}{n}\log \mu(C_n) \to h(\mu)$, a.e. (μ). By the Birkhoff Ergodic Theorem we expect $\frac{1}{n}\log|\mathrm{diam}(C_n)| \sim -\frac{1}{n}\log|(T^n)'(x)| \to \int \log|T'| d\mu$, a.e. (μ) $\square$

5.2 Multifractal Analysis. For a measure μ on a set X we can ask about the set of points x for which the limit

$$d_\mu(x) = \lim_{r \to 0} \frac{\log \mu(B(x,r))}{\log r}$$

exists. Let $X_\alpha = \{x : \text{ the limit } d_\mu(x) = \alpha\}$ be the set for which the limit exists, and equals α. There is a natural decomposition of the set X by "level sets":

$$X = \bigcup_{-\infty < \alpha < \infty} X_\alpha \cup \{x \in X \,|\, d_\mu(x) \text{ does not exist}\}\,.$$

To study this decomposition one defines the following:

Definition. The *dimension spectrum* is a function $f_\mu : \mathbb{R} \to [0, d]$ given by $f_\mu(\alpha) = \dim_H(X_\alpha)$, i.e., the Hausdorff dimension of the set X_α.

The "multifractal analysis" of the measure μ describes the size of the sets X_α through the behaviour of the function f_μ.

Example. Let us consider an iterated function scheme $T_1, \dots, T_k$ with similarities satisfying the open set condition. Consider the Bernoulli measure μ associated with the vector $(p_1, \dots, p_k)$. We have already seen that:

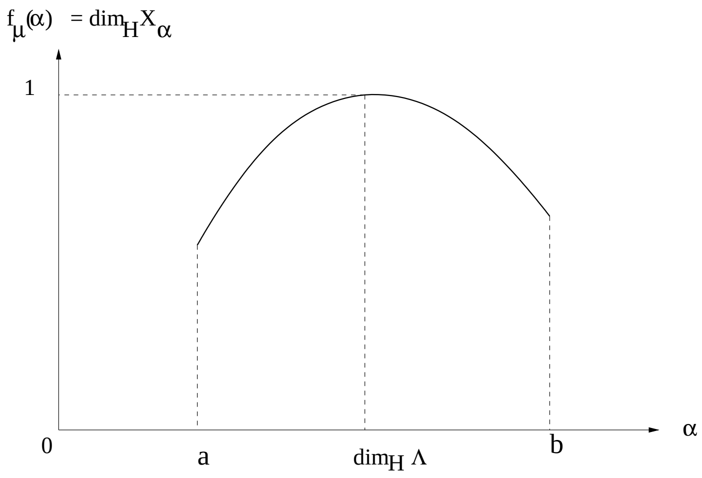
(1) $d_\mu(x)$ exists for a.e. (μ) x and is equal to $\dim_H(\mu)$. (In this particular case, this limit is equal to $\frac{\sum_{i=1}^k p_i \log p_i}{\sum_{i=1}^k p_i \log r_i}$).

We claim that the following is also true.

(2) Except in the very special case $p_i = r_i^{\dim_H(\Lambda)}$, for $i = 1, \dots, k$, there is an interval (a, b) containing $\dim_H(\Lambda)$ such that $f_\mu : (a, b) \to \mathbb{R}$ is analytic.

Sketch proof of (2). For each α, we can write

$$X_\alpha = \Pi \left\{ \underline{x} \in \Sigma : \lim_{n \to +\infty} \frac{\sum_{j=1}^n \log p_{x_j}}{\sum_{j=1}^n \log r_{x_j}} = \alpha \right\}.$$



Multifractal analysis describes the size of sets X_α for which the pointwise dimension is exactly α.

For each $q \in \mathbb{R}$, we can choose $T(q) \in \mathbb{R}$ such that $P(-T(q)\log|r_{x_0}|+q\log p_{x_0}) = 0$. There exists an associated Bernoulli measure ν_q and constants $C_1, C_2 > 0$ such that

$$C_1 \leq \frac{\nu_q([i_1, \cdots, i_n])}{\prod_{i=0}^{n-1} \exp\left(-T(q)\log r_{x_i} + q\log p_{x_i}\right)} \leq C_2. \tag{5.1}$$

Furthermore, we associate to q the particular value

$$\alpha(q) = \frac{\int \log p_{x_0} d\nu_q}{\int \log r_{x_0} d\nu_q}.$$

For a.e. (ν_q) $x \in X_{\alpha(q)}$ we have that $d_{\nu_q}(x) = \alpha(q)$ by the Birkhoff ergodic theorem and the definition of X_α. In particular, $\nu_q(X_\alpha) = 1$.[4] If $(r_1, \ldots, r_k) \neq (p_1, \ldots, p_k)$ then $f_\nu(\alpha)$ and $T(q)$ are strictly convex (and are Legendre transforms of each other).

We then claim that:

(a) $\alpha(q)$ is analytic
(b) $f_\nu(\alpha(q)) = (dim_H X_{\alpha(q)}) = T(q) + q\alpha(q)$.

and then (2) follows.

For part (a) observe that since $P(\cdot)$ is analytic, we deduce from the Implicit Function Theorem that the function $T(q)$ is analytic as a function of q. Observe that $T(0) = \dim_H X$. We can check by direct computation that $T'(q) \leq 0$ and $T''(q) \geq 0$.

Part (b) follows from the observation that $d_{\nu_q}(x) = T(q)+q\alpha(q)$ for a.e. $x \in K_\alpha$ and $\bar{d}_{\nu_q}(x) = T(q) + q\alpha(q)$ for all $x \in K_\alpha$ by (5.1). We then apply Theorem 5.1 and Theorem 5.2. □

Example: Expanding maps. Let $T\colon I \to I$ be an expanding transformation on the unit interval I. Let μ be a T-invariant ergodic probability measure. We say that μ is a *Gibbs measure* if $\phi(x) = \log \frac{d\mu T}{d\mu}$ is piecewise C^1 (or merely Hölder continuous would suffice). The most familiar example of a Gibbs measure is given by the following.

[4]We can also identify $\alpha(q) = -T'(q)$, then it has a range $[\alpha_1, \alpha_2] \subset \mathbb{R}^+$.

Proposition 5.5 ('Folklore Lemma'). *There is a unique absolutely continuous invariant probability measure ν (i.e., we can write $d\nu(x) = \rho(x)dx$).*

The main result is the following.

Proposition 5.6. *Assume that μ is a Gibbs measure (but not ν):*

1. *The pointwise dimension $d_\mu(x)$ exists for μ-almost every $x \in I$. Moreover, $d_\mu(x) = d_\mu \equiv h_\mu(T)/\int_X \log|T'|\, d\mu$ for μ-almost every $x \in I$.*
2. *The function $f_\mu(\alpha)$ is smooth and strictly convex on some interval $(\alpha_{\min}, \alpha_{\max})$ containing d_μ.* $\blacksquare$

Let ψ be a positive function defined by $\log\psi = \phi - P(\phi)$, where $P(\phi)$ denotes the pressure of ϕ. Clearly ψ is a Hölder continuous function on I such that $P(\log\psi) = 0$ and μ is also the equilibrium state for $\log\psi$. We define the two parameter family of Hölder continuous functions $\phi_{q,t} = -t\log|T'| + q\log\psi$. Define the function $t(q)$ by requiring that $P(\phi_{q,t(q)}) = 0$ and let μ_q be the equilibrium state for $\phi_{q,t(q)}$

5.3 Computing Lyapunov exponents. In many examples, the Lyapunov exponents $\int \log|T'(x)|d\mu(x)$ can be computed in much the same way that Hausdorff dimension was. More precisely, this integral can be approximated by periodic orbit estimates. In the interests of definiteness, consider the absolutely continuous T-invariant measure ν.

For definiteness, let us consider the case of the absolutely continuous invariant measure ν. We construct the family of approximating measures by a more elaborate regrouping of the periodic points to define new invariant probability measures. Let λ_n be the sequence of numbers given by

$$\lambda_n = \frac{\sum_{\substack{\underline{k}=(k_1,\dots,k_m)\\ k_1+\dots+k_m\le n}} \frac{(-1)^m r(\underline{k})}{m!}\left(\sum_{\substack{i=1,\dots,m\\ x\in\mathrm{Fix}(T^{k_i})}} k_i \log|T'(x)|\right)}{\sum_{\substack{\underline{k}=(k_1,\dots,k_m)\\ k_1+\dots+k_m\le n}} \frac{(-1)^m r(\underline{k})}{m!}\left(\sum_{\substack{i=1,\dots,m\\ x\in\mathrm{Fix}(T^{k_i})}} k_i\right)}$$

where we write

$$r(\underline{k}) = \prod_{j=1}^{m} \sum_{z\in\mathrm{Fix}(T^{k_j})} \frac{1}{k_j|(T^{k_j})'(z) - 1|}.$$

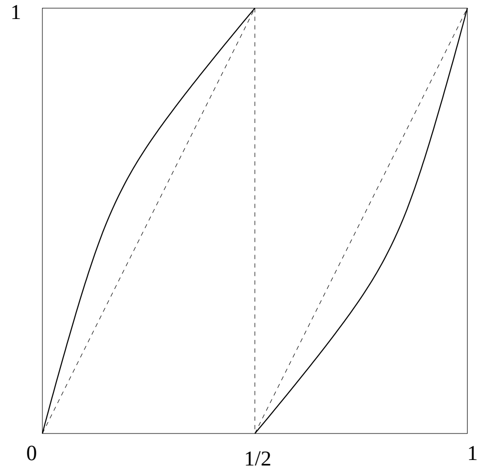
and $\mathrm{Fix}(T^n) = \{x \in [0,1] : T^n x = x\}$.
We have the following superexponentially converging estimate.

Theorem 5.7. *If $T : [0,1] \to [0,1]$ is a C^ω piecewise expanding Markov map with absolutely continuous invariant measure μ then there exists $C > 0$ and $0 < \theta < 1$ with $|\lambda_n - \int \log|T'|\, d\nu| \le C\theta^{n^2}$*

Example. Consider the family $T_{\frac{1}{4\pi}} : [0,1] \to [0,1]$ defined by

$$T_{\frac{1}{4\pi}}(x) = 2x + \varepsilon \sin 2\pi x \ (\mathrm{mod}\ 1),$$

for $-\frac{1}{2\pi} < \varepsilon < \frac{1}{2\pi}$.



A plot of the non-linear analytic expanding map of the interval $T_{\frac{1}{4\pi}}(x) = 2x + \varepsilon \sin 2\pi x \pmod 1$

We can estimate the Lyapunov exponent $\int \log |T'_{1/4\pi}|\, d\nu$ in terms of the estimates

$$\lambda_n \to \int \log |T'_{1/4\pi}|\, d\nu \text{ [super-exponential rate]}$$

n	using λ_n
6	0.6837719
7	0.68377196
8	0.68377196024
9	0.6837719602421451
10	0.6837719602421451396
11	0.683771960242145139619160
12	0.6837719602421451396191607l

6. Classic results: Projections, Slices and translations

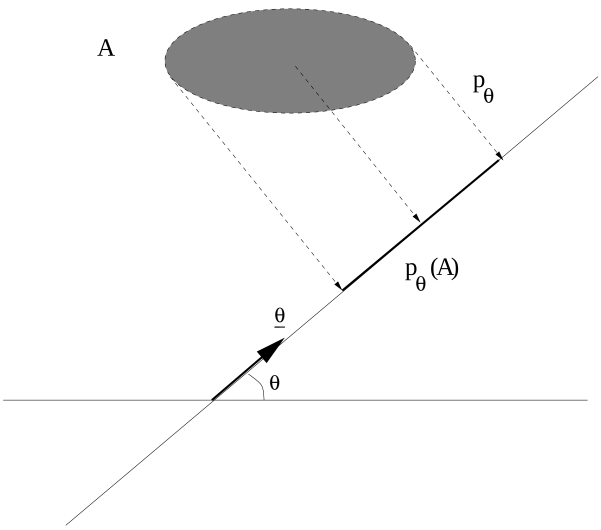
6.1 The Projection Theorem. We begin with one of the classical projection theorems. Let $A \subseteq \mathbb{R}^2$ and $p_\theta : \mathbb{R}^2 \to \mathbb{R}$ correspond to the linear projection onto the line at an angle θ to the x axis. More precisely, let $\underline{\theta} = (\cos\theta, \sin\theta)$ and for $\underline{x} = (x, y)$ we write $\underline{x} \cdot \underline{\theta} = (x\cos\theta + y\sin\theta)$

$$p_\theta : \mathbb{R}^2 \to \mathbb{R}$$
$$p_\theta(x, y) = \underline{x} \cdot \underline{\theta}$$

Let l denote one dimensional Lebesgue measure on the real line. We begin with the following result which shows that *if the set is small enough there is no drop in the Hausdorff dimension for typical directions.*

Theorem 6.1 (Projection Theorem). *Let $A \subset \mathbb{R}^2$ and $\dim_H A = s$.*

(1) *If $s \le 1$ then for almost all θ, $\dim_H p_\theta(A) = \dim_H A$*
(2) *If $s > 1$ then for almost all θ, $l(p_\theta(A)) > 0$.*



The set A is projected onto a one dimension line at an angle θ to the x-axis.

This result was first proved in 1954 by Marstrand. Kaufmann introduced an alternative method, which we will follow.

We begin with a preliminary lemma.

Lemma 6.2.

(1) *If $H^t(X) > 0$ there there exists a measure on μ on X such that*

$$\int_X \int_X \frac{d\mu(x)d\mu(y)}{|x-y|^t} < +\infty;$$

(2) *Conversely, if μ is a measure such that*

$$\int_X \int_X \frac{d\mu(x)d\mu(y)}{|x-y|^s} < +\infty$$

then $\dim_H(\mu) \ge s$.

Proof of Lemma 6.2. Assume that $H^t(X) > 0$. We require the following fact: *There exists a compact set $K \subset X$ with $0 < H^t(K) < +\infty$ and $b > 0$ such that* $H^t(K \cap B(x,r)) \leq br^t$ (We omit the proof).

Let $\mu = H^t|K$ be the restriction to K. For each $x \in K$ we define $\phi(x) = \int_K \int_K \frac{d\mu(x)d\mu(y)}{|x-y|^t}$. We can then bound:

$$\begin{aligned}\phi(x) &= \int_{|x-y|\leq 1} \frac{d\mu(x)d\mu(y)}{|x-y|^t} + \int |x-y| \geq 1 \frac{d\mu(x)d\mu(y)}{|x-y|^t} \\ &\leq \sum_{n=1}^{\infty} \frac{\mu(B(x, \frac{1}{2^n}))}{2^{n(t)}} + \mu(\mathbb{R}^n) \leq C\end{aligned}$$

for some constant $C > 0$. Thus we have that

$$\int_X \int_X \frac{d\mu(x)d\mu(y)}{|x-y|^t} = \int_X \int_X (\phi(x))\, d\mu(x) \leq C.$$

This completes the proof of Part (1).

To prove part (2), let us denote $\phi(y) = \int \frac{d\mu(x)}{|x-y|^s} \in L^1(\mu)$. In particular, by choosing $M > 0$ sufficiently large the set $A_M = \{y : \phi(y) \leq M\}$ we have that $\mu(A_M) > 0$. Let $\nu = \mu|A_M$ be the restriction to A_M. Then for all $x \in A$ and $r > 0$ we have

$$M \geq \int_{A_M} \frac{d\nu(x)}{|x-y|^s} \geq \int_{B(x,r)} \frac{d\nu(x)}{|x-y|^s} \geq \frac{1}{r^s}\nu(B(x,r))$$

In particular, $\nu(B(x,r)) \leq Mr^s$ for all $r > 0$. Thus by the Mass Distribution Principle we have that $\dim_H(A) \geq s$. This completes the proof. □

Proof of Theorem 6.1. For part (1), let $A \subset \mathbb{R}^2$ where $\dim A = s < 1$. Let $\epsilon > 0$ and denote $t = s - \epsilon$. From the definition of Hausdorff dimension we know that $H^t(A) > 0$. Thus by Lemma 6.1 there exists a probability measure $\mu \subset \mathbb{R}^2$ on A such that

$$\int_A \int_A \frac{\mathrm{d}\mu(x)\mathrm{d}\mu(y)}{|x-y|^t} < \infty.$$

We define $\mu_\theta = p_\theta \mu$ to be the projection of the measure μ onto the line $\mathbb{R}$ (i.e., $\mu_\theta(I) = \mu(p - \theta^{-1}I)$ for any interval $I \subset \mathbb{R}$). In particular,

$$\mu_\theta([a,b]) = \mu(A \cap p_\theta^{-1}([a,b])) = \mu\{x \in A : a \leq x \cdot \underline{\theta} \leq b\}.$$

To show that for a given θ we have that $\dim_H(p_\theta A) > t$ it suffices to show that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\mathrm{d}\mu_\theta(u)\mathrm{d}\mu_\theta(v)}{|u-v|^t} < \infty, \tag{6.1}$$

the result then follows by Lemma 6.1. In particular, if we can show that

$$I := \int_0^{\pi} \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\mathrm{d}\mu_\theta(u)\mathrm{d}\mu_\theta(v)}{|u-v|^t} \right) \mathrm{d}\theta < \infty \tag{6.2}$$

then by Fubini's Theorem we have for almost all θ the inner integral (6.1) is finite, and the result follows. From the definition of μ_θ we can rewrite this as

$$I = \int_0^\pi \int_A \int_A \frac{\mathrm{d}\mu(x)\mathrm{d}\mu(y)\mathrm{d}\theta}{|x \cdot \underline{\theta} - y \cdot \underline{\theta}|^t} = \left(\int_0^\pi \frac{\mathrm{d}\theta}{|\underline{\theta}.\underline{\tau}|^t} \right) \int_A \int_A \frac{\mathrm{d}\mu(x)\mathrm{d}\mu(y)}{|x-y|^t}$$

