2.5. The relaxation time approximation. The Boltzmann integral as a current in **k**-space

The relaxation and fluctuation characteristics introduced in sect. 2.4 enable a semiquantitative description to be given for the change in the distribution during the scattering process. But they do not, in general, allow the Boltzmann integral given by eq. (2.41) to be represented in the "τ-approximation", i.e., in the form

$$\frac{\partial}{\partial t}\delta f(\boldsymbol{k}) = -\frac{\delta f(\boldsymbol{k})}{\tau}. \tag{2.78}$$

A number of cases are known, however, in which the representation given by eq. (2.78) is feasible. Moreover, it has sometimes been found possible to represent the Boltzmann integral given by eq. (2.41) in differential form, which, though more complex than eq. (2.78), leads, nevertheless, to substantial simplifications.

Localized perturbations of the type given by eq. (2.46)
The possibility of writing the Boltzmann integral in the form of eq. (2.78) follows directly from the definition (2.41), since $\delta f(\boldsymbol{k}') \equiv 0$. At the same time, a comparison of eq. (2.41) with eq. (2.48) indicates that $\tau = \tau_0$. Whether or not a perturbation can be regarded as being localized depends on whether collisions can be neglected in which $\boldsymbol{k}$ and $\boldsymbol{k}'$ both lie within the perturbed region, compared to collisions that give rise to exchange between the perturbed and unperturbed regions. For this reason, a perturbation that formally is not described by eq. (2.46) can sometimes be regarded as a localized one (see sect. 7.1).

Elastic scattering in the isotropic model
Making use of the properties of spherical harmonics, it can be shown that for such scattering the "τ-approximation" is valid for perturbations of the type

$$\delta f(\boldsymbol{k}) = \sum_{m=-l}^{l} C_m(\varepsilon)\, Y_{lm}(\boldsymbol{n}) \quad (l \neq 0), \tag{2.79}$$

where Y_{lm} are spherical harmonics and C_m are arbitrary functions of ε. Perturbations with $l = 0$ that depend only on ε do not relax in perfectly elastic scattering. For perturbations described by eq. (2.79)

$$\frac{\partial}{\partial t}\delta f(\boldsymbol{k}) = -\frac{1}{\tau_l(\varepsilon)}\delta f(\boldsymbol{k}), \tag{2.80}$$

where τ_l has the form

$$\frac{1}{\tau_l} = \sum_{\boldsymbol{k}'} W_{\boldsymbol{k}\to\boldsymbol{k}'}\left[1 - P_l(\cos\theta)\right]. \tag{2.81}$$

In scattering by impurities these times are related to the weighted cross sections, given by eq. (2.21), as follows:

$$1/\tau_l = Nv\sigma_l, \quad l = 0, 1, 2, \ldots. \tag{2.82}$$

The time τ_1, determined by eq. (2.81), coincides with the momentum relaxation time defined above, and

$$\frac{1}{\tau_2} = \frac{3}{2}\frac{1}{\tau_\perp}. \tag{2.83}$$

The validity of eq. (2.80) is readily evident for $l = 1$ and $l = 2$ if we note that these perturbations can also be written as

$$\delta f_k = C_i(\varepsilon)\, k_i \qquad\qquad (l = 1), \tag{2.84}$$

$$= C_{ij}(\varepsilon)\left(k_i k_j - \tfrac{1}{3}\delta_{ij}k^2\right) \quad (l = 2). \tag{2.85}$$

When, for example, $l = 1$

$$\frac{\partial}{\partial t}\delta f(\boldsymbol{k}) = C_i\langle k_i - k_i'\rangle_W = -C_i\frac{k_i}{\tau_1} = -\frac{\delta f(\boldsymbol{k})}{\tau_1}. \tag{2.86}$$

Perturbations with $l = 1$ are set up in a weak electric field $\boldsymbol{E}$ when δf is proportional to the field; in this case $C_i \propto E_i$. Hence the time τ_1 appears in a description of electrical conductivity.

Perturbations with $l = 2$, proportional to the square of the field, occur, for example, in photo-excitation of carriers by polarized light. Here $C_{ij} \propto h_i h_j$, where $\boldsymbol{h}$ is a unit vector defining the polarization of the light (see sect. 13.6). A perturbation of this type is also set up by the joint action of two weak fields $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$ of different frequencies ν_1 and ν_2. In this case, perturbations of the distribution function, oscillating at the frequencies $\nu_1 \pm \nu_2$, have $C_{ij} \propto E_{1i}E_{2j}$. It is clear from these examples that $\tau_\perp$ is the time during which a memory of the light polarization is retained by the photoexcited electrons. This time also determines the nonlinear effects of frequency mixing.

