Geometry in Physics

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Differential geometry and topology are about mathematics of objects that are, in a sense, 'smooth'. These can be objects admitting an intuitive or visual understanding – curves, surfaces, and the like – or much more abstract objects such as high dimensional groups, bundle spaces, etc. While differential geometry and topology, respectively, are overlapping fields, the perspective at which they look at geometric structures are different: **differential topology** puts an emphasis on global properties. Pictorially speaking, it operates in a world made of amorphous or jelly-like objects whose properties do not change upon continuous deformation. Questions asked include: is an object compact (or infinitely extended), does it have holes, how is it embedded into a host space, etc?

In contrast, **differential geometry** puts more weight on the actual look of geometric structures. (In differential geometry a coffee mug and a donut are *not* equivalent objects, as they would be in differential topology.) It often cares a about distances, local curvature, the area of surfaces, etc. Differential geometry heavily relies on the fact that any smooth object, looks locally *flat* (you just have to get close enough.) Mathematically speaking, smooth objects can be locally modelled in terms of suitably constructed linear spaces, and this is why the concepts of linear algebra are of paramount importance in this discipline. However, at some point one will want to explore how these flat approximations change upon variation of the reference point. 'Variations' belong to the department of calculus, and this simple observations shows that differential geometry will to a large extend be about the synthesis of linear algebra and calculus.

In the next section, we will begin by introducing the necessary foundations of linear algebra, notably tensor algebra. Building on these structures, we will then advance to introduce elements of calculus.

1.1 Exterior Algebra

In this section, we introduce the elements of (multi)linear algebra relevant to this course. Throughout this chapter, V will be an \mathbb{R} -vector space of finite dimension n.

1.1.1 Dual Basis

Let V^* be the **dual vector space** of V, i.e. the space of all linear mappings from V to the real numbers:

$$V^* = \{\phi : V \to \mathbb{R}, \text{ linear}\}$$



Figure 1.1 The arena of differential topology and geometry. The magnified area illustrates that a smooth geometric structure looks locally flat. Further discussion, see text

Let $\{e_i\}$ a basis of V, and $\{e^i\}$ be the associated basis of V^* . The latter is defined by $\forall i, j = 1, \ldots, N$: $e^i(e_j) = \delta^i_j$. A change of basis $\{e_i\} \mapsto \{e'_i\}$ is defined by a linear transformation $A \in GL(n)$. Specifically, with

$$e'_{i} = (A^{-1})^{j}{}_{i}e_{j}, \tag{1.1}$$

the components v^i of a vector $v = v^i e_i$ transform as $v^{i\prime} = A^i_{\ j} v^j$. We denote this transformation behaviour, i.e. transformation under the matrix $\{A^i_j\}$ representing A, as '**contravariant**' transformation. Accordingly, the components $\{v^i\}$ are denoted contravariant components. By (general) conventions, contravariant components carry their indices upstairs, as superscripts. Similarly, the components of linear transformations are written as $\{A^i_j\}$, with the contravariant indices (upstairs) to the left of the covariant indices (downstairs). For the systematics of this convention, see below.

The dual basis transforms by the transpose of the inverse of A, i.e. $e^{i\prime} = A^i_{\ j} e^j$, implying that the components w_i of a general element $w \equiv w_i e^i \in V^*$ transform as $w'_i = (A^{-1})^j_{\ i} w_j$. Transformation under the matrix $(A^{-1})^j_{\ i}$ is denoted as **covariant** transformation. Covariant components carry their indices downstairs, as subscripts.

INFO Recall, that for a general vector space, V, there is no canonical mapping $V \rightarrow V^*$ to its **dual space**. Such mappings require additional mathematical structure. This additional information may either lie in the choice of a basis $\{e_i\}$ in V. As discussed above, this fixes a basis $\{e^i\}$ in V. A canonical (and basis-independent) mapping also exists if V comes with a scalar product, \langle , \rangle :

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 $V \times V \to \mathbb{R}, (v, v') \mapsto \langle v, v' \rangle$. For then we may assign to $v \in V$ a dual vector v^* defined by the condition $\forall w \in V : v^*(w) = \langle v, w \rangle$. If the coordinate representation of the scalar product reads $\langle v, v' \rangle = v^i g_{ij} v'^j$, the dual vector has components $v_i^* = v^j g_{ji}$.

Conceptually, contravariant (covariant) transformation are the ways by which elements of vector spaces (their dual spaces) transform under the representation of a linear transformation in the vector space.

1.1.2 Tensors

Tensors (latin: *tendo* – I span) are the most general objects of multilinear algebra. Loosely speaking, tensors generalize the concept of matrices (or linear maps), to maps that are 'multilinear'. To define what is meant by this, we need to recapitulate that the **tensor product** $V \otimes V'$ of two vector spaces V and V' is the set of all (formal) combinations $v \otimes v'$ subject to the following rules:

 $\triangleright v \otimes (v' + w') \equiv v \otimes v' + v \otimes w'$ $\triangleright (v + w) \otimes v' \equiv v \otimes v' + w \otimes v'$ $\triangleright c(v \otimes w) \equiv (cv) \otimes w \equiv v \otimes (cw).$

Here $c \in \mathbb{R}$, $v, w \in V$ and $v', w' \in V'$. We have written ' \equiv ', because the above relations *define* what is meant by addition and multiplication by scalars in $V \otimes V'$; with these definitions $V \otimes V'$ becomes a vector space, often called the **tensor space** $V \otimes V'$. In an obvious manner, the definition can be generalized to tensor products of higher order, $V \otimes V' \otimes V'' \otimes \ldots$.

We now consider the specific tensor product

$$T_p^q(V) = (\otimes^q V) \otimes (\otimes^p V^*), \tag{1.2}$$

where we defined the shorthand notation $\otimes^q V \equiv \underbrace{V \otimes \cdots \otimes V}_q$. Its elements are called tensors

of degree (q, p). Now, a dual vector is something that maps vectors into the reals. Conversely, we may think of a vector as something that maps dual vectors (or linear forms) into the reals. By extension, we may think of a tensor $\phi \in T_p^q$ as an object mapping q linear forms and p vectors into the reals:

$$\phi: \quad (\otimes^q V^*) \otimes (\otimes^p V) \to \mathbb{R}, \\ (v'_1, \dots, v'_a, v_1, \dots, v_p) \mapsto \phi(v'_1, \dots, v'_a, v_1, \dots, v_p).$$

By construction, these maps are **multilinear**, i.e. they are linear in each argument, $\phi(\ldots, v + w, \ldots) = \phi(\ldots, v, \ldots) + \phi(\ldots, w, \ldots)$ and $\phi(\ldots, cv, \ldots) = c\phi(\ldots, v, \ldots)$.

The tensors form a linear space by themselves: with $\phi, \phi' \in T_p^q(V)$, and $X \in (\otimes^q V^*) \otimes (\otimes^p V)$, we define $(\phi + \phi')(X) = \phi(X) + \phi(X')$ through the sum of images, and $\phi(cX) = c\phi(X)$. Given a basis $\{e_i\}$ of V, the vectors

$$e_{i_1} \otimes \cdots \otimes e_{i_q} \otimes e^{j_1} \otimes \cdots \otimes e^{j_p}, \qquad i_1, \dots, j_p = 1, \dots, N,$$

form a natural basis of tensor space. A tensor $\phi \in T^q_p(V)$ can then be expanded as

$$\phi = \sum_{i_1,\dots,j_p=1}^{N} \phi^{i_1,\dots,i_p}_{j_1,\dots,j_p} e_{i_1} \otimes \dots \otimes e_{i_q} \otimes e^{j_1} \otimes \dots \otimes e^{j_p}.$$
(1.3)

A few examples:

- $\triangleright T_0^1$ is the space of **vectors**, and
- \triangleright T_1^0 the space of linear forms.
- $\triangleright T_1^1$ is the space of linear maps, or **matrices**. (Think about this point!) Notice that the contravariant indices generally appear to the left of the covariant indices; we have used this convention before when we wrote A^i_{i} .
- $\triangleright T_2^0$ is the space of **bilinear forms**.
- \triangleright T_N^0 contains the **determinants** as special elements (see below.)

Generally, a tensor of 'valence' (q, p) is characterized by the q contravariant and the p covariant indices of the constituting vectors/co-vectors. It may thus be characterized as a 'mixed' tensor (if $q, p \neq 0$) that is contravariant of degree q and covariant of degree p. In the **physics literature**, tensors are often identified with their components, $\phi \leftrightarrow \{\phi^{i_1,\ldots,i_p}\}$ which are then – rather implicitly – characterized by their transformation behavior.

The list above may illustrate, that tensor space is sufficiently general to encompass practically all relevant objects of (multi)linear algebra.

1.1.3 Alternating forms

In our applications below, we will not always have to work in full tensor space. However, there is one subspace of $T_p^0(V)$, the so-called space of alternating forms, that will play a very important role throughout:

Let $\Lambda^p V^* \subset T^0_p(V)$ be the set of p-linear real valued alternating forms:

$$\Lambda^p V^* = \{ \phi : \otimes^p V \to \mathbb{R}, \text{ multilinear \& alternating} \}.$$
(1.4)

Here, 'alternating' means that $\phi(\ldots, v_i, \ldots, v_j, \ldots) = -\phi(\ldots, v_j, \ldots, v_i, \ldots)$. A few remarks on this definition:

- \triangleright The sum of two alternating forms is again an alternating form, i.e. Λ^p is a (real) vector space (a subspace of $T_p^0(V)$.)
- $\triangleright \Lambda^p V^*$ is the *p*-th completely antisymmetric tensor power of V^* , $\Lambda^p V^* = (\otimes_1^p V^*)_{asym}$.
- $\triangleright \Lambda^1 V = V^* \text{ and } \Lambda^0 V \equiv \mathbb{R}.$

$$\triangleright \Lambda^{p>n}V = 0$$

- $\triangleright \dim \Lambda^p V^* = \binom{n}{p}.$
- \triangleright Elements of $\Lambda^{p}V^{*}$ are called forms (of degree p).

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1.1.4 The wedge product

Importantly, alternating forms can be multiplied with each other, to yield new alternating forms. Given a p-form and a q-form, we define this so-called **wedge product (exterior product)** by

$$\wedge : \Lambda^{p} V^{*} \otimes \Lambda^{q} V^{*} \rightarrow \Lambda^{p+q} V^{*},$$

$$(\phi, \psi) \mapsto \phi \wedge \psi,$$

$$(\phi \wedge \psi)(v_{1}, \dots, v_{p+q}) \equiv \frac{1}{p!q!} \sum_{P \in S_{p+q}} \operatorname{sgn} P \phi(v_{P1}, \dots, v_{Pp}) \psi(v_{P(P+1)}, \dots, v_{P(p+q)}).$$

For example, for p = q = 1, $(\phi \land \psi)(v, w) = \phi(v)\psi(w) - \phi(w)\psi(v)$. For p = 0 and q = 1, $\phi \land \psi(v) = \phi \cdot \psi(v)$, etc. Important properties of the wedge product include ($\phi \in \Lambda^p V^*, \psi \in \Lambda^q V^*, \lambda \in \Lambda^r V^*, c \in \mathbb{R}$):

- \triangleright bilinearity, i.e. $(\phi_1 + \phi_2) \land \psi = \phi_1 \land \psi + \phi_2 \land \psi$ and $(c\phi) \land \psi = c(\phi \land \psi)$.
- \triangleright associativity, i.e. $\phi \land (\psi \land \lambda) = (\phi \land \psi) \land \lambda \equiv \phi \land \psi \land \lambda$.
- \triangleright graded commutativity, $\phi \land \psi = (-)^{pq} \psi \land \phi$.

INFO A (real) algebra is an \mathbb{R} -vector space W with a product operation '.'

$$W imes W o W,$$

 $u, v \mapsto u \cdot v,$

subject to the following conditions $(u, v, w \in W, c \in \mathbb{R})$:

 $\triangleright (u+v) \cdot w = u \cdot w + v \cdot w,$ $\triangleright u \cdot (v+w) = u \cdot v + u \cdot w,$ $\triangleright c(v \cdot w) = (cv) \cdot w + v \cdot (cw).$

The direct sum of vector spaces

$$\Lambda V^* \equiv \bigoplus_{p=0}^n \Lambda^p V^* \tag{1.5}$$

together with the wedge product defines a real algebra, the so-called an **exterior algebra** or **Grassmann algebra**. We have $\dim \Lambda V^* = \sum_{p=0}^n \binom{n}{p} = 2^n$. A basis of $\Lambda^p V^*$ is given by all forms of the type

$$e^{i_1} \wedge \dots \wedge e^{i_p}, \qquad 1 \le i_1 < \dots < i_p \le n.$$

To see this, notice that i) these forms are clearly alternating, i.e. (for fixed p) they belong to $\Lambda^p V$, they are b) linearly independent, and c) there are 2^n of them. These three criteria guarantee the basis-property. Any p-form can be represented in the above basis as

$$\phi = \sum_{i_1 < \dots < i_p} \phi_{i_1,\dots,i_p} e^{i_1} \wedge \dots \wedge e^{i_p}, \tag{1.6}$$

where the coefficients $\phi_{i_1,\ldots,i_p} \in \mathbb{R}$ obtain as $\phi_{i_1,\ldots,i_p} = \phi(e_{i_1},\ldots,e_{i_p})$. Alternatively (yet equivalantly), ϕ may be represented by the unrestricted sum

$$\phi = \frac{1}{p!} \sum_{i_1, \dots, i_p} \phi_{i_1, \dots, i_p} e^{i_1} \wedge \dots \wedge e^{i_p},$$

where ϕ_{i_1,\ldots,i_p} is totally antisymmetric in its indices. By construction, the components of a p-form transform as a covariant tensor of degree p. Indeed, it is straightforward to verify that under the basis transformation $\phi_{i_1,\ldots,i_p} \mapsto \phi'_{i_1,\ldots,i_p}$, where $\phi'_{i_1,\ldots,i_p} = (A^{-1T})_{i_1}^{j_1} \ldots (A^{-1T})_{i_p}^{j_p} \phi_{j_1,\ldots,j_p}$.

1.1.5 Inner derivative

For any $v\in V$ there is a mapping $i_v:\Lambda^pV^*\to\Lambda^{p-1}V^*$ lowering the tensor degree by one. It is defined by

$$(i_v\phi)(v_1,\ldots,v_{p-1}) \equiv \phi(v,v_1,\ldots,v_{p-1}).$$

The components of $i_v \phi$ obtain by 'contraction' of the components of ϕ :

$$(i_v\phi)_{i_1,\dots,i_{p-1}} = v^i\phi_{i,i_1,\dots,i_{p-1}}$$

Properties of the inner derivative:

- $\triangleright i_v$ is a linear mapping, $i_v(\phi + \phi') = i_v \phi + i_v \phi'$.
- \triangleright It is also linear in its 'parametric argument', $i_{v+w} = i_v + i_w$.
- \triangleright i_v obeys the (graded) Leibnitz rule:

$$i_v(\phi \wedge \psi) = (i_v \phi) \wedge \psi + (-)^p \phi \wedge (i_v) \psi, \qquad \phi \in \Lambda^p V^*.$$
(1.7)

It is for this rule that we call i_v a 'derivative'.

 \triangleright The inner derivative is 'antisymmetric' in the sense that $i_v \circ i_w = -i_w \circ i_v$, in particular, $(i_v)^2 = 0$.

1.1.6 Pullback

Given a form $\phi \in \Lambda^p V^*$ and a linear map $F: W \to V$ we may define a form $(F^*\phi) \in \Lambda^p W$ by the **pullback** operation,

$$F^* : \Lambda^p V^* \to \Lambda^p W^*,$$

$$\phi \mapsto F^* \phi,$$

$$(F^* \phi)(w_1, \dots, w_p) \equiv \phi(Fw_1, \dots, Fw_p).$$
(1.8)

To obtain a component–representation of the pullback, we consider bases $\{e_i\}$ and $\{f_i\}$ of V and W, respectively. The map F is defined by a component matrix (mind the 'horizontal' positioning of indices, $F^j_{\ i} = (F^T)_i^{\ i}$.)

$$Ff_i = F^j_{\ i}e_j.$$

1.2 Differential forms in \mathbb{R}^n

It is then straightforward to verify that the action on the dual basis is given by

$$F^*e^i = F^i{}_j f^j$$

(Cf. with the component representation of basis changes, (1.1).) The action of F^* on general forms may be obtained by using the rules

 $\triangleright F^* \text{ is linear.}$ $\triangleright F^*(\phi \land \psi) = (F^*\phi) \land (F^*\psi),$ $\triangleright (F \circ G)^* = G^* \circ F^*.$

The first two properties state that F^* is a (Grassmann)algebra homomorphism, i.e. a linear map between algebras, compatible with the product operation.

1.1.7 Orientation

Given a basis $\{e_i\}$ of a vector space V, we may define a top-dimensional form $\omega \equiv e^1 \wedge e^2 \wedge \cdots \wedge e^n \in \Lambda^n V^*$. Take another basis $\{e'_i\}$. We say that the two bases are oriented in the same way if $\omega(e'_1, e'_2, \ldots, e'_n) > 0$. One verifies that (a) orientation defines an equivalence relation and (b) that there are only two equivalence classes. A vector space together with a choice of orientation is called an oriented vector space.

EXERCISE Prove the statements above. To this end, show that $\omega(e'_1, e'_2, \dots, e'_n) = \det(A^{-1})$, where A is the matrix mediating the basis change, i.e. $e'_i = (A^{-1T})_i^{\ j} e_j$. Reduce the identification of equivalence classes to a classification of the sign of determinants.

1.2 Differential forms in \mathbb{R}^n

In this section, we generalize the concepts of forms to *differential forms*, that is forms ϕ_x continuously depending on a parameter x. Throughout this section, we assume $x \in U$, where U is an open subset of \mathbb{R}^n . This will later be generalized to forms defined in different spaces.

1.2.1 Tangent vectors

Let $U \subset \mathbb{R}^n$ be open in \mathbb{R}^n . For practical purposes, we will often think of U as parameterized by **coordinates**. A system of coordinates is defined by a diffeomorphism (an bijective map such that the map and its inverse are differentiable),

$$\begin{array}{rcl} x:K & \to & U, \\ (x^1,\ldots,x^n) & \mapsto & x(x^1,\ldots,x^n), \end{array}$$
(1.9)

where the coordinate domain $K \subset \mathbb{R}^n$ is an open subset by itself (cf. Fig. 1.2.)¹ If no confusion is possible, we do not explicitly discriminate between elements $x \in U$ and their representation in terms of an *n*-component coordinate vector.

¹ At this stage, the distinction K vs. U – both open subsets of \mathbb{R}^n – may appear tautological. However, the usefulness of the distinction between these two sets will become evident in chapter 2.



Figure 1.2 An open subset of \mathbb{R}^n and its coordinate representation

EXAMPLE For a chosen basis in \mathbb{R}^n , each element $x \in U$ may be canonically associated to an n-component coordinate vector (x^1, \ldots, x^n) . We then need not distinguish between K and U and $x : U \to U$ is the identity mapping.

EXAMPLE let $U = \mathbb{R}^2 - \{\text{negative } x \text{ axis}\}$. With $K =]0, r[\times]0, 2\pi[$ a system of polar coordinates is defined by the diffeomorphism $K \to U, (r, \phi) \mapsto (r \cos \phi, r \sin \phi) = x(r, \phi)$.

The **tangent space** of U at x is defined by

$$T_x U = \{x\} \times \mathbb{R}^n = \{(x,\xi) | \xi \in \mathbb{R}^n\}.$$
 (1.10)

Through the definition $a(x,\xi) + b(x,\eta) \equiv (x,a\xi + b\eta)$, $a,b \in \mathbb{R}$, T_xU becomes a real *n*-dimensional vector space. Note that each x carries its own copy of \mathbb{R}^n ; For $x \neq y$, T_xU and T_yU are to be considered as independent vector spaces. Also, the space \mathbb{R}^n

attached to $x \in U$ should not be confused with the host space $\mathbb{R}^n \supset U$ in which U lives.

EXAMPLE Consider a curve $\gamma : [0,1] \to U$. Monitoring its velocity, $d_t\gamma(t) \equiv \dot{\gamma}$, we obtain entries $(\gamma, \dot{\gamma}) \in T_{\gamma}U$ of tangent space. Even if U is 'small', the velocities $\dot{\gamma}$ may be arbitrarily large, i.e. the space of velocities \mathbb{R}^n is really quite different from the base U (not to mention that the latter doesn't carry a vector space structure.)

Taking the union of all tangent spaces, we obtain the so-called **tangent bundle** of U,

$$TU \equiv \bigcup_{x \in U} T_x U = \{(x,\xi) | x \in U, \xi \in \mathbb{R}^n\} = U \times \mathbb{R}^n.$$
(1.11)

Notice that $TU \subset \mathbb{R}^{2n}$ is open in \mathbb{R}^{2n} . It is therefore clear that we may contemplate differentiable mapping to and from the tangent bundle. Throughout, we will assume all mappings to be sufficiently smooth to permit all derivative operations we will want to consider.

PHYSICS (M) Assume that U defines the set of n generalized coordinates of a mechanical system. The Lagrangian function $L(x, \dot{x})$ takes coordinates and generalized velocities as its arguments. Now, we have just seen that $(x, \dot{x}) \in T_x U$, i.e.