However, we know by (6.1) that the second part of this last term is finite. Thus it remains to show that,

$$\int_0^\pi \frac{\mathrm{d}\theta}{|\underline{\theta}.\underline{\tau}|^t} < \infty.$$

We know that,

$$\int_0^\pi \frac{\mathrm{d}\theta}{|\underline{\tau} \cdot \underline{\theta}|^t} = \int_0^\pi \frac{\mathrm{d}\theta}{|\cos(\tau - \theta)|^t}$$

The derivative of $\cos(\tau - \theta)$ is bounded away from 0 when $\cos(\tau - \theta)$ is equal to 0 so when $|\cos(\tau - \theta)|^t$ is close to 0 it can be bounded below by Cx^t for some $C > 0$. Since $t < 1$ this means

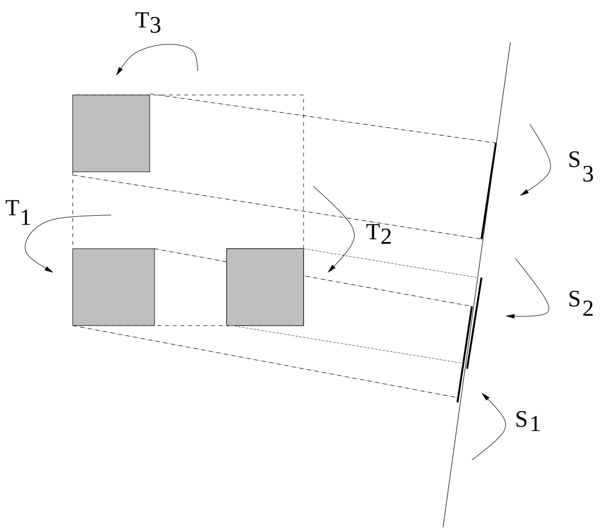
$$\int_0^\pi \frac{\mathrm{d}\theta}{|\cos(\tau - \theta)|^t} < \infty.$$

Thus $I < \infty$ for any $t < s$ and so the proof is complete. We omit the proof of part (2) $\square$

Example. Consider the iterated function scheme in $\mathbb{R}^2$ given by contractions T_1, T_2, T_3 of the form

$$\begin{aligned} T_1(x, y) &= (x/3, y/3) \\ T_2(x, y) &= (x/3, y/3) + (0, 1) \\ T_3(x, y) &= (x/3, y/3) + (1, 0) \end{aligned}$$

and let $\Lambda \subset \mathbb{R}^2$ be the associated Limit set. Since the iterated function scheme holds we know that this set has Hausdorff dimension $\dim_H(\Lambda) = 1$.



For the iterated function scheme T_1, T_2, T_3 we know the Hausdorff Dimension of the limit set (since Moran's Theorem applies). Thus for "typical" λ be know the Hausdorff Dimension of the limit set for S_1, S_2, S_3.

Consider the projection $p_\theta : \mathbb{R}^2 \to \mathbb{R}$ onto the line at an angle θ. The image limit set $p_\theta(\Lambda) \subset \mathbb{R}$ is the limit set for the iterated function scheme on $\mathbb{R}$ given by contractions T_1, T_2, T_3 of the form

$$S_1(x) = x/3$$
$$S_2(x) = x/3 + 1$$
$$S_3(x) = x/3 + \lambda$$

(up to scaling the line by $\cos\theta$) where $\lambda = \tan\theta$ on the real line. Let us denote $\Lambda_\lambda = p_\theta(\Lambda)$.

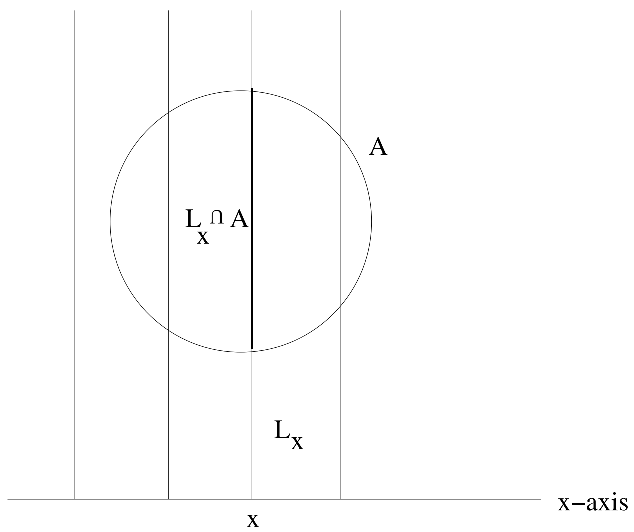
The open set condition does not apply in this case. However, from Theorem 6.1 we can deduce that for a.e. λ (or equivalently for a.e. θ) we have that $\dim_H(\Lambda) = 1$. Clearly, this cannot be true for all λ. For example, when $\lambda = 0$ then $S_1 = S_2$ and the iterated function scheme has a limit set consisting only of a Cantor set (the middle $(1 - 2\lambda)$ Cantor set) with Hausdorff Dimension $-\log 2/\log\lambda$.

There is a natural generalization to projections $p : \mathbb{R}^n \to \mathbb{R}^m$.

6.2 The Slice Theorem. Assume that $A \subset \mathbb{R}^2$ has dimension $\dim_H(A)$. Let $L_x = \{(x, y) : y \in \mathbb{R}\}$ be a vertical line. We can make the following assertion about the dimension of a typical intersection $A \cap L_x$.

The next theorem shows that *if the set is large enough then typical slices have dimensions that drop by at least* 1.

Theorem 6.3 (Marstrand's Slice Theorem). *Assume that* $\dim_H(A) \geq 1$*, then for almost every* $x \in \mathbb{R}$ *we have that* $\dim_H(A \cap L_x) \leq \dim_H(A) - 1$.



For a typical vertical slice through a large set A the dimension of the slice drops by at least 1.

We begin with a preliminary result

Lemma 6.4. *For $1 \leq \alpha \leq 2$ we can write*

$$H^\alpha(A) \geq \int H^{\alpha-1}(A \cap L_x)dx$$

Proof of Lemma 6.4. Given $\epsilon, \delta > 0$, let $\{U_i\}$ be an open cover of A with $\mathrm{diam}(U_i) < \epsilon$ and such that

$$\sum_i \mathrm{diam}(U_i) \leq H^\alpha_\epsilon(A) + \delta.$$

We can cover each U_i by a square $I_i \times J_i$ aligned with the axes (whose sides are of length l_i at most the diameter of U_i, i.e., $\mathrm{diam}(U_i) < \epsilon$).

Consider a function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = \sum_i \chi_{I_i \times J_i}(x, y) l_i^{\alpha-2},$$

where

$$\chi_{I_i \times J_i}(x, y) = \begin{cases} 1 & \text{if } x \in I_i, y \in J_i \\ 0 & \text{otherwise.} \end{cases}$$

The sets $\{L_x \cap (I_i \times J_i)\}$ form a cover for $L_x \cap A$ of diameter $\epsilon > 0$. Thus using this cover we have that

$$H^{\alpha-1}_\epsilon(A \cap L_x) \leq \sum_{\{i \,:\, x \in I_i\}} l_i^{\alpha-1}. \tag{6.3}$$

For a fixed x we have

$$\int_{-\infty}^{\infty} f(x, y)dy = \int_{-\infty}^{\infty} \left(\sum_i \chi_{I_i \times J_i}(x, y) l_i^{\alpha-2} \right) dy = \epsilon \sum_{\{i \,:\, x \in I_i\}} l_i^{\alpha-1}$$

Thus we have that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)dxdy = \epsilon \int_{-\infty}^{\infty} \left(\sum_{i \,:\, x \in I_i} l_i^{\alpha-1} \right) dx$$

In particular, by (6.3) we have that

$$\begin{aligned} \int H^{\alpha-1}(A \cap L_x)dx &\leq \int_{-\infty}^{\infty} \left(\sum_{i \,:\, x \in I_i} l_i^{\alpha-1} \right) dx = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)dxdy \\ &\leq \sum_i l_i^{\alpha-2} l_i^2 = \sum_i l_i^\alpha \\ &\leq H^\alpha(A) + \delta \end{aligned}$$

using that $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)dxdy = \sum_i \mathrm{Area}(I_i \times J_i) l_i^{\alpha-2}$.

Letting $\delta \to 0$ gives

$$\int H^{\alpha-1}_\epsilon(A \cap L_x)dx \leq H^\alpha_\epsilon(A)$$

Letting $\epsilon \to 0$ gives that $H_\epsilon^{\alpha-1}(A) \nearrow H_\epsilon^{\alpha-1}(A)$ and so

$$\int H^{\alpha-1}(A \cap L_x)dx \leq H^\alpha(A)$$

This completes the proof of the lemma. $\square$

Proof of Theorem 6.3. Let $\alpha > \dim_H(A)$ then by Lemma 6.3:

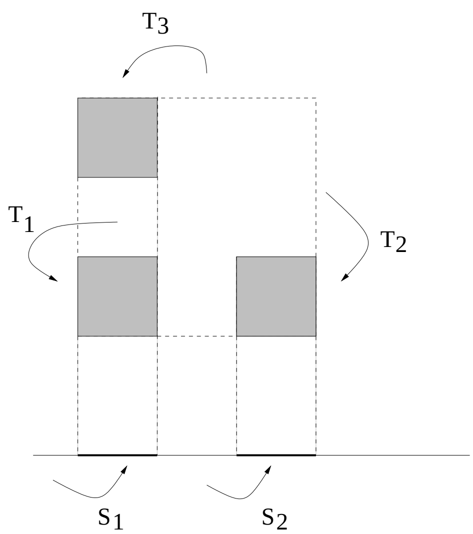
$$0 = H^\alpha(A) = \int_{-\infty}^{\infty} H^{\alpha-1}(A \cap L_x)dx.$$

Thus, by Fubini's Theorem $H^{\alpha-1}(A \cap L_x) = 0$ for a.e. x. In particular, $\dim_H(A \cap L_x) \leq \alpha - 1$ for such x, as required. $\square$

Example. Fix $\frac{1}{3} < \lambda < \frac{1}{2}$. Consider the iterated function scheme in $\mathbb{R}^2$ given by contractions T_1, T_2, T_3 of the form

$$\begin{aligned} T_1(x,y) &= (\lambda x, \lambda y) \\ T_2(x,y) &= (\lambda x, \lambda y) + (0,1) \\ T_3(x,y) &= (\lambda x, y) + (1,0) \end{aligned}$$

and let $\Lambda \subset \mathbb{R}^2$ be the associated Limit set. Since $\lambda < \frac{1}{2}$ the Open Set Condition holds and by Moran's Theorem we know that the Limit set Λ has Hausdorff dimension $\dim_H(\Lambda) = -\frac{\log 3}{\log \lambda} > 1$. Let us take the vertical slices $L_x \cap \Lambda$ through this limit set.



The dimension drop on typical slices is strictly greater than 1.

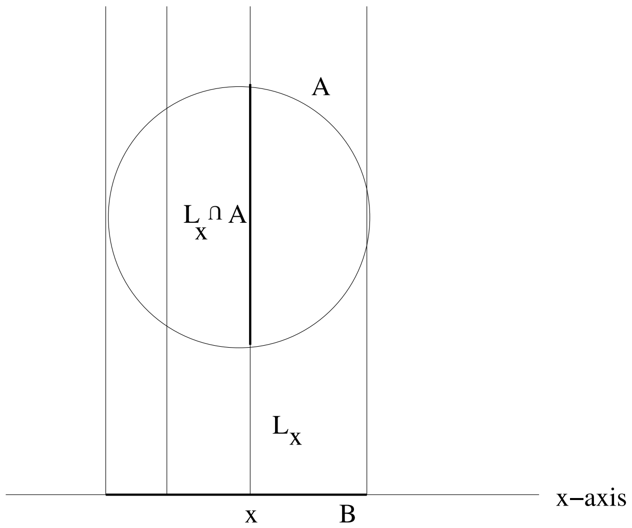
The projection onto the x-axis is a middle $(1 - 2\lambda)$ Cantor set X. For $x \in X$ the Haudorff Dimension $\dim_H(L_x \cap \Lambda)$ is in the range $[0, -\frac{\log 2}{\log \lambda}]$. However, X has zero measure. On the complement $\mathbb{R} - X$ we have that $L_x \cap \Lambda = \emptyset$. In particular, $\dim_H(L_x \cap \Lambda) = 0 < \dim_H(\Lambda) - 1$ (a strict inequality).

Assume that $A \subset \mathbb{R}^2$ has dimension $\dim_H(A)$. Again, let $L_x = \{(x,y) : y \in \mathbb{R}\}$ be a vertical line. The following relates $\dim_H(A)$ to typical values $\dim_H(A \cap L_x)$ for a typical x, with respect to a more general measure μ.

Theorem 6.5 (Generalized Marstrand's Slice Theorem). *Let $B \subset \mathbb{R}$. Assume that μ is a probability measure on B and $C > 0$ with $\mu(I) \leq C(\mathrm{diam}(I))^\alpha$, for intervals $I \subset \mathbb{R}$. If $A \subset \mathbb{R}^2$ then*

$$\dim_H(A) \geq \alpha + \dim_H(A \cap L_x)$$

the for almost every $x \in B$ with respect to μ.



For a typical vertical slice through a large set A (relative to a measure μ on B) the dimension of the slice drops by at least the value α (depending on the measure μ).

Proof. The proof is similar to that of Theorem 6.3. Fix $\gamma > \dim_H(A)$. If we can show that

$$\int H^{\gamma-\alpha}(A \cap L_x) d\mu(x) < +\infty$$

then by Fubini's Theorem $H^{\gamma-\alpha}(A \cap L_x) < +\infty$ for a.e. (μ) x. In particular, $\dim_H(A \cap L_x) \leq \gamma - \alpha$ for a.e. (μ) x, by definition.

We can cover B by squares $I_i \times J_i$ aligned with the axes whose side lengths l_i satisfy $\sum_i l_i^\gamma < \epsilon$. If we define

$$f(x, y) = \sum_i \chi_{I_i \times J_i}(x, y) l_i^{\gamma-\alpha-1}$$

then we can write

$$\begin{aligned} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dy d\mu(x) &= \sum_i l_i^{\gamma-\alpha-1} \mathrm{diam}(A_i) \mu(B_i) \\ &\leq C \sum_i l_i^\alpha \leq C\epsilon \end{aligned} \tag{6.4}$$

We can denote

$$Q_i^x = \begin{cases} J_i & \text{if } x \in I_i \\ \emptyset & \text{otherwise} \end{cases}$$

then these sets form cover of $F \cap L_x$. By Fubini's theorem we can interchange integrals and write

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dyd\mu(x) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dyd\mu(x)\\ &= \int \left(\sum_i \text{diam } (Q_i^x)^{\gamma-\alpha}\right) d\mu(x) \qquad (6.5)\\ &\geq \int H_\epsilon^{\gamma-\alpha}(L_x \cap F)d\mu(x).\end{aligned}$$

Thus by (6.4) and (6.5):

$$0 \leq \int H_\epsilon^{\gamma-\alpha}(L_x \cap F)d\mu(x) \leq C\epsilon$$

Finally, letting $\delta \to 0$ gives

$$\int H_\epsilon^{\gamma-\alpha}(F \cap L_x)dx \leq H_\epsilon^{\gamma-\alpha}(F),$$

and then letting $\epsilon \to 0$ gives

$$\int H^{\gamma-\alpha}(F \cap L_x)dx = 0.$$

Thus Fubini's Theorem gives that the integrand is finite almost everywhere, i.e., $H^{\gamma-\alpha}(F \cap L_x) = 0$ for a.e. (μ) x. In particular, $\dim_H(A \cap L_x) \leq \gamma - \alpha$ for a.e. (μ) x. Since γ can be chosen arbitrarily close to $dim_H(A)$ this completes the proof. □

The slicing theorems generalize to k-dimensional slices of sets in R^n.

6.3 Falconer's Theorem. We shall formulate a simple version of this result in one dimension, although a version is valid in arbitrary dimensions.

Let us fix $0 < \lambda < \frac{1}{2}$. We want to consider affine maps $T_i : \mathbb{R} \to \mathbb{R}$ $(i = 1, \ldots, k)$ of the real line $\mathbb{R}$ defined by $T_i x = \lambda x + b_i$, for $i = 1, \ldots, k$, where $b_1, \ldots, b_k \in \mathbb{R}$. Let us use the notation $\underline{b} = (b_1, \ldots, b_k) \in \mathbb{R}^k$ and then let us denote by $\Lambda_{\underline{b}}$ the associated limit set.

Theorem 6.6 (Falconer's Theorem). *For almost all* $\underline{b} = (b_1, \ldots, b_k) \in \mathbb{R}^k$ *we have that* $\dim_H \Lambda_{\underline{b}} = -\log k/\log\lambda$.

Of course, this if $T_1, \ldots, T_k$ satisfy the Open Set Condition then the formula for Hausdorff Dimension automatically holds by Moran's Theorem.