Small-angle scattering
When the changes in all three components of the momentum are small in each scattering event, the motion of the electron in $\boldsymbol{k}$-space consists of diffusion. Therefore, the rate of change of the distribution function due to collisions is the divergence of the diffusion current:

$$\frac{\partial}{\partial t}f(\boldsymbol{k}) = -\frac{\partial'}{\partial k_i}J_i(\boldsymbol{k}), \tag{2.87}$$

$$J_i(\boldsymbol{k}) = -A_i(\boldsymbol{k})\,f(\boldsymbol{k}) - D_{ij}(\boldsymbol{k})\frac{\partial}{\partial k_j}f(\boldsymbol{k}). \tag{2.88}$$

Here D_{ij} is the diffusion tensor given by eq. (2.59), and the vector of dynamic

friction A_i is expressed in terms of this tensor and of the momentum loss rate vector given by eq. (2.50), namely

$$A_i(k) = P_i(k) + \frac{\partial}{\partial k_j} D_{ij}(k). \tag{2.89}$$

In the isotropic case, according to eqs. (2.53), (2.61), and (2.62), the flux J_i is determined by the times τ_1, $\tau_\perp$ and $\tau_\parallel$.

The presence in eq. (2.88) of a nonderivative term (in contrast to the customary diffusion flux in *r*-space) is due to the fact that in equilibrium, when the flux should vanish, the distribution $f_T(k)$ is not a constant, but depends upon k.

Quasi-elastic scattering

Since the changes in a component of the momentum may be large in an event of quasi-elastic scattering, the motion of an electron in momentum space is not always diffusive. However, if we are only interested in the position of the electron along the energy axis, then the electron travels in a manner resembling diffusion along this axis. Therefore, the rate of change of the distribution with respect to energy,

$$f(\varepsilon) = \overline{f(k)}^{\,\varepsilon}, \tag{2.90}$$

is the divergence of the diffusion current

$$\frac{\partial}{\partial t} f(\varepsilon) = -\frac{1}{g(\varepsilon)} \frac{\partial}{\partial \varepsilon} [g(\varepsilon) \, J(\varepsilon)], \tag{2.91}$$

$$J(\varepsilon) = -A(\varepsilon) \, f(\varepsilon) - D(\varepsilon) \frac{\partial}{\partial \varepsilon} f(\varepsilon). \tag{2.92}$$

Here $g(\varepsilon)$ is the density of states, eq. (1.9), and the coefficients A and D are related to the averaged quantities given by eq. (2.67): D is the coefficient of energy diffusion and $A(\varepsilon)$ is the so-called coefficient of dynamic friction:

$$A(\varepsilon) = Q(\varepsilon) + \frac{1}{g(\varepsilon)} \frac{\partial}{\partial \varepsilon} [g(\varepsilon) \, D(\varepsilon)]. \tag{2.93}$$

If the scattering is from a thermal bath at the temperature T, it follows from the principle of detailed balance (see sect. 2.2) that, for the equilibrium distribution $f_T(k)$ at the same temperature, the currents given by eq. (2.88) and (2.92) vanish. (If we had not resorted to the principle of detailed balance, we could only have come to the conclusion that the divergence of these currents equals zero.) Consequently,

$$A_i(k) = (\hbar^2/mT) D_{ij}(k) \, k_j, \tag{2.94}$$

$$A(\varepsilon) = D(\varepsilon)/T. \tag{2.95}$$

For the isotropic case, it follows from eq. (2.94) that

$$A_i(k) = \frac{\varepsilon}{T} \frac{k_i}{\tau_\parallel(\varepsilon)}.$$ (2.96)

Since, on the other hand, we have eq. (2.89) for A_i, the following relation between the relaxation times for small-angle scattering of a test particle by a thermal bath is obtained:

$$\frac{1}{\tau_1(\varepsilon)} = \frac{\varepsilon}{T} \frac{1}{\tau_\parallel(\varepsilon)} + \frac{\partial}{\partial \varepsilon}\left[\varepsilon\left(\frac{1}{\tau_\parallel(\varepsilon)} - \frac{1}{2\tau_\perp(\varepsilon)}\right)\right].$$ (2.97)

Randomizing scattering
This term refers to the scattering whose probability has the following properties:

$$W_{k \to k'} = W_{k \to -k'} = W_{-k \to k'}.$$ (2.98)

In such a scattering a test electron at the point k goes, after being scattered, to the point k' or the point $-k'$ with equal probability, the velocities $v_{k'}$ and $v_{-k'}$ at these points being equal and opposite. Therefore, the average velocity of the test electrons vanishes even after the first collision. It is readily evident from eq. (2.43) that the scattering and transport times coincide for randomizing scattering,

$$\tau_0(k) = \tau_1(k),$$ (2.99)

both for Boltzmann and Fermi gases. A special case of randomizing scattering is elastic isotropic scattering with a scattering cross section independent of θ.

If the small perturbation δf of the distribution function is odd, i.e.,

$$\delta f(-k) = -\delta f(k)$$ (2.100)

(when, for instance, it is due to a weak electric field), then, as can be seen from eq. (2.41), we have eq. (2.78) with $\tau = \tau_0$.

Hence, for randomizing scattering and a small antisymmetric perturbation to the distribution, a "τ-approximation" with the scattering time τ_0 is valid. For a Boltzmann gas, eq. (2.78) is simply a consequence of the fact that the scattering-in term vanishes.