Figure 1.3 Visualizaton of a coordinate frame



It is important not to get confused about the following point: for a given curve x = x(t), the generalized derivatives $\dot{x}(t)$ are dependent variables and $L(x, \dot{x})$ is determined by the coordinate curves x(t). However, the function L as such is defined without reference to specific curves. The way it is defined, it is a function of the 2n variables $(x, \xi) \in T_x U$.

1.2.2 Vector fields and frames

A smooth mapping

$$v: U \to TU$$

$$x \mapsto (x, \xi(x)) \equiv v(x)$$
(1.12)

is called a **vector field** on U.

Consider a set of n vector fields $\{b_i\}$. We call this set a **frame** (or n-frame), if $(\forall x \in U : \{b_i(x)\})$ linear independent.) The n vectors $\{b_i(x)\}$ thus form a basis of T_xU , smoothly depending on x.

For example, a set of coordinates x^1, \ldots, x^n parameterizing U induces a frame often denoted $(\partial/\partial x^1, \ldots, \partial/\partial x^n)$ or $(\partial_1, \ldots, \partial_n)$ for brevity. It is defined by the n vector fields

$$\frac{\partial}{\partial x^i}\Big|_x \equiv \left(x, \frac{\partial x(x^1, \dots, x^n)}{\partial x^i}\right), \qquad i = 1, \dots, n.$$

Notice that the notation hints at a connection (vectors \leftrightarrow derivatives of functions), a point to which we will return below.

Frames provide the natural language in which to formulate the concept of 'moving' coordinate systems pervasive in physical applications.

EXAMPLE Let $U \subset \mathbb{R}^2$ be parameterized by cartesian coordinates $x = (x^1, x^2)$. Then $\partial_1 = (x, (1, 0))$ and $\partial_2 = (x, (0, 1))$, i.e. $\partial_{1,2}$ are vector fields locally tangent to the axes of a cartesian coordinate system.

Now consider the same set parameterized by **polar coordinates**, $x = (r \cos \phi, r \sin \phi)$. Then $\partial_r = (x, (\cos \phi, \sin \phi))$ and $\partial_{\phi} = (x, -r \sin \phi, r \cos \phi)$ are tangent to the axes of a polar coordinate system. (To obtain a proper orthonormal frame, we would need to normalize these vector fields (see section 1.3.)

Every vector $v(x) \in T_x U$ may be decomposed with respect to a frame as

$$v(x) = \sum_{i} v^{i}(x)b_{i}(x),$$

and every vector field v admits the global decomposition $v = \sum_i v^i b_i$ where $v^i : U \to \mathbb{R}$ are smooth functions. Expansions of this type usually appear in connection with coordinate systems (frames). For example, the curve $(\gamma, \dot{\gamma})$ may be represented as

$$(\gamma, \dot{\gamma}) = \sum_{i} \dot{\gamma}^{i} \frac{\partial}{\partial x^{i}} \big|_{\gamma},$$

where $\gamma^i=\gamma^i(t)$ is the coordinate representation of the curve.

The connection between two frames $\{b_i\}$ and $\{b'_i\}$ is given by

$$b'_i(x) = \sum_i (A^{-1T})_i{}^j(x)b_j(x)$$

where $A(x) \in GL(n)$ is a group-valued smooth function – frames transform covariantly. (Notice that $A^{-1}(x)$ is the inverse of the matrix A(x), and *not* the inverse of the function $x \mapsto A$.) Specifically, the transformation between the coordinate frames of two systems $\{x^i\}$ and $\{y^i\}$ follows from

$$rac{\partial x}{\partial y^i} = \sum_j rac{\partial x}{\partial x^j} rac{\partial x^j}{\partial y^i}.$$

This means,

$$\frac{\partial}{\partial y^i} = \frac{\partial x^j}{\partial y^i} \frac{\partial}{\partial x^j},$$

implying that $(A^{-1})^{j}{}_{i}=\frac{\partial x^{j}}{\partial y^{i}}.$

1.2.3 The tangent mapping

Let $U \subset \mathbb{R}^n$ and $v \subset \mathbb{R}^m$ be open subsets equipped with coordinates (x^1, \ldots, x^n) and (y^1, \ldots, y^m) , respectively. Further, let

$$F: U \to V$$

$$x \mapsto y = F(x),$$

$$(x^{1}, \dots, x^{n}) \mapsto (F^{1}(x^{1}, \dots, x^{n}), \dots, F^{m}(x^{1}, \dots, x^{n}))$$

$$(1.13)$$



Figure 1.4 On the definition of the tangent mapping

be a smooth map. The map F gives rise to an induced map $T_xF: T_xU \to T_{F(x)}V$ of tangent spaces which may be defined as follows: Let $\gamma: [0,1] \to U$ be a curve such that $v = (x,\xi) = (\gamma(t), \dot{\gamma}(t))$. We then define the **tangent map** through

$$T_x F v = (F(\gamma(t)), d_t (F \circ \gamma)(t)) \in T_{F(x)} V.$$

To make this a proper definition, we need to show that the r.h.s. is independent of the particular choice of γ . To this end, recall that $\dot{\gamma} = d_t \gamma^i \frac{\partial}{\partial x^i} = v^i \frac{\partial}{\partial x^i}$, where γ^i are the coordinates of the curve γ . Similarly, $d_t(F \circ \gamma) = d_t(F \circ \gamma)^j \frac{\partial}{\partial y^j}$. However, by the chain rule, $d_t(F \circ \gamma)^j = \frac{\partial F^j}{\partial x^i} d_t \gamma^i = \frac{\partial F^j}{\partial x^i} v^i$. We thus have

$$v = v^i \frac{\partial}{\partial x^i}, \qquad (T_x F)v = \frac{\partial F^j}{\partial x^i} v^i \frac{\partial}{\partial y^j},$$
(1.14)

which is manifestly independent of the representing curve. For fixed x, the tangent map is a linear map $T_xF: T_xU \to T_{F(x)}V$. This is why we do not put its argument, v, in brackets, i.e. T_xFv instead of $T_xF(v)$.

Given two maps $U \xrightarrow{F} V \xrightarrow{G} W$, we have $T_x(G \circ F) = T_{F(x)}G \circ T_xF$. In the literature, the map TF is alternatively denoted as F_* .

1.2.4 Differential forms

A differential form of degree p (or p-form, for short) is a map ϕ that assigns to every $x \in U$ a covariant tensor of degree p, $\phi_x \in \Lambda^p T_x U$. Given p vector fields, $v_1, \ldots, v_p, \phi_x(v_1(x), \ldots, v_p(x))$ is a real number which we require to depend smoothly on x. We denote by $\Lambda^p U$ the set of all p-forms on U. (Not to be confused with the set of p-forms of a single vector space discussed above.) With the obvious linear structure,

$$(a\phi + b\psi)_x \equiv a\phi_x + b\psi_x$$

 $\Lambda^p U$ becomes an infinite dimensional real vector space. (Notice that $\Lambda^0 U$ is the set of real-valued functions on U.) We finally define

$$\Lambda U \equiv \bigoplus_{p=0}^{n} \Lambda^{p} U,$$

the **exterior algebra** of forms. The **wedge product** of forms and the **inner derivative** with respect to a vector field are defined point–wise. They have the same algebraic properties as their ancestors defined in the previous section and will be denoted by the same symbols.

INFO **Differential forms in physics** – **why?** In Physics, we tend to associate everything that comes with a sense of 'magnitude and direction' with a vector. From a computational point of view, this identification is (mostly) o.k., conceptually, however, it may obscure the true identity of physical objects. Many of our accustomed 'vector fields' aren't vector fields at all. And if they are not, they are usually differential forms in disguise.

Let us illustrate this point on examples: consider the concept of 'force, i.e. one of the most elementary paradigms of a 'vector' F in physics teaching. However, force isn't a vector at all! What is more, the non-vectorial nature of force is not rooted in some abstract mathematical thinking but rather in considerations close to experimentation! For force is measured by measuring the amount of *work*, W, needed to move a test charge along a small curve segment. Now, locally, a curve γ can be identified with its tangent vector $\dot{\gamma}$, and work is a scalar. This means that force is something that assigns to a vector ($\dot{\gamma}$) a scalar (W). In other words, **force is a one form**. In physics, we often write this as $W = F \cdot \dot{\gamma} \delta t$, where δt is a reference time interval. This notation hints at the existence of a scalar product. Indeed, a scalar product is required to identify the 'true force', a linear form in dual space, with a vector. However, as long as we think about forces as differential forms, no reference to scalar products is needed.

Another example is current (density). In physics, we identify this with a vector field, much like force above. However, this isn't how currents are measured. Current densities are determined by fixing a small surface area in space, and counting the number of charges that pass through that area per unit time. Now, a small surface may be represented as the parallelogram spanned by two vectors v, w, i.e. current density is something that assigns to two vectors a number (of charges). In other words, current density is a two-form. (It truly is a two form (rather than a general covariant 2-tensor) for the area of the surface is given by the anti-symmetric combination $v \times w$. Think about this point, or see below.) In electrodynamics we allow for time varying currents. It then becomes useful to upgrade current to a three form, whose components determine the number of charges associated to a space-time 'box' spanned by a spatial area and a certain extension in time.

The simplest 'non-trivial' forms are the differential one-forms. Locally, a one-form maps tangent vectors into the reals: $\phi_x : T_x U \to \mathbb{R}$. Thus, ϕ_x is an element of the dual space of tangent space, the so-called **cotangent space**, $(T_x U)^*$. For a given basis $(b_1(x), \ldots, b_n(x))$ of $T_x M$,² the cotangent space is spanned by $(b^1(x), \ldots, b^n(x))$. The induced set of 1-forms (b^1, \ldots, b^n)

 $^{^2}$ Depending on the context, we sometimes indicate the x-dependence of mathematical objects, F, by a bracket notation, F'(x)', and sometimes by subscripts F'_x .

defines a frame of the **cotangent bundle**, i.e. of the unification $(TU)^* \equiv \bigcup_x (T_xU)^*$. The connection between a cotangent frame $\{b^i\}$ and another one $\{b'^i\}$ is given by

$$b'^i(x) = \sum_{j=1}^n \gamma^i_{\ j}(x) b^j(x)$$

where $\gamma^{-1}(x)$ is the matrix (function) generating the change of frames $\{b_i\} \to \{b'_i\}$.

For a given coordinate system $x = x(x^1, ..., x^n)$ consider the coordinate frame $\{\partial_i\}$. Its dual frame will be denoted by $\{dx^i\}$. The action of the dual frame is defined by

$$dx^{i}\left(\frac{\partial}{\partial x^{j}}\right) = \delta^{i}_{\ j}.$$
(1.15)

Under a change of coordinate systems, $\{x^i\} \rightarrow \{y^i\}$,

$$dx^i \to dy^i = \sum_j \frac{\partial y^i}{\partial x^j} dx^j,$$
 (1.16)

i.e. in this case, $\gamma^i{}_j=\frac{\partial y^i}{\partial x^j}.$ (Notice the mnemonic character of this formula.)

By analogy to (1.6), we expand a general *p*-form as

$$\phi = \sum_{i_1,\dots,i_p} \phi_{i_1,\dots,i_p} \, b^{i_1} \wedge \dots \wedge b^{i_p},\tag{1.17}$$

where $\phi_{i_1,...,i_p}(x)$ is a continuous function antisymmetric in its indices. Specifically, for the coordinate forms introduced in (1.15), this expansion assumes the form

$$\phi = \sum_{i_1,\dots,i_p} \phi_{i_1,\dots,i_p} \, dx^{i_1} \wedge \dots \wedge dx^{i_p}.$$
(1.18)

PHYSICS (M) Above, we had argued that the Lagrangian of a mechanical system should be interpreted as a function $L: TU \to \mathbb{R}$ on the tangent bundle. But where is **Hamiltonian mechanics** defined? The Hamiltonian is a function H = (q, p) of coordinates and momenta. The latter are defined through the relation $p_i = \frac{\partial L(q, q)}{\partial q^i}$. The positioning of the indices suggests, that the components op_i do not transform as the components of a vector. Indeed, consider a coordinate transformation $q'^i = q'^i(q)$. We have $\dot{q}'^i = \frac{\partial q'^i}{\partial q^j} \dot{q}^j$, which shows that velocities transform covariantly with the matrix $A^i_{\ j} = \frac{\partial q'^i}{\partial q^j}$. The momenta, however, transform as

$$p'_{i} = \frac{\partial L(q(q'), \dot{q}(q', \dot{q}'))}{\partial \dot{q}'^{i}} = \frac{\partial L(q(q'), \dot{q}(q', \dot{q}'))}{\partial \dot{q}^{j}} \frac{\partial \dot{q}^{j}}{\partial \dot{q}'^{i}} = \frac{\partial q^{j}}{\partial q'^{i}} p_{j}$$

where we noted that $\dot{q}^j = \frac{\partial q^j}{\partial q'^i} \dot{q}'^i$ implies $\frac{\partial \dot{q}^j}{\partial \dot{q}'^i} = \frac{\partial q^j}{\partial q'^i}$. This is covariant transformation behavior under the matrix $(A^{-1})^i_{\ j} = \frac{\partial q^j}{\partial q'^i}$. We are thus led to the conclusions that

The momenta p_i of a mechanical system are the components of a covariant rank 1 tensor. Accordingly, the Hamiltonian H(q, p) is a function on the cotangent bundle TU^* .

We can push this interpretation a little bit further: recall (cf. info block on p ??) that a vector space V can be canonically identified with its dual, V^* , iff V comes with a scalar product. This is relevant to our present discussion, for

the Lagrangian of a mechanical systems defined a scalar product on tangent space, TU.

This follows from the fact that the kinetic energy T in L = T - U usually has the structure $T = \frac{1}{2}\dot{q}^i F_{ij}(q)\dot{q}^j$ of a bilinear form in velocities. This gives $p_i = F_{ij}\dot{q}^j$, (where the \dot{q}^j on the r.h.s. needs to be expressed as a function of q and p).

1.2.5 Pullback of differential forms

Let $F: U \to V$ be a smooth map as defined in (1.13). We may then define a pullback operation

$$F^*: \Lambda^p V^* \to \Lambda^p U^*.$$

For $\psi \in \Lambda^p V^*$ it is defined by

$$(F^*\psi)_x(v_1,\ldots,v_p) = \psi_{F(x)}(F_*v_1,\ldots,F_*v_p)$$

For fixed x, this operation reduces to the vector space pullback operation defined in section 1.1.6. In the specific case p = 0, ψ is a function and $F^*\psi = \psi \circ F$. Straightforward generalization of the identities discussed in section 1.1.6 obtains

 $\label{eq:rescaled} \begin{array}{l} \triangleright \ F^* \ \text{is linear,} \\ \triangleright \ F^*(\phi \wedge \psi) = (F^*\phi) \wedge (F^*\psi), \\ \triangleright \ (G \circ F)^* = F^* \circ G^*. \end{array}$

or

Let $\{y^j\}$ and $\{x^i\}$ be coordinate systems of V and U, resp. (Notice that TU and TV need not have the same dimensionality!). Then

$$(F^*dy^j)\left(\frac{\partial}{\partial x^i}\right) = dy^j\left(F_*\left(\frac{\partial}{\partial x^i}\right)\right) = dy^j\left(\frac{\partial F^l}{\partial x^i}\frac{\partial}{\partial y^l}\right) = \frac{\partial F^j}{\partial x^i},$$

$$(1.19)$$

EXAMPLE In the particular case $\dim U = \dim V = n$,³ and $\operatorname{degree}(\psi) = n$ (a top-dimensional form), ψ can be written as

$$\psi = g dy^1 \wedge \dots \wedge dy^n$$

where $g:V \to \mathbb{R}$ is a smooth function. The pullback of ψ is then given by

$$F^*\psi = (g \circ F) \det\left(\frac{\partial F}{\partial x}\right) dx^1 \wedge \dots \wedge dx^n,$$

³ By dimU we refer to the dimension of the embedding vector space $\mathbb{R}^n \supset U$.

where $\partial F/\partial x$ is shorthand for the matrix $\{\partial F^j/\partial x^i\}$. The appearance of a determinant signals that top-dimensional differential forms have something to do with 'integration' and 'volume'.

1.2.6 Exterior derivative

The exterior derivative

$$\begin{aligned} d: \Lambda^p U &\to \Lambda^{p+1} U, \\ \phi &\mapsto d\phi, \end{aligned}$$

is a mapping that increases the degree of forms by one. It is one of the most important operations in the calculus of differential forms. To mention but one of its applications, the exterior derivative encompasses the operations of vector analysis, div, grad, curl, and will be the key to understand these operators in a coherent fashion.

To start with, consider the particular case of 0-forms, i.e. smooth functions $f: U \to \mathbb{R}$. The 1-form df is defined as

$$df = \sum_{i} \frac{\partial f}{\partial x^{i}} dx^{i}.$$
 (1.20)

A number of remarks on this definition:

- ▷ Eq. (1.20) is consistent with our earlier notation dx^j for the duals of $\frac{\partial}{\partial x^f}$. Indeed, for $f = x^j$ a coordinate function, $dx^j = \frac{\partial x^j}{\partial x^i} dx^i = dx^j$.
- Eq. (1.20) actually is a definition (is independent of the choice of coordinates.) For a different coordinate system, {yⁱ},

$$df = \frac{\partial f}{\partial y^j} dy^j = \frac{\partial f}{\partial x^i} \frac{\partial x^i}{\partial y^j} \frac{\partial y^j}{\partial x^k} dx^k = \frac{\partial f}{\partial x^i} dx^i,$$

where we used the chain rule and the transformation behavior of the dx^{j} 's Eq. (1.16).

 \triangleright The evaluation of $d\!f$ on a vector field $v=v^i\frac{\partial}{\partial x^i}$ obtains

$$dfv = \frac{\partial f}{\partial x^i}v^i,$$

i.e. the **directional derivative** of f in the direction identified by the vector components v^i . Specifically, for $v^i = d_t \gamma^i$, defined by the velocity of a curve, $df d_t \gamma = \frac{\partial f}{\partial x^i} d_t \gamma^i = d_t (f \circ \gamma)$. \triangleright The operation d defined by (1.20) obeys the Leibnitz rule,

$$d(fg) = dfg + gdf,$$

and is linear

$$d(af + bg) = adf + bdg, \qquad a, b = \text{const.}.$$

To generalize Eq. (1.20) to forms of arbitrary degree p, we need the following

Theorem: There is precisely one map $d: \Lambda U \to \Lambda U$ with the following properties:

 \triangleright d obeys the (graded) Leibnitz rule

$$d(\phi \wedge \psi) = (d\phi) \wedge \psi + (-)^p \phi \wedge (d\psi), \qquad (1.21)$$

where p is the degree of ϕ .

 $\triangleright d$ is linear

$$d(a\phi + b\psi) = ad\phi + bd\psi, \qquad a, b = \text{const.}$$

 $\begin{array}{l} \triangleright \ d \text{ is nilpotent, } d^2 = 0. \\ \triangleright \ d(\Lambda^p U) \subset \Lambda^{p+1} U. \\ \triangleright \ d(\Lambda^0 U) \text{ is given by the definition above.} \end{array}$

To prove this statement, we represent an arbitrary form $\phi \in \Lambda^p U$ as an unrestricted sum,

$$\phi = \frac{1}{p!} \sum_{i_1, \dots, i_p} \phi_{i_1, \dots, i_p} dx^{i_1} \wedge \dots \wedge dx^{i_p}.$$

Leibnitz rule and nilpotency then imply

$$d\phi = \frac{1}{p!} \sum_{i_1,\dots,i_p} (d\phi_{i_1,\dots,i_p}) dx^{i_1} \wedge \dots \wedge dx^{i_p}, \qquad (1.22)$$

i.e. we have found a unique representation. However, it still needs to be shown that this representation meets all the criteria listed above.

As for 1.), let

$$\phi = \frac{1}{p!} \sum_{i_1,\dots,i_p} \phi_{i_1,\dots,i_p} dx^{i_1} \wedge \dots \wedge dx^{i_p}, \qquad \psi = \frac{1}{q!} \sum_{i_1,\dots,i_q} \psi_{j_1,\dots,j_q} dx^{j_1} \wedge \dots \wedge dx^{j_q}.$$

Then,

$$\begin{aligned} d(\phi \wedge \psi) &= \frac{1}{p!q!} \sum_{\substack{i_1, \dots, i_p \\ j_1, \dots, j_p}} d(\phi_{i_1, \dots, i_p} \psi_{j_1, \dots, j_q}) dx^{i_1} \wedge \dots \wedge dx^{i_p} \wedge dx^{j_1} \wedge \dots \wedge dx^{j_q} = \\ &= \frac{1}{p!q!} \sum_{\substack{i_1, \dots, i_p \\ j_1, \dots, j_p}} \left[(d\phi_{i_1, \dots, i_p}) \psi_{j_1, \dots, j_p}) + \phi_{i_1, \dots, i_p} d\psi_{j_1, \dots, j_q} \right] dx^{i_1} \wedge \dots \wedge dx^{i_p} \wedge dx^{j_1} \wedge \dots \wedge dx^{j_q} = \\ &= (d\phi) \wedge \psi + (-)^p \phi \wedge d\psi, \end{aligned}$$

where we used the chain rule, and the relation $d\psi \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p} = (-)^p dx^{i_1} \wedge \cdots \wedge dx^{i_p} \wedge d\psi$. Property 3. follows straightforwardly from the symmetry/antisymmetry of second derivatives $\partial^2_{x^i x^j} \phi_{i_1,\ldots,i_p}$ /differentials $dx^i \wedge dx^j$.