Proof of Theorem 6.6. Let $U \subset \mathbb{R}$ be an open set chosen such that $T_i U \subset U$ for all $1 \leq i \leq k$. Given $\delta > 0$ we can choose n sufficiently large that $\lambda^n \text{diam}(U) \leq \delta$. Let us cover $\Lambda_{\underline{b}}$ by open sets $\{T_{\underline{i}}(U) : |\underline{i}| = n\}$. Given $s > 0$ can estimate

$$H_\delta^s(\Lambda_{\underline{b}}) \leq \sum_{|\underline{i}|=n} \text{diam}(U_{\underline{i}}) \leq (k\lambda^s)^n$$

In particular, for any $s > -\log k/\log\lambda$ we have that $(k\lambda^s) < 1$ and so we deduce that $\dim_H \Lambda_{\underline{b}} \leq s|$. In particular, $\dim_H \Lambda_{\underline{b}} \leq -\log k/\log\lambda$.

On the other hand, let us consider the Bernoulli measure $\nu = (\frac{1}{k}, \cdots, \frac{1}{k})^{\mathbb{Z}^+}$ on the associate sequence space $\Sigma = \{1, \dots, k\}^{\mathbb{Z}^+}$. Let $\pi_{\underline{b}} : \Sigma \to \Lambda_{\underline{b}}$ be the natural coding given by $\pi_{\underline{b}}(\underline{i}) = \lim_{n\to+\infty} T_{i_0} \cdots T_{i_n}(0)$. We can consider the associated measure $\mu_{\underline{b}} = \pi_{\underline{b}}\nu$ (i.e., $\mu_{\underline{b}}(I) = (\pi_{\underline{i}}^{-1} I))$. Let us fix $s > -\log k / \log \lambda$ For any $R > 0$ we can write

$$\int_{|\underline{b}|\le R} \left(\int_{\Lambda_{\underline{b}}} \int_{|\underline{b}|\le R} \frac{d\mu_(\underline{b})(x) d\mu(\underline{b})}{|x-y|^s} \right) d\underline{b} = \int_{|\underline{b}|\le R} \left(\int_{\Sigma} \int_{\Sigma} \frac{d\nu(\underline{i}) d\nu(\underline{j})}{|\pi_{\underline{b}}(\underline{i}) - \pi_{\underline{b}}(\underline{j})|^s} \right) d\underline{b},$$

where we integrate over the ball of radius R with respect to lebesgue measure. Moreover, using Fubini's theorem we can reverse the order of the integrals in the last expression to get

$$\int_{\Sigma} \int_{\Sigma} \left(\int_{|\underline{b}|\le R} \frac{d\underline{b}}{|\pi_{\underline{b}}(\underline{i}) - \pi_{\underline{b}}(\underline{j})|^s} \right) d\nu(\underline{i}) d\nu(\underline{j}) \tag{6.6}$$

If the sequences $\underline{i}$, $\underline{j}$ agree in the first n spaces (but differ in the $(n+1)$st place) then we can write

$$\pi_{\underline{b}}(\underline{i}) - \pi_{\underline{b}}(\underline{i}) = \lambda^{n+1} \left((b_{i_{n+1}} - b_{j_{n+1}}) + \sum_{m=1}^{\infty} \lambda^m (b_{i_{n+m+1}} - b_{j_{n+m+1}}) \right)$$

where $b_{i_{n+1}} \neq b_{j_{n+1}}$ are distinct elements from $\{b_1, \dots, b_k\}$. In particular, differentiating in the direction corresponding to $b_{i_{n+1}}$ (whilst fixing the other directions) we see that

$$\begin{aligned} \left| \frac{\partial(\pi_{\underline{b}}(\underline{i}) - \pi_{\underline{b}}(\underline{i}))}{\partial b_{i_{n+1}}} \right| &= \lambda^{n+1} \left| \left(1 + \sum_{m=1}^{\infty} \lambda^m \frac{\partial(b_{i_{n+m+1}} - b_{j_{n+m+1}})}{\partial b_{i_{n+1}}} \right) \right| \\ &\ge \lambda^{n+1} \left(1 - \sum_{m=1}^{\infty} \lambda^m \right) \ge C\lambda^{n+1} \end{aligned}$$

for some $C > 0$. We can then write

$$\int_{|\underline{b}|\le R} \frac{d\underline{b}}{|\pi_{\underline{b}}(\underline{i}) - \pi_{\underline{b}}(\underline{i})|^s} \le D\lambda^{-s(n+1)} \tag{6.7}$$

for some $D > 0$. Substituting (6.7) into (6.6) we have that

$$\begin{aligned} \int_{\underline{b}} \left(\int_{\Lambda_{\underline{b}}} \int_{\Lambda_{\underline{b}}} \frac{d\mu_{\underline{b}}(x) d\mu_{\underline{b}}(y)}{|x-y|^s} \right) d\underline{b} &\le C^s \int_{\Sigma} \left(\sum_{n=1}^{\infty} \sum_{i_0,\dots,i_n} \underbrace{\mu[i_0, \dots, i_n]}_{=(\frac{1}{k})^{n+1}} \lambda^{-s(n+1)} \right) d\mu(\underline{i}) \\ &\le C^s \sum_{n=1}^{\infty} \left(\frac{\lambda^{-s}}{k} \right)^{n+1} < +\infty \end{aligned}$$

By Fubini's Theorem we deduce that for almost every $\underline{b}$ we have that the integrand is finite almost everywhere, i.e.,

$$\int_{\Lambda_{\underline{b}}} \int_{\Lambda_{\underline{b}}} \frac{d\mu_{\underline{b}}(x) d\mu_{\underline{b}}(y)}{|x-y|^s} < +\infty$$

provided $s < -\log k / \log \lambda$. In particular, we deduce from lemma 6.2 that for such $\underline{b}$ we have $\dim_H(\Lambda_{\underline{b}}) > s$. Since s can be chosen arbitrarily close to $-\log k / \log \lambda$ the result follows. □

This Theorem also has a natural extension to higher dimensions.

7. One dimensional Iterated Function Schemes with overlaps

In this chapter we shall consider one dimensional iterated function schemes with over laps (i.e., such that the Open set condition fails). In this context we will concentrate on two particular examples. We will be interested in: the Hausdorff dimension of the limit set; and the properties of naturally associated measures (absolute continuity, dimension, etc.), The key tool in our study here is the application of the so called "transversality method" which helps in showing certain integrals are finite. We have already seen this in another guise, in the proofs in the previous chapter.

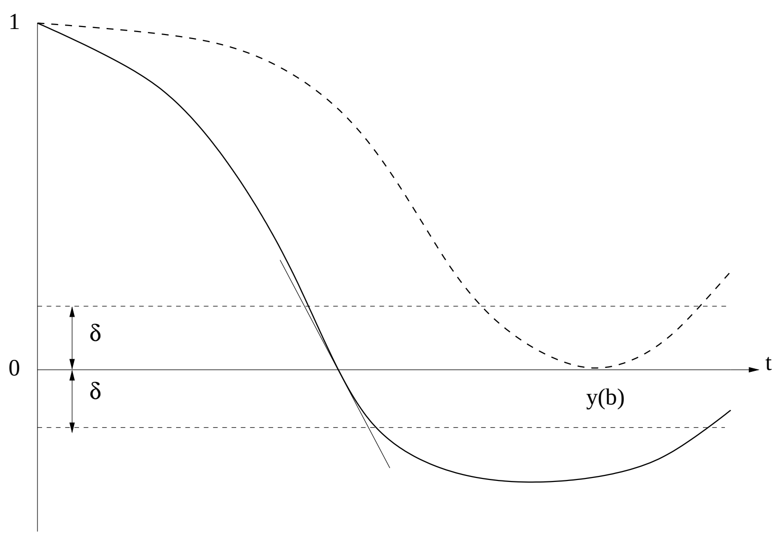
7.1 Transversality: Properties of Power Series. A general result about when specific power series satisfy a transversality condition is given. Let $\mathcal{F}_b$ be a family of analytic functions such that $f(0) = 1$ and whose coefficients are real numbers that lie all in an interval $[-b, b]$, for some $b > 0$, i.e.,

$$\mathcal{F}_b = \left\{ f(t) = 1 + \sum_{k=1}^{\infty} c_k t^k \,:\, c_k \in [-b, b] \right\}.$$

In practise, we shall only need to consider the case where $b \in \mathbb{N}$. Of course, every function $f \in \mathcal{F}_b$ converges on the interval $(-1, 1)$.[5] We now define,

$$y(b) = \min\{x > 0 : \exists f \in \mathcal{F}_b \text{ where } f(x) = f'(x) = 0\},$$

i.e., the first occurrence of a double zero for any function $\mathcal{F}_b$.



The dotted line shows the function which has the first double zero (at $y(b)$). Any other function which gets δ-close to the horizontal axis before $y(b) - \epsilon$ must have slope at least δ (in modulus).

[5]Of course, the power series converges on the unit disk D on the complex plane. As an aside, we recall that any analytic function $F : D \to \mathbb{C}$ which is simple (i.e., it is one-one onto its image) must necessarily have a bound on its coefficients of the form $|c_k| \leq k$ (Bieberbach Conjecture)

The basic idea is that we can deal with real valued functions $f \in \mathcal{F}_b$ on an interval $[0, y(b) - \epsilon]$, for any $\delta > 0$, which have the property that when they cross the x-axis their slope has to be bounded away from zero. For example, when $\delta > 0$ a function is said to be δ-transversal if whenever its graph comes within δ of t-axis then its slope is at most $-\delta$ or at least δ (i.e, $|f(t)| \leq \delta$ implies $|f'(t)| \geq \delta$). In particular, given $\epsilon > 0$ we can find $\delta = \delta(\epsilon)$ such that every $f \in \mathcal{F}_b$ is δ-transversal on $[0, y(b) - \epsilon]$.

Claim. *It is possible to numerically compute $y(1) \approx 0.649\ldots$ and also to show that $y(2) = 0.5$.*

Example. Consider the series $f(t) = 1 - \sum_{k=1}^{\infty} t^k = 1 - \frac{t}{1-t} \in \mathcal{F}_1$ (with $b = 1$). The first zero is at $t = \frac{1}{2} < y(1)$ but the derivative $f'(t) = -\frac{1}{(1-t)^2}$ takes the value $f'(\frac{1}{2}) = -4 < 0$.

Approach to Claim. To illustrate the method consider the case $b = 1$. The basic idea is to consider functions $h \in \mathcal{F}_1$ of the special form

$$h(x) = 1 - \underbrace{\sum_{i=1}^{k-1} x^i}_{\frac{x - x^{k+1}}{1-x}} + a_k x^k + \underbrace{\sum_{i=k+1}^{\infty} x^i}_{\frac{x^{k+1}}{1-x}} \tag{7.1}$$

with $|a_k| \leq 1$. We claim that if we can find any such function, a value $0 < x_0 < 1$ and $0 < \delta < 1$ such that $h(x_0) > \delta$ and $h'(x_0) < -\delta$ then $y(1) \geq x_0$. More precisely, for $f \in \mathcal{F}_b$ we have that if $g(x) < \delta$ then $g'(x) < -\delta$.

Observation: By construction $h''(x)$ is a power series with at most one sign change, and thus has at most one zero on $(0, 1)$. In particular, $h(x) > \delta$ and $h'(x) < -\delta$ for all $0 \leq x \leq x_0$.

There are two cases to consider:

(1) If $k = 1$ then $h'(0) = a_1$. In particular, $h'(0) < h'(x_0) < -\delta$ (by the observation above); and
(2) If $k \neq 1$ Then $h'(0) = -1 < -\delta$.

Let $g \in \mathcal{F}_b$ and let

$$f(t) := g(t) - h(t) = 1 + \sum_{i=1}^{k-1} \underbrace{(b_n - 1)}_{c_i \geq 0} t^i - + \underbrace{(a_k - b_k)}_{c_k} t^k - \sum_{i=l+1}^{\infty} \underbrace{(1 - b_i)}_{c_i \geq 0} t^i. \tag{7.2}$$

Since for $0 \leq x \leq x_0$ we have $h(x) > \delta$ then if $g(x) < \delta$ we have that $f(x) = g(x) - h(x) < 0$. However, because of the particular form of $f(x)$ in (7.2), with positive coefficients followed by negative coefficients, one easily sees that $f(x) < 0$ implies $f'(x) = g'(x) - h'(x) < 0$. Finally, since by the observation $h'(x) < -\delta$ we deduce that $g'(x) < -\delta$, as required.

In particular, if let

$$h(x) = 1 - x - x^2 - x^3 + \frac{1}{2}x^4 + \sum_{i=5}^{\infty} x^i$$

then one can check that $h(2^{-2/3}) > 0.07$ and $h'(2^{-2/3}) < -0.09$ and so $y(1) \geq 2^{-2/3}$ A more sophisticated choice of $h(x)$ leads to the better bounds described above. $\square$

A general result shows the following.

Proposition 7.1. *The function $y : [1, \infty) \to [0, 1]$ is strictly decreasing, continuous and piecewise algebraic function. Moreover,*

(1) $y(b) \geq (\sqrt{b} + 1)^{-1}$ *for* $1 \leq b < 3 + \sqrt{8}$*; and*
(2) $y(b) = (\sqrt{b} + 1)^{-1}$ *for* $b \geq 3 + \sqrt{8}$

The proof uses a variation on the proof of the claim above.

The following technical corollary is crucial when trying to use the transversality technique to calculate the dimension or measure of the limit sets for self-similar sets.

Proposition 7.2 ("Transversality Lemma"). *Let $b > 0$.*

(1) *Given $0 < s < 1$ there exists $K > 0$ such that*
$$\int_0^{y(b)} \frac{d\lambda}{|f(\lambda)|^s} \leq K,$$
for all $f \in \mathcal{F}_b$;
(2) *There exists $C > 0$ such that,*
$$Leb\{0 \leq \lambda \leq y(b) \,:\, |f(\lambda)| \leq \epsilon\} \leq C\epsilon.$$
for all $f \in \mathcal{F}_b$ and all sufficiently small $\epsilon > 0$.

Proof. To see part (1), we can write
$$[0, y(b)] = \underbrace{\{x \in [0, y(b)] \,:\, |f(x)| > \delta\}}_{=:S_1} \cup \underbrace{\{x \in [0, y(b)] \,:\, |f'(x)| > \delta\}}_{=:S_2}.$$

In particular, we can bound
$$\begin{aligned}\int_0^{y(b)} \frac{\mathrm{d}\lambda}{|f(\lambda)|^s} &\leq \int_{S_1} \frac{\mathrm{d}\lambda}{|f(\lambda)|^s} + \int_{S_2} \frac{\mathrm{d}\lambda}{|f(\lambda)|^s} \\ &\leq \frac{1}{\delta^s} + \frac{1}{\delta^s}\end{aligned}$$

For part (2) we need only observe that if $|f(x)| \leq \epsilon \leq \delta$ then x is contained in an interval I upon which $-\epsilon \leq f(t) \leq \epsilon$ is monotone and, by δ-transversality, we have that $|f'(t)| \geq \delta$. In particular, the length of I is at most $(2/\delta)\epsilon$ and I contains a zero. The result easily follows form the observation that the number of zeros of f is uniformly bounded. (For example, by Jenson's formula from complex analysis the number $n(x_0)$ of zeros $z_1, \cdots, z_{n(x_0)}$ (ordered by modulus) of $f(z)$ with $|z_i| < x_0$ satisfies
$$\prod_{i=1}^{n(x_0)} \frac{x_0}{|z_i|} = \exp\left(\int_0^{2\pi} \log|f(re^{i\theta})| d\theta\right) \leq 1 + \frac{bx_0}{1 - x_0}$$
and we also have
$$\prod_{i=1}^{n(x_0)} \frac{x_0}{|z_i|} \geq \prod_{i=1}^{n(x_0-\epsilon)} \frac{x_0}{|z_i|} \geq \left(\frac{x_0 - \epsilon}{x_0}\right)^{n(x_0-\epsilon)}.$$
Comparing these two expressions gives a uniform bound. □

The first part is extremely useful when proving theorems involving generic conclusions. The second part is useful in the case when we wish to show that a class of self-similar sets have positive Lebesgue measure for almost all parameter values.

7.2 The $\{0,1,3\}$-Problem. We want to describe the dimension of certain self-similar sets where the images of the similarities overlap. Given $0 < \lambda < 1$, let $\{T_0, T_1, T_2\}$ be an iterated function scheme on $\mathbb{R}$ where,

$$\begin{aligned} T_0(x) &= \lambda x \\ T_1(x) &= \lambda x + 1 \\ T_2(x) &= \lambda x + 3. \end{aligned}$$

Observe that:

(i) For $\lambda \in (0, \frac{1}{4})$ the Open Set Condition holds (since $T_i([0,1]) \cap T_j([0,1]) = \emptyset$, for $i \neq j$) and the dimension of the associated limit set $\Lambda(\lambda)$ is $\dim_H \Lambda(\lambda) = \dim_B \Lambda(\lambda) = -\frac{\log 3}{\log \lambda}$, by Moran's Theorem.

(ii) When $\lambda \in (\frac{1}{4}, \frac{1}{3})$ the Open Set Condition does not hold, and we only know that $\dim_H \Lambda(\lambda) \leq \dim_B \Lambda(\lambda) \leq -\frac{\log 3}{\log \lambda}$.

The problem of whether $\dim_H \Lambda(\lambda) = -\frac{\log 3}{\log \lambda}$ holds for a specific value of λ is far from well understood, in general. This class of problems was studied by Keane, Smorodinsky and Solomyak. In particular they showed:

(iii) For $\frac{2}{5} < \lambda < 1$ we have that $\Lambda(\lambda)$ is an open interval.

A generic description of the behaviour of $\dim_H(\Lambda(\lambda))$ in the region $(\frac{1}{4}, \frac{1}{3})$ is given by the following result.