Importantly, pullback and exterior derivative commute,

$$F^* \circ d = d \circ F^*, \tag{1.23}$$

for any smooth function F. (The proof amounts to a straightforward application of the chain rule.)

INFO Let us briefly address the connection between **exterior differentiation and vector calculus** alluded to in the beginning of the section. The discussion below relies on an identification of coand contravariant components that makes sense only in cartesian coordinates. It is, thus, limited in scope and merely meant to hint at a number of connections whose coordinate invariant (geometric) meaning will be discussed somewhat further down.

Consider a one-form $\phi = \phi_i dx^i$, i.e. an object characterized by d (covariant) components. Its exterior derivative can be written as

$$d\phi = \frac{\partial \phi_i}{\partial x^j} dx^j \wedge dx^i = \frac{1}{2} \left(\frac{\partial \phi_i}{\partial x^j} - \frac{\partial \phi_j}{\partial x^i} \right) dx^j \wedge dx^i.$$

Considering the case n = 3 and interpreting the coefficients ϕ_i , as components of a 'vector field' v^i ,⁴ we are led to identify the three coefficients of the form $d\phi$ as the components of the **curl** of $\{\phi_i\}$.

Similarly, identifying the three components of the two-form $\psi = \epsilon_{ijk}\psi^i dx^j \wedge dx^k$ (ϵ_{ijk} is the fully antisymmetric tensor) with a vector field, $\psi^i \leftrightarrow v^i$, we find $d\psi = \left(\frac{\partial}{\partial x^i}\psi_i\right) dx^1 \wedge dx^2 \wedge dx^3$, i.e. the three-form $d\psi$ is defined by the **divergence** $\nabla \cdot \mathbf{v} = \partial_i v^i$.

Finally, for a function ϕ , we have $d\phi = \partial_i \phi dx^i$, i.e. (the above naive identification understood) a 'vector field' whose components are defined by the **gradient** $\nabla \phi = \{\partial_i \phi\}$. (For a coordinate invariant representation of the vector differential operators, we refer to section xx.)

Notice that in the present context relations such as $\nabla \cdot \nabla \times \mathbf{v} = 0$ or $\nabla \times \nabla f = 0$ all follow from the nilpotency of the exterior derivative, $d^2 = 0$.

PHYSICS (E) The differential forms of electrodynamics. One of the objectives of the present course is to formulate electrodynamics in a geometrically oriented manner. Why this geometric approach? If one is primarily interested in applied electrodynamics (i.e. the solution of Maxwell's equations in a concrete setting), the geometric formulation isn't of that much value: calculations are generally formulated in a specific coordinate system, and once one is at this stage, it's all down to the solution of differential equations. The strengths of the geometric approach are more of conceptual nature. Specifically, it will enable us to

- understand the framework of physical foundations needed to formulate electrodynamics. Do we need a metric (the notion of 'distances') to formulate electrodynamics? How does the structure of electrodynamics emerge from the condition of relativistic invariance? These questions and others are best addressed in a geometry oriented framework.
- ▷ formulate the equations of electrodynamics in a very concise, and easy to memorize manner. This compact formulation is not only of aesthetic value, rather, it connects to the third and perhaps most important aspect:
- understand electrodynamics as a representative of a family of theories known as gauge theories. The interpretation of electrodynamics as a gauge theory has paved the way to one of the most rewarding and important developments in modern theoretical physics, the understanding of fundamental 'forces' electromagnetism, weak and strong interactions, and gravity as part of one unifying scheme.

We will here formulate the geometric view of electrodynamics in a 'bottom up' approach. That is, we will introduce the basic objects of the theory, fields, currents, etc. first on a purely formal level. The actual meaning of the definitions will then gradually get disclosed as we go along.

⁴ The naive identification $\phi_i \leftrightarrow v^i$ does not make much sense, unless we are working in cartesian coordinates. That's why the expressions derived here hold only in cartesian frames.

Electrodynamics is formulated in (4 = 3 + 1)-dimensional space, three space dimensions and one time dimension. For the time being, we identify this space with \mathbb{R}^4 . Unless stated otherwise, cartesian coordinates $(x^0, x^1, x^2, x^3, x^4)$ will be understood, where $x^0 = ct$ measures time, and c is the speed of light.

Let us begin by introducing the 'sources' of the theory: we define the current 3-form, $j \in \Lambda^3 \mathbb{R}^4$,

$$j = \frac{1}{3!} \epsilon_{\mu\nu\lambda\rho} j^{\mu} dx^{\nu} \wedge dx^{\lambda} \wedge dx^{\rho}, \qquad (1.24)$$

where $\epsilon_{\mu\nu\lambda\rho}$ is the fully antisymmetric tensor in four dimensions.⁵ We may rewrite this as

$$\begin{split} j = j^0 dx^1 \wedge dx^2 \wedge dx^3 - \\ j^1 dx^2 \wedge dx^3 \wedge dx^0 - \\ j^2 dx^3 \wedge dx^1 \wedge dx^0 - \\ j^3 dx^1 \wedge dx^2 \wedge dx^0. \end{split}$$

We tentatively identify $j^{1,2,3}$ with the vectorial components of the current density familiar from electrodynamics. This means that, say, j^1 will be the number of charges flowing through surface elements in the (23)-plane during a given time interval. Comparing with our heuristic discussion in the beginning of the chapter, we have extended the definition by a 'dynamical component', i.e. j^1 actually measures the number of charges associated with a 'space-time box' spanned by a spatial area in the (23) plane and a stretch in the 0-direction (time.) To be more pre-



cise, take a triplet of vectors $(\Delta x^2 e_2, \Delta x^3 e_3, \Delta x^0 e_0)$, anchored at a space time point (x^0, x^1, x^2, x^3) . The value obtained by evaluating the current form on these arguments, $j(\Delta x^2 e_2, \Delta e^3 e_3, c\Delta t e_0)$, then is the number of charges passing through the area $\Delta x^2 \Delta x^3$ at (x^1, x^2, x^3) during time interval $[t, t + \Delta t]$, where $\Delta t = \Delta x^0/c$ (see the figure, where the solid arrows represent the time line of particles passing through the shaded spatial area in time Δt .) Similarly, j^0 will be interpreted as (c times) the number of charges in a spatial box spanned by the (1, 2, 3)-coordinates. Thus, $j^0 = c\rho$, where ρ is the physical charge density.

On physical grounds, we need to impose a **continuity equation**, $\partial_{\mu}j^{\mu} = \partial_{t}\rho + \sum_{i=1}^{3} \partial_{x_{i}}j^{i} = \partial_{\mu}j^{\mu} = 0.^{6}$ This condition is equivalent to the vanishing of the exterior derivative,

$$dj = 0.$$
 (1.25)

To check this, compute

$$dj = \frac{1}{3!} \epsilon_{\mu\nu\lambda\rho} \partial_{\tau} j^{\mu} dx^{\tau} \wedge dx^{\nu} \wedge dx^{\lambda} \wedge dx^{\rho} =$$

$$= \frac{1}{3!} \epsilon_{\mu\nu\lambda\rho} \partial_{\tau} j^{\mu} \epsilon^{\tau\nu\lambda\rho} dx^{0} \wedge dx^{1} \wedge dx^{2} \wedge dx^{3} = \partial_{\mu} j^{\mu} dx^{0} \wedge dx^{1} \wedge dx^{2} \wedge dx^{3},$$

where the second equality follows from the antisymmetry of the wedge product and the third from the properties of the antisymmetric tensor (see below), $\epsilon_{\mu\nu\lambda\rho}\epsilon^{\tau\nu\lambda\rho} = 3!\delta^{\tau}_{\mu}$.

⁵ I.e. $\epsilon_{\mu\nu\lambda\rho}$ vanishes if any two of its indices are identical. Otherwise, it equals the sign of the permutation $(0, 1, 2, 3) \rightarrow (\epsilon, \mu, \nu, \sigma)$.

⁶ Following standard conventions, indices i, j, k, \ldots are 'spatial' and run from 1 to 3, while μ, ν, ρ are space-time indices running from zero to three.

1.2 Differential forms in \mathbb{R}^n



Figure 1.5 Cartoon on the measurement prescriptions for electric fields (left) and magnetic fields (right). Discussion, see text.

Next in our list of definitions are the **electric and magnetic fields**. The **electric field** bears similarity to a *force*. At least, it is measured like a force is, that is by displacement of a test particle along (small) stretches in space and recording the corresponding work. Much like a force, the electric field is something that converts a 'vector' (infinitesimal curve segment) into a number (work). We thus describe the field in terms of a one-form

$$E = E_i dx^i, (1.26)$$

where $E_i = E_i(x,t)$ are the time dependent coefficients of the field strength. In cartesian coordinates (and assuming the standard scalar product), these can be identified with the components of the electric field 'vector'. The **magnetic field**, in contrast, is not measured like a force. Rather, one measures the magnetic *flux* threading spatial areas (e.g. by measuring the torque exerted on current loops.) In analogy to our discussion on page **??** we thus define the magnetic field as a two-form

$$B = B_1 dx^2 \wedge dx^3 + B_2 dx^3 \wedge dx^1 + B_3 dx^1 \wedge dx^2.$$
(1.27)

Attentive readers may wonder how this expression fits into our previous discussion of coordinate representations of differential forms: as it is written, B is not of the canonical form $B = B_{ij}dx^i \wedge dx^j$. On a similar note, one may ask how the coefficients B_i above relate to the familiar magnetic field 'vector' B_v of electrodynamics (the subscript v serves to distinguish B_v from the differential form.) The first thing to notice is that B_v is not a conventional vector, rather it is a **pseudovector**, or **axial vector**. A pseudovector is a vector that transforms conventionally under spatial rotation. However, under parity non-conserving operations (reflection, for example), it transforms like a vector, *plus* it changes sign (see Fig. 1.6.) Usually, pseudovectors obtain by taking the cross product of conventional vectors. Familiar examples include angular momentum, l, $(l = r \times p)$ and the magnetic field, B_v .

To understand how this relates to the two-form introduced above, let us rewrite the latter as $B = \frac{1}{2}B^i\epsilon_{ijk}dx^i \wedge dx^k$, where B^i transforms contravariantly (like a vector.) Comparison with B shows that, e.g., $B_1 = B^1\epsilon_{123} = \tilde{B}^1$. I.e. in a given basis, $B_1 = B^1$ and we may identify these with the components of the field (pseudo)vector B_v . However, under a basis change, $B^i \to A^i{}_jB^j$ transforms like a vector, while (check it!) $\epsilon_{ijk} \to \epsilon_{ijk} \det(A)$. This means $B_i \to A^i{}_jB_j \det(A)$. For rotations, $\det A = 1$ and $\{B_i\}$ transforms like a vector. However, for reflections $\det A = -1$, we pick up an additional sign change. Thus, $\{B_i\}$ transforms like a pseudovector, and may be identified with the magnetic field strength (pseudo)vector B_v . The distinction between pseudo- and conventional vectors can be avoided, if we identify B with what it actually is, a differential two-form.



Figure 1.6 On the axial nature of the magnetic field. Spatial reflection at a plane leads to reflection of the field vector plus a sign change. In contrast, ordinary vectors (such as the current density generating the field) just reflect.

We combine the components of the electric field, E, and the magnetic field, B, into the so-called field strength tensor,

$$F \equiv E \wedge dx^0 + B \equiv F_{\mu\nu} dx^{\mu} \wedge dx^{\nu}.$$
(1.28)

Comparison with the component representations in (1.26) and (1.27) shows that

 $\frac{1}{c}$

$$\{F_{\mu\nu}\} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3\\ E_1 & 0 & B_3 & -B_2\\ E_2 & -B_3 & 0 & B_1\\ E_3 & B_2 & -B_1 & 0 \end{pmatrix}.$$
 (1.29)

It is straightforward to verify that the **homogeneous Maxwell equations**, commonly written as (CGS units)

$$\label{eq:phi} \begin{split} \nabla \cdot B &= 0, \\ \frac{\partial}{\partial t} B + \nabla \times E &= 0, \end{split}$$

are equivalent to the closedness of the field strength tensor,

$$dF = 0. (1.30)$$

We next turn to the discussion of the two partner fields of E and B, the electric D-field and the magnetic H-field, respectively. In vacuum, these are usually identified with E and B, i.e. E = D and B = H, resp. However, in general, the fields are different and this shows quite explicitly in the present formalism. In principle, D and H can be introduced with reference to a measurement prescription, much like we did above with E and B. However, this discussion would lead us too far astray and we here introduce these fields pragmatically. That is, we require that D and H satisfy a differential equation whose component-representation equals the inhomogeneous Maxwell equations. To this end, we define the two form D by

$$D = D_1 dx^2 \wedge dx^3 + D_2 dx^3 \wedge dx^1 + D_3 dx^1 \wedge dx^2,$$
(1.31)

similar in structure to the magnetic (!) form B. The field H is defined as a one-form,

$$H = H_i dx^i. aga{1.32}$$

The components $\{D_i\}$ and $\{H_i\}$ appearing in these definitions are two be identified with the 'vector' components in the standard theory where, again, the appearance of covariant indices indicates that the vectorial interpretation becomes problematic in non-metric environments. We now define the differential two-form

$$G \equiv -H \wedge dt + D = G_{\mu\nu} dx^{\mu} \wedge dx^{\nu}, \tag{1.33}$$

with component representation

$$\{G_{\mu\nu}\} = \begin{pmatrix} 0 & H_1 & H_2 & H_3 \\ -H_1 & 0 & D_3 & -D_2 \\ -H_2 & -D_3 & 0 & D_1 \\ -H_3 & D_2 & -D_1 & 0 \end{pmatrix}.$$
 (1.34)

With these definitions it is straightforward to verify that the **inhomogeneous Maxwell equations** assume the form

$$dG = j. \tag{1.35}$$

What the present discussion does *not* tell us is how the two main players in the theory, the covariant tensors F and G are connected to each other. To establish this connection, we need additional structure, viz. a metric, and this is a point to which we will return below.

To conclude this section let us briefly address the coupling between electromagnetic fields and matter. This coupling is mediated by the **Lorentz force** acting on charges q, which, in standard notation, assumes the form $F = q(E + v \times B)$. Here, v is the velocity vector, and all other quantities are vectorial, too. Translating to forms, this reads

$$F = q(E - i_v B), \tag{1.36}$$

where F is the force one-form.

INFO The general fully antisymmetric tensor or Levi-Civita symbol or ϵ -tensor is a mixed tensor defined by

$$\epsilon_{\nu_1,\ldots,\nu_n}^{\mu_1,\ldots,\mu_n} = \begin{cases} +1 & ,(\mu_1,\ldots,\mu_n) \text{ an even permutation of } (\nu_1,\ldots,\nu_n), \\ -1 & ,(\mu_1,\ldots,\mu_n) \text{ an odd permutation of } (\nu_1,\ldots,\nu_n), \\ 0 & , & \text{else.} \end{cases}$$
(1.37)

In the particular case $(\mu_1, \ldots, \mu_n) = (1, \ldots, n)$ one abbreviates the notation to $\epsilon_{\nu_1, \ldots, \nu_n}^{1, \ldots, n} \equiv \epsilon_{\nu_1, \ldots, \nu_n}$. Similarly, $\epsilon_{1, \ldots, n}^{\mu_1, \ldots, \mu_n} \equiv \epsilon^{\mu_1, \ldots, \mu_n}$. Important identities fulfilled by the ϵ -tensor include $\epsilon_{\mu_1, \ldots, \mu_n} \epsilon^{\mu_1, \ldots, \mu_n} \epsilon^{\mu', \mu_2, \ldots, \mu_n} \epsilon^{\mu', \mu_2, \ldots, \mu_n} = (n-1)! \delta_{\mu}^{\mu'}$.

1.2.7 Poincaré Lemma

Forms ϕ which are annihilated by the exterior derivative, $d\phi = 0$, are called **closed**. For example, every *n*-form defined on $U \subset \mathbb{R}^n$ is closed. Also, forms $\phi = d\kappa$ that can be written as exterior derivatives of another form κ — so called **exact** forms — are closed. One may wonder whether *every* closed form is exact, $d\phi = 0 \stackrel{?}{\Rightarrow} \phi = d\kappa$. The answer to this question is negative; in general closedness does not imply exactness.

EXAMPLE On $U = \mathbb{R}^2 - \{0\}$ consider the form

$$\psi = \frac{xdy - ydx}{x^2 + y^2}.$$

It is straightforward to verify that $d\psi = 0$. Nonetheless, ψ is not exact: one may verify that $\psi = d \arctan(y/x)$, everywhere where the defining function exists. However the function $\arctan(y/x)$ is ill-defined on the positive *x*-axis, i.e. ψ can not be represented as the exterior derivative of a smooth function on the entire domain of definition of ψ . (A more direct way to see this is to notice that on $\mathbb{R}^2 - \{0\}, \ \psi = d\phi$, where ϕ is the polar angle. The latter 'jumps' at the positive *x*-axis by 2π , i.e. it is not well defined on all of *U*.)

The conditions under which a closed form is exact are stated by the

Theorem (Poincaré Lemma): On a star-shaped⁷ open subset $U \subset \mathbb{R}^n$ a form $\phi \in \Lambda^p U$ is exact if and only if it is closed.

The Lemma is proven by explicit construction. We here restrict ourselves to the case of one-forms, $\phi \in \Lambda^1 U$ and assume that the reference point establishing the star-shapedness of U is at the origin. Let the one-form $\phi = \phi_i dx^i$ be closed. Then, the zero-form (function) $f(x) = \int_0^1 dt \sum_i \phi_i(tx) x^i$ satisfies the equation $df = \phi$, i.e. we have shown that ϕ is exact. (Notice that the existence of the integration path $tx, t \in [0, 1]$ relies on the star-shapedness of the domain.) Indeed,

$$df = \int_0^1 dt \left[t \frac{\partial \phi_i(y)}{\partial y^j} \Big|_{y=tx} dx^j x^i + \phi_i(tx) dx^i \right] = \\ = \int_0^1 dt \left[t d_t \phi_i(tx) dx^i + \phi_i(tx) dx^i + t \underbrace{\left(\frac{\partial \phi_i}{\partial y^j} - \frac{\partial \phi_j}{\partial y^i} \right)}_{0 \ (d\phi=0)} dx^j dx^i \right] = \\ = \phi_i(x) dx^i,$$

where in the last line we have integrated by parts. (Notice that $\int_0^1 dt \,\phi_i(tx)x^i = \int_0^1 dt \,\phi_i(tx)d_t(tx) = \int dx^i \phi_i$ coincides with the standard vector-analysis definition of the line-integral of the 'irrotational field $\{\phi_i\}'$ along the straight line from the origin to x. The proof of the Lemma for forms of higher degree p > 1 is similar.

INFO The above example, and the proof of the Lemma suggest a connection (exactness \leftrightarrow geometry). This connection is the subject of **cohomology theory**.

1.2.8 Integration of forms

Orientation of open subsets $U \subset \mathbb{R}^n$

We generalize the concept of orientation introduced in section 1.1.7 to open subsets $U \subset \mathbb{R}^n$. Consider a no-where vanishing form $\omega \in \Lambda^n U$, i.e. a form such that for any frame (b_1, \ldots, b_n) ,

⁷ A subset $U \subset \mathbb{R}^n$ is star-shaped if there is a point $x \in U$ such that any other $y \in U$ is connected to x by a straight line.

1.2 Differential forms in \mathbb{R}^n

 $\forall x \in U : \omega_x(b_1(x), \dots, b_n(x)) \neq 0$. We call the frame (b_1, \dots, b_n) oriented if

$$\forall x \in U, \qquad \omega_x(b_1(x), \dots, b_n(x)) > 0.$$
(1.38)

A set of coordinates (x^1, \ldots, x^n) is called oriented if the associated frames $(\partial/\partial_{x^1}, \ldots, \partial/\partial_{x^n})$ are oriented.

Conversely, an orientation of U may be introduced by declaring

$$\omega = dx^1 \wedge \dots \wedge dx^r$$

to be an orienting form. Finally, two forms ω_1 and ω_2 define the same orientation iff they differ by a positive function $\omega_1 = f\omega_2$.

Integration of *n*-forms

Let $K \subset U$ be a sufficiently (in the sense that all regular integrals we are going to consider exist) regular subset. Let (x^1, \ldots, x^n) be an oriented coordinate system on U. An arbitrary *n*-form $\phi \in \Lambda^n U$ may then be written as $\phi = f dx^1 \wedge \cdots \wedge dx^n$, where f is a smooth function given by

$$f(x) = \phi_x \left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n} \right).$$

The integral of the n-form ϕ is then defined as

$$\int_{K} \phi \equiv \int_{K} f(x) \, dx^{1} \dots dx^{n}, \qquad \phi = f(x) \, dx^{1} \wedge \dots \wedge dx^{n}, \tag{1.39}$$

where the notation $\int_{K} (\dots) dx^1 \dots dx^n$ (no wedges between differentials) refers to the integral of standard calculus over the domain of coordinates spanning K.

Under a change of coordinates, $(x^1, \ldots, x^n) \to (y^1, \ldots, y^n)$, the integral changes as

$$\int_{K,x} \phi \to \operatorname{sgn} \operatorname{det} \left(\frac{\partial x^i}{\partial y^j} \right) \, \int_{K,y} \phi,$$

where $\int_{K,x} \phi$ is shorthand for the evaluation of the integral in the coordinate representation x. The sign factor sgn det $(\partial x^i/\partial y^j)$ has to be read as

$$\operatorname{sgn} \det \left(\frac{\partial x^i}{\partial y^j} \right) = \frac{\left| \det \left(\frac{\partial x^i}{\partial y^j} \right) \right|}{\det \left(\frac{\partial x^i}{\partial y^j} \right)},$$

where the modulus of the determinant in the numerator comes from the variable change in the standard integral and the determinant in the denominator reflects is due to $\phi = f(x)dx^1 \wedge \cdots \wedge dx^n = f(x(y)) \det(\partial x^i/\partial y^j) dy^1 \wedge \cdots \wedge dy^n$. This results tells us that (a) the integral is invariant under an orientation preserving change of coordinates (the definition of the integral canonical) while (b) it changes sign under a change of orientation.

Let $F: V \to U, y \mapsto F(y) \equiv x$ be a **diffeomorphism**, i.e. a bijective map such that F and F^{-1} are smooth. We also assume that F is orientation preserving, i.e. that $dy^1 \wedge \cdots \wedge dy^n \in \Lambda^n V$



Figure 1.7 On the definition of oriented *p*-dimensional surfaces embedded in \mathbb{R}^n

and $F^*(dx^1 \wedge \cdots \wedge dx^n) \in \Lambda^n V$ define the same orientation. This is equivalent to the condition (why?) det $(dx^i/dy^j) > 0$. We then have

$$\int_{F^{-1}(K)} F^* \phi = \int_K \phi.$$
 (1.40)

The proof follows from $F^*(f(x) dx^1 \wedge \cdots \wedge dx^n) = \det(dx^i/dy^j) f(x(y)) dy^1 \wedge \cdots \wedge dy^n$, the definition of the integral Eq. (1.39), and the standard calculus rules for variable changes.

Integration of *p*-forms

A *p*-dimensional **oriented surface** $T \subset \mathbb{R}^n$ is defined by a smooth parameter representation $Q: K \to T, (\tau^1, \ldots, \tau^p) \mapsto Q(\tau^1, \ldots, \tau^p)$ where $K \subset V \subset \mathbb{R}^p$, and V is an oriented open subset of \mathbb{R}^p , i.e. a subset equipped with an oriented coordinate system.

The integral of a p-form over the surface T is defined by

$$\int_{T} \phi = \int_{K} Q^* \phi.$$
(1.41)

On the rhs we have an integral over a p-form in p-dimensional space to which we may apply the definition (1.39). If p = n, we may choose a parameter representation such that V = U, K = T and Q =inclusion map. With this choice, Eq. (1.41) reduces to the n-dimensional case discussed above. Also, a change of parameters is given by an orientation preserving map in parameter space V. To this parameter change we may apply our analysis above (identification p = n understood). This shows that the definition (1.41) is parameter independent.

Examples:

 \triangleright For p = 1, T is a **curve** in \mathbb{R}^n . The integral of a one form $\phi = \phi_i dx^i$ along T is defined by

$$\int_{T} \phi = \int_{[0,1]} Q^* \phi = \int_{[0,1]} \phi_i(Q(\tau)) \frac{\partial Q^i}{\partial \tau} d\tau, \qquad (1.42)$$

where we assumed a parameter representation $Q : [0,1] \to T, \tau \to Q(\tau)$. The last expression conforms with the standard calculus definition of line integrals provided we identify the components of the form ϕ_i with a vector field.

▷ For p = 2 and n = 3 we have a surface in three–dimensional space. Assuming a parameter representation $Q : [a,b] \times [c,d] \rightarrow T, (\tau^1,\tau^2) \rightarrow Q(\tau^1,\tau^2)$, and a form $\phi = v^1 dx^2 \wedge dx^3 + v^2 dx^3 \wedge dx^1 + v^3 dx^1 \wedge dx^2 = \frac{1}{2} \epsilon_{ijk} v^i dx^j \wedge dx^k$, we get

$$\int_{T} \phi = \int_{[a,b] \times [c,d]} Q^{*} \phi =$$

$$= \frac{1}{2} \int_{[a,b] \times [c,d]} \epsilon_{ijk} v^{i} (Q(\tau^{1},\tau^{2})) \left(\frac{\partial Q^{j}}{\partial \tau^{1}} \frac{\partial Q^{k}}{\partial \tau^{2}} - \frac{\partial Q^{j}}{\partial \tau^{2}} \frac{\partial Q^{k}}{\partial \tau^{1}} \right) d\tau^{1} d\tau^{2} =$$

$$= \int_{[a,b] \times [c,d]} \epsilon_{ijk} v^{i} (Q(\tau^{1},\tau^{2})) \frac{\partial Q^{j}}{\partial \tau^{1}} \frac{\partial Q^{k}}{\partial \tau^{2}} d\tau^{1} d\tau^{2}.$$
(1.43)

In the standard notation of calculus, this would read $\int d\tau^1 d\tau^2 v \cdot (\partial_{\tau^1} Q \times \partial_{\tau^2} Q) = \int dS n \cdot v$, where $dS n = \partial_{\tau^1} Q \times \partial_{\tau^2} Q$ is 'surface element \times normal vector field'.

Cells and chains

A **p-cell** σ of in \mathbb{R}^p is a triple $\sigma = (D, Q, Or)$ consisting of (cf. Fig. 1.8)

 \triangleright a convex polyhedron $D \subset \mathbb{R}^p$,

 \triangleright a differentiable mapping (parameterization) $Q: D \to K \subset \mathbb{R}^n$, and

 \triangleright an orientation (denoted 'Or') of \mathbb{R}^p .

Using the language of cells, our previous definition of the integral of a p-form assumes the form

$$\int_{\sigma} \phi = \int_{D} Q^* \phi,$$

where we have written \int_{σ} (instead of $\int_{Q(D)}$ for the integral over the cell. The *p*-cell differing from σ in the choice of orientation is denoted $-\sigma$. If no confusion is possible, we will designate cells $\sigma = (D, Q, \text{Or})$ just by reference to their base-polyhedron D, or to their image Q(D). (E.g. what we mean when we speak of the 1-'cell' [a, b] is (i) the interval [a, b] plus (ii) the image Q([a, b]) with (iii) some choice of orientation.)

EXAMPLE Let σ be the **unit-circle**, S^1 in two-dimensional space. We may represent σ by a 1-cell in \mathbb{R}^2 whose base polyhedron is the interval $I = [0, 2\pi]$ of (oriented) \mathbb{R}^1 and the map $Q : [0, 2\pi] \to \mathbb{R}^2$, $t \mapsto (\cos t, \sin t)$ The integral of the 1-form $\phi = x^1 dx^2 \in \Lambda^1 \mathbb{R}^2$ over σ then evaluates to

$$\int_{\sigma} \phi = \int_{I} Q^* \phi = \int_{I} \cos t \, d(\sin t) = \int_{0}^{2\pi} \cos^2 t \, dt = \pi$$

Now, let σ be the **unit-disk**, D^2 in two dimensional space. We describe σ by

$$Q: D \equiv [0,1] \times [0,2\pi] \rightarrow D^2,$$

(r, \phi) \mapsto (r\cos \phi, r\sin \phi)

The integral of the one–form $d(x^1 dx^2) = dx^1 \wedge dx^2$ over σ is given by

$$\int_{D^2} dx^1 \wedge dx^2 = \int_D Q^* (dx^1 \wedge dx^2) = \int \underbrace{\det\left(\frac{d(x^1, x^2)}{d(r, \phi)}\right)}_r dr \wedge d\phi = \int_0^1 dr \int_0^{2\pi} r = \pi.$$

We thus observe $\int_{S^1} x^1 dx^2 = \int_{D^2} d(x^1 x^2)$, a manifestation of Stokes theorem \ldots



Figure 1.8 On the concept of oriented cells and their boundaries. Discussion, see text

As a generalization of a single cell, we introduce chains. A p-chain is a formal sum

$$c = m_1 \sigma_1 + \dots + m_r \sigma_r,$$

where σ_i are *p*-cells and the 'multiplicities' $m_i \in \mathbb{Z}$ integer valued coefficients. Introducing the natural identifications

 $> m_1\sigma_1 + m_2\sigma_2 = m_2\sigma_2 + m_1\sigma_1,$ $> 0\sigma = 0,$ > c + 0 = c, $> m_1\sigma + m_2\sigma = (m_1 + m_2)\sigma,$ $> (m_1\sigma_1 + m_2\sigma_2) + (m'_1\sigma'_1 + m'_1\sigma'_2) = m_1\sigma_1 + m_2\sigma_2 + m'_1\sigma'_1 + m'_2\sigma'_2,$

the space of p-chains, C_p , becomes a linear space.⁸

Let σ be a p-cell. Its **boundary** $\partial \sigma$ is a (p-1)-chain in which may be defined as follows: Consider the p-1 dimensional faces D_i of the polyhedron D underlying σ . The mappings $Q_i: D_i \to \mathbb{R}^n$ are the restrictions of the parent map $Q: D \to \mathbb{R}^n$ to the faces D_i . The faces D_i inherit their orientation from that of D. To see this, let (e_1, \ldots, e_p) be a positively oriented frame of D. At a point $x_i \in D_i$ consider a vector n normal and outwardly directed (w.r.t. the bulk of D.) A frame (f_1, \ldots, f_{p-1}) is positively oriented, if (n, f_1, f_{p-1}) is oriented in the same way as (e_1, \ldots, e_p) . We thus define⁹

$$\partial \sigma = \sum_{i} \sigma_{i},$$

where $\sigma_i = (D_i, Q|_{D_i}, Or_i)$ and Or_i is the induced orientation. The collection of faces, D_i , defines the boundary of the base polyhedron D:

$$\partial D = \sum_i D_i.$$

(

 $^{^8}$ Strictly speaking, the integer-valuedness of the coefficients of elementary cells implies that C_p is an abelian group (rather than the real vector space we would have obtained were the coefficients arbitrary.)

⁹ These definitions work for (p > 1)-cells. To include the case p = 1 into our definition, we agree that a 0-chain is a collection of points with multiplicities. The boundary $\partial \sigma$ of a 1-cell defined in terms of a line segment \overrightarrow{AB} from a point A to B is B - A.

The boundary of a *p*-chain $\sigma = \sum_i m_i \sigma_i$ is defined as $\partial \sigma = \sum_i m_i \partial \sigma_i$. We may, thus, interpret ∂ as a linear operator, the **boundary operator** mapping *p* chains onto p-1 chains:

$$\begin{array}{rcl} \partial: C_p & \to & C_{p-1} \\ \sigma & \mapsto & \partial \sigma. \end{array} \tag{1.45}$$

Let $\phi \in \Lambda^p U$ be a *p*-form and $\sigma = \sum m_i \sigma_i \in C_p$ be a *p*-chain. The integral of ϕ over σ is defined as

$$\int_{\sigma} \phi \equiv \sum_{i} m_{i} \int_{\sigma_{i}} \phi.$$
(1.46)

1.2.9 Stokes theorem

Theorem (Stokes): Let $\sigma \in C_{p+1}$ be an arbitrary (p+1)-chain and $\phi \in \Lambda^p U$ be an arbitrary p-form on U. Then,

$$\int_{\partial\sigma} \omega = \int_{\sigma} d\omega.$$
(1.47)

Examples

Before proving this theorem, we go through a number of examples. (For simplicity, we assume that $\sigma = (D, Q, \text{Or})$ is an elementary cell, where $D = [0, 1]^{p+1}$ is a (k+1)-dimensional unitcube and $Q(D) \subset U$ lies in an open subset $U \subset \mathbb{R}^n$. We assume cartesian coordinates on U.)

- ▷ p = 0, n = 1: Consider the case where $Q : D \to Q(D)$ is just the inclusion mapping embedding the line segment D = [0, 1] into \mathbb{R} . The boundary $\partial D = 1 - 0$ is a zero-chain containing the two terminating points 1 and 0. Then, $\int_{\partial [0,1]} \phi = \phi(1) - \phi(0)$ and $\int_{[0,1]} d\phi = \int_{[0,1]} \partial_x \phi dx$, where ϕ is any 0-form (function), i.e. Stokes theorem reduces to the well known integral formula of one-dimensional calculus.
- $\triangleright p = 0$ arbitrary *n*: The cell $\sigma = ([0, 1], Q, Or)$ defines a curve $\gamma = Q([0, 1])$ in \mathbb{R}^n . Stokes theorem assumes the form

$$\phi(Q(1)) - \phi(Q(0)) = \int_{\gamma} d\phi = \int_{[1,0]} d(\phi \circ Q) = \int_0^1 \sum_i \frac{\partial \phi}{\partial x^i} \Big|_{Q(s)} \dot{Q}^i(s) ds,$$

i.e. it relates the line integral of a 'gradient field' $\partial_i \phi$ to the value of the 'potential' ϕ at the boundary points.

 $\triangleright p = 1, n = 3$: The cell $\sigma = ([0, 1] \times [0, 1], Q, Or)$ represents a smooth surface embedded into \mathbb{R}^3 . With $\phi = \phi_i dx^i$, the integral

$$\int_{\partial\sigma} \phi = \int_{\partial\sigma} \phi_i \, dx^i = \int_{\partial[0,1]^2} \phi_i(Q(\tau)) \frac{\partial Q^i}{\partial\tau} \, d\tau$$

reduces to the line integral over the boundary $\partial([0,1]^2) = [0,1] \times \{0\} + \{1\} \times [0,1] - [0,1] \times \{1\} - \{0\} \times [0,1]$ of the square $[0,1]^2$ (cf. Eq. (1.42)). With $d\phi_i dx^i = \frac{\partial \phi_i}{\partial x^j} dx^j \wedge dx^i$, we have (cf. Eq. (1.43))

$$\int_{\sigma} d\phi = \int_{\sigma} \frac{\partial \phi_i}{\partial x^j} \, dx^j \wedge dx^i = \int_{[0,1]^2} \epsilon^{kji} \frac{\partial (\phi \circ Q)^j}{\partial x^j} \, \epsilon^{klm} \frac{\partial Q^l}{\partial \tau^1} \frac{\partial Q^m}{d\tau^2} \, d\tau^1 d\tau^2.$$

Note that (in the conventional notation of vector calculus) $\epsilon^{kji}\frac{\partial \phi^i}{\partial x^j} = (\nabla \times \phi)^k$, i.e. we rediscover the formula

$$\int_{\gamma} ds \cdot \phi = \int_{\sigma} dS \cdot (\nabla \times \phi),$$

(known in calculus as Stokes law.)

ightarrow p = 2, n = 3: The cell $\sigma = ([0, 1]^3, Q, \operatorname{Or})$ defines a three dimensional 'volume' $Q([0, 1]^3)$ in three-dimensional space. Its boundary $\partial \sigma$ is a smooth surface. Let $\phi \equiv \frac{1}{2}\phi_{ij} dx^i \wedge dx^j$ be a two-form. Its exterior derivative is given by $d\phi = \frac{1}{2}\epsilon^{kij}\frac{\partial\phi_{ij}}{\partial x^k} dx^k \wedge dx^i \wedge dx^j$ and the l.h.s. of Stokes theorem assumes the form

$$\int_{\partial\sigma} \phi = \frac{1}{2} \int_{\partial\sigma} \phi_{ij} \, dx^i \wedge dx^j = \int_{\partial([0,1]^3)} (\phi \circ Q)_{ij} \frac{\partial Q^i}{\partial \tau^1} \frac{\partial Q^j}{\partial \tau^2} \, d\tau^1 d\tau^2.$$

The r.h.s. is given by

$$\int_{\sigma} d\phi = \frac{1}{2} \int_{\sigma} \epsilon^{kij} \frac{\partial \phi_{ij}}{\partial x^k} \, dx^k \wedge dx^i \wedge dx^j = \frac{1}{2} \int_{[0,1]^3} \epsilon^{kij} \frac{\partial (\phi \circ Q)_{ij}}{\partial x^k} \det\left(\frac{\partial Q}{\partial \tau}\right) \, d\tau^1 d\tau^2 d\tau^3.$$

Identifying the three independent components of ϕ with a vector according to $\phi_{ij} = \epsilon_{ijk}v^k$, we have $\frac{1}{2}\epsilon^{kij}\frac{\partial\phi_{ij}}{\partial x^k} = \nabla \cdot v$ and Stokes theorem reduces to **Gauß law** of calculus,

$$\int_{\partial \sigma} dS \cdot v = \int_{\sigma} dV (\nabla \cdot v).$$

Proof of Stokes theorem

Thanks to Eq. (1.46) it suffices to prove Stokes theorem for individual cells.

To start with, let us assume that the cell $\sigma = ([0,1]^{p+1}, Q, \operatorname{Or})$ has a unit-(p+1)-cube as its base. (Later on, we will relax that assumption.) Consider $[0,1]^{p+1}$ to be dissected into $N \gg 1$ small cubes r_i of volume $1/N \ll 1$. Then, $\sigma = \sum_i \sigma_i$ where $\sigma_i = (r_i, Q, \operatorname{Or})$ and $\partial \sigma = \sum_i \partial \sigma_i$. Since $\int_{\sigma} d\phi = \sum_i \int_{\sigma_i} d\phi$ and $\int_{\partial \sigma} \phi = \sum \int_{\partial \sigma_i} \phi$, it is sufficient to prove Stokes theorem for the small 'micro-cells'.

Without loss of generality, we consider the corner-cube $r_1 \equiv ([0, \epsilon]^{p+1}, Q, \text{Or})$, where $\epsilon = N^{-\frac{1}{p+1}}$. For (notational) simplicity, we set p = 1; the proof for general p is absolutely analogous. By definition,

$$\int_{r_1} d\phi = \int_{[0,\epsilon]^2} Q^* d\phi = \int_{[0,\epsilon]^2} dQ^* \phi.$$

1.3 Metric

Assuming that the 1-form ϕ is given by $\phi = \phi_i dx^i$, we have

$$\begin{split} dQ^*\phi &= dQ^*(\phi_i dx^i) = d((\phi \circ Q)_i d(\underbrace{x^i \circ Q}_{Q^i})) = d(\phi \circ Q)_i \wedge dQ^i = \\ &= \left[\frac{\partial(\phi \circ Q)_i}{\partial \tau^1} \frac{\partial Q^i}{\partial \tau^2} - \frac{\partial(\phi \circ Q)_i}{\partial \tau^2} \frac{\partial Q^i}{\partial \tau^1} \right] d\tau^1 \wedge d\tau^2. \end{split}$$

We thus have

$$\begin{split} \int_{[0,\epsilon]^2} dQ^* \phi &= \int_0^\epsilon d\tau^1 d\tau^2 \left[\frac{\partial (\phi \circ Q)_i}{\partial \tau^1} \frac{\partial Q^i}{\partial \tau^2} - \frac{\partial (\phi \circ Q)_i}{\partial \tau^2} \frac{\partial Q^i}{\partial \tau^1} \right] = \\ &= \epsilon^2 \left[\frac{\partial (\phi \circ Q)_i}{\partial \tau^1} \frac{\partial Q^i}{\partial \tau^2} - \frac{\partial (\phi \circ Q)_i}{\partial \tau^2} \frac{\partial Q^i}{\partial \tau^1} \right] (0,0) + \mathcal{O}(\epsilon^3). \end{split}$$

We want to relate this expression to $\int_{\partial r_1} \phi = \int_{\partial [0,\epsilon]^2} Q^* \phi = \int_{\partial [0,\epsilon]^2} (\phi \circ Q)_i dQ^i$. Considering the first of the four stretches contributing to the boundary $\partial [0,\epsilon]^2 = [0,\epsilon] \times \{0\} + \{\epsilon\} \times [0,\epsilon] - [0,\epsilon] \times \{\epsilon\} - \{0\} \times [0,\epsilon]$ we have

$$\int_0^{\epsilon} \left((\phi \circ Q)_i \frac{\partial Q^i}{\partial \tau^1} \right) (\tau^1, 0) \, d\tau^1 \simeq \epsilon \left((\phi \circ Q)_i \frac{\partial Q^i}{\partial \tau^1} \right) (0, 0) + \mathcal{O}(\epsilon^2).$$

Evaluating the three other contributions in the same manner and adding up, we obtain

$$\begin{split} \int_{\partial r_1} \phi &= \epsilon \left(\left((\phi \circ Q)_i \frac{\partial Q^i}{\partial \tau^1} \right) (0,0) + \left((\phi \circ Q)_i \frac{\partial Q^i}{\partial \tau^2} \right) (0,\epsilon) - \right. \\ &- \left((\phi \circ Q)_i \frac{\partial Q^i}{\partial \tau^1} \right) (0,\epsilon) - \left((\phi \circ Q)_i \frac{\partial Q^i}{\partial \tau^2} \right) (0,0) \right) + \mathcal{O}(\epsilon^2) = \\ &= -\epsilon^2 \left(\frac{\partial}{\partial \tau^2} \left((\phi \circ Q)_i \frac{\partial Q^i}{\partial \tau^1} \right) - \frac{\partial}{\partial \tau^1} \left((\phi \circ Q)_i \frac{\partial Q^i}{\partial \tau^2} \right) \right) (0,0) + \mathcal{O}(\epsilon^3) = \\ &= \epsilon^2 \left(\frac{\partial (\phi \circ Q)_i}{\partial \tau^1} \frac{\partial Q^i}{\partial \tau^2} - \frac{\partial (\phi \circ Q)_i}{\partial \tau^2} \frac{\partial Q^i}{\partial \tau^1} \right) (0,0) + \mathcal{O}(\epsilon^3), \end{split}$$

i.e. the same expression as above. This proves Stokes theorem for an individual micro-cell and, thus, (see the argument given above) for a d-cube.