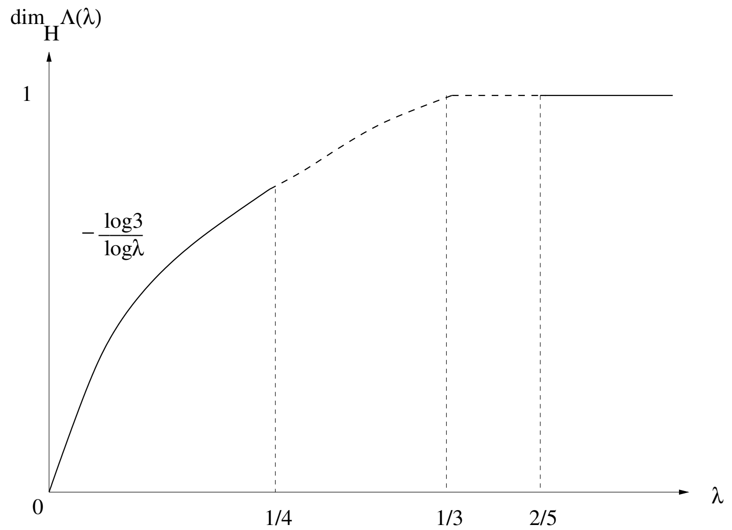
Theorem 7.3.

(a) *For almost all* $\lambda \in (\frac{1}{4}, \frac{1}{3}]$,

$$\dim_H \Lambda(\lambda) = \dim_B \Lambda(\lambda) = -\frac{\log 3}{\log \lambda};$$

and

(b) *There is a dense set of values* $\mathcal{D} \subset (\frac{1}{4}, \frac{1}{3}]$ *such that for* $\lambda \in \mathcal{D}$ *we have that* $\dim_H \Lambda(\lambda) \leq \dim_B \Lambda(\lambda) < -\frac{\log 3}{\log \lambda}$



In the range $0 < \lambda \leq \frac{1}{4}$ we always have $\dim_H \Lambda(\lambda) = -\log 3/\log \lambda$; but for $\frac{1}{4} < \lambda \leq \frac{1}{3}$ we only know the result for a.e. λ; for $\frac{2}{5} < \Lambda < 1$ we always have $\dim_H \Lambda(\lambda) = 1$.

Proof. To prove part (a), it is first easy to see from the definitions that $\dim_H \Lambda(\lambda) \leq \dim_B \Lambda(\lambda) \leq -\frac{\log 3}{\log \lambda}$. We now consider the opposite inequality. Let $\mu = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})^{\mathbb{Z}^+}$ be the usual $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$-Bernoulli measure on the space of sequences $\Sigma = \{0, 1, 2\}^{\mathbb{Z}^+}$. For any $0 < \lambda < 1$ we can define the map $\Pi_\lambda : \Sigma \to \mathbb{R}$ by

$$\Pi_\lambda(\underline{i}) = \sum_{k=0}^{\infty} i_k \lambda^k.$$

Thus on each possible attractor $\Lambda(\lambda)$ a self-similar measure ν_λ can be defined by $\nu_\lambda = \mu \circ \Pi_\lambda^{-1}$. Given $\epsilon > 0$ let $s_\epsilon(\lambda) = -\frac{\log 3}{\log(\lambda + \epsilon)}$. Note that the proof can be completed (as in the proofs in the previous chapter) if it can be shown that,

$$I = \int_{\frac{1}{4}}^{\frac{1}{3}} \left(\iint \frac{\mathrm{d}\nu_\lambda(x) \mathrm{d}\nu_\lambda(y)}{|x-y|^{s_\epsilon(\lambda)}} \right) \mathrm{d}\lambda < \infty$$

for all $\epsilon > 0$. In particular, the finiteness of the integrand, for almost all λ, allows us to deduce that for those values $\dim_H \Lambda(\lambda) \geq s_\epsilon(\lambda)$. Since the value of $\epsilon > 0$ is arbitrary, we get the lower bound $\dim_H \Lambda(\lambda) \geq -\frac{\log 3}{\log \lambda}$.

Using the map Π_λ the inner two integrals can be rewritten in terms of the measure μ on Σ and we can rewrite the last expressions as

$$I = \int_{\frac{1}{4}}^{\frac{1}{3}} \left(\iint \frac{\mathrm{d}\mu(\underline{i}) \mathrm{d}\mu(\underline{j})}{|\Pi_\lambda(\underline{i}) - \Pi_\lambda(\underline{j})|^{s_\epsilon(\lambda)}} \right) \mathrm{d}\lambda.$$

We then turn I into a product of two expressions. More precisely, let $t = \max_{\frac{1}{4} \leq \lambda \frac{1}{3}} s_\epsilon(\lambda)$ and note that $t < 1$. In particular, if $\underline{i} \neq \underline{j}$ then they agree until the $|\underline{i} \wedge \underline{j}|$-th term and we can write

$$\begin{aligned} |\Pi_\lambda(\underline{i}) - \Pi_\lambda(\underline{j})|^{s_\epsilon(\lambda)} &= \lambda^{|\underline{i} \wedge \underline{j}| s_\epsilon(\lambda)} \left(\sum_{k=0}^{\infty} a_k \lambda^k \right)^{s_\epsilon(\lambda)} \\ &\geq \left(\frac{1}{3} + \epsilon \right)^{s_\epsilon(\lambda) |\underline{i} \wedge \underline{j}|} \left(\sum_{k=0}^{\infty} a_k \lambda^k \right)^{t}, \end{aligned}$$

where $\{a_k\}_{k \in \mathbb{Z}^+}$ is the sequence $a_k := i_{k+|\underline{i} \wedge \underline{j}|} - j_{k+|\underline{i} \wedge \underline{j}|} \in \{0, \pm 1, \pm 2, \pm 3\}$ and $a_0 \neq 0$. Substituting this back into the integrand in I and using Fubini's Theorem we get

$$I \leq \int_\Sigma \int_\Sigma \frac{\mathrm{d}\mu(\underline{i}) \mathrm{d}\mu(\underline{j})}{\left(\frac{1}{3} + \epsilon\right)^{|\underline{i} \wedge \underline{j}|}} \left(\int_{\frac{1}{4}}^{\frac{1}{3}} \frac{\mathrm{d}\lambda}{\left(\sum_{k=0}^{\infty} a_k \lambda^k\right)^t} \right). \tag{7.3}$$

We can estimate the first integral in (7.3) by

$$\begin{aligned} \iint \frac{\mathrm{d}\mu(\underline{i}) \mathrm{d}\mu(\underline{j})}{\left(\frac{1}{3} + \epsilon\right)^{|\underline{i} \wedge \underline{j}|}} &\leq \sum_{k=0}^{\infty} \sum_{[i_0, i_1, \ldots, i_{k-1}]} \frac{\mu([i_0, i_1, \ldots, i_{k-1}])}{\left(\frac{1}{3} + \epsilon\right)^k} \\ &= \sum_{k=0}^{\infty} \frac{\frac{1}{3}^k}{\left(\frac{1}{3} + \epsilon\right)^k} < \infty. \end{aligned}$$

Thus to show that $I < \infty$ it remains to bound the second integral in (7.3) by

$$\int \frac{d\lambda}{\left(\sum_{k=0}^{\infty} a_k \lambda^k\right)^t} < \infty$$

for any sequence $\{a_k\}_{k\in\mathbb{Z}^+}$, where $a_k \in \{0, \pm1, \pm2\}$ and $a_0 \neq 0$. Let $f(\lambda) = 1 + \sum_{k=0}^{\infty} \left(\frac{a_k}{a_0}\right) \lambda^k$ then we can apply part (1) of Proposition 7.1 to deduce that the integral is finite, since $y(2) \geq \frac{1}{3}$.

To prove part (b), we need only observe that if for some n we can find distinct $(i_1, \ldots, i_n), (j_1, \ldots, j_n) \in \{0, 1\}^n$ such that

$$\sum_{k=1}^{n} i_k \lambda^k = \sum_{k=1}^{n} j_k \lambda^k$$

then at the n-th level of the construction at least two of the 2^n intervals of length λ^n coincide. In particular, it is easy to see that

$$\dim_B(\Lambda(\lambda)) \leq -\frac{n-1}{n} \frac{\log 3}{\log \lambda}.$$

It is then an easy to matter to show that the set $\mathcal{D}$ of such λ is dense in $(\frac{1}{4}, \frac{1}{3})$. □

Remark. It is also possible to show a corresponding result where generic λ is understood in a topological sense: for λ is a dense G_δ set (i.e., a countable intersection of open dense sets).

Remark. Of course one can prove somewhat similar results where $\{0, 1, 3\}$ is replaced by some other finite set of numbers. These are usually called *deleted digit expansions.*

7.3 The Erdös-Solomyak Theorem. We recall some results about the properties of self-similar measures. Let $\lambda \in (0, 1)$. We let,

$$T_0(x) = \lambda x$$
$$T_1(x) = \lambda x + 1.$$

Let $\nu = \nu_\lambda$ be a measure such that for all $J \subset \left[0, \frac{1}{1-\lambda}\right]$,

$$\nu(J) = \frac{1}{2}\nu(T_0^{-1}(J)) + \frac{1}{2}\nu(T_1^{-1}(J)). \tag{7.4}$$

In fact, is unique probability measure satisfying this identity called the *self-similar measure.* Equivalently, we say this is a Bernoulli convolution with respect to $\underline{p} = (\frac{1}{2}, \frac{1}{2})$.

In particular, we wish to know whether the measures ν_λ are absolutely continuous or not (i.e., whenever B is a Borel set with $\mathrm{Leb}(B) = 0$ then $\nu_\lambda(B) = 0$). To begin with, it is an easy exercise to see that if $0 < \lambda < \frac{1}{2}$ then the Iterated Function Scheme $\{T_0, T_1\}$ satisfies the Open Set Condition, thus $\Lambda(\lambda)$ is a Cantor set with

$$\dim_H(\Lambda(\lambda)) = -\frac{\log 3}{\log \lambda},$$

by Moran's Theorem and, in particular, has zero Lebesgue measure. Thus ν_λ is singular with respect to Lebesgue measure.

Jessen-Wintner Theorem. *The measure ν_λ is either absolutely continuous or singular with respect to Lebesgue measure Leb (i.e, either every set B with $Leb(B) = 0$ satisfies $\nu_\lambda(B) = 0$, or there exists a set B with $Leb(B) = 0$ and $\nu_\lambda(B) = 1$).*

Proof. Every measure ν_λ can be written in the form $\nu_\lambda = \nu^{abs} + \nu^{sing}$, where $\nu^{abs} << Leb$ and $\nu^{sing} \perp Leb$ (This is the Lebesgue decomposition theorem). However, substituting into (7.4) we see that both ν^{abs} and ν^{sing} satisfy the identity. By uniqueness we have that one of them must be zero. □

Next we recall one of the classical theorems in Harmonic Analysis. Let us define the Fourier transform $\widehat{\nu} : \mathbb{R} \to \mathbb{R}$ by

$$\widehat{\nu}(u) = \int e^{iut} d\nu(t), \text{ for } u \in \mathbb{R}.$$

The following result describes the behaviour of $\widehat{\nu}(u)$ as $|u| \to +\infty$.

Riemann-Lebesgue Theorem. *If the measure ν is absolutely continuous then $\widehat{\nu}(u) \to 0$ as $|u| \to +\infty$.*

We can use the Riemann-Lebesgue Theorem to show that for some value of $\lambda \in [\frac{1}{2}, 1]$ the measure ν_λ is singular.

Pisot Numbers. We recall that $\theta > 1$ is an algebraic integer if it is a zero of a polynomial $P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ with $a_{n-1}, \ldots, a_0 \in \mathbb{Z}$. Let $\theta_1, \ldots, \theta_{n-1} \in \mathbb{C}$ be the other roots of $P(x)$. We call λ a *Pisot Number* if $|\theta_1|, \cdots, |\theta_{n-1}| < 1$.

Clearly, there are at most countably many Pisot numbers (since there are at most countably many such polynomials $P(x)$). The smallest Pisot numbers are $\theta = 1.3247\cdots$ (which is a root for $x^3 - x - 1$) and $\theta = 1.3802\cdots$ (which is a root for $x^3 - x - 1$). However, perhaps the most important feature of these numbers is the following:

$$\min_{k \in \mathbb{N}} |\theta^n - k| = O(\Theta^n) \text{ as } n \to +\infty$$

where $\Theta = \max\{|\theta_1|, \ldots, |\theta_{n-1}|\} < 1$.

The following highly influential Theorem was published by Erdös in 1939.

Erdös's Theorem. *If $\theta := 1/\lambda$ is a Pisot number then the measure ν_λ is singular.*

Proof. This is based on the study of the Fourier Transform of the measure ν_λ. In fact, if we let $\delta(x)$ be the Dirac measure on $x \in \mathbb{R}$ then

$$\frac{1}{2^n} \sum_{i_1 \ldots i_n \in \{0,1\}} \delta\left(\sum_{j=1}^{n} i_j \lambda^j\right) \to \nu_\lambda$$

(where convergence is in the weak star topology) as $n \to +\infty$, and so we can write

$$\widehat{\nu}_\lambda(u) := \int_{-\infty}^{\infty} e^{itx} d\nu_\lambda(x) = \lim_{n \to \infty} \prod_{k=0}^{n} \left(\frac{e^{-iu\lambda^k} + e^{iu\lambda^k}}{2}\right)$$

For a Pisot number θ we can choose for each $n \geq 1$ a natural number $k_n \in \mathbb{N}$ such that $|\theta^n - k_n| = O(\Theta^{-n})$. In particular, if we let $u \in \mathbb{N}$ then we can show that there exists $c > 0$ such that

$$\prod_{k=0}^{n} \left(\frac{e^{-iu\lambda^k} + e^{iu\lambda^k}}{2} \right) > c \text{ for all } n \geq 0$$

In particular, we can bound $\inf_{m \in \mathbb{N}} \nu_\lambda(m) > 0$. Thus $\nu_\lambda(u) \not\to 0$ as $u \to +\infty$. By the Riemann-Lebesgue Lemma ν_λ is not absolutely continuous. Thus, by the Jessen-Wintner theorem, we deduce that ν_λ is singular. $\square$

Erdös also showed the following:

(i) If $\lambda = 2^{-1/k}$, for some $k \geq 1$, then ν_λ is absolutely continuous; and
(ii) There exists $\epsilon > 0$ such that for almost all $\lambda \in [1-\epsilon, 1]$ the measure ν_λ is absolutely continuous.

He went onto conjecture that for almost all $\lambda \in [\frac{1}{2}, 1]$ the measure is absolutely continuous. This was eventually proved in 1995 by Solomyak:

Erdös-Solomyak Theorem. *For almost all $\lambda \in [\frac{1}{2}, 1]$ the measure ν_λ is absolutely continuous.*

There is a useful criteria for the measure ν_λ to be absolutely continuous.

Absolute Continuity Lemma. *The measure ν_λ is absolutely continuous if*

$$\int \left(\liminf_{r \to 0} \frac{\nu_\lambda(B(x,r))}{2r} \right) d\nu_\lambda(x) < \infty.$$

Proof of the Absolute Continuity theorem. From the hypotheses we see that for a.e. (ν_λ) x we have that $\underline{D}(x) := \left(\liminf_{r \to 0} \frac{\nu_\lambda(B(x,r))}{2r} \right) < +\infty$. It therefore suffices to show that if $\mathrm{leb}(A) = 0$ and $u > 0$, then the set $X_u := \{x \in A : \underline{D}(x) \leq u\}$ satisfies $\nu_\lambda(X_u) = 0$.

Let us fix $\epsilon > 0$. For each $x \in X_u$ we can choose a sequence $r_i \searrow 0$ with $\mu(B(x, r_i))/2r_i \leq u + \epsilon$. Let us denote $A = X_u$. By the Besicovitch covering lemma, we can choose a cover $\{B_i\}$ with is a union of two families $\{B_i^{(0)}\} \cup \{B_i^{(1)}\}$ (each of which consists of balls which are pairwise disjoint). In particular, let us assume that $\mu(\cup_i B_i^{(0)}) > \frac{1}{2}$. In particular, we can bound

$$\mu(A - \cup_i B_i^{(0)}) \leq \mu(A) - \mu(\cup_i B_i^{(0)}) \leq \frac{1}{2}\mu(A),$$

for $\eta > 0$. We can proceed inductively, replacing A by $A - \cup_i B_i^{(0)}$. Finally, taking the union of the families of balls at each step we arrive at a countable family of balls $\{B_i\}$ such that:

(1) $\mu(X_u - \cup_i B_i) = 0$; and
(2) $\mu(B_i) \leq (u + \epsilon)\lambda(B_i) = (u + \epsilon)2r_i$

In particular,

$$\mu(X_u) \leq \sum_i \mu(B_i) \leq (u + \epsilon) \sum_i \lambda(B_i) \leq (u + \epsilon)(\mathrm{leb}(X_u) + \epsilon).$$

In particular, since $\epsilon > 0$ is arbitrary we have that $\mu(X_u) \leq u\mathrm{leb}(X_u) = 0$. $\square$

We follow a variation on Solomyak's original proof (due to Peres and Soloymak) which makes use of this lemma.

Proof of the Erdös-Solomyak Theorem. We will also let $\mu = (\frac{1}{2}, \frac{1}{2})^{\mathbb{Z}^+}$ be the usual $(\frac{1}{2}, \frac{1}{2})$-Bernoulli measure defined on the sequence space, $\Sigma = \{0,1\}^{BbbZ^+}$. As usual, we let $\Pi_\lambda : \{0,1\}^{\mathbb{N}} \to \mathbb{R}$ be defined by,

$$\Pi_\lambda(\underline{i}) = \sum_{n=0}^{\infty} i_n \lambda^n.$$

We can also write $\nu_\lambda = \Pi_\Lambda \mu$ (i.e., $\nu_\lambda(B) = \mu(\Pi_\Lambda^{-1} B)$ for all intervals $B \subset \mathbb{R}$).

To begin with, we want to show that ν_λ is absolutely continuous for a.e. $\lambda \in \left(\frac{1}{2}, y(2)\right)$, where $y(2) = 0.68\cdots$. In this case, it is sufficient to show for any $\epsilon > 0$

$$I = \int_{\frac{1}{2}+\epsilon}^{y(2)} \left(\int \liminf_{r \to 0} \frac{\nu_\lambda(B(x,r))}{2r} \mathrm{d}\nu_\lambda(x) \right) \mathrm{d}\lambda < \infty.$$

In particular, since $\epsilon > 0$ is arbitrary we can then deduce that for almost every $\lambda \in \left(\frac{1}{2}, y(2)\right)$ we have that the integrand is finite. Thus for such λ we can apply the previous lemma to deduce that ν_λ is absolutely continuous, as required.

The first step is to apply Fatou's Lemma (to move the lim inf outside of the integral) and then reformulate the integral in terms of integrals on the sequence space Σ. Thus

$$\begin{aligned} I &\leq \liminf_{r \to 0} \frac{1}{2r} \int_{\frac{1}{2}+\epsilon}^{y(2)} \left(\int \nu_\lambda(B(x,r)) \mathrm{d}\nu_\lambda(x) \right) \mathrm{d}\lambda \\ &\leq \liminf_{r \to 0} \frac{1}{2r} \int_{\frac{1}{2}+\epsilon} \left(\int_{\Sigma}^{y(2)} \int_{\Sigma} {}_{\{\omega, \tau : |\Pi_\lambda(\omega) - \Pi_\lambda(\tau)| \leq r\}} \mathrm{d}\mu(\omega) \mathrm{d}\mu(\tau) \right) \mathrm{d}\lambda. \end{aligned}$$

Applying Fubini's Theorem bounds I (to switch the oder of the integrals) gives

$$I \leq \liminf_{r \to 0} \frac{1}{2r} \int_\Sigma \int_\Sigma Leb \left\{ \lambda \in \left(\frac{1}{2} + \epsilon, y(2) \right) \ : \ |\Pi_\lambda(\omega) - \Pi_\lambda(\tau)| \leq r \right\} \mathrm{d}\mu(\omega) \mathrm{d}\mu(\tau).$$

To simplify this bound observe that

$$|\Pi_\lambda(\omega) - \Pi_\lambda(\tau)| = \lambda^{|\omega \wedge \tau|} g(\lambda)$$

where $g(\lambda) \in \mathcal{F}_\lambda$ for all $\omega, \tau \in \Sigma$. Thus by definition of $y(2)$ and Proposition 7.2 we have that

$$Leb \left\{ \lambda \in \left(\frac{1}{2} + \epsilon, y(2) \right) : \ |\Pi_\lambda(\omega) - \Pi_\lambda(\tau)| \leq r \right\} \leq 2C \left(\frac{1}{3} + \epsilon \right)^{|\omega \wedge \tau|} r$$

for some $C > 0$. This allows us to bound:

$$\begin{aligned} I &\leq C \int_\Sigma \int_\Sigma \left(\frac{1}{2} + \epsilon \right)^{-|\omega \wedge \tau|} \mathrm{d}\mu(\omega) \mathrm{d}(\tau) \\ &\leq C \int_\Sigma \left(\sum_{n=0}^{\infty} \frac{1}{2^n} \left(\frac{1}{2} + \epsilon \right)^{-n} \right) \mathrm{d}(\tau) < +\infty \end{aligned}$$

which can be seen to be finite by simply integrating on the shift space. Since $\epsilon > 0$ is arbitrary, this shows that ν_λ is absolutely continuous for a.e. $\lambda \in [\frac{1}{2}, y(2)]$.

We shall just sketch how to extend this result to the larger interval $[\frac{1}{2}, 1]$. Recall from the proof of Erdös's theorem that the Fourier transform of the measure ν_λ takes the form

$$\widehat{\nu}_\lambda(u) = \prod_{k=0}^{\infty} \left(\frac{e^{-iu\lambda^k} + e^{iu\lambda^k}}{2} \right)$$

and then we can write

$$\widehat{\nu}_\lambda(u) = \underbrace{\prod_{\substack{k=0 \\ k \neq 2 \ (\mathrm{mod})\ 3}}^{\infty} \left(\frac{e^{-iu\lambda^k} + e^{iu\lambda^k}}{2} \right)}_{=:\widehat{\nu}'_\lambda} \times \prod_{\substack{k=0 \\ k = 2 \ (\mathrm{mod})\ 3}}^{\infty} \left(\frac{e^{-iu\lambda^k} + e^{iu\lambda^k}}{2} \right).$$

Absolute continuity of ν'_λ would imply absolute continuity of ν_λ (since it is a classical fact that convolving an absolutely continuous measure with another measure gives an absolutely continuous measure again). However, modifying the above proof we can replace $\mathcal{F}_b$ be $\mathcal{F}'_b \subset \mathcal{F}_b$ in which the coefficients satisfy $c_{3i+1}c_{3i+2} = 0$ for all $i \geq 0$. For such sequences one can show that the region of transversality can be extended as far as $x_0 = 1/\sqrt{2}$ and so we can deduce that ν_λ is absolutely continuous for a.e. $\frac{1}{2} < \lambda < \frac{1}{\sqrt{2}}$. Finally, since we can write $\widehat{\nu}_\lambda(u) = \widehat{\nu}_{\lambda^2}(u)\widehat{\nu}_{\lambda^2}(\lambda u)$ we can deduce that ν_λ is also absolutely continuous for a.e. $\frac{1}{\sqrt{2}} < \lambda < \frac{1}{2^{1/4}}$. Proceeding inductively completes the proof. □

Remark. The original proof of Solomyak used another result from Fourier analysis: If $\widehat{\nu}_\lambda \in L^2(\mathbb{R})$ then ν_λ is absolutely continuous and the Radon-Nikodym derivative $\frac{d\nu_\lambda}{dx} \in L^2(\mathbb{R})$. In particular, he showed the stronger result that for a.e. $\frac{1}{2} < \lambda < 1$ one has $\frac{d\nu_\lambda}{dx} \in L^2(\mathbb{R})$.

Remark. It is also possible to show that for a.e. λ we have $\frac{d\nu_\lambda}{dx} > 0$ for a.e. $x \in [-\frac{1}{1-\lambda}, \frac{1}{1-\lambda}]$.

7.4 Dimension of the measures ν_λ. Unlike the case of the $\{0, 1, 3\}$-problem, the limit set in the above example is an interval and thus its Hausdorff dimension holds no mystery. However, the dimension of the measure is still of some interest. We shall consider the slightly more general of different Bernoulli measures. Let $\underline{p} = (p_0, p_1)$ be a probability vector (i.e., $0 < p_0, p_1 < 1$ and let $p_0 + p_1 = 1$).

Let $\nu_\lambda = \nu_\lambda^{p_0,p_1}$ now denote the unique probability measure such that

$$\nu_\lambda(J) = p_0\nu_\lambda(T_0^{-1}(J)) + p_1\nu_\lambda(T_1^{-1}(J)).$$

for all $J \subset \left[0, \frac{1}{1-\lambda}\right]$.

The main result on these measures is the following.

Theorem 7.4.

(1) *For almost all $\lambda \in [\frac{1}{2}, y(1) = 0.649\ldots]$,*

$$\dim_H \nu_\lambda^{(p_0,p_1)} = \min\left(\frac{p_0 \log p_0 + p_1 \log p_1}{\log \lambda}, 1 \right).$$

(2) *For almost all $\lambda \in [p_0^{p_0}p_1^{p_1}, y(1) = 0.649]$ we have that ν_λ is absolutely continuous.*

Unfortunately, it is not possible to move past the upper bound $y(1)$ on these intervals using properties of the Fourier transform $\widehat{\nu}_\lambda$ (as in the previous section) because this function is not as well behaved in the case of general (p_0, p_1) as it was in the specific case of $(\frac{1}{2}, \frac{1}{2})$ in the Erdös-Solomyak Theorem.

Proof. We shall show the lower bound on the dimension of the measure in part (1). The proof of Part (2) is similar to that in the special case $p_0 = p_1 = \frac{1}{2}$.

We let $\mu = \mu_{p_0,p_1} = (p_0, p_1)^{\mathbb{Z}^+}$ denote the usual (p_0, p_1)-Bernoulli measure defined on the sequence space, $\Sigma = \{0,1\}^{\mathbb{Z}^+}$. We again let $\Pi_\lambda : \Sigma \to \mathbb{R}$ be defined by,

$$\Pi_\lambda(\underline{i}) = \sum_{n=0}^{\infty} i_n \lambda^n.$$

As usual, we have that $\nu_\lambda^{(p_0,p_1)} = \mu^{(p_0,p_1)} \circ \Pi_\lambda^{-1}$. We shall use the following lemma.

Claim. *For any $\alpha \in (0,1]$ we have that for almost all $\lambda \in [0.5, y(1) = 0.649\ldots]$*

$$\dim \nu_\lambda^{(p_0,p_1)} \geq \min\left(\frac{\log((p_0^{\alpha+1} + p_1^{\alpha+1})^{\frac{1}{\alpha}})}{\log \lambda}, 1 \right).$$

Proof of Claim. Fix (p_0, p_1) and let $\epsilon > 0$. For brevity of notation we denote $d(\alpha, \epsilon) = (p_0^{\alpha+1} + p_1^{\alpha+1} + \epsilon)^{\frac{1}{\alpha}}$. Let us write $S_\epsilon(\lambda) = \min\left(\frac{\log(d(\alpha,\epsilon))}{\log \lambda}, 1 - \epsilon\right)$. We can first rewrite

$$I = \int_{0.5}^{y(1)} \int \left(\int \frac{d\nu_\lambda(x)}{|x-y|^{S_\epsilon(\lambda)}} \right)^\alpha d\nu_\lambda(y) d\lambda = \int_{0.5}^{y(1)} \int \left(\int \frac{d\mu(\underline{i})}{|\Pi_\lambda(\underline{i}) - \Pi_\lambda(\underline{j})|^{S_\epsilon(y)}} \right)^\alpha d\mu(\underline{j}) d\lambda.$$

To prove the claim it suffices to show that $I < +\infty$. Next we apply Fubini's theorem and Hölder's inequality $\int f^\alpha \leq C(\int f)^\alpha$ for $\alpha \in (0,1]$) to get

$$\begin{aligned} I &\leq C \int \left(\int_{0.5}^{y(1)} \int \frac{d\mu(\underline{i}) d\lambda}{|\Pi_\lambda(\underline{i}) - \Pi_\lambda(\underline{j})|^{s_\epsilon(\lambda)}} \right)^\alpha d\mu(\underline{j}) \\ &\leq C_1 \int \left(\int_{0.5}^{y(1)} \int \frac{d\mu(\underline{i}) d\lambda}{\left(\lambda^{|\underline{i} \wedge \underline{j}|} \, |a_0 + \sum_{n=1}^\infty a_n \lambda^n|^{s_\epsilon(\lambda)}\right)^\alpha} \right) d\mu(\underline{j}), \end{aligned}$$

for some $C_1 > 0$, where $a_n \in \{-1, 0, 1\}$ for $n \geq 1$ and $a_0 \in \{-1, 1\}$. By transversality,

$$\begin{aligned} I \leq & C_1 \int \left(\int_{0.5}^{y(1)} \int \frac{d\mu(\underline{i}) d\lambda}{\left(d(\alpha,\epsilon)^{|\underline{i} \wedge \underline{j}|} \, |a_0 + \sum_{n=1}^\infty a_n \lambda^n|\right)^{s_\epsilon(\lambda)}} \right)^\alpha d\mu(\underline{j}) \\ \leq & C_1 \int \left(\int_{0.5}^{y(1)} \frac{d\lambda}{|a_0 + \sum_{n=1}^\infty a_n \lambda^n|^{s_\epsilon(\lambda)}} \int \frac{d\mu(\underline{i})}{d(\alpha,\epsilon)^{|\underline{i} \wedge \underline{j}|}} \right)^\alpha d\mu(\underline{j}) \\ \leq & C_2 \int \left(\int \frac{d\mu(\underline{i})}{d(\alpha,\epsilon)^{|\underline{i} \wedge \underline{j}|}} \right)^\alpha d\mu(\underline{j}) \\ \leq & C_2 \int \left(\sum_{k=0}^\infty \frac{\mu(W_{\omega,k})}{d(\alpha,\epsilon)^k} \right)^\alpha d\mu(\omega) < +\infty \end{aligned}$$

for some $C_2 > 0$. Consider the inequality $(\sum_i b_i)^\alpha \leq \sum_i b_i^\alpha$ for $b_i > 0$ and $\alpha \in (0,1]$, then

$$\begin{aligned} I &\leq C_2 \sum_{k=0}^{\infty} \sum_{w \in W_k} \frac{\mu(W)^{\alpha+1}}{d(\alpha,\epsilon)^{\alpha k}} \\ &\leq C_2 \sum_{k=0}^{\infty} d(\alpha,\epsilon)^{-\alpha k} (p_0^{\alpha+1} + p_1^{\alpha+1})^k. \end{aligned}$$

Thus since $d(\alpha,\epsilon)^\alpha > p_0^{\alpha+1} + p_1^{\alpha+1}$ we have $I < \infty$ and hence, since the integrand must be finite almost everywhere, we deduce that

$$\dim \nu_\lambda \geq \min\left(\frac{d(\alpha,\epsilon)}{\log \lambda}, 1-\epsilon\right)$$

for almost all $\lambda \in [\frac{1}{2}, y(2)]$. To complete the proof of the claim we let $\epsilon = \frac{1}{n}$ for $n \in \mathbb{N}$ and let $n \to \infty$. □

To complete the proof of the Theorem we let $\alpha_n = \frac{1}{n}$ for $n \in$ and observe that,

$$\lim_{n\to\infty} \frac{\log(p_0^{\alpha_n+1} + p_1^{\alpha_n+1})}{\alpha_n \log \lambda} = \frac{p_0 \log p_0 + p_1 \log p_1}{\log \lambda}.$$

7.5 The $\{0,1,3\}$ problem revisited: the measure ν_λ. Finally, We can also consider the question of absolute continuity for the $\{0,1,3\}$ problem in the region $\lambda \in [\frac{1}{3}, y(2)]$. Let ν_λ be defined as before. The analogue of the Erdös-Solomyak theorem is the following.

Theorem 7.5. *For a.e. $\lambda \in [\frac{1}{3}, y(2)]$ the measure ν_λ is absolutely continuous. In particular, $\Lambda(\lambda)$ has positive Lebesgue measure.*

This result was also proved by Solomyak. The method of proof is very similar to that in the case of section 7.3 and we only outline the main steps. Thus to show that ν_λ is absolutely continuous for a.e. $\lambda \in \left(\frac{1}{3}, y(3)\right)$ it is sufficient to show for any $\epsilon > 0$

$$I = \int_{\frac{1}{3}+\epsilon}^{y(2)} \left(\int \liminf_{r\to 0} \frac{\nu_\lambda(B(x,r))}{2r} \mathrm{d}\nu_\lambda(x) \right) \mathrm{d}\lambda < \infty.$$

The first step is to apply Fatou's Lemma (to take the lim inf outside of the integral) and to rewrite this as an integral on Σ. Thus

$$\begin{aligned} I &\leq \liminf_{r\to 0} \frac{1}{2r} \int_{\frac{1}{3}+\epsilon}^{y(2)} \left(\int \nu_\lambda(B(x,r)) \mathrm{d}\nu_\lambda(x) \right) \mathrm{d}\lambda \\ &\leq \liminf_{r\to 0} \frac{1}{2r} \int_{\frac{1}{3}+\epsilon} \left(\int_\Sigma^{y(2)} \int_\Sigma {}_{\{\omega,\tau : |\Pi_\lambda(\omega) - \Pi_\lambda(\tau)| \leq r\}} \mathrm{d}\mu(\omega) \mathrm{d}\mu(\tau) \right) \mathrm{d}\lambda. \end{aligned}$$

Applying Fubini's Theorem (to switch the order of the integrals) gives

$$I \leq \liminf_{r\to 0} \frac{1}{2r} \int_\Sigma \int_\Sigma L\left\{ \lambda \in \left(\frac{1}{3} + \epsilon, y(2)\right) : |\Pi_\lambda(\omega) - \Pi_\lambda(\tau)| \leq r \right\} \mathrm{d}\mu(\omega) \mathrm{d}\mu(\tau).$$

As usual, one can write

$$|\Pi_\lambda(\omega) - \Pi_\lambda(\tau)| = \lambda^{|\omega\wedge\tau|} g(\lambda)$$

where $g(\lambda) \in \mathcal{F}_2$ for all $\omega, \tau \in \Sigma$. Thus transversality gives that

$$Leb\left\{\lambda \in \left(\frac{1}{3} + \epsilon, y(2)\right) : |\Pi_\lambda(\omega) - \Pi_\lambda(\tau)| \leq r\right\} \leq C\left(\frac{1}{3} + \epsilon\right)^{|\omega\wedge\tau|} r$$

for some $C > 0$. This gives,

$$I \leq \frac{C}{2}\int_\Sigma \int_\Sigma \left(\frac{1}{3} + \epsilon\right)^{-|\omega\wedge\tau|} \mathrm{d}\mu(\omega)\mathrm{d}(\tau) < +\infty$$

which can easily be seen to be finite, as in the earlier proofs.