INFO The proof of the general case proceeds as follows: A d-simplex is the volume spanned by d + 1 linearly independent points (a line in one dimension, a triangle in two dimensions, a tetrad in three dimensions, etc.) One may show that (a) a d-cube may be diffeomorphically mapped onto a d-simplex. This implies that Stokes theorem holds for cells whose underlying base polyhedron is a simplex. Finally, one may show that (b) any polyhedron may be decomposed into simplices, i.e. is a chain whose elementary cells are simplices. As Stokes theorem trivially carries over from cells to chains one has, thus, proven it for the general case.

1.3 Metric

In our so far discussion, notions like 'length' or 'distances' – concepts of paramount importance in elementary geometry – where not an issue. Yet, the moment we want to say something about

the actual shape of geometric structures, means to measure length have to be introduced. In this section, we will introduce the necessary mathematical background and various follow up concepts built on it. Specifically, we will reformulate the standard operations of vector analysis in a manner not tied to specific coordinate systems. For example, we will learn to recognize the all-infamous expression of the 'Laplace operator in spherical coordinates' as a special case of a structure that not difficult to conceptualize.

1.3.1 Reminder: Metric on vector spaces

As before, V is an n-dimensional \mathbb{R} -vector space. A (pseudo)metric on V is a bilinear form

$$\begin{array}{rcl} g:V\times V & \rightarrow & \mathbb{R}, \\ (v,w) & \mapsto & g(v,w) \equiv \langle v,\omega \rangle \end{array}$$

which is symmetric, g(v, w) = g(w, v) and non-degenerate: $\forall w \in V, g(v, w) = 0 \Rightarrow v = 0$. Occasionally, we will use the notation (V, g) to designate the pair (vector space, its metric).

Due to its linearity, the full information on g is stored in its value on the basis vectors e_i ,

$$g_{ij} \equiv g(e_i, e_j).$$

(Indeed, g may be represented as $g = g_{ij}e^i \otimes e^j \in T_2^0(V)$.) The matrix $\{g_{ij}\}$ is symmetric and has non-vanishing determinant, $\det(g_{ij}) \neq 0$. Under a change of basis, $e'_i = (A^{-1})^j_{\ i}e_j$ it transforms covariantly, $g'_{ii'} = (A^{-1})^j_{\ i}(A^{-1})^{j'}_{\ i'}g_{i'j'}$, or

$$g'_{ii'} = (A^{-1T}gA^{-1})_{ii'}.$$

Being a symmetric bilinear form, the metric may be diagonalized, i.e. a basis $\{\tilde{e}_i\}$ exists, wherein $g_{ij} = g(\tilde{e}_i, \tilde{e}_j) \propto \delta_{ij}$. Introducing **orthonormalized basis vectors** by $\theta_i = \tilde{e}_i/(g(\tilde{e}_i, \tilde{e}_i)^{1/2})$, the matrix representing g assumes the form

$$g = \eta \equiv \operatorname{diag}(\underbrace{1, \dots, 1}_{r}, \underbrace{-1, \dots, -1}_{n-r}),$$
(1.48)

where we have ordered the basis according to the signature of the eigenvalues. The set of transformations A leaving the metric form-invariant, $A^{-1T}\eta A^{-1} = \eta$ defines the (pseudo)orthogonal group, O(r, n-r). The difference, 2r-n between the number of positive and negative eigenvalues is called the **signature** of the metric. (According to the theorem of Sylvester), the signature is invariant under changes of basis. A metric with signature n is called **positive definite**.

Finally, a metric may be *defined* by choosing any basis and declaring it to be orthonormal, $g(\theta_i, \theta_j) \equiv \eta_{ij}$.

1.3.2 Induced metric on dual space

Let $\{\theta_i\}$ be an orthonormal basis of V and $\{\theta^i\}$ be the corresponding dual basis. We define a metric $\overset{*}{g}$ on V^* by requiring $\overset{*}{g}(\theta^i, \theta^j) \equiv \eta^{ij}$. Here, $\{\eta^{ij}\}$ is the inverse of the matrix $\{\eta_{ij}\}$.¹⁰

 $^{^{10}\,}$ The distinction has only notational significance: $\{\eta_{ij}\}$ is self inverse, i.e. $\eta_{ij}=\eta^{ij}.$

Now, let $\{e_i\}$ be an arbitrary oriented basis and $\{e^i\}$ its dual. The matrix elements of the metric and the dual metric are defined as, respectively, $g_{ij} \equiv g(e_i, e_j)$ and $g^{ij} \equiv g^*(e^i, e^j)$. (Mind the position of the indices!) By construction, one is inverse to the other,

1

$$g_{ik}g^{kj} = \delta^j_{\ i}.\tag{1.49}$$

Canonical isomorphism $V \rightarrow V^*$

While V and V^* are isomorphic to each other (by virtue of the mapping $e_i \mapsto e^i$) the isomorphy between the two spaces is, in general, not canonical; it relies on the choice of the basis $\{e_i\}$. However, for a vector space with a metric g, there is a **basis-invariant connection to dual space**: to each vector v, we may assign a dual vector v^* by requiring that $\forall w \in V : v^*(w) \stackrel{!}{=} g(v, w)$. We thus have a canonical mapping:

$$I: V \to V^*,$$

$$v \mapsto v^* \equiv g(v, .).$$
(1.50)

In physics, this mapping is called raising or lowering of indices: For a basis of vectors $\{e_i\}$, we have

$$J(e_i) = g_{ij}e^j$$

For a vector $v = v^i e_i$, the components of the corresponding dual vector $J(v) \equiv v_i e^i$ obtain as $v_i = g_{ij}v^j$, i.e. again by lowering indices.

Volume form

If $\{\theta_i\}$ is an oriented orthonormal basis, the *n*-form

$$\omega \equiv \theta^1 \wedge \dots \wedge \theta^n \tag{1.51}$$

is called a **volume form**. Its value $\omega(v_1, \ldots, v_n)$ is the volume of the parallel epiped spanned by the vectors v_1, \ldots, v_n .

As a second important application of the general basis, we derive a general representation of the volume form. Let the mapping $\{\theta_i\} \rightarrow \{e_i\}$ be given by $\theta_i = (A^{-1})^j_{\ i} e_j$. Then, we have the component representation of the metric

$$\eta^{ij} = A^i_{\ k} g^{kl} (A^T)^{\ j}_l. \tag{1.52}$$

Taking the determinant of this relation, using that $\det A > 0$ (orientation!), and noting that $\det\{g^{ij}\}/\det\{\eta^{ij}\} = |\det\{g^{ij}\}|$, we obtain the relation

$$\det A = |g|^{1/2},\tag{1.53}$$

where we introduced the notation

$$g \equiv \det\{g_{ij}\} = (\det\{g^{ij}\})^{-1}.$$

At the same time, we know that the representation of the volume form in the new basis reads as $\omega = \det A e^1 \wedge \cdots \wedge e^n$, or

$$\omega = |g|^{1/2} e^1 \wedge \dots \wedge e^n.$$
(1.54)

1.3.3 Hodge star

We begin this section with the observation that the two vector spaces $\Lambda^p V^*$ and $\Lambda^{n-p} V^*$ have the same dimensionality

$$\dim \Lambda^p V^* = \dim \Lambda^{n-p} V^* = \begin{pmatrix} n \\ p \end{pmatrix}.$$

By virtue of the metric, a canonical isomorphism between the two may be constructed. This mapping, the so-called Hodge star is constructed as follows: Starting from the elementary scalar product of 1-forms, $g^{ij} = \stackrel{*}{g}(e^i, e^j) \equiv \langle e^i, e^j \rangle$, a scalar product of p-forms may be defined by

$$\langle e^{i_1} \wedge \dots \wedge e^{i_p}, e^{j_1}, \wedge \dots \wedge e^{j_p} \rangle \equiv \det\{\langle e^{i_k}, e^{j_l} \rangle\} = \det\{g^{i_k j_l}\}$$

Since every form $\phi \in \Lambda^p V^*$ may be obtained by linear combination of basis-forms $e^{i_1}, \wedge \cdots \wedge e^{i_p}$ this formula indeed defines a scalar product on all of $\Lambda^p V^*$. Indeed, noting that for any matrix $X = \{X^{ij}\}, \det X = \epsilon_{i_1,\dots,i_n}^{1,\dots,n} X^{i_1 1} \dots X^{i_n n}$, i.e.

$$\det\{g^{i_k j_l}\} = \epsilon^{i_1, \dots, i_p}_{k_1, \dots, k_p} g^{k_1 j_1} \dots g^{k_p j_p},$$

we obtain the explicit representation

*

$$\langle \phi, \psi \rangle = \frac{1}{p!} \phi^{i_1, \dots, i_p} \psi_{i_1, \dots, i_p}.$$
(1.55)

(Notice that the r.h.s. of this expression may be equivalently written as $\langle \phi, \psi \rangle = \frac{1}{p!} \phi_{i_1,...,i_p} \psi^{i_1,...,i_p}$. The **Hodge star** is now defined as follows

$$\begin{array}{cccc} : \Lambda^p V^* & \to & \Lambda^{n-p} V^*, \\ \alpha & \mapsto & *\alpha, \end{array}$$

$$\forall \beta \in \Lambda^p V^* : \ \langle \beta, \alpha \rangle \omega \doteq \beta \land (*\alpha). \tag{1.56}$$

To see that this relation uniquely defines an (n - p)-form, we identify the components $(*\alpha)_{i_{p+1},\ldots,i_n}$ of the target form $*\alpha$. To this end, we consider the particular case, $\beta = e^1 \wedge \cdots \wedge e^p$. Separate evaluation of the two sides of the definition obtains

$$\begin{split} \omega \langle e^1 \wedge \dots \wedge e^p, \alpha \rangle &= \frac{\omega}{p!} \alpha_{i_1, \dots, i_p} \langle e^1 \wedge \dots \wedge e^p | e^{i_1} \wedge \dots \wedge e^{i_p} \rangle = \\ &= \frac{\omega}{p!} \epsilon_{j_1, \dots, j_p}^{i_1, \dots, j_p} g^{1j_1} \dots g^{pj_p} \alpha_{i_1, \dots, i_p} = \omega g^{1i_1} \dots g^{pi_p} \alpha_{i_1, \dots, i_p} = \omega \alpha^{1, \dots, p} \\ e^1 \wedge \dots \wedge e^p \wedge (*\alpha) &= \frac{1}{(n-p)!} (*\alpha)_{i_{p+1}, \dots, i_n} e^1 \wedge \dots \wedge e^p \wedge e^{i_{p+1}} \wedge \dots \wedge e^{i_n} = \omega \frac{(*\alpha)_{p+1, \dots, n}}{|g|^{1/2}} \end{split}$$

1.3 Metric

Comparing these results we arrive at the equation

$$(*\alpha)_{p+1,\dots,n} = |g|^{1/2} \alpha^{1,\dots,p} = \frac{|g|^{1/2}}{p!} \epsilon_{i_1,\dots,i_p,p+1,\dots,n} \alpha^{i_1,\dots,i_p}.$$

It is clear from the construction that this result holds for arbitrary index configurations, i.e. we have obtained the **coordinate representation of the Hodge star**

$$(*\alpha)_{i_{p+1},\dots,i_n} = \frac{|g|^{1/2}}{p!} \epsilon_{i_1,\dots,i_n} \alpha^{i_1,\dots,i_p} \ . \tag{1.57}$$

In words: the coefficients of the Hodge'd form obtain by (i) raising the p indices of the coefficients of the original form, (ii) contracting these coefficients with the first p indices of the ϵ -tensor, and (iii) multiplying all that by $\sqrt{|g|}/p!$.

We derive two important properties of the star: The **star operation is compatible with the scalar product**,

$$\forall \phi, \psi \in \Lambda^p V^* : \langle \phi, \psi \rangle = \operatorname{sgn} g \langle *\phi, *\psi \rangle , \qquad (1.58)$$

and it is self-involutary in the sense that

$$\forall \phi \in \Lambda^p V^* : \quad * * \phi = \operatorname{sgn} g \left(- \right)^{p(n-p)} \phi.$$
(1.59)

INFO The proof of Eqs. (1.58) and (1.59): The first relation is proven by brute force computation:

$$\begin{split} \langle *\phi, *\psi \rangle &= \frac{1}{(n-p)!} (*\phi)_{i_{p+1},\dots,i_n} (*\psi)^{i_{p+1},\dots,i_n} = \\ &= \frac{|g|}{(n-p)! (p!)^2} \epsilon_{i_1,\dots,i_n} \phi^{i_1,\dots,i_p} \psi_{l_1,\dots,l_p} \underbrace{\epsilon_{j_1,\dots,j_n} g^{l_1 j_1} \dots g^{l_p j_p} g^{i_{p+1} j_{p+1}} \dots g^{i_n j_n}}_{g^{-1} \, \epsilon^{l_1 \dots l_p i_{p+1} \dots i_n}} = \\ &= \frac{\operatorname{sgn} g}{(p!)^2} \epsilon_{i_1,\dots,i_p}^{i_1,\dots,i_p} \psi_{l_1,\dots,l_p} = \frac{\operatorname{sgn} g}{p!} \phi^{i_1,\dots,i_p} \psi_{i_1,\dots,i_p} = \operatorname{sgn} g \, \langle \phi, \psi \rangle. \end{split}$$

To prove the second relation, we consider two forms $\phi \in \Lambda^p V^*$ and $\psi \in \Lambda^{n-p} V^*$. Then,

$$\begin{split} \omega \langle *\psi, \phi \rangle &= \quad *\psi \wedge *\phi = (-)^{p(n-p)} *\phi \wedge *\psi = (-)^{p(n-p)} \omega \langle *\phi, \psi \rangle = \\ &= \quad (-)^{p(n-p)} \operatorname{sgn} g \langle *\phi, *\psi \rangle = (-)^{p(n-p)} \operatorname{sgn} g \langle *\psi, *\phi \rangle. \end{split}$$

Holding for every ψ this relation implies (1.59).

1.3.4 Isometries

As a final metric concept of linear algebra we introduce the notion of isometries. Let V and V' be two vector spaces with metrics g and g', respectively. An **isometry** $F: V \to V'$ is a linear mapping that conforms with the metric:

$$\forall v, w \in V: \quad g(v, w) = g'(Fv, Fw). \tag{1.60}$$

EXAMPLE (i) The set of isometries of \mathbb{R}^4 with **Minkowski metric** $\eta = \text{diag}(1, -1, -1, -1)$ is the Lorentz group SO(3, 1). (ii) The canonical mapping $J : V \to V^*$ is an isometry of (V, g) and $(V^*, \overset{*}{g})$.

1.3.5 Metric structures on open subsets of \mathbb{R}^n

Let $U \subset \mathbb{R}^n$ be open in \mathbb{R}^n . A **metric on** U is a collection of vector space metrics

$$g_x: T_xU \times T_xU \to \mathbb{R},$$

smoothly depending on x. To characterize the metric we may choose a frame $\{b_i\}$. This obtains a matrix–valued function

$$g_{ij}(x) \equiv g_x(b_i(x), b_j(x)).$$

The change of one frame to another, $b'_i = (A^{-1})^j_{\ i} b_j$ transforms the metric as $g_{ij}(x) \to g'_{ij}(x) = (A^{-1T})^k_i(x)g_{kl}(x)(A^{-1})^l_{\ j}(x)$. Similarly to our discussion before, we define an induced metric $*g_x$ on co-tangent space by requiring that $*g_x(b^i_x, b^j_x) \equiv g^{ij}(x)$ be inverse to the matrix $\{g_{ij}(x)\}$.

EXAMPLE Let $U = \mathbb{R}^3 - (\text{negative } x - \text{axis})$. Expressed in the cartesian frame $\frac{\partial}{\partial x^i}$, the Euclidean metric assumes the form $g_{ij} = \delta_{ij}$. Now consider the coordinate frame corresponding to polar coordinates, $(\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \phi})$. Transformation formulae such as

$$\frac{\partial}{\partial r} = \frac{\partial x^1}{\partial r}\frac{\partial}{\partial x^1} + \frac{\partial x^2}{\partial r}\frac{\partial}{\partial x^2} + \frac{\partial x^3}{\partial r}\frac{\partial}{\partial x^3} = \sin\theta\cos\phi\frac{\partial}{\partial x^1} + \sin\theta\sin\phi\frac{\partial}{\partial x^2} + \cos\theta\frac{\partial}{\partial x^3}$$

obtain the matrix

$$A^{-1} = \begin{pmatrix} \sin\theta\cos\phi & -r\sin\theta\sin\phi & r\cos\theta\cos\phi\\ \sin\theta\sin\phi & r\sin\theta\cos\phi & r\cos\theta\sin\phi\\ \cos\theta & 0 & -r\sin\phi \end{pmatrix}$$

Expressed in the polar frame, the metric then assumes the form

$$g = A^{-1T} \mathbf{1} A^{-1} = \{g_{ij}\} = \begin{pmatrix} 1 & & \\ & r^2 \sin^2 \theta & \\ & & r^2 \end{pmatrix}.$$

The dual frames corresponding to cartesian and polar coordinates on \mathbb{R}^3 -(negative *x*-axis) are given by, respectively, (dx^1, dx^2, dx^3) and $(dr, d\phi, d\theta)$. The dual metric in polar coordinates reads

$$\{g^{ij}\} = \begin{pmatrix} 1 & & \\ & r^{-2}\sin^{-2}\theta & \\ & & r^{-2} \end{pmatrix}.$$

Also notice that $\sqrt{|g|} = r^2 \sin \theta$, i.e. the volume form

$$\omega = dx^1 \wedge dx^2 \wedge dx^3 = r^2 \sin \theta dr \wedge d\phi \wedge d\theta$$

transforms in the manner familiar from standard calculus.

Now, let $(U \subset \mathbb{R}^n, g)$ and $(U' \subset \mathbb{R}^m, g')$ be two metric spaces and $F : U \to U'$ a smooth mapping. F is an isometry, if it conforms with the metric, i.e. fulfills the condition

$$\forall x \in U, \forall v, w \in T_x U: g_x(v, w) \stackrel{!}{=} g'_{F(x)}(F_* v, F_* w).$$

In other words, for all $x \in U$, the mapping $T_x F$ must be an isometry between the two spaces $(T_x U, g_x)$ and $(T_{F(x)}U', g'_{F(x)})$. Chosing coordinates $\{x^i\}$ and $\{y^j\}$ on U and U', respectively,
defining $g_{ij} = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$ and $g'_{ij} = g'(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j})$, and using Eq. (1.14), using Eq. (1.14), one obtains the condition

$$g_{ij} \stackrel{!}{=} \frac{\partial F^l}{\partial x^i} \frac{\partial F^k}{\partial x^j} g'_{lk}.$$
 (1.61)

A space (U,g) is called **flat** if an isometry $(U,g) \to (V,\eta)$ onto a subset $V \subset \mathbb{R}^n$ with metric η exists. Otherwise, it is called curved.¹¹

Turning to the other operations introduced in section 1.3.2, the **volume form** and the **Hodge** star are defined locally: $(*\phi)_x = *\phi_x$, and $\omega = \sqrt{|g|}b^1 \wedge \cdots \wedge b^n$. It is natural to extend the scalar product introduced in section 1.3.3 by integration of the local scalar products against the volume form:

$$\begin{array}{rcl} \langle \;,\; \rangle : \Lambda^p U \times \Lambda^p U &\to & \mathbb{R}, \\ & (\phi, \psi) &\mapsto & \langle \phi, \psi \rangle \equiv \int_U \langle \phi_x, \psi_x \rangle \, \omega. \end{array}$$

Comparison with Eq. (1.56) then obtains the important relation

$$\langle \phi, \psi \rangle = \int_{U} \phi \wedge *\psi.$$
(1.62)

Given two p-forms on a metric space, this defines a natural way to produce a number. On p xx below, we will discuss applications of this prescription in physics.

1.3.6 Holonomic and orthonormal frames

In this section we focus on dual frames. A dual frame may have two distinguished properties: An **orthonormal frame** is one wherein

$$g^{ij} = \eta^{ij}.$$

It is always possible to find an orthonormal frame; just subject the symmetric matrix g^{ij} to a Gram–Schmidt orthonormalization procedure.

A holonomic frame is one whose basis forms β^i are exact, i.e. a frame for which n functions x^i exist such that $\beta^i = dx^i$. The linear independence of the β^i implies that the functions x^i form a system of coordinates. Open subsets of \mathbb{R}^n may always be parameterized by global systems of coordinates, i.e. holonomic frames exist. In a holonomic frame,

$$g = dx^i g_{ij} dx^j, \qquad g^* = \frac{\partial}{\partial x^i} g^{ij} \frac{\partial}{\partial x^j}.$$

It turns out, however, that it is not always possible to find **frames that are both orthonormal** and holonomic. Rather,

¹¹ The definition of 'curved subsets of \mathbb{R}^{n} ' makes mathematical sense but doesn't seem to be a very natural context. The intuitive meaning of curvature will become more transparent once we have introduced differentiable manifolds.

the neccessary and sufficient condition ensuring the existence of orthonormal and holonomic frames is that the space (U,g) must be flat.