Finally, we can consider a general Bernoulli measure $\mu = (p_0, p_1, p_2)^{\mathbb{Z}^+}$ on Σ and associate the probability measure $\nu_\lambda^{p_0,p_1,p_2} = \Pi_\lambda \mu$. In particular, $\nu = \nu_\lambda^{p_0,p_1,p_2}$ will be the self-similar measure such

$$\nu(J) = p_0\nu(T_0^{-1}(J)) + p_1\nu(T_1^{-1}(J)) + p_2\nu(T_2^{-1}(J)),$$

that for all $J \subset \left[0, \frac{1}{1-\lambda}\right]$.

The analogue of Theorem 7.4 is the following:

Theorem 7.5.

(1) *For almost all* $\lambda \in [\frac{1}{3}, y(2) = 0.5]$,

$$\dim_H \nu_\lambda^{(p_0,p_1,p_2)} = \min\left(\frac{p_0 \log p_0 + p_1 \log p_1 + p_2 \log p_2}{\log \lambda}, 1\right).$$

(2) *For almost all* $\lambda \in [p_0^{p_0} p_1^{p_1} p_2^{p_2}, y(2) = 0.5]$ *we have that* ν_λ *is absolutely continuous.*

8. Iterated function schemes with overlaps: Higher dimensions

We now turn to the study of Iterated Function systems in $\mathbb{R}^2$. The starting point is the study of classical Sierpinski carpets. However, we want to modify the construction to allow for overlaps (i.e., where the Open Set Condition fails) by increasing the scaling factor λ. This can be viewed as a multidimensional version of the results from the previous chapter. More precisely, for some range of scaling values we can study the Hausdorff dimension of the limit set for typical values (as in the $\{0, 1, 3\}$-problem) and for another range of scaling values we can study the Lebesgue measure on the limit set (as in the Erdös problem).

8.1 Fat Sierpinski Gaskets. Let $0 < \lambda < 1$ and natural numbers $n > k$. We consider a family of n contractions given by,

$$T_i(x, y) = (\lambda x, \lambda y) + (c_i^{(1)}, c_i^{(2)}),$$

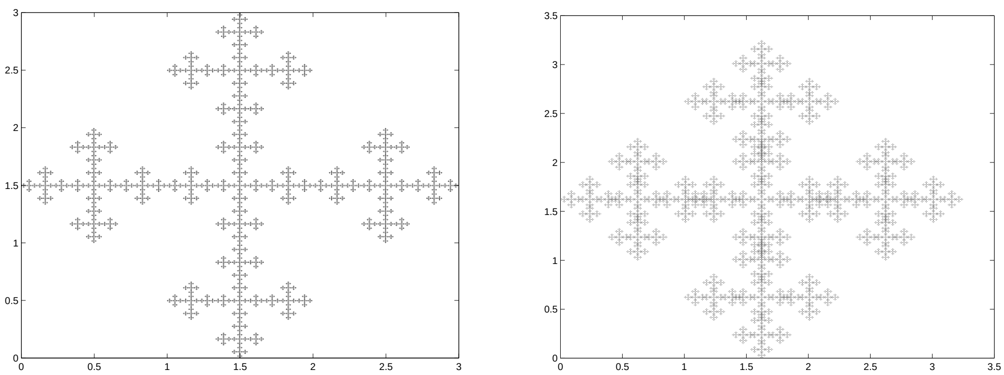
$i = 0, \dots, n-1$ where $(c_i^{(1)}, c_i^{(2)}) \in \{(j, l) \in \mathbb{Z}^2 : 0 \leq j, l \leq k-1\}$ are n distinct points in a $k \times k$ grid. If $\lambda \in (0, \frac{1}{k}]$ then it immediately follows from Moran's Theorem that the attractor $\Lambda(\lambda)$ has dimension $-\frac{\log n}{\log \lambda}$.

Example 1. Our first example is the fat Sierpiński carpet. Here we take $n = 8$ and $k = 3$ and choose $c_0 = (0, 0), c_1 = (0, 1), c_2 = (0, 2), c_3 = (1, 0), c_4 = (1, 2), c_5 = (2, 0), c_6 = (2, 1), c_7 = (2, 2)$. In Theorem 8.1, we can take $s = \left(\frac{2}{3}\right)^{\frac{2}{3}} 0.338\dots$. Thus we have that for almost all $\lambda \in [\frac{1}{3}, 0.338\cdots]$ that

$$\dim_H \Lambda(\lambda) = -\frac{\log 8}{\log \lambda}.$$

Example 2. Our next example is the Vicsek set. Here we take $n = 5$ and $k = 3$ and $c_0 = (1, 0), c_1 = (0, 1), c_2 = (1, 1), c_3 = (2, 1), c_4 = (1, 2)$. We can take $s = \left(\frac{3}{5}\right)^{\frac{3}{5}} \left(\frac{1}{5}\right)^{\frac{2}{5}} = 0.3866\dots$. Thus we have that for almost all $\lambda \in [\frac{1}{3}, 0.386]$ that

$$\dim_H \Lambda(\lambda) = -\frac{\log 5}{\log \lambda}.$$



The Vicsek cross (with $\lambda = \frac{1}{3}$) and the Fat Vicsek (with $\lambda = 0.386$)

Our main results are rather similar in nature to those in the last chapter. However, our approach requires a detailed study of the measures supported on fat Sierpiński carpets.

Theorem 8.1. *There exists $\frac{1}{k} \leq s \leq \frac{1}{\sqrt{n}}$ such that for almost all $\lambda \in (\frac{1}{k}, s)$ we have,*

$$\dim_H \Lambda(\lambda) = -\frac{\log n}{\log \lambda}.$$

There are a dense sets of values in $(\frac{1}{k}, \frac{1}{\sqrt{n}}]$ where this inequality is strict.

Of course, for Theorem 8.1 to have any value we need to give an explicit estimate for s which, in most cases, satisfies $s > \frac{1}{k}$. Let denote the number of images in the jth row by

$$n_j = \mathrm{Card}\{1 \leq l \leq k : c_i^{(2)} = j\},$$

for $1 \leq j \leq n$. If we assume that each $n_i \geq 1$ then, as we see from the proof, we can take

$$s = \min\left\{ \frac{1}{n}\left(\prod_{j=1}^{k} n_j^{n_j}\right), \left(\prod_{j=1}^{k} n_j^{-n_j}\right)^{\frac{1}{n}} \right\}.$$

It should be noted that if all the values of n_j are the same then $s = \frac{1}{k}$ and then Theorem 8.1 yields no new information.

8.2 Measures on Fat Sierpinski Carpets. As usual, upper bounds on the Hausdorff Dimension are easier. In particular, it follows immediately from a consideration of covers that $\dim_H \Lambda(\lambda) \leq \dim_B \Lambda(\lambda) \leq -\frac{\log n}{\log \lambda}$. Moreover, for the sets which we consider an argument analagous to that in the previous chapter that there are a dense sets of values $\lambda \in (\frac{1}{k}, \frac{1}{\sqrt{n}}]$ where this inequality is strict.

To complete the proof Theorem 8.1 by the now tried and tested method of studying measures supported on the fat Sierpiński carpets and using these to get lower bounds on $\dim_H \Lambda(\lambda)$. More precisely, let μ be a shift invariant ergodic measure defined on $\Sigma_n = \{1, \cdots, n\}^{\mathbb{Z}^+}$ and define a map $\Pi_\lambda : \Sigma_n \to \Lambda(\lambda)$ by,

$$\Pi_\lambda(\underline{i}) = \lim_{j\to\infty} T_{i_0} \circ \cdots \circ T_{i_{n-1}}(0,0) = \sum_{j=0}^{n} c_{i_j} \lambda^j.$$

Thus we can define a measure ν_λ supported on $\Lambda(\lambda)$ by $\nu_\lambda = \mu \Pi_\lambda^{-1}$ (i.e., $\nu_\lambda(A) = \mu(\Pi_\lambda^{-1} A)$, for Borel sets $A \subset \mathbb{R}$). We also introduce a map $p : \Sigma_n \to \Sigma_k$ which is given by,

$$p(i_0, i_1, \dots) = (c_{i_0}^{(2)}, c_{i_1}^{(2)}, \dots)$$

(i.e., we associated to symbol i the label for the vertical coordinate of $(c_i^{(1)}, c_i^{(2)})$).

We define a shift invariant measure $\overline{\mu}$ on Σ_k by $\overline{\mu} = \mu p^{-1}$ (i.e., $\overline{\mu}(B) = \mu(p^{-1}B)$, for Borel sets $B \subset \Sigma_k$). We have already defined the entropies $h(\mu)$ and $h(\overline{\mu})$ (in a previous chapter) and we can obtain the following technical estimates on the Hausdorff Dimension of the measure of ν_λ.

Proposition 8.2. *For almost all* $\lambda \in \left[\frac{1}{k}, \frac{1}{\sqrt{n}}\right]$ *we have that,*

(1) $\dim_H(\nu_\lambda) = -\frac{h(\mu)}{\log\lambda}$ *if* $\max\left\{-\frac{h(\overline{\mu})}{\log\lambda}, -\frac{h(\mu)-h(\overline{\mu})}{\log\lambda}\right\} \leq 1$;

(2) $\dim_H(\nu_\lambda) \in \left[\min\left\{1 - \frac{h(\overline{\mu})}{\log\lambda}, 1 - \frac{h(\mu)-h(\overline{\mu})}{\log\lambda}\right\}, -\frac{h(\mu)}{\log\lambda}\right]$ *otherwise.*

Example (Bernoulli measure). In fact, for the proof of Theorem 8.1, it suffices to consider only Bernoulli measures. If $\mu = (\frac{1}{n}, \cdots, \frac{1}{n})^{\mathbb{Z}^+}$ then $h(\mu) = \log n$. If there are $n_1, \cdots, n_k$ squares in the k-rows then $\overline{\mu} = (\frac{n_1}{n}, \cdots, \frac{n_k}{n})^{\mathbb{Z}^+}$ and

$$h(\overline{\mu}) = -\sum_i \frac{n_i}{n}\log\frac{n_i}{n} = \log n - \frac{1}{n}\sum_i n_i \log n_i.$$

The rest of this section is devoted to the proof of this Proposition. In the next section we shall deduce Theorem 8.1. For $\xi \in \Sigma$ we define μ_ξ to be the conditional (probability) measure on $p^{-1}(\xi)$ defined

$$\mu(A) = \int_{\Sigma_k} \mu_\xi(A \cap p^{-1}\xi)\mathrm{d}\overline{\mu}(\xi),$$

for any Borel set $A \subseteq \Sigma_n$. Let $\mathcal{B}(\Sigma_n)$ and $\mathcal{B}(\Sigma_k)$ denote the Borel sigma algebras for Σ_n and Σ_k, respectively. Let $\mathcal{A} = p^{-1}\mathcal{B}(\Sigma_k) \subset \mathcal{B}(\Sigma_n)$ be the corresponding σ-invariant sub-sigma algebra on Σ_n. In particular, this is a smaller sigma algebra which cannot distinguish between symbols in $\{0, 1, \ldots, n-1\}$ that project under p to the same symbol in Σ_k.

We recall the following result:

Ledrappier-Young Lemma. *For* μ *almost every* $\underline{x} \in \Sigma_n$

$$\lim_{N\to\infty} -\frac{\log(\mu_\xi([x_0, \ldots, x_{N-1}]))}{N} = h(\mu) - h(\overline{\mu}) := h(\mu|\mathcal{A}).$$

Proof. We omit the proof in the general case, but observe that for Bernoulli measures it is fairly straight forward to see this. In particular, for a.e. (μ), $x \in \Sigma_n$ the symbols in $p^{-1}(i)$ occur with frequency $\frac{n_i}{n}$ and have associated weight $\frac{n_i}{n}$. Thus the limit is

$$h(\mu|\mathcal{A}) = \frac{n_i}{n}\log\left(\frac{n_i}{n}\right),$$

as required. □

Let us define $\overline{\Pi}_\lambda : \Sigma_k \to \mathbb{R}$ by

$$\overline{\Pi}_\lambda(\underline{i}) = \sum_{j=0}^{\infty} c_{i_j}^{(2)}\lambda^j.$$

In particular, $\overline{\Pi}_\lambda$ corresponds to mapping sequences from Σ_k to points on $\mathbb{R}$ by first mapping the sequence $\underline{i}$ to the limit set $\Lambda(\lambda) \subset \mathbb{R}^2$ followed by the horizontal projection of $\Lambda(\lambda)$ to the y-axis. For any sequence $\xi \in \Sigma_k$ it is convenient to write

$y_\xi = \overline{\Pi_\lambda}(\xi)$. It is easy to see that $\Pi_\lambda(p^{-1}\xi) \subset \Lambda(\lambda) \subset \mathbb{R}^2$ is actually the part of the limit set $\Lambda(\lambda)$ lying on the horizontal line $L_{y_\xi} := \{(x,y) : y = y_\xi\}$.[6]

We define two new measures. Firstly, $\overline{\nu}_\lambda = \overline{\mu} \circ \overline{\Pi}_\lambda$ on the vertical axis $\mathbb{R}$ and, secondly, on the horizontal axis $\nu_{\lambda,\xi} = \mu_\xi \circ \Pi_\lambda^{-1}$ on $\Lambda(\lambda) \cap L_{y_\xi}$. The following lemma allows us to relate the dimensions of these various measures.

Lemma 8.3. *Let $s \geq 0$. If for a.e. $(\overline{\mu})$ $\xi \in \Sigma_k$ we have that $\dim_H \nu_{\lambda,\xi} \geq s$ then*

$$\dim_H \nu_\lambda \geq \dim_H \overline{\nu_\lambda} + s.$$

Proof. Let $A \subseteq \mathbb{R}^2$ be any Borel set such that $\nu_\lambda(A) = 1$. It follows that $\mu(\Pi_\lambda^{-1}(A)) = 1$ and thus by the decomposition of μ, we have that

$$1 = \mu(\Pi_\lambda^{-1}(A)) = \int \mu_\xi(\Pi_\lambda^{-1} A \cap p^{-1}\xi) \mathrm{d}\overline{\mu}(\xi).$$

Thus for a.e. $(\overline{\mu})$ $\xi \in \Sigma_k$ we have $\mu_\xi(\Pi_\lambda^{-1}(A) \cap p^{-1}\xi) = 1$ and, hence, again from the definitions, $\nu_{\lambda,\xi}(A \cap L_{\Pi_\lambda(\xi)}) = 1$. However, $\dim \nu_{\lambda,\xi} \geq s$ for a.e.$(\overline{\mu})$ ξ and thus $\dim_H(A \cap L_{\Pi_\lambda(\xi)}) \geq s$ for a.e.$(\overline{\mu})$ ξ. In particular, $\dim_H(A \cap L_y) \geq s$ for a.e. $(\overline{\nu}_\lambda)$ y. By applying Marstrand's Slicing Theorem to the set $B = \{y : \dim_H(A \cap L_y) \geq s\}$, which is of full $\overline{\nu}_\lambda$ measure, we deduce that $\dim A \geq s + \dim \overline{\nu}_\lambda$. Since this holds for all Borel sets A where $\nu_\lambda(A) = 1$we conclude that $\dim \nu_\lambda(A) \geq s + \dim \overline{\nu}_\lambda$. □

Since $\overline{\nu}_\lambda$ is a measure on the real line, its properties are better understood. In particular, we have the following result.

Lemma 8.4. *For almost all $\lambda \in \left[\frac{1}{k}, y(k-1)\right]$ we have that*

$$\dim(\overline{\nu}_\lambda) = \min\left(1, -\frac{h(\overline{\mu})}{\log \lambda}\right).$$

Proof. The proof makes use of transversality and the Shannon-McMullen-Brieman theorem, and follows the general lines of Theorem 7.3.

Firstly, it is easy to see from the definitions that $\dim_H \Lambda(\lambda) \leq \dim_B \Lambda(\lambda) \leq -\frac{h(\overline{\mu})}{\log \lambda}$. We now consider the opposite inequality. Given $\epsilon > 0$ let $s_\epsilon(\lambda) = -\frac{h(\overline{\mu})}{\log(\lambda+\epsilon)}$. Note that the proof can be completed (as in the proofs in the previous chapters) if it can be shown that,

$$I = \int_{\frac{1}{k}}^{y(k-1)} \left(\iint \frac{\mathrm{d}\nu_\lambda(x)\mathrm{d}\nu_\lambda(y)}{|x-y|^{s_\epsilon(\lambda)}} \right) \mathrm{d}\lambda < \infty,$$

---

[6] To see this, let $\omega \in \Sigma_n$ satisfy $p\omega = \xi$ then we know that $c_{\omega_i}^{(2)} = \xi_i$. Thus if we consider

$$\Pi_\lambda(\omega) = \sum_{i=0}^{n} (c_{\omega_i}^{(1)}, c_{\omega_i}^{(2)})\lambda^i = \sum_{i=0}^{\infty} (c_{\omega_i}^{(1)}, c_{\xi_i}^{(2)})\lambda^i$$

then the y-co-ordinate of $\Pi_\lambda(\omega)$ is equal to $\overline{\Pi_\lambda}(\xi)$. Thus any point in $(x,y) \in \Pi_\lambda(p^{-1}\xi)$ lies on the line $y = y_\xi = \overline{\Pi_\lambda}(\xi)$ which we denote $L_{\overline{\Pi_\lambda}(\xi)}$.

for all $\epsilon > 0$. In particular, the finiteness of the integrand, for almost all λ, allows us to deduce that for these values $\dim_H \Lambda(\lambda) \geq s_\epsilon(\lambda)$. Since the value of $\epsilon > 0$ is arbitrary, we get the required lower bound $\dim_H \Lambda(\lambda) \geq -\frac{h(\overline{\mu})}{\log \lambda}$.

The inner two integrals can be rewritten in terms of the measure μ on Σ and we can rewrite this as

$$I = \int_{\frac{1}{k}}^{y(k-1)} \left(\iint \frac{\mathrm{d}\mu(\underline{i})\mathrm{d}\mu(\underline{j})}{|\overline{\Pi}_\lambda(\underline{i}) - \overline{\Pi}_\lambda(\underline{j})|^{s_\epsilon(\lambda)}} \right) \mathrm{d}\lambda.$$

Let $t = \max_{\frac{1}{k} \leq \lambda \leq y(k-1)} s_\epsilon(\lambda)$ and note that $t < 1$. In particular, if $\underline{i} \neq \underline{j}$ then they agree until the $|\underline{i} \wedge \underline{j}|$-th term and we can write

$$\begin{aligned} |\Pi_\lambda(\underline{i}) - \Pi_\lambda(\underline{j})|^{s_\epsilon(\lambda)} &= \lambda^{|\underline{i} \wedge \underline{j}| s_\epsilon(\lambda)} \left(\sum_{k=0}^{\infty} a_k \lambda^k \right)^{s_\epsilon(\lambda)} \\ &\geq \left(e^{-h(\overline{\mu})} + \epsilon \right)^{s_\epsilon(\lambda) |\underline{i} \wedge \underline{j}|} \left(\sum_{k=0}^{\infty} a_k \lambda^k \right)^t, \end{aligned}$$

where $\{a_k\}_{k \in \mathbb{Z}^+}$ is the sequence $a_k := i_{k+|\underline{i} \wedge \underline{j}|} - j_{k+|\underline{i} \wedge \underline{j}|} \in \{0, \pm 1, \ldots, \pm(k-1)\}$ and $a_0 \neq 0$. Substituting this back into the integrand in I and using Fubini's Theorem we get

$$I \leq \int_\Sigma \int_\Sigma \frac{\mathrm{d}\mu(\underline{i})\mathrm{d}\mu(\underline{j})}{\left(e^{-h(\overline{\mu})} + \epsilon \right)^{|\underline{i} \wedge \underline{j}|}} \left(\int_{\frac{1}{k}}^{y(k-1)} \frac{\mathrm{d}\lambda}{\left(\sum_{k=0}^{\infty} a_k \lambda^k \right)^t} \right). \tag{8.1}$$

We can estimate the first integral in (8.1) by

$$\begin{aligned} \int \int \frac{\mathrm{d}\mu(\underline{i})\mathrm{d}\mu(\underline{j})}{\left(e^{-h(\overline{\mu})} + \epsilon \right)^{|\underline{i} \wedge \underline{j}|}} &\leq \sum_{m=0}^{\infty} \sum_{[i_0, i_1, \ldots, i_{k-1}]} \frac{\mu([i_0, i_1, \ldots, i_{m-1}])}{\left(e^{-h(\overline{\mu})} + \epsilon \right)^m} \\ &= \sum_{m=0}^{\infty} \frac{e^{-mh(\overline{\mu})}}{\left(e^{-h(\overline{\mu})} + \epsilon \right)^m} < \infty. \end{aligned}$$

Thus to show that $I < \infty$ it remains to bound the second integral in (8.1) by

$$\int \frac{\mathrm{d}\lambda}{\left(\sum_{k=0}^{\infty} a_k \lambda^k \right)^t} < \infty$$

for any sequence $\{a_k\}_{k \in \mathbb{Z}^+}$, where $a_k \in \{0, \pm 1, \ldots \pm (k-1)\}$ and $a_0 \neq 0$. Let $f(\lambda) = 1 + \sum_{k=0}^{\infty} \left(\frac{a_k}{a_0} \right) \lambda^k$ then we can apply part (1) of Proposition 7.1 to deduce that the integral is finite. $\square$

The next lemma allows us to associate to the measure $\overline{\nu}_\lambda$ a set $Y \subset \mathbb{R}$.

Lemma 8.5. *For almost every $\lambda \in [\frac{1}{k}, y(k-1)]$ there exists a set $Y \subset \mathbb{R}$ with $\dim_H(Y) = \dim_H(\overline{\nu}_\lambda)$ such that for any $\xi \in (\overline{\Pi}_\lambda)^{-1} Y \subset \Sigma_k$ we can bound*

$$\dim_H(\nu_{\lambda,\xi}) \geq \min \left\{ -\frac{h(\nu|\mathcal{A})}{\log \lambda}, 1 \right\}.$$

Proof. Given $\delta > 0$, it is enough to show that for almost all $\lambda \in [\frac{1}{k}, y(1)]$ there exists a set $X = X_\delta \subset \Sigma_k$ with $\overline{\mu}(X) \geq 1 - \delta$ and such that for any $\xi \in X$, $\dim_H(\nu_{\xi,\lambda}) \geq \frac{-h(\mu|\mathcal{A})}{\log \lambda}$. In particular, we can take $Y = \cap_{n=1}^\infty X_{\frac{1}{n}}$.

Fix $\epsilon, \epsilon' > 0$. By Ergorov's Theorem there exist sets $X_{\epsilon'} \subset \Sigma_k$ and a constant $K > 0$ such that:

(1) $\overline{\mu}(X_{\epsilon'}) > 1 - \epsilon'$; and
(2) for any $\underline{\xi} \in X_{\epsilon'}$ there exists $Y_{\epsilon'}$ such that for any $\underline{x} \in X_{\epsilon'}$ we can bound

$$\mu_\xi[x_0, \dots, x_N] \leq K \exp\left(-\left(h(\mu|\mathcal{A}) - \epsilon\right) N\right), \text{ for } N \geq 1.$$

Let us denote $s = s_\epsilon(\lambda) = -\frac{h(\mu|\mathcal{A})}{\log \lambda} - 2\epsilon$. We want to consider the measure $\overline{\mu}$ restricted to $X_{\epsilon'}$ and the measure $\nu_{\lambda,\xi}$ restricted to $\Pi_\lambda(Y_{\epsilon'}) \cap L_\xi$, where $\xi \in X_{\epsilon'}$. This allows us to use the explicit bound in (2). Consider the multiple integral

$$I = \int_{\frac{1}{k}}^{y(k-1)} \int_{X_{\epsilon'}} \left(\int_{\Pi_\lambda Y_{\epsilon'}} \int_{\Pi_\lambda Y_{\epsilon'}} \frac{d\nu_{\xi,\lambda}(x) d\nu_{\xi,\lambda}(y)}{|x-y|^s} \right) d\overline{\mu}(\xi) d\lambda$$

We want to prove finiteness of this integral by lifting $\nu_{\xi,\lambda}$ to μ_ξ on $p^{-1}\xi$ and then using Fubini's Theorem to rewrite the integral as:

$$\begin{aligned}
I &= \int_{X_{\epsilon'}} \int_{Y_{\epsilon'}} \int_{Y_{\epsilon'}} \int_{\frac{1}{k}}^{y(k-1)} \frac{d\lambda}{|\Pi_\lambda(\underline{i}) - \Pi_\lambda(\underline{j})|^s} d\mu_\xi(\underline{i}) d\mu_\xi(\underline{j}) d\overline{\mu}(\xi) \\
&= \int_{X_{\epsilon'}} \int_{Y_{\epsilon'}} \int_{Y_{\epsilon'}} \int_{\frac{1}{k}}^{y(k-1)} \frac{d\lambda}{|\sum_{n=1}^\infty (i_n - j_n)\lambda^n|^s} d\mu_\xi(\underline{i}) d\mu_\xi(\underline{j}) d\overline{\mu}(\xi) \\
&= \int_{X_{\epsilon'}} \int_{Y_{\epsilon'}} \int_{X_{\epsilon'}} \int_{\frac{1}{k}}^{y(k-1)} \frac{d\lambda}{e^{(h(\mu|\mathcal{A})-2\epsilon)|\underline{i}\wedge\underline{j}|} |\sum_{n=0}^\infty a_n \lambda^n|^s} d\mu_\xi(\underline{i}) d\mu_\xi(\underline{j}) d\overline{\mu}(\xi)
\end{aligned}$$

where we have that $a_n \in \{0, \pm 1, \dots, \pm(k-1)\}$ and $a_0 \neq 0$. Thus we can use transversality to write

$$\begin{aligned}
I &\leq C \int_{X_{\epsilon'}} \int_{Y_{\epsilon'}} \int_{Y_{\epsilon'}} e^{-(h(\mu|\mathcal{A})+2\epsilon)\underline{i}\wedge\underline{j}} d\mu_\xi(\underline{i}) d\mu_\xi(\underline{j}) d\overline{\mu}(\xi) \\
&\leq C \sum_{m=0}^\infty e^{-m(h(\mu|\mathcal{A})+2\epsilon)} (\mu_\xi \times \mu_\xi) \left(\{(\underline{i}, \underline{j}) \in Y_{\epsilon'} \times Y_{\epsilon'} : i_a = j_b, 0 \leq a \leq m\} \right) \\
&\leq CK \sum_{m=0}^\infty e^{-m(h(\mu|\mathcal{A})+2\epsilon)} e^{(h(\mu|\mathcal{A})+\epsilon)m} < +\infty.
\end{aligned}$$

In particular, from this we deduce that that for almost every $\lambda \in [\frac{1}{k}, y(k-1)]$, there is a set $Y = Y(\lambda) \subset \Pi_\lambda(X)$ of $\overline{\nu}$ measure $1 - \epsilon'$ such that for $y \in Y$ one can choose $\xi \in \overline{\Pi}_\lambda^{-1}(y)$ such that

$$\int_{\Pi_\lambda Y_{\epsilon'}} \int_{\Pi_\lambda Y_{\epsilon'}} \frac{d\nu_{\lambda,\xi}(x) d\nu_{\xi,\lambda}(y)}{|x-y|^s} < +\infty.$$

By results in a previous chapter, this allows us deduce that $\dim_H(\nu_{\lambda,\xi}) \geq s$. Finally, since $\epsilon > 0$ was arbitrary, the result follows. $\square$

Proof of Proposition 8.2. By combining the estimates in Lemma 8.4 and 8.5 and the Marstrand Slicing Lemma we can see that for almost every $\lambda \in [\frac{1}{k}, y(k-1)]$

$$\dim_H \nu_\lambda \geq \min\left\{-\frac{h(\mu|\mathcal{A})}{\log\lambda}, 1\right\} + \min\left(1, -\frac{h(\overline{\mu})}{\log\lambda}\right).$$

Thus if $-\frac{h(\mu|\mathcal{A})}{\log\lambda} < 1$ and $-\frac{h(\overline{\mu})}{\log\lambda} < 1$ we have that

$$\dim \nu_\lambda \geq -\frac{h(\mu|\mathcal{A})}{\log\lambda} - \frac{h(\overline{\mu})}{\log\lambda}$$

for almost every $\lambda \in [\frac{1}{k}, b_{k-1}]$. However, from the definitions:

$$h(\mu) = h(\overline{\mu}) + h(\mu|\mathcal{A})$$

and thus for almost every $\lambda \in [\frac{1}{k}, b_{k-1}]$ we have,

$$\dim \nu_\lambda \geq -\frac{h(\mu)}{\log\lambda}.$$

This completes the proof of Proposition 8.2. □

8.3 Proof of Theorem 8.1. To prove Theorem 8.1 it remains to apply Proposition 8.2 with a suitable choice of μ to get the lower bound.

More precisely, let μ denote the Bernoulli measure $\mu = \left(\frac{1}{n}, \ldots, \frac{1}{n}\right)^{\mathbb{Z}^+}$. Thus $h(\mu) = \log n$. We saw before that transversality gives $b_{k-1} \geq (1+\sqrt{k-1})^{-1}$ and thus since $k < n$ we have that $y(k-1) \geq (1+\sqrt{k-1})^{-1} \geq n^{-\frac{1}{2}}$. We need to find conditions on λ such that $-\log\lambda \geq h(\overline{\mu})$ and $-\log\lambda \geq h(\mu|\mathcal{A})$ and then we can calculate

$$\begin{aligned}
h(\overline{\mu}) = -\sum_{i=0}^{k-1} \frac{n_i}{n}\log\left(\frac{n_i}{n}\right) &= -\frac{1}{n}\sum_{i=0}^{k-1}(n_i\log n_i - n_i\log n) \\
&= -\frac{1}{n}\sum_{i=0}^{k-1}\log n_i^{n_i} + \log n \\
&= -\frac{1}{n}\left(\log\prod_{i=0}^{k-1} n_i^{n_i}\right) + \log n \\
&= -\log\left(\frac{\prod_{i=0}^{k-1}(n_i^{n_i})^{\frac{1}{n}}}{n}\right).
\end{aligned}$$

We can write

$$h(\mu|\mathcal{A}) = \sum_{i=0}^{k-1}\frac{n_i}{n}\log n_i = \log\left(\prod_{i=0}^{k-1} n_i^{n_i}\right)^{\frac{1}{n}}.$$

Thus, if we choose

$$s = \min\left\{\frac{1}{n}\left(\prod_{i=0}^{k-1} n_i^{n_i}\right)^{\frac{1}{n}}, \left(\prod_{i=0}^{k-1} n_i^{-n_i}\right)^{\frac{1}{n}}\right\}$$

then for almost every $\lambda \in [\frac{1}{k}, s]$ we have that,

$$\dim_H \nu \geq -\frac{h(\mu)}{\log \lambda} = -\frac{\log n}{\log \lambda}.$$

In particular, for almost every $\lambda \in [\frac{1}{k}, s]$ we have that

$$\dim_H \Lambda(\lambda) \geq -\frac{\log n}{\log \lambda},$$

as required. □

8.4 Fat Sierpinski Carpets. As the value of λ increases the limit set $\Lambda(\lambda)$ becomes larger. Eventually, we have a similar type of result where for typical λ the set $\Lambda(\lambda)$ has positive measure.

More precisely, we have the following result we obtain concerning the two dimensional measure of the attractor.

Theorem 8.6. *There exists* $\frac{1}{\sqrt{n}} \leq t \leq y(k-1)$ *such that for almost all* $\lambda \in [t, y(k-1)]$ *we have that* $\text{leb}(\Lambda(\lambda)) > 0$.

Examples. For the Sierpinski Carpet, we can take $t = 0.357\ldots$. For the Vicsek cross we can take and $t = 0.4541$.

The following simple lemma shows how we can show absolute continuity of ν_λ using absolute continuity of the conditional measures.

Lemma 8.7. *If* $\overline{\nu}_\lambda$ *is absolutely continuous and* $\nu_{\lambda,\xi}$ *is absolutely continuous for a.e.* $(\overline{\mu})$ ξ *then* ν_λ *is absolutely continuous.*

Proof. Let $A \subset \mathbb{R}^2$ be any set such that $\text{Leb}(A) = 0$. We need to show that $\nu_\lambda(A) = 0$. Using the definiton of ν_λ and the decomposition of μ we get that

$$\nu_\lambda(A) = \mu(\Pi_\lambda^{-1} A) = \int_{\Sigma_k} \mu_\xi(\Pi_\lambda^{-1} A \cap p^{-1}\xi) d\overline{\mu}(\xi).$$

From the definition of $\nu_{\xi,\lambda}$ we have that

$$\mu_\xi(\Pi_\lambda^{-1} A \cap p^{-1}\xi) = \nu_{\lambda,\xi}(\Pi_\lambda(\Pi_\lambda^{-1} A \cap p^{-1}\xi)).$$

Since $\text{Leb}(A) = 0$, we know that the set $\{y \in \mathbb{R} : \mathit{Leb}(L_y \cap A) > 0\}$ has zero Lebesgue measure. Thus from the absolute continuity of ν_λ we have

$$\overline{\mu}\{\xi \in \Sigma_k : \text{Leb}(L_{\Pi_\lambda \xi} \cap A) > 0\} = \overline{\nu}_\lambda\{y \in \mathbb{R} : \text{Leb}(L_y \cap A) > 0\} = 0.$$

Since $\nu_{\lambda,\xi}$ is absolutely continuous for $\overline{\mu}$ almost all ξ we know that $\nu_{\lambda,\xi}(\Pi_\lambda(\Pi_\lambda^{-1} A \cap p^{-1}\xi)) = 0$ for $\overline{\mu}$ almost all ξ. Thus we have that $\nu_\lambda(A) = 0$, as required. □

We now need to determine when the measures $\overline{\nu}_\lambda$ and $\nu_{\lambda,\xi}$ are absolutely continuous. The following result concerning $\overline{\nu}_\lambda$ is useful.

Lemma 8.8. *For almost all $\lambda \in [e^{-h(\overline{\mu})}, b_{k-1}]$ the measure $\overline{\nu}_\lambda$ is absolutely continuous with respect to one dimensional Lebesgue measure.*

Proof. We omit the proof since it is similar to the proof of the next lemma. □

Of course, it is possible that $e^{-h(\overline{\mu})} > b_{k-1}$. In this case the lemma does not give any new information. We now prove a result about the absolute continuity of measures supported on the fibre.

lemma 8.9. *For almost all λ in*

$$\left\{ \lambda \in \left[\frac{1}{k}, b_{k-1}\right] \ : h(\mu|\mathcal{A}) > -\log\lambda \right\}$$

there exists a set $X \subseteq \Sigma_k$ such that $\overline{\mu}(X) = 1$ and for any $\xi \in X$ the measure $\nu_{\lambda,\epsilon}$ is absolutely continuous on $L_{\overline{\Pi}_\lambda(\xi)}$.