To see this, consider the holonomic representation $g = dx^i g_{ij} dx^j$ and ask for a transformation onto new coordinates $\{x^i\}$ to $\{y^j\}$ such that $g = dy^i \eta_{ij} dy^j$ be an orthonormal representation. Substituting $dx^i = \frac{\partial dx^i}{\partial y^j} dy^j$ into the defining equation of the metric, we obtain $g = \frac{\partial x^i}{\partial y^l} \frac{\partial x^j}{\partial y^k} g_{ij} dy^l dy^k \stackrel{!}{=} dy^l \eta_{lk} dy^k$. Comparison with (1.61) we see that the coordinate transformation must be an isometry.

EXAMPLE We may consider a piece of the unit sphere as parameterized by, say, the angular values $0 < \theta < 45 \deg$ and $0 < \phi < 90 \deg$. The metric on the sphere obtains by projection of the Euclidean metric of \mathbb{R}^3 onto the submanifold r = 1, $g^{ij} = \operatorname{diag}(\sin^{-2}\theta, 1)$. (Focusing on the coordinate space we may, thus, think of our patch of the sphere as an open subset of \mathbb{R}^2 (the coordinate space) endowed with a non-Euclidean metric.) We conclude that $(\sin\theta d\phi, d\theta)$ is an orthonormal frame. Since $d(\sin\theta d\phi) = \cos\theta d\theta \wedge d\phi \neq 0$ it is, however, not holonomic.

1.3.7 Laplacian

Coderivative

As before, $U \subset \mathbb{R}^n$ is an open oriented subset of \mathbb{R}^n with metric g. In the linear algebra of metric spaces, taking the adjoint of an operator A is an important operation: $\langle Av, w \rangle = \langle v, A^{\dagger}w \rangle'$. Presently, the exterior derivative, d, is our most important 'linear operator'. It is, thus, natural to ask for an operator, δ , that is adjoint to 'd' in the sense that

$$\forall \phi \in \Lambda^{p-1} U, \psi \in \Lambda^p U : \langle d\phi, \psi \rangle \stackrel{!}{=} \langle f, \delta \psi \rangle + \dots,$$

where the ellipses stand for boundary terms $\int_{\partial U}(...)$ (which vanish if U is boundaryless or ϕ, ψ have compact support inside U.) Clearly, δ must be an operator that *decreases* the degree of forms by one. An explicit formula for δ may be obtained by noting that

$$\begin{aligned} \langle d\phi,\psi\rangle &= \int_U d\phi\wedge *\psi = -(-)^{p-1} \int_U \phi\wedge d*\psi \stackrel{(1.59)}{=} \\ &= (-)^p \operatorname{sgn} g(-)^{(p-1)(n-p+1)} \int_U \phi\wedge **d*\psi = \operatorname{sgn} g(-)^{(p+1)n+1} \langle \phi, *d*\psi \rangle, \end{aligned}$$

where in the second equality we integrated by parts (ignoring surface terms). This leads to the identification of the **coderivative**, a differential operator that *lowers* the degree of forms by one:

$$\begin{split} \delta &: \Lambda^p U \quad \to \quad \Lambda^{p-1} U, \\ \phi \quad \mapsto \quad \delta \phi \equiv \, \mathrm{sgn} \, g(-)^{n(p+1)+1} * d * \phi. \end{split}$$

Two more remarks on the coderivative:

▷ It squares to zero, $\delta\delta \propto (*d*)(*d*) \propto *dd* = 0$.

 \triangleright Applied to a one-form $\phi \equiv \phi_i dx^i$, it obtains (exercise)

$$\delta\phi = -\frac{1}{|g|^{1/2}} \frac{\partial}{\partial x^i} (|g|^{1/2} \phi^i).$$
(1.63)

Laplacian and the operations of vector analysis

We next combine exterior derivative and coderivative to define a second order differential operator,

$$\begin{aligned} \Delta : \Lambda^{p}U &\to \Lambda^{p}U, \\ \phi &\mapsto \Delta\phi \equiv -(d\delta + \delta d)\phi. \end{aligned} \tag{1.64}$$

By construction, this operator is self adjoint,

$$\langle \Delta \phi, \psi \rangle = \langle \phi, \Delta \psi \rangle$$

If the metric is positive definite it is called the the Laplacian. If r = n - 1 (cf. Eq. (1.48)) it is called the d'Alambert operator or wave operator (and usually denoted by \Box .)

Using Eq. (1.63), it is straightforward to verify that the action of Δ on 0-forms (functions) $f \in \Lambda^0 U$ is given by

$$\Delta f = \frac{1}{|g|^{1/2}} \frac{\partial}{\partial x^i} \left(|g|^{1/2} g^{ij} \frac{\partial}{\partial x^j} f \right).$$
(1.65)

EXAMPLE Substitution of the spherical metric discussed in the example on p 34, it is straightforward to verify that the **Laplacian in three dimensional spherical coordinates** assumes the form

$$\Delta = \frac{1}{r^2} \partial_r r^2 \partial_r + \frac{1}{r^2 \sin \theta} \partial_\theta \sin \theta \partial_\theta + \frac{1}{r^2 \sin^2 \theta} \partial^2 \phi$$

We are now, at last, in a position to establish contact with the operations of vector analysis. Let $f \in \Lambda^0 U$ be a function. Its **gradient** is the vector field

grad
$$f \equiv J^{-1}df = \left(g^{ij}\frac{\partial}{\partial x^j}f\right)\frac{\partial}{\partial x^i}.$$
 (1.66)

Let $v = v^i \frac{\partial}{\partial x^i}$ be a vector field defined on an open subset $U \subset \mathbb{R}^3$. Its **divergence** is defined as

$$\operatorname{div} v \equiv -\delta J v = \frac{1}{|g|^{1/2}} \frac{\partial}{\partial x^i} |g|^{1/2} v^i.$$
(1.67)

Finally, let $v = v^i \frac{\partial}{\partial x^i}$ be a vector field defined on an open subset $U \subset \mathbb{R}^3$ of *three-dimensional* space. Its **curl** is defined as

$$\operatorname{curl} v \equiv J^{-1} * dJv = \frac{1}{|g|^{1/2}} \epsilon_{ijk} \left(\frac{\partial}{\partial x^i} v^j\right) \frac{\partial}{\partial x^k}.$$
(1.68)

Exterior Calculus

PHYSICS (E) Let us get back to our discussion of Maxwell theory. On p 19 we had introduced two distinct two-forms, F and G, containing the electromagnetic fields as coefficients. However, the connection between these two objects was left open. On the other hand, a connection of some sort must exist, for we know that in vacuum the electric field E (entering F) and the displacement field D (entering G) are not distinct. (And analogously for B and H.) Indeed, there exists a relation between F and G, and it is provided by the metric.

To see this, chose an orthonormal frame wherein the Minkovski metric assumes the form

$$\eta = \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 & \\ & & & -1 \end{pmatrix}.$$
 (1.69)

Assuming vacuum, E = B and B = H, and using Eq. (1.57), it is then straightforward to verify that the 2 form F and the (4-2)-form G are related through

$$G = *F. (1.70)$$

How does this equation behave under a **coordinate transformation** of Minkovski space? Under a general transformation, the components of F transform as those of a second rank covariant tensor. The equation dF = 0 is invariant under such transformations. However, the equation G = *F involves the Hodge star, i.e. an operation depending on a metric. It is straightforward to verify (do it) remains form-invariant only under isometric coordinate transformations. Restricting ourselves to linear coordinate transformations, this identifies the invariance group of Maxwell theory as the **Poincaré group**, i.e. the group of linear isometries of Minkowski space. The subgroup of the Poincaré group stabilizing at least one point (i.e. discarding translations of space) is the **Lorentz group**.

The Maxwell equations now assume the form d * F = j and dF = 0, resp. In the traditional formulation of electromagnetism, the solution of these equations is facilitated by the introduction of a 'vector potential'. Let us formulate the ensuing equations in the language of differential forms:

On open subsets of \mathbb{R}^4 , the closed form F may be generated from a **potential one-form**, A,

$$F = dA, \tag{1.71}$$

whereupon the inhomogeneous Maxwell equations assume the form

$$d * dA = j. \tag{1.72}$$

Now, rather than working with the second order differential operator d*d, it would be nicer to express Maxwell theory in terms of the self adjoint Laplacian, Δ . To this end, we act on the inhomogeneous Maxwell equation with a Hodge star to obtain $*d * dA = \delta dA = *j$. We now require A to obey the **Lorentz gauge** condition $\delta A = 0$.¹² A potential obeying the Lorentz condition can always be found by applying a **gauge transformation** $A \to A' \equiv A + df$, where f is a 0-form (a function). The

 12 In a component notation, $A=A_{\mu}dx^{\mu}$ and

$$\begin{split} \delta A &= *d * A = *d \left(\frac{1}{3!} \epsilon_{\mu\nu\sigma\tau} A^{\mu} dx^{\nu} \wedge dx^{\sigma} \wedge dx^{\tau} \right) = \\ &= \frac{1}{3!} \left(* \epsilon_{\mu\nu\sigma\tau} \partial_{\rho} A^{\mu} dx^{\rho} \wedge dx^{\nu} \wedge dx^{\sigma} \wedge dx^{\tau} \right) = \partial_{\rho} A^{\rho}, \end{split}$$

which is the familiar expression for the Lorentz gauge.

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condition $\delta A' = 0$ then translates to $\delta df = \Delta f \stackrel{!}{=} -\delta A$, i.e. a linear differential equation that can be solved. In the Lorentz gauge, the inhomogeneous Maxwell equations assume the form

$$-\Box A = *j,\tag{1.73}$$

where $\Box = -(\delta d + d\delta)$.

So far, we have achieved little more than a reformulation of known equations. However, as we are going to discuss next, the metric structures introduced above enable us to interpret Maxwell theory from an entirely new perspective: it will turn out that the equations of electromagnetism can be 'derived' entirely on the basis of geometric reasoning, i.e. without reference to physical input!

Much like Newton's equations are equations of motions for point particles, the Maxwell equations (1.73) can be interpreted as equations of motions for a field, A. It is then natural to ask whether these equations, too, can be obtained from a Lagrangian variational principle. What we need to formulate a **variational principle** is an action functional S[A], whose variation $\delta S[A]/\delta A = 0$ obtains Eq. (1.73) as its Euler-Lagrange equation.

At first sight, one may feel at a loss as to how to construct a suitable action. It turns out, however, that geometric principles almost uniquely determine the form of the action functional: let's postulate that our action be as simple as possible, i.e. a low order polynomial in the degrees of freedom of the theory, the potential, A. Now, to construct an action, we need something to integrate over, that is 4-forms. Now, our so-fare development of the theory is based on the differential forms, A, F, and j. Out of these, we can construct the 4-forms $F \wedge F$, $F \wedge *F$ and $j \wedge A$. The first of these is exact, $F \wedge F = dA \wedge dA = d(A \wedge dA)$ and hence vanishes under integration. Thus, a natural candidate of an action reads

$$S[A] = \int \left(c_1 F \wedge *F + c_2 j \wedge A \right),$$

where c_i are constants.¹³

Let us now see what we get upon variation of the action. Substituting $A \to A + a$ into the action we obtain

$$S[A+a] = \int \left(c_1(dA+da) \wedge *(dA+da) + c_2j \wedge (A+a) \right).$$

Expanding to first order in a and using the symmetry of the scalar product $\int \phi \wedge *\psi = \int \psi \wedge *\phi$, we arrive at

$$S[A+a] - S[A] = \int a \wedge (2c_1d * dA - c_2j).$$

Stationarity of the integral is equivalent to the condition $2c_1d * dA - c_2j = 0$. Comparison with Eq. (1.72) shows that this condition is equivalent to the Maxwell equation, provided we set $c_1 = c_2/2$. Summarizing, we have seen that the structure of the Maxwell equations – which entails the entire body of electromagnetic phenomena – follows largely from purely geometric reasoning!

1.4 Gauge theory

In this section, we will apply the body of mathematical structures introduced above to one of the most important paradigmes of modern physics, gauge theory. Gauge principles are of enormously

¹³ In principle, one might allow $c_i = c_i(A)$ to be functions of A. However, this would be at odds with our principle of 'maximal simplicity'.

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general validity, and it stands to reason that this is due to their geometric origin. This chapter aims to introduce the basic ideas behind gauge theory, within the framework of the mathematical theory developed thus far. In fact, we will soon see that a more complete coverage of gauge theories, notably the discussion of non-abelian gauge theory, requires the introduction of more mathematical structure: (Lie) group theory, the theory of differential manifolds, and bundle theory. Our present discussion will be heuristic in that we touch these concepts, without any ambition of mathematical rigor. In a sense, the discussion of this section is meant to *motivate* the mathematical contents of the chapters to follow.

1.4.1 Field matter coupling in classical and quantum mechanics (reminder)

Gauge theory is about the coupling of matter to so-called gauge fields. According to the modern views of physics, the latter mediate forces (electromagnetic, strong, weak, and gravitational), i.e. what we are really up to is a description of matter and its interactions. The most basic paradigm of gauge theory is the coupling of (charged) matter to the electromagnetic field. We here recapitulate the traditional description of field/matter coupling, both in classical and quantum mechanics. (Readers familiar with the coupling of classical and quantum point particles to the electromagnetic field may skip this section.)

Consider the Lagrangian of a charged point particle coupled to the electromagnetic field. Representing the latter by a four-potential with components $\{A^{\mu}\} = (\phi, A^{i})$ the corresponding **Lagrangian function** is given by (we set the particle charge to unity)¹⁴

$$L = \frac{m}{2}\dot{x}^i \dot{x}^i - \phi + \dot{x}^i A^i.$$

Exercise: consider the Euler-Lagrange equations $(d_t\partial_{\dot{x}^i}L - \partial_{x^i})L = 0$ to verify that you obtain the Newton equation of a particle subject to the Lorentz force, $m\ddot{x} = E + v \times B$, where $E = -\nabla \phi - \partial_t A$ and $B = \nabla \times A$.

The canonical momentum is then given by $p_i = \partial_{x^i} L = mx^i + A^i$, which implies the Hamilton function

$$H = \frac{1}{2m}(p - A)_{i}(p - A)_{i} + \phi.$$

We may now quantize the theory to obtain the Schrödinger equation ($\hbar = 1$ throughout)

$$\left[i\partial_t - \frac{1}{2m}(-i\nabla - A)_i(-i\nabla - A)_i - \phi\right]\psi(x,t) = 0.$$
(1.74)

What happens to this equation under a gauge transformation, $A \to A + \nabla \theta$, $\phi \to \phi - \partial_t \theta$? Substitution of the transformed fields into (1.74) obtains a changed Schrödinger equation. However, the gauge dependent contributions get removed if the gauge the wave function as $\psi(x,t) \to$

 $^{^{14}\,}$ In this section, we work in a non-relativistic setting. We assume a standard Euclidean metric and do not pay attention to the co- or contravariance of indices.

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 $\psi(x,t)e^{i\theta(x,t)}.$ We thus conclude that a gauge transformation in quantum mechanics is defined by

$$\begin{aligned} A &\to A + \nabla \theta, \\ \phi &\to \phi - \partial_t \phi, \\ \psi &\to e^{if} \psi. \end{aligned} \tag{1.75}$$

1.4.2 Gauge theory: general setup

The take home message of the previous section is that gauge transformations act on the \mathbb{C} -valued functions of quantum mechanics through the multiplication by phases. Formally, this defines a U(1)-action in \mathbb{C} . Let us now anticipate a little to say that the states relevant to the more complex gauge theories of physics will take values in higher dimensional vector spaces \mathbb{C}^n . Natural extensions of the (gauge) group action will then be $\mathbb{U}(n)$ actions or SU(n) actions. We thus anticipate that a **minimal arena of gauge theory** will involve

- ▷ a domain of space time formally an open subset $U \subset \mathbb{R}^d$, where d = 4 corresponds to (3+1)-dimensional space time.¹⁵
- ▷ A 'bundle' of vector spaces $B \equiv \bigcup_{x \in U} V_x$, where $V_x \simeq V$ and V is an *r*-dimensional real or complex vector space.¹⁶
- ▷ A transformation group G (gauge group) acting in the spaces V_x . Often, this will be a normpreserving group, i.e. G = U(n) or SU(n) for complex vector spaces and G = O(n) or SO(n) for real vector spaces.
- \triangleright A (matter) field, i.e. a map $\Phi: U \to B, x \mapsto \Phi(x)$. (This is the generalization of a 'wave function'.)
- Some dynamical input (the generalization of a Hamiltonian.) At first sight, the choice the dynamical model appears to be solely determined by the physics of the system at hand. However, we will see momentarily that important (physical!) features of the system follow entirely on the basis of geometric considerations. Notably, we will be forced to introduced a structure known in mathematics as a connection, and in physics as a gauge field.

 $^{^{15}}$ In condensed matter physics, one is often interested in cases d<4.

 $^{^{16}}$ This is our second example of a vector bundle. (The tangent bundle TU was the first.) For the general theory of bundle spaces, see chapter ...

Let us try to demystify the last point in the list above. Physical models generally involve the comparison of states at nearby points. For example the derivative operation in a quantum Hamiltonian, $\partial_x \psi(x,t)$ 'compares' wave function amplitudes at two infinitesimally close points. In other words, we will want to take 'derivatives' of states. Now, it is clear how to take the derivative of a real scalar field $V = \mathbb{R}$: just form the quotients





where γ is a curve in U locally tangent to the direction in which we want to differentiate. However, things start to get problematic when $\Phi_x \in V_x$ and $V_x \simeq V$ is a higher dimensional vector space. The problem now is that $\Phi_{\gamma(\epsilon)} \in V_{\gamma(\epsilon)}$ and $\Phi_{\gamma(0)} \in V_{\gamma(0)}$ live in *different* vector spaces. But how do we compare vectors of different spaces? Certainly, the naive formula (1.76) won't work anymore. For the concrete evaluation of the expression above requires the introduction of components, i.e. the choice of bases of the two spaces. The change of the basis in only one of the spaces would change the outcome of the derivative (cf. the figure above) which shows that (1.76) is a meaningless expression.

We wish to postulate the freedom to independently choose a basis at different points in space times an integral part of the theory. (For otherwise, we would need to come up with some principle that synchronizes bases uniformly in space and time. This would amount to an 'instantaneous action at the distance' a concept generally deemed as problematic.) Still, we need some extra structure that will enable us to compare fields at different points. The idea is to introduce a principle that determines when two fields Φ_x and $\Phi_{x'}$ are to be identified. This principle must be gauge invariant in that identical fields remain identical after two independent changes of bases at x and x'. A change of basis at x is mediated by an element of the gauge group $g_x \in G$. Here, g_x is to be interpreted as a linear transformation $g_x : V_x \to V_x$ acting in the field space at x. The components of the field in the new representation will be denoted by $g_x \Phi_x$.

In mathematics, the principle establishing a gauge covariant relation between fields at different points is called a **connection**. The idea of a connection can be introduced in different ways. We here start by defining an operation called **parallel transport**. Parallel transport will assign to each $\Phi_x \in V_x$ and each curve γ connecting x and x' an element $\Gamma[\gamma]\Phi_x \in V_{x'}$ which we interpret as the result of 'transporting' the field Φ_x along γ to the space $V_{x'}$. In view of the isomorphy $V_x \simeq V \simeq V_{x'}$, we may think of $\Gamma[\gamma] \in G$ as an element of the gauge group. Importantly, parallel transport is defined so as to commute with gauge transformations, which is to say that the operation of parallel transport must not depend on the bases used to represent the spaces V_x and $V_{x'}$, resp.

In formulas the condition of gauge covariance is expressed as follows: subject Φ_x to a gauge transformation to obtain $g_x \Phi_x$. Parallel translation will yield $\Gamma[\gamma]g_x \Phi_x$. This has to be equal to

the result $g_{x'}\Gamma[\gamma]\Phi_x$ obtained if we first parallel transport and only then gauge transform. We are thus lead to the condition

$$\Gamma[\gamma] = g_{x'} \Gamma[\gamma] g_x^{-1}. \tag{1.77}$$

As usual, conditions of this type are easiest to interpret for curve segments of infinitesimal length. For such curves, $\Gamma[\gamma] \simeq id$. will be close to the group identity, and $\Gamma[\gamma] - id$. will be approximately and element of the **Lie algebra**, g.¹⁷ Infinitesimal parallel transport will thus be a prescription assigning to an infinitesimally short segment (represented by a tangent vector) an element close to the group identity (represented by a Lie algebra element). In other words,



Infinitesimal parallel transport is described by a g-valued one-form, A, on U.