Proof. It suffices to show that given $\epsilon' > 0$, there exists a set $X_{\epsilon'} \subseteq \Sigma_k$ such that $\overline{\mu}(X_{\epsilon'}) \geq 1-\epsilon'$ and for any $\xi \in X_{\epsilon'}$ there exists a set $Y_{\epsilon',\xi} \subset L_{\overline{\Pi}_\lambda(\xi)}$ where $\mu_\xi(Y'_\epsilon) \geq 1-\epsilon'$ and $\nu_{\lambda,\epsilon}$ is absolutely continuous on $Y_{\epsilon',\xi}$. We can then take $X = \cap_{N=1}^\infty X_{\frac{1}{N}}$.

Let $\epsilon, \epsilon' > 0$. From Ergorov's Theorem we know that there exists $K > 0$ and a set $X_{\epsilon'} \subseteq \Sigma_k$ such that $\overline{\mu}(X_{\epsilon'})$ and for $\xi \in X_{\epsilon'}$ there exists $Y_{\epsilon',\xi} \subseteq p^{-1}\xi$ with $\mu_\xi(Y_{\epsilon',\xi}) > 1-\epsilon'$ and for $\underline{x} \in Y_{\epsilon',\xi}$ we have that

$$\mu_\xi[x_0, \dots, x_{N-1}] \leq K \exp\left(-\left(h(\mu|\mathcal{A}) - \epsilon\right) N\right), \text{ for } N \geq 1.$$

We recall that to show that $\nu_{\xi,\lambda}$ is absolutely continuous it suffices to show that $\underline{D}(\nu_{\xi,\lambda})(x)$ is finite, for a.e.$(\nu_{\xi,\lambda})$ $x \in \Pi_\lambda Y_{\epsilon',\xi}$. In particular, it suffices to show that

$$\int_{\Pi_\lambda Y_{\epsilon',\xi}} \underline{D}(\nu_{\xi,\lambda})(x) d\nu_{\xi,\lambda}(x) < +\infty.$$

Moreover, to show that for almost every λ there exists a set of ξ of $\overline{\mu}$ measure at least $1-\epsilon'$ such that $\nu_{\xi,\lambda}$ is absolutely continuous, it suffices to show that

$$I := \int_t^{b_y(k-1)} \int_{X_{\epsilon'}} \left(\int_{\Pi_\lambda Y_{\epsilon',\xi}} \underline{D}(\nu_{\xi,\lambda})(x) d\nu_{\xi,\lambda}(x) \right) d\overline{\mu}(\xi) d\lambda < +\infty,$$

providing t is sufficiently large. We take $t > e^{h(\mu|\mathcal{A})+2\epsilon}$. For $\omega, \tau \in p^{-1}\xi$ we define

$$\phi_r(\omega,\tau) = \{\lambda : |\Pi_\lambda(\omega) - \Pi_\lambda(\tau)| \leq r\},$$

for $r > 0$. We start by lifting to the shift space, applying Fatou's Lemma and Fubini's Theorem

$$\begin{aligned} I &\leq \liminf_{r\to 0} \frac{1}{2r} \int_t^{b_y(k-1)} \int_{X_{\epsilon'}} \int_{Y_{\epsilon',\xi}} \int_{Y_{\epsilon',\xi}} (\omega,\tau) \mu_\xi(\omega) d\mu_\xi(\tau) d\overline{\mu}(\xi) d\lambda \\ &\leq \liminf_{r\to 0} \frac{1}{2r} \int_{X_{\epsilon'}} \int_{Y_{\epsilon',\xi}} \int_{Y_{\epsilon',\xi}} \mathrm{leb}(\phi_r(\omega,\tau)) d\mu_\xi(\omega) d\mu_\xi(\tau) d\overline{\mu}(\xi), \end{aligned}$$

where is the characteristic function for $\{(\omega, \tau) : |\Pi_\lambda(\omega) - \Pi_\lambda(\tau)| \leq r\}$. We can deduce

$$\begin{aligned} I &\leq C \int_{X_{\epsilon'}} \int_{Y_{\epsilon',\xi}} \int_{Y_{\epsilon',\xi}} t^{-|\omega\wedge\tau|} d\mu_\xi(\omega) d\mu_\xi(\tau) d\overline{\mu}(\xi) \\ &\leq C \int_{X_{\epsilon'}} \int_{Y_{\epsilon',\xi}} \int_{Y_{\epsilon',\xi}} e^{-|\omega\wedge\tau|(h(\mu|\mathcal{A})+2\epsilon)} d\mu_\xi(\omega) d\mu_\xi(\tau) d\overline{\mu}(\xi) \\ &\leq C \int_{X_{\epsilon'}} \sum_{m=0}^{\infty} e^{-m(h(\mu|\mathcal{A})+2\epsilon)} (\mu_\xi \times \mu_\xi) (\Delta_m) \, d\overline{\mu}(\xi) \\ &\leq CK \sum_{m=0}^{\infty} e^{-m(h(\mu|\mathcal{A})+2\epsilon)} e^{m(h(\mu|\mathcal{A})+\epsilon)} < \infty, \end{aligned}$$

where $\Delta_m = \{(\tau, \omega) \in Y_{\epsilon',\xi} \times Y_{\epsilon',\xi} : \omega_1 = \tau_1, \ldots, \omega_m = \tau_m\}$. This completes the proof. □

We can give an explicit value for t by,

$$t = \sup \left\{ \prod_{j=1}^{k} n_j^{-q_j} : \sum_{j=1}^{k} q_j \log \left(\frac{q_j}{n_j} \right) = 0, \sum_{j=1}^{k} q_j = 1 \text{ and } q_j \geq 0 \right\}.$$

Of course is possible that in some examples $t \geq y(k-1)$, in which case Theorem 8.6 tells us nothing new.

Proof of Theorem 8.6. Of course, to prove Theorem 8.6 we want to use Lemma 8.7 once we know that $\overline{\nu}_\lambda$ and $\lambda_{\xi,\lambda}$ are absolutely continuous. It remains to relate the value of t to the entropies in Lemma 8.8 and Lemma 8.9. Let $\underline{q} = (q_0, \ldots, q_{k-1})$ be a probability vector. Let $p_i = \frac{q_{p(i)}}{n_{p(i)}}$ for $i = 1, \ldots, n$ and μ be the $\underline{p}$-Bernoulli measure on Σ_n. If we let $\overline{\mu} = \mu p^{-1}$ then we have that

$$h(\overline{\mu}) = \sum_{i=0}^{k-1} \text{ and } h(\mu|\mathcal{A}) = \sum_{i=0}^{k-1} q_i \log n_i.$$

If we let t be defined as above then for $\epsilon > 0$ let $\underline{q}$ satisfy $\sum_{j=0}^{k-1} n_j^{-q_j} \geq t - \epsilon$ then for any $\lambda \geq t - \epsilon$ we have that $-\log \lambda \leq h(\overline{\mu}) = h(\mu|\mathcal{A})$. Thus for almost every $\lambda \geq t - \epsilon$ the measure ν_λ is absolutely continuous and hence $\mathrm{Leb}(\Lambda(\lambda)) > 0$. The proof is completed by letting $\epsilon \to 0$. □

Example: Higher dimension. The results in this chapter can be generalised without difficulty to higher dimensional setting. We consider two such setting in $\mathbb{R}^3$. Firstly we consider the Sierpiński tetrahedron. This consists of the following four similarities.

$$\begin{aligned} T_0(x, y, z) &= \lambda(x, y, z) + (0, 0, 0) \\ T_1(x, y, z) &= \lambda(x, y, z) + (1, 0, 0) \\ T_2(x, y, z) &= \lambda(x, y, z) + (0, 1, 0) \\ T_3(x, y, z) &= \lambda(x, y, z) + (0, 0, 1). \end{aligned}$$

In the case where $\lambda = \frac{1}{2}$ this iterated function system would satisfy the open set condition and the attractor, $\Lambda(\lambda)$ would have dimension $\frac{\log 4}{\log 2} = 2$. We consider the case when $\lambda > \frac{1}{2}$.

$$\dim \Lambda(\lambda) = -\frac{\log 4}{\log \lambda}$$

for almost every $\lambda \in [0.5.0.569\ldots]$.

The menger sponge is another example of a self-similar set in $\mathbb{R}^3$. In the standard case it consists of 20 contractions of ratio $\frac{1}{3}$. The values of c_i consists of all triples of $(x, y, z) \in (0, 1, 2)^3$ where at most one of x, y of z takes the value 1. If we consider the case where the contraction ratio (λ) are bigger than $\frac{1}{3}$ we have that $\dim \Lambda(\lambda) = -\frac{\log 20}{\log \lambda}$ for almost all $\lambda \leq 0.348$ and that $\Lambda(\lambda)$ has positive measure for almost every $\lambda \geq 0.393$.

8.5 Limits sets with positive measure and no interior. Consider the following problem (posed by Peres and Solomyak): *Can one find examples of self-similar sets with positive Lebesgue measure, but with no interior?*

A variant of the method in the preceding section leads to families of examples of such sets.

The construction. Let $\underline{t} = (t_1, t_2)$ with $0 \leq t_1, t_2 \leq 1$. We consider ten similarities (with the same contraction rate $\frac{1}{3}$) given by

$$\begin{aligned}
T_0(x,y) &= \left(\frac{1}{3}x, \frac{1}{3}y\right) & T_5(x,y) &= \left(\frac{1}{3} + \frac{1}{3}x, \frac{1}{3}y + 1\right) \\
T_1(x,y) &= \left(\frac{1}{3}x, \frac{1}{3}y + t_1\right) & T_6(x,y) &= \left(\frac{2}{3} + \frac{1}{3}x, \frac{1}{3}y\right) \\
T_2(x,y) &= \left(\frac{1}{3}x, \frac{1}{3}y + t_2\right) & T_7(x,y) &= \left(\frac{2}{3} + \frac{1}{3}x, \frac{1}{3}y + t_1\right) \\
T_3(x,y) &= \left(\frac{1}{3}x, \frac{1}{3}y + 1\right) & T_8(x,y) &= \left(\frac{2}{3} + \frac{1}{3}x, \frac{1}{3}y + t_2\right) \\
T_4(x,y) &= \left(\frac{1}{3} + \frac{1}{3}x, \frac{1}{3}y\right) & T_9(x,y) &= \left(\frac{2}{3} + \frac{1}{3}x, \frac{1}{3}y + 1\right).
\end{aligned}$$

This construction is similar in spirit to those in the previous section. To see that the associated limit set $\Lambda_{\underline{t}}$ has empty interior, we need only observe that the intersection of $\Lambda_{\underline{t}}$ with each of vertical lines $\{(k + \frac{1}{2})3^{-n}\} \times \mathbb{R}$, with $n \geq 0$ and $0 \leq k \leq 3^n - 1$ has zero measure. It remains to show that typically $\Lambda_{\underline{t}}$ has positive measure.

Let $\Sigma_{10} = \{1, 2, \cdots, 10\}^{\mathbb{Z}^+}$ denote the full shift on 10 symbols and let $\Pi_{\underline{t}} : \Sigma_{10} \to \Lambda_{\underline{t}}$ be the usual projection map. Let

$$\mu = \left(\frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{6}, \frac{1}{6}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}\right)^{\mathbb{Z}^+}$$

be a Bernoulli measure on Σ_{10}. To show that $\Lambda_{\underline{t}}$ has non-zero Lebesgue measure it suffices to show that $\nu := \mu \Pi_{\underline{t}}^{-1}$ is absolutely continuous. By construction, ν projects to Lebesgue measure on the unit interval in the x-axis, thus it suffices to show the conditional measure $\nu_{\underline{t},x}$ on Lebesgue almost every vertical line $\{x\} \times \mathbb{R}$ is absolutely continuous.

Let $\Sigma_3 = \{1,2,3\}^{\mathbb{Z}^+}$ be a full shift on 3 symbols corresponding to coding the horizontal coordinate. As before, there is a natural map $p : \Sigma_{10} \to \Sigma_3$ corresponding to the map on symbols given by

$$\begin{aligned} p(1) = p(2) = p(3) = p(4) &= 1 \\ p(5) = p(6) &= 2 \\ p(7) = p(8) = p(9) = p(1) &= 3. \end{aligned}$$

Then $\mu p^{-1} = \overline{\mu} = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)^{\mathbb{Z}^+}$ is the Bernoulli measure on Σ_3. Given $\xi \in \Sigma_3$ let μ_ξ denote the induced measure on $p^{-1}(\xi)$. Clearly, if $\Pi_{\underline{t},\xi} : p^{-1}(\xi) \to \{x\} \times \mathbb{R}$ is the restriction of $\Pi_{\underline{t}}$, then by construction $\mu_\xi \Pi_{\underline{t},\xi}^{-1} = \nu_{\underline{t},x}$. We also let $\pi : \Sigma_3 \to [0,1]$ be the natural projection from Σ_3 to the x-axis given by

$$\pi(\xi) = \sum_{n=0}^{\infty} \xi_n \left(\frac{1}{3}\right)^{n+1}.$$

The analogue of transversality is the following:

Lemma 8.10. *There exists $C > 0$ such that*

$$\Delta_\xi(r; \omega, \tau) := \mathrm{Leb}\{\underline{t} \in [0,1]^2 \,:\, |\Pi_{\underline{t},\xi}(\omega) - \Pi_{\underline{t},\xi}(\tau)| \le r\} \le C 3^{|\omega \wedge \tau|} r, \ \textit{for } r > 0.$$

Proof. Let $\omega, \tau \in p^{-1}(\xi)$ with $|\omega \wedge \tau| = n$ (i.e., $\tau_i = \omega_i$ for $i < n$ and $\tau_n \neq \omega_n$). Since $\omega, \tau \in p^{-1}(\xi)$ we have $i(\omega_n) = i(\tau_n)$ for all n, and $\Pi_{\underline{t},\xi}(\omega) - \Pi_{\underline{t},\xi}(\tau) = (0, \phi_{\underline{t},\xi}(\omega,\tau))$, where

$$\phi_{\underline{t},\xi}(\omega,\tau) = 3^{-n} \left((t_{j(\omega_n)} - t_{j(\tau_n)}) + \sum_{k=1}^{\infty} 3^{-k} (t_{j(\omega_{k+n})} - t_{j(\tau_{k+n})}) \right)$$

and $j|_{\{0,4,6\}} \equiv 0, j|_{\{1,7\}} \equiv 1, j|_{\{2,8\}} \equiv 2, j|_{\{3,5,9\}} \equiv 3$, and $t_0 = 0, t_3 = 1$ for convenience. If $\{j(\omega_n), j(\tau_n)\} = \{0,3\}$, then

$$|\phi_{\underline{t},\xi}(\omega,\tau)| \ge 3^{-n} \left(1 - \sum_{k=1}^{\infty} 3^{-k}\right) = 3^{-n}/2,$$

in view of $t_j \in \{0,1\}$ for all j, and (1) follows. Otherwise, let $j \in \{j(\omega_n), j(\tau_n)\} \cap \{1,2\}$. Then

$$\left| \frac{\partial \phi_{\underline{t},\xi}(\omega,\tau)}{\partial t_j} \right| \ge 3^{-n} \left(1 - \sum_{k=1}^{\infty} 3^{-k}\right) = 3^{-n}/2,$$

which also implies (1). $\square$

Now we use Lemma 8.11 to prove that $\nu_{\underline{t},x}$ is absolutely continuous for a.e. x. For a sequence $\xi \in \Sigma_3$ we define $n_i(\xi)$ to be the number of i's in the first n terms of ξ. By the Strong Law of Large Numbers, given $\epsilon, \delta > 0$ we can use Egorov's theorem to choose a set $X \subset [0,1]$ of measure $\mathrm{leb}(X) > 1 - \epsilon$ (equivalently $\overline{\mu}(\pi^{-1}X) > 1 - \epsilon$)

such that there exists $N \in \mathbb{N}$ where for $n \geq N$, $n_i(\xi) \geq \left(\frac{1}{3} - \delta\right)^n$, for $i = 0, 1, 2$. We can bound

$$\begin{aligned}
&\int_{[0,1]^2} \int_X \left(\int_{\{x\}\times\mathbb{R}} \underline{D}(\nu_{\underline{t},x})(y) d\nu_{\underline{t},x}(y) \right) d(\mathrm{leb})(x) d\underline{t} \\
&\leq \liminf_{r\to 0} \frac{1}{2r} \int_{\pi^{-1}X} \left(\int_{p^{-1}(\xi)} \int_{p^{-1}(\xi)} \Delta_\xi(r;\omega,\tau) d\mu_\xi(\omega) d\mu_\xi(\tau) \right) d\overline{\mu}(\xi) \\
&\leq C \int_{\pi^{-1}X} \left(\sum_{n=0}^{\infty} \sum_{\tau_0,\dots,\tau_{n-1}} \mu_\xi[\tau_0,\dots,\tau_{n-1}]^2 3^n \right) d\overline{\mu}(\xi) \\
&\leq C \int_{\pi^{-1}X} \left(\sum_{n=0}^{\infty} 4^{-n_0(\xi)} 2^{-n_1(\xi)} 4^{-n_2(\xi)} 3^n \right) d\overline{\mu}(\xi) \\
&\leq CC_1 + C \sum_{n=N}^{\infty} \left(4^{-(\frac{2}{3}-2\delta)} 2^{-(\frac{1}{3}-\delta)} 3 \right)^n,
\end{aligned}$$

for some $C_1 > 0$ bounding the first N terms of the series, and observe that the series is finite for δ sufficiently small. This implies the absolute continuity for a.e. $\underline{t}$.

We have proved the following result.

Theorem 8.12. *For almost every $\underline{t} \in [0,1]^2$ the limit set $\Lambda_{\underline{t}}$ has positive Lebesgue measure and empty interior.*