Let us now derive more concrete expressions for the parallel transportation of fields. To this end, let $\Phi(t) = \Gamma(\gamma(t))\Phi(0)$ denote the fields obtained by parallel translation along a curve $\gamma(t)$. We then have

$$\Phi(t+\epsilon) = \Gamma(\gamma(t+\epsilon))\Phi(0) = \Gamma(\gamma(t+\epsilon))(\Gamma(\gamma(t)))^{-1}\Gamma(\gamma(t))\Phi(0) = \Gamma(\gamma(t+\epsilon))(\Gamma(\gamma(t)))^{-1}\Phi(t).$$

Taylor expansion to first order obtains

$$\Phi(t+\epsilon) = \Phi(t) - \epsilon A(d_t \gamma(t)) \Phi(t) + \mathcal{O}(\epsilon^2), \qquad (1.78)$$

where $A(d_t\gamma((t)) = -d_\epsilon|_{\epsilon=0}\Gamma(\gamma(t+\epsilon))(\Gamma(\gamma(t)))^{-1} \in \mathfrak{g}$ is the Lie algebra element obtained by evaluating the one form A on the tangent vector $d_t\gamma(t)$. Taking the limit $\epsilon \to 0$, we obtain a differential equation for parallel transport

$$d_t \Phi(t) = -A(d_t \gamma(t))\Phi(t). \tag{1.79}$$

Having expressed parallel transport in terms of a \mathfrak{g} -valued one-form, the question arises what conditions gauge invariance imply on this form. Comparing with (1.77) and denoting the gauge transformed connection form by A', we obtain (all equalities up to first order in ϵ)

$$\Phi'(t+\epsilon) \equiv [\mathrm{id.} - \epsilon A'(d_t\gamma)] \Phi'(t) =$$

= $g(t+\epsilon)\Phi(t+\epsilon) = g(t+\epsilon) [\mathrm{id.} - \epsilon A(d_t\gamma)] \Phi(t) =$
= $g(t+\epsilon) [\mathrm{id.} - \epsilon A(d_t\gamma)] g^{-1}(t)\Phi'(t) =$
[$\mathrm{id.} + \epsilon((d_tg(t))g^{-1}(t) - g(t)A(d_t\gamma_t))g^{-1}(t)] \Phi'(t).$

¹⁷ Referring for a more substantial discussion to chapter xx below, we note that the Lie algebra of a Lie group G is the space of all 'tangent vectors' $d_t |_{t=0} g(t)$ where g(t) is a smooth curve in g with g(0) = id.

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where g(t) is shorthand for $g(\gamma(t))$. Comparing terms, and using that $(d_t g(t))g^{-1} = -g(t)d_tg^{-1}(t)$, we arrive at the identification

$$A' = gAg^{-1} + gdg^{-1}, (1.80)$$

where gdg^{-1} is the g-valued one form defined by $(gdg^{-1})(v) = g(\gamma(0))d_t|_{t=0}g^{-1}(\gamma(t))$ where $d_t\gamma(t) = v$.

Notice what happens in the case G = U(1) and $\mathfrak{g} = i\mathbb{R}$ relevant to conventional quantum mechanics. In this case, writing $g = e^{i\theta}$, where θ is a real valued function on U, the differential form $gdg^{-1} = -id\theta$ collapses to a real valued form. Also $g^{-1}Ag = A$, on account of the commutativity of the group. This leads to the transformation law $A' = A - id\theta$ reminiscent of the transformation behavior of the electromagnetic potential. (The extra *i* appearing in this relation is a matter of convention.) This suggests a tentative identification

connection form (mathematics) = gauge potential (physics).

Eq. (1.79) describes the infinitesimal variant of parallel transport. Mathematically, this is a system of ordinary linear differential equations with time dependent coefficients. Equations of this type can be solved in terms of so-called path ordered exponentials: Let us define the generalized exponential series

$$\Gamma[\gamma_t] \equiv P \exp\left(-\int_{\gamma_t} A\right) \equiv \sum_{j=1}^{\infty} (-)^j \int_0^t dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{j-1}} dt_j A(\dot{\gamma}(t_1)) A(\dot{\gamma}(t_2)) \dots A(\dot{\gamma}_{t_j}),$$
(1.81)

where γ_t is a shorthand for the extension of a curve $\gamma = \{\gamma(s) | s \in [0,1]\}$ up to the parameter value s = t.

The series is constructed so as to solve the differential equation

$$d_t P \exp\left(-\int_{\gamma_t} A\right) = A(\dot{\gamma}(t)) P \exp\left(-\int_{\gamma(t)} A\right),$$

with initial condition $P \exp\left(-\int_{\gamma_0} A\right) = \mathrm{id.}$. Consequently

$$\Phi(t) = P \exp\left(-\int_{\gamma_t} A\right) \Phi(0)$$

describes the parallel transport of $\Phi(0)$ along curve segments of finite length.

INFO In the abelian case G = U(1) relevant to electrodynamics, the (matrices representing the) elements $A(\dot{\gamma})$ at different times commute. In this case,

$$\int_0^t dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{j-1}} dt_j A(\dot{\gamma}(t_1)) A(\dot{\gamma}(t_2)) \dots A(\dot{\gamma}_{t_j}) = \frac{1}{j!} \left(\int_0^{\gamma_t} dt A(\dot{\gamma}) \right)^j$$

and $\Gamma[\gamma_t] = \exp\left(-\int_0^{\gamma_t} dt A(\dot{\gamma})\right)$ collapses to an ordinary exponential. In components, this may be written as

$$\Gamma[\gamma] = e^{-\int_0^t ds \, A_\mu(\gamma(s))\dot{\gamma}^\mu(s)}$$

1.4 Gauge theory

1.4.3 Field strength

Our discussion above shows that a connection naturally brings about an object, A, behaving similar to a generalized potential. This being so, one may wonder whether the 'field strength' corresponding to the potential carries geometric meaning, too. As we are going to show next, the answer is affirmative.

Consider a connection as represented by its connection one-form, A. The ensuing parallel transporters $\Gamma[\gamma]$ generally depend on the curve, i.e. parallel transport along two curves connecting two points x and x' will not, in general give identical results. Equivalently, $\Gamma[\gamma]$ may differ from unity, even if γ is closed. To understand the consequences, let us consider the case of the abelian group G = U(1) relevant to quantum electrodynamics. In this, case, (see info section above), parallel transport around a closed loop in space-time can be written as

$$\Gamma[\gamma] = e^{-\int_{\gamma} A} = e^{-\int_{S(\gamma)} dA} = e^{-\int_{S(\gamma)} F},$$

where $S(\gamma)$ may be any surface surrounded by γ . This shows that the existence of a non-trivial parallel transporter around a closed loop is equivalent to the presence of a non-vanishing field strength form. (Readers familiar with quantum mechanics may interpret this phenomenon as a manifestation of the Aharonov-Bohm effect: a non-vanishing Aharonov-Bohm phase along a closed loop (in space) is indicative of a magnetic field penetrating the loop.)

How does the concepts of a 'field strength' generalize to the non-abelian case? As a result of a somewhat tedious calculation one finds that the non-abelian generalization of F is given by

$$F = dA + A \wedge A. \tag{1.82}$$

The \mathfrak{g} -valued components of the two-form F are given by

$$F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} + [A_{\mu}, A_{\nu}],$$

where [X,Y] = XY - YX is the matrix commutator. Under a gauge transformation $A \rightarrow gAg^{-1} + gdg^{-1}$. Substituting this into (1.82), we readily obtain

$$F' = gFg^{-1}. (1.83)$$

INFO To prove Eq. (1.82), we consider an infinitesimal curve of length ϵ . The area bounded by the curve will then be of $\mathcal{O}(\epsilon^2)$. We wish to identify contributions to the path ordered exponential of this order. A glance at the abelian expression $\exp(-\int_{S(\gamma)} F) = 1 + F \times \mathcal{O}(\epsilon^2) + \mathcal{O}(\epsilon^3)$ shows that this is sufficient to identify the generalization of F.

We thus expand

$$\Gamma[\gamma] = \mathrm{id.} + \int_0^1 dt_1 A(\dot{\gamma}(t_1)) + \int_0^1 dt_1 \int_0^{t_1} dt_2 A(\dot{\gamma}(t_1)) A(\dot{\gamma}(t_2)) + \mathcal{O}(\epsilon^3).$$

The term of first order in A is readily identified as $\int_{\gamma} A = \int_{S(\gamma)} dA$, which is of $\mathcal{O}(\epsilon^2)$. Turning to the second term, we represent the product of matrices

$$A(\dot{\gamma}(t_1))A(\dot{\gamma}(t_2)) = \frac{1}{2} \left([A(\dot{\gamma}(t_1)), A(\dot{\gamma}(t_2))]_+ + [A(\dot{\gamma}(t_1)), A(\dot{\gamma}(t_2))]_- \right)$$

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as a sum of a symmetrized and an anti-symmetrized contribution. Here, $[A, B]_{\pm} = AB \pm BA$. The symmetric contribution evaluates to

$$\int_0^1 dt_1 \int_0^{t_1} dt_2 \left[A(\dot{\gamma}(t_1)), A(\dot{\gamma}(t_2)) \right]_+ = \left(\int_0^1 dt \, A(t) \right)^2 = \mathcal{O}(\epsilon^4),$$

and can be discarded. Turning to the antisymmetric contribution, we obtain

$$\begin{split} &\int_{0}^{1} dt_{1} \int_{0}^{t_{1}} dt_{2} \left[A(\dot{\gamma}(t_{1})), A(\dot{\gamma}(t_{2})) \right]_{+} \\ &= \frac{1}{2} A_{\mu} A_{\nu} \int_{0}^{1} dt_{1} \int_{0}^{t_{1}} dt_{2} \left(\dot{\gamma}^{\mu}(t_{1}) \dot{\gamma}^{\nu}(t_{2}) - \dot{\gamma}^{\nu}(t_{1}) \dot{\gamma}^{\mu}(t_{2}) \right) + \mathcal{O}(\epsilon^{3}) = \\ &= \frac{1}{2} A_{\mu} A_{\nu} \int_{0}^{1} dt_{1} \left(\dot{\gamma}^{\mu}(t_{1}) \gamma^{\nu}(t_{1}) - \dot{\gamma}^{\nu}(t_{1}) \gamma^{\mu}(t_{1}) \right) + \mathcal{O}(\epsilon^{3}) = \\ &= \frac{1}{2} A_{\mu} A_{\nu} \int_{\gamma} \left(d\gamma^{\mu} \gamma^{\nu} - d\gamma^{\nu} \gamma^{\mu} \right) + \mathcal{O}(\epsilon^{3}) = \\ &= \frac{1}{2} A_{\mu} A_{\nu} \int_{S(\gamma)} \left(d\gamma^{\mu} \wedge d\gamma^{\nu} - d\gamma^{\nu} \wedge d\gamma^{\mu} \right) + \mathcal{O}(\epsilon^{3}) = \\ &= \int_{S(\gamma)} A \wedge A + \mathcal{O}(\epsilon^{3}). \end{split}$$

In the crucial first equality, we noted that for an infinitesimal curve, $A(\dot{\gamma}(t)) = (A_{\mu})_{\gamma(t)} \dot{\gamma}^{\nu}(t) \simeq (A_{\mu})_{\gamma(0)} \dot{\gamma}^{\nu}(t) \equiv A_{\mu} \dot{\gamma}^{\nu}(t)$, i.e. the coefficients of the potential form can be pulled out of the integral. Combining terms, we arrive at

$$\Gamma[\gamma] = \mathrm{id.} + \int_{S(\gamma)} \left(dA + A \wedge A \right) + \mathcal{O}(\epsilon^3)$$

Comparing with the abelian expression, we obtain (1.82) for the non-abelian generalization of the field strength form.

The discussion above illustrates the appearance of maps carrying **representations different** from the fundamental group representation of G in V: Let us assume that we are interested in the variation of a smooth map $\Phi: U \to X$. Here, X = B corresponds to a V-valued function. However, we may also choose to consider forms $X = \Lambda^p U$, or just ordinary functions $X = \mathbb{R}$. These maps generally carry a representation of the group G whose specifics depend on the target space and on the definition of the map. For X = V, this representation will be the fundamental representation considered above, $\Phi_x \to g_x \Phi_x$, where $g_x = \{g_{x,ij}\}$ is the matrix representing $g_x \in U$. For $X = \Lambda^2 U$, we may encounter other representations. For example, for $\Phi = F$, the field strength form, $F_x \to gF_xg^{-1}$ transforms according to the adjoint representation, cf. Eq. (1.83). Finally, for $X = \mathbb{R}$, Φ does not transform under G, transformation behavior which we may formally assign to the singlet representation.

It is straightforward to generalize the notion of parallel transport to objects transforming according to arbitrary group representations. For example, for an object transforming according to the adjoint representation, the analog of (1.78) reads

$$\Phi(t+\epsilon) = \Phi(t) - \epsilon \left[A(d_t \gamma(t)) \Phi(t) - \Phi(t) A(d_t \gamma(t)) \right] + \mathcal{O}(\epsilon^2),$$

which immediately leads to

$$d_t \Phi(t) = -[A(d_t \gamma), \Phi(t)]. \tag{1.84}$$

The generalization to objects transforming under yet different representations of G should be straightforward.

1.4.4 Exterior covariant derivative

With the notion of parallel transport in place, we are now in a position to define a meaningful derivative operation. The idea simply is to measure variations of objects defined in U in terms of deviations from the parallel transported objects. Consider, thus, a curve $\gamma(t)$, as before. The derivative of a function along γ is described by the differential quotient

$$D\Phi\big|_{\gamma(t)}(\dot{\gamma}(t)) \equiv \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left[\Phi_{\gamma(t+\epsilon)} - \Phi_{\gamma(t)} + A(\dot{\gamma}(t))\Phi_{\gamma(t)} \right]$$

Here, both $\Phi_{\gamma(t+\epsilon)}$ and the parallel transport of $\Phi_{\gamma(t)}$, i.e. $\Phi_{\gamma(t)} + A(\dot{\gamma}(t))\Phi_{\gamma(t)}$ are considered to be elements of $V_{\gamma(t+\epsilon)}$, and we assumed Φ to transform under the fundamental representation of G. This expression defines the so-called **(exterior) covariant derivative** of Φ along $\dot{\gamma}$. The general covariant derivative (prior to reference to a direction of differentiation) is defined as

$$D\Phi \equiv d\Phi + A \wedge \Phi, \tag{1.85}$$

where in the case of a V_x -valued function (zero-form), the wedge product reduces to the conventional product between the 'matrix' A and the 'vector' Φ . In components:

$$(D\Phi)^i \equiv d\Phi^i + A^i{}_j \wedge \Phi^j.$$

The wedge product becomes important, once we generalize to the covariant derivative of differential forms. For example, for a two-form transforming under the **adjoint representation**, the covariant derivative reads (cf. Eq. (1.84)),

$$D\Phi = d\Phi - A \wedge \Phi + \Phi \wedge A. \tag{1.86}$$

Let us discuss the most important properties of the covariant derivative:

▷ By design, the covariant derivative is compatible with gauge transformations: with

$$\Phi' = g\Phi,$$

$$A' = gAg^{-1} + gdg^{-1}$$

we have

$$D'\Phi' = g(D\Phi),$$

where $D' \equiv d + A' \wedge$. This is checked by direct substitution of the definitions. \triangleright The covariant derivative obeys the **Leibniz rule**

$$D(\Psi \wedge \Phi) = D\Psi \wedge \Phi + (-)^{q}\Psi \wedge D\Phi.$$

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 \triangleright Unlike with $d^2 = 0$, the covariant derivative is **not nilpotent**. Rather, one may check by straightforward substitution that

$$D^2 = F \wedge,$$

where F is the field strength form (1.82).

 \triangleright Finally, let us consider the covariant derivative of F itself. The field strength transforms under the adjoint representation, (1.83), which means that the covariant derivative $DF = dF + A \wedge F - F \wedge A$ is well defined. Substituting the definition (1.82), we readily obtain the **Bianchi identity**,

$$DF = 0, (1.87)$$

which generalizes the homogeneous Maxwell equations to general gauge theories.

The covariant derivative plays a very important role in physics. Important applications include all areas of **gauge theory**, and **general relativity**. In the latter context, the role of the connection is assumed by the so-called Riemannian connection associated to the curvature of space time. We will return to this point in chapter xx below.

1.5 Summary and outlook

In this chapter, we have introduced a minimal framework of mathematical operations relevant to differential geometry: we discussed the (exterior) multilinear algebra, and its generalization to alternating and locally linear maps on the tangent bundle of open subsets of \mathbb{R}^n , i.e. the apparatus of differential forms. We learned how to differentiate and integrate differential forms, thus generalizing the basic operations of 'vector analysis'. Finally, we introduced the concept of a metric as an important means to characterize geometric structures.

The mathematical framework introduced above is powerful enough to describe various applications of physics in a unified and efficient way. We have seen how to formulate the foundations of classical mechanics and electrodynamics in a 'coordinate invariant' way. Here, the notion of coordinate invariance implies three major advantages: (i) the underlying structure of the theory becomes maximally visible, i.e. formulas aren't cluttered with indices, etc., (ii) the change(ability) between different coordinate systems is exposed in transparent terms, and (iii) formulas relating to specific coordinate systems (think of the formula (1.65)) are formulated so as to expose the underlying conceptual structure. (You can't say *this* about the standard formula of the Laplacian in spherical coordinates.)

However, in our discussion of the final 'example' – gauge theory – we were clearly pushing limits, and several important limitations of our so far theory became evident. What we need, at least, is an extension from geometry on 'open subsets of \mathbb{R}^n to more general geometric structures. Second, we should like to give the notion of 'bundles' – i.e. mathematical constructs where some mathematical structure is locally attached to each point of a base structure – a more precise definition. And thirdly, the ubiquitous appearance of continuous groups in physical applications calls for a geometry oriented discussion of group structures. In the following chapters we will discuss these concepts in turn.

In this chapter, we will learn how to describe the geometry of structures that cannot be identified with open subsets of \mathbb{R}^n . Objects of this type are pervasive both in mathematics, and in physical applications. In fact, it is the lack of an identification with a single 'coordinate domain' in \mathbb{R}^n that makes a geometric structure interesting. Prominent examples of such 'manifolds' include spheres, tori, the celebrated Moebius strip, continuous groups, and many more.

2.1 Basic structures

2.1.1 Differentiable manifolds

Spaces M which *locally* (yet not necessarily globally) look like open subsets of \mathbb{R}^n are called manifolds. The precise meaning of the notion "look like" is provided by the following definition: A **chart** of a manifold M is a pair (U, α) , where $U \subset M$ is an open (!) subset of M and

$$\begin{array}{rcl} \alpha: U & \to & \alpha(U) \subset \mathbb{R}^n \\ & x & \mapsto & \alpha(x) \end{array}$$

is a homeomorphism (α is invertible and both α and α^{-1} are continuous) of U onto an image $\alpha(U) \subset \mathbb{R}^n$ which is open in \mathbb{R}^n . Notice that the definition above requires the existence of a certain amount of mathematical structure: We rely on the existence of 'open' subsets and 'continous' mappings. This means that M must, at least, be a topological space. More precisely, M must be a Hausdorff space¹ whose topology is generated by a countable basis. Loosely speaking, the Hausdorffness of M means that it is a topological space (the notion of openness and continuity exists) on which we may meaningfully identify distinct points.

The chart assigns to any point $x \in U \subset M$ a set of *n*-coordinates $\alpha^i(x)$, the coordinates of x with respect to the chart α . If we are working with a definite chart we will, to avoid excessive notation, often use the alternative designation $x^i(x)$, or just x^i . Given the notion of charts, we are able to define **topological manifolds**. A topological manifold M is a Hausdorff space with countable basis such that every point of M lies in a coordinate neighbourhood, i.e. in the domain of definition U of a chart. A collection of charts (U_r, α_r) such that $\bigcup_r U_r = M$ covers M is called an **atlas** of M.

¹ A **Hausdorff space** is a topological space for which any two distinct points possess disjoint neighbourhoods. (Exercise: look up the definitions of topological spaces, bases of topologies, and neighbourhoods.)



Figure 2.1 On the definition of topological manifolds. Discussion, see text.

INFO Minimal manifolds as defined above are called **topological manifolds**. However, most manifolds that are encountered in (physical) practice may be embedded into some sufficiently high dimensional \mathbb{R}^n . (Do not confuse the notions 'embedding' and 'identifying'. I.e. we may think of the two-sphere as a subset of \mathbb{R}^3 , it is, however, not possible to identify it with a subspace of \mathbb{R}^2 .) In such cases, the manifolds inherits its topology from the standard topology of \mathbb{R}^n , and we need not worry about topological subtleties.

Consider now two charts (U_1, α_1) and (U_2, α_2) with non-empty intersection $U_1 \cap U_2$. Each $x \in U_1 \cap U_2$ then possesses two coordinate representations $x_1 \equiv \alpha_1(x)$ and $x_2 \equiv \alpha_2(x)$. These coordinates are related to each other by the map $\alpha_2 \circ \alpha_1^{-1}$, i.e.

$$\begin{aligned} \alpha_2 \circ \alpha_1^{-1} : \alpha_1(U_1 \cap U_2) &\to & \alpha_2(U_1 \cap U_2), \\ x_1 &\mapsto & x_2 = \alpha_2 \circ \alpha_1^{-1}(x_1), \end{aligned}$$

or, in components, $x_2^i = \alpha_2^i(\alpha_1^{-1}(x_1^1, \ldots, x_1^n))$. The coordinate transformation $\alpha_2 \circ \alpha_1^{-1}$ defines a homeomorphism between the open subsets $\alpha_1(U_1 \cap U_2)$ and $\alpha_2(U_1 \cap U_2)$. If, in addition, all coordinate transformations of a given atlas are C^{∞} , the atlas is called a C^{∞} -atlas. (In practice, we will exclusively deal with C^{∞} -systems.)

EXAMPLE The most elementary example of a manifold is an **open subset** $U \subset \mathbb{R}^n$. It may be covered by a one-atlas chart containing just (U, id_U) .

EXAMPLE Consider the **two-sphere** $S^2 \subset \mathbb{R}^3$, i.e. the set of all points $x \in \mathbb{R}^3$ fulfilling the condition (Euclidean metric in \mathbb{R}^3) $(x^1)^2 + (x^2)^2 + (x^3)^2 = 1$. We cover S^2 by two charts-domains,

2.1 Basic structures

 $U_1 \equiv \{x \in S^2 | x^3 > -1\}$ (S^2 – south pole) and $U_2 \equiv \{x \in S^2 | x^3 < 1\}$ (S^2 – north pole). The two coordinate mappings (aka stereographic projections of the sphere) α_1 are defined by

$$\begin{aligned} \alpha_1(x^1, x^2, x^3) &= \frac{1}{1+x^3}(x^1, x^2) \in \alpha_1(U_1) \subset \mathbb{R}^2, \qquad x^3 > -1\\ \alpha_2(x^1, x^2, x^3) &= \frac{1}{1-x^3}(x^1, x^2) \in \alpha_2(U_2) \subset \mathbb{R}^2, \qquad x^3 < 1. \end{aligned}$$

If the union of two atlases of M, $\{(U_i, \alpha_i)\}$ and $\{(V_i, \beta_i)\}$ is again an atlas, the two parent atlases are called compatible. Compatibility of atlases defines an equivalence relation. Individual equivalence classes of this relation are called **differentiable structures**. I.e. a differentiable structure on M contains a maximum set of mutually compatible atlases. A manifold M equipped with a differentiable structure is called a **differentiable manifold**. (Throughout we will refer to differentiable manifolds just as 'manifolds'.)

EXAMPLE Let $M = \mathbb{R}$ be equipped with the standard topology of \mathbb{R} and a differentiable structure be defined by the one-chart atlas $\{(\mathbb{R}, \alpha_1)\}$, where $\alpha_1(x) = x$. Another differentiable structure is defined by $\{(\mathbb{R}, \alpha_2)\}$, where $\alpha_2(x) = x^3$. These two atlases indeed belong to different differentiable structures. For, $\alpha_1 \circ \alpha_2^{-1} : x \to x^{1/3}$ is not differentiable at x = 0.

2.1.2 Differentiable mappings

A function $f: M \to \mathbb{R}$ is called a **differentiable function** (at $x \in M$) if for any chart $U \ni x$, the function $f \circ \alpha^{-1} : \alpha(U) \subset \mathbb{R}^n \to \mathbb{R}$ is differentiable in the ordinary sense of calculus, i.e. $f(x^1, \ldots, x^n)$ has to be a differentiable function at $\alpha(x)$ (cf. Fig. 2.2, top.) It does, in fact, suffices to verify differentiability for just one chart of M's differentiable structure. For with any other chart, β , $f \circ \beta^{-1} = (f \circ \alpha^{-1}) \circ (\alpha \circ \beta^{-1})$ and differentiability follows from the differentiability of the two constituent maps. The algebra of differentiable functions of M is called $C^{\infty}(M)$.

More generally, we will want to consider maps

$$F: M \to M'$$

between differentiable manifolds M and M' of dimensions n and n', resp. (cf. Fig. 2.2, center part.) Let, $x \in M$, $U \ni x$ the domain of a chart and $U' \ni x' \equiv F(x)$ be a chart of M' containing x's image. The function F is **differentiable** at x if $\alpha' \circ F \circ \alpha^{-1} : \alpha(U) \subset \mathbb{R}^n \to \alpha'(F(U)) \subset \mathbb{R}^{n'}$ is differentiable in the sense of ordinary calculus (i.e. $F^i(x^1, \ldots, x^n) \equiv \alpha'^i(F(x^1, \ldots, x^n))$), $i = 1, \ldots, n'$ are differentiable at $\alpha(x)$.) The set of all smooth mappings $F : M \to M'$ will be designated by $\mathcal{C}^{\infty}(M, M')$.

The map F is a **diffeomorphism** if it is invertible and both F and F^{-1} are differentiable. (What this means is that for any two chart domains, $\alpha' \circ F \circ \alpha^{-1}$ diffeomorphically maps $\alpha(U)$ onto $\alpha'(F(U))$. If a diffeomorphism $F : M \to M'$ exists, the two manifolds M and M' are **diffeomorphic**. In this case, of course, dim $M = \dim M'$.

INFO The definitions above provide the link to the mathematical apparatus developed in the previous chapter. By virtue of charts, maps between manifolds may be locally reduced to maps between open subsets of \mathbb{R}^n (viz. the maps expressed in terms of local coordinates.) It may happen, though, that



Figure 2.2 On the definition of continuous maps of manifolds into the reals, or between manifolds. Discussion, see text.

a map meaningfully defined for a local coordinate neighbourhood defies extension to the entire atlas covering M. Examples will be encountered below.

2.1.3 Submanifolds

A subset N of an n-dimensional manifold M is called a q-dimensional **submanifold** of M if for each point $x_0 \in N$ there is a chart (U, α) of M such that $x_0 \in U$ and for all $x \in U \cap N$,

$$\alpha(x) = (x^1, \dots, x^q, a^{q+1}, \dots, a^n),$$

with $\alpha^{q+1}, \ldots, a^n$ fixed. Defining $\overline{U} = U \cap N$ and $\overline{\alpha} : \overline{U} \to \mathbb{R}^q, \overline{\alpha}(x) = (x^1, \ldots, x^q)$, we obtain a chart $(\overline{U}, \overline{\alpha})$ of the *q*-dimensional manifold N. A (compatible) collection of such charts defines a differentiable structure of N.

EXAMPLE An open subset $U \subset M$ is an *n*-dimensional submanifold of M. We may think of a collection of isolated points in M as a zero-dimensional manifold.

EXAMPLE Let $f^i: M \to \mathbb{R}$, i = 1, ..., p be a family of p functions on M. The set of solutions of the equations $f^1(x) = \cdots = f^p(x) = \text{const.}$ defines a p-dimensional submanifold on M if the map $M \to \mathbb{R}^p, x \mapsto (f^1(x), \ldots, f^p(x))$ has rank p.

2.2 Tangent space

2.2 Tangent space

In this section, we will generalize the notion of tangent space as introduced in the previous chapter to the tangent space of a manifold. To make geometric sense, this definition will have to be independent of the chosen atlas of the manifold.

2.2.1 Tangent vectors

The notation $\frac{\partial}{\partial x^i}$ introduced in section 1.2.1 to designate coordinate vector fields suggests an interpretation of vector fields as differential operators. (Indeed, the components v^i of $v = v^i \frac{\partial}{\partial x^i}$ where obtained by *differentiating* the coordinate functions in the direction of v.) This view turns out to be very convenient when working on manifolds. In the following, we will give the derivative-interpretation of vectors the status of a precise definition.

Consider a curve $\gamma : [-a, a] \to M$ such that $\gamma(0) = x$. To define a vector v_x tangent to the manifold at $x \in M$ we take directional derivatives of functions $f : M \to \mathbb{R}$ at x in the direction identified by γ . I.e. the action of the tangent vector v_x identified by the curve on a function f is defined by

$$v_x(f) \equiv d_t \Big|_{t=0} f(\gamma(t)).$$

Of course there are other curves γ' such that $\gamma'(t_0) = \gamma(0)$ and $d_t f(\gamma(t)) \big|_{t=0} = d_t f(\gamma'(t)) \big|_{t=t_0}$. All these curves are 'tangent' to each other at x and will generate the same tangent vector action. It is thus appropriate to identify a **tangent vectors** v_x at x as equivalence classes of curves tangent to each other at x.

For a given chart (U, α) , the components v_x^i of the vector v_x are obtained by letting v_x act on the coordinate functions:

$$v_x^i \equiv v_x(x^i) = d_t \gamma^i(t) \big|_{t=0},$$

where $\gamma^i \equiv \alpha^i \circ \gamma$. According to the chain rule, the action of the vector on a general function is then given by

$$v_x(f) = v_x^i \frac{\partial f}{\partial x^i},\tag{2.1}$$

where $\bar{f} = f \circ \alpha^{-1} : \alpha(U) \subset \mathbb{R}^n \to \mathbb{R}$ is a real valued function of n variables and $\partial_{x^i} \bar{f}$ its partial derivative. In a notation emphasizing v_x 's action as a differentiable operator we have

$$v_x = v_x^i \frac{\partial}{\partial x^i}.$$

Notice the analogy to our earlier definition in section 1.2.1; the action of a vector is defined by taking directional derivatives in the direction identified by its components.

The definition above is coordinate independent and conforms with our earlier definition of tangent vectors. At the same time, however, it appears to be somewhat un-natural to link the definition of vector (differential operators) to a set of curves.² Indeed, there exists an alternative

² Cf. the introduction of partial derivatives in ordinary calculus: while the action of a partial derivative operation on a function is *defined* in terms of a curve (i.e. the curve identifying the direction of the derivative), in practice one mostly applies partial derivatives without explicit reference to curves.

definition of vectors which does not build on the notion of curves: A **tangent vector** v_x at $x \in M$ is a derivation, i.e. a linear map from the space of smooth functions³ defined on some open neighbourhood of x into the real numbers fulfilling the conditions

$$v_x(af + bg) = av_x f + bv_x g, \quad a, b \in \mathbb{R}, f, g \text{ functions}, \quad \text{linearity},$$

 $v_x(fg) = fv_x g + (v_x f)g, \qquad \qquad \text{Leibnitz rule}.$

To see that this definition is equivalent to the one given above, let y be a point infinitesimally close to x. We may then Taylor expand

$$f(y) = f(x) + (\alpha(y) - \alpha(x))^i \left. \frac{\partial \bar{f}}{\partial x^i} \right|_x$$

Again defining the components v_x^i of the vector (in the chart α) as

$$v_x^i = v_x \alpha^i$$

and taking the limit $y \to x$ we find that the action of the vector on f is given by (2.1).

We may finally relate the definitions of tangent vectors given above to (the mathematical formulation of) a definition pervasive in the physics literature. Let (α, U) and (α', U') be two charts such that $x \in U \cap U'$. The action of a tangent vector v_x on a function then affords the two representations

$$v_x f = v_x^i \left. \frac{\partial f}{\partial x^i} \right|_{\alpha(x)} = v_x^{i\prime} \left. \frac{\partial f'}{\partial x^{\prime i}} \right|_{\alpha'(x)}$$

where $\bar{f}' = f \circ \alpha'^{-1} = \bar{f} \circ (\alpha \circ \alpha'^{-1})$. Using the abbreviated notation $(\alpha' \circ \alpha^{-1})(x) = x'(x)$ we have

$$\frac{\partial \bar{f}}{\partial x^i} = \frac{\partial \bar{f}}{\partial x'^j} \frac{\partial x'^j}{\partial x^i},$$

we obtain the transformation law of vector components

$$v_x^{\prime i} = \frac{\partial x^{\prime i}}{\partial x^j} v_x^j.$$
(2.2)

This leads us to yet another possibility to define **tangent vectors**: A tangent vector v_x is described by a triple (U, α, V) , where $V \in \mathbb{R}^n$ is an *n*-component object (containing the components of v_x in the chart α .) The triples (U, α, V) and (U', α', V') describe the same tangent vector iff the components are related by $V'^i = \frac{\partial x'^i}{\partial x^j} V^j$.

We have, thus introduced three different (yet equivalent) ways of defining tangent vectors on manifolds:

> Tangent vectors as equivalence classes of curves, as

derivative operations acting on (germs of) functions, and

▷ a definition in terms of (contravariant) transformation behaviour of components.

 $^{^3}$ To be precise, vectors map so-called germs of functions into the reals. A germ of a function is obtained by identifying functions for which a neighbourhood around x exists in which the reference functions coincide.

2.2 Tangent space

2.2.2 Tangent space

Defining a linear structure in the obvious manner, $(av_x + bw_x)(f) = av_x(f) + bw_x(f)$, the set of all tangent vectors at x becomes a linear space, the **tangent space** T_xM . The union $\bigcup_{x \in M} T_xM \equiv TM$ defines the **tangent bundle** of the manifold. Notice that

The tangent bundle, TM of an $n\mbox{-dimensional}$ manifold, M, is a $2n\mbox{-dimensional}$ manifold by itself.

For, in a chart domain of M with coordinates $\{x^i\}$, the elements of TM may be identified in terms of coordinates $\{x^1, \ldots, x^n, v^1, \ldots, v^n\}$, where the 'vectorial components' of T_xM are parameterized of $v = v^i \frac{\partial}{\partial x^i}$. In a similar manner, we may introduce the **cotangent bundle** as the space $TM^* \equiv \bigcup_{x \in M} T_x^*M$, where T_x^*M is the dual space of T_xM . Again, TM^* is a 2n-dimensional manifold. Below, we will see that it comes with a very interesting mathematical structure.

A vector field on the domain of a chart, (U, α) is a smooth mapping

$$\begin{array}{rccc} v:U & \to & TU, \\ & x & \mapsto & v^i_x \frac{\partial}{\partial x^i} \in T_x U \end{array}$$

A vector field on the entire manifold is obtained by extending this mapping from a single chart to an entire atlas and requiring the obvious compatibility relation, $x \in U \cap U' \Rightarrow v_x = v'_x$. (In coordinates, this condition reads as (2.2).)

The set of all smooth vector fields on M is denoted by vect(M). For $v \in vect(M)$ and $f \in C^{\infty}M$, the action of the vector field on the function obtains another function, v(f), defined by

$$(v(f))(x) = v_x(f).$$

A frame on (a subset of) M is a set (b_1, \ldots, b_n) of n vector fields linearly independent at each point of their definition. A coordinate system (U, α) defines a local frame $(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n})$. However, in general no frame extensible to all of M exists. If such a frame exists, the manifold is called **parallelizable**.

EXAMPLE Open subsets of \mathbb{R}^n , Lie groups (see next chapter), and certain spheres S^1, S^3, S^7 are examples of parallelizable manifolds. Non-parallelizable are all other spheres, the Moebius strip and many others more.

2.2.3 Tangent mapping

For a smooth map $F: M \to M'$ between two differentiable manifolds, the tangent mapping may be defined by straightforward extension of our definition in section 1.2.3. For $x \in M$, we

define

$$\begin{array}{rccc} TF_x:T_xM & \to & T_{F(x)}M', \\ v_x & \mapsto & TF_x(v_x), \end{array}$$

$$[TF_x(v_x)]f \equiv v_x(f \circ F).$$

For two coordinate systems (U, α) , and (U', α') covering $x \in U$ and the image point $F(x) \in U'$, respectively, the components $(TF_x(v_x))^i$ of the vector $(TF_x)(v_x)$ obtain as

$$(TF_x(v_x))^i = \frac{\partial F^i}{\partial x^j} v^j,$$

where $\bar{F} = \alpha' \circ F \circ \alpha^{-1}$. The tangent mapping of the composition of two maps $G \circ F$ evaluates to

$$T(G \circ F)_x = TG_{F(x)} \circ TF_x.$$

2.2.4 Differential Forms

Differential forms are defined by straightforward generalization of our earlier definition of differential forms on open subsets of \mathbb{R}^n : A p-form ϕ on a differentiable manifold maps $x \in M$ to $\phi_x \in \Lambda^p(T_xM)^*$. The x-dependence is required to be smooth, i.e. for $v_1, \ldots, v_p \in \text{vect}(M)$, $\phi_x(v_1(x), \ldots, v_p(x))$ is a smooth function of x. The vector space of p-forms is denoted by $\Lambda^p M$ and the algebra of forms of general degree by

$$\Lambda M \equiv \bigoplus_{p=0}^{n} \Lambda^{p} M.$$

The mathematics of differential forms on manifolds completely parallels that of forms on open subsets of \mathbb{R}^n . Specifically,

- ▷ The wedge product of differential forms and the inner product of a vector field and a form are defined as in section 1.2.4.
- \triangleright A (dual) n-frame is a set of n linearly independent 1-forms. On a coordinate domain, the coordinate forms (dx^1, \ldots, dx^n) locally define a frame, dual to the coordinate vector fields $(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n})$. In the domain of overlap of two charts, a p-form affords the two alternative coordinate representations

$$\phi = \frac{1}{p!}\phi_{i_1,\dots,i_p}dx^{i_1}\wedge\dots\wedge dx^{i_p},$$

$$\phi = \frac{1}{p!}\phi'_{i_1,\dots,i_p}dx'^{i_1}\wedge\dots\wedge dx'^{i_p},$$

$$\phi'_{i_1,\dots,i_p} = \phi_{j_1,\dots,j_p}\frac{\partial x^{j_1}}{\partial x'^{i_1}}\dots\frac{\partial x^{j_p}}{\partial x'^{i_p}}.$$

- \triangleright For a given chart, the **exterior derivative** of a *p*-form is defined as in Eq. (1.22). The coordinate invariance of that definition pertains to manifolds, i.e. the definition of the exterior derivative does not depend on the chosen chart; given an atlas, $d\phi$ may be defined on the entire manifold.
- ▷ The **pullback** of a differential form under a smooth mapping between manifolds is defined as before. Again, pullback and exterior derivative commute.

The one mathematical concept whose generalization from open subsets of \mathbb{R}^n to manifolds requires some thought is Poincaré's lemma. As a generalization of our earlier definition of starshaped subsets of \mathbb{R}^n , we define the notion of 'contractible manifolds': A manifold M is called **contractible** contractible manifold if the identity mapping $M \to M, x \mapsto x$ may be continuously deformed to a constant map, $M \to M, x \mapsto x_0, x_0 \in M$ fixed. In other words, there has to exist a family of continuous mappings,

$$F: [0,1] \times M \rightarrow M, (t,x) \mapsto F(x,t),$$

such that F(x, 1) = x and $F(x, 0) = x_0$. For fixed x, F(x, t) defines a curve starting at x_0 and ending at x. (Exercise: show that \mathbb{R}^n is contractible, while $\mathbb{R}^n - \{0\}$ is not.)

Poincaré's lemma now states that on a contractible manifold a *p*-form is exact if and only if it is closed.

2.2.5 Lie derivative

A vector field implies the notion of 'transport' on a manifold. The idea is to trace the behavior of mathematical objects — functions, forms, vectors, etc. — as one 'flows' along the directions specified by the reference field. In this section, we define and explore the properties of the ensuing derivative operation, the Lie derivative.

The flow of a vector field

Given a vector field, v, we may attribute to each point $x \in M$ a curve whose tangent vector at x equals v_x . The union of all these curves defines the flow of the vector field.

More precisely, we wish to introduce a one-parameter group of diffeomorphisms,

$$\begin{split} \Phi : V &\to U, \\ (x,\tau) &\mapsto \Phi_{\tau}(x), \end{split}$$

where $\tau \in \mathbb{R}$ parameterizes the one-parameter group for fixed $x \in M$. Think of $\Phi(x, \tau)$ as a curve parameterized by τ . We parameterize this curve such that $\Phi_0(x) = x$. For each domain of a chart, U, and $x \in U$, we further require that $V = \operatorname{supp}(\Phi) \cap \{x\} \times \mathbb{R} = \{x\} \times \operatorname{interval} \ni \{(x,0)\}$, i.e. for each x the parameter interval of the one-parameter group is finite. Φ is a group of diffeomorphisms in the sense that

$$\Phi_{\tau+\tau'}(x) = \Phi_{\tau}(\Phi_{\tau'}(x)).$$

Each map Φ defines a vector field $v \in vect(M)$,

$$\begin{array}{rcl} v: M & \to & TM, \\ x & \mapsto & (x, d_\tau \big|_{\tau=0} \Phi_\tau(x)), \end{array}$$

i.e. x is mapped onto the tangent vector of the curve $\Phi_{\tau}(x)$ at $\tau = 0$. Conversely, each vector field v defines a one-parameter group of diffeomorphisms, the **flow of a vector field**. The flow — graphically, the trajectories traced out by a swarm of particles whose velocities at x(t) equal $v_{x(t)}$ — is defined by (see the figure above)

$$\forall x \in M : d_{\tau} \Phi_{\tau}(x) \stackrel{!}{=} v_{\Phi_{\tau}(x)}.$$

Here, we interpret $\Phi_{\tau} : \mathcal{C}^{\infty}(M) \to \mathcal{C}^{\infty}(M), f \mapsto \Phi_{\tau}(f)$ as a map between functions on M, where $(\Phi_{\tau}(f))(x) \equiv f(\Phi_{\tau}(x))$. The equation above then reads as $(d_{\tau}\Phi_{\tau})(f) = d_{\tau}f(\Phi_{\tau}) = \frac{\partial f}{\partial x^{i}}d_{\tau}\Phi_{\tau}^{i} = \frac{\partial f}{\partial x^{i}}v^{i}$. With a local decomposition $v = v^{i}\frac{\partial}{\partial x^{i}}$ this translates to the set of first order ordinary differential equations,

$$i = 1, \ldots, n: d_{\tau} \Phi^i_{\tau}(x) \stackrel{!}{=} v^i_{\Phi_{\tau}(x)}$$

Together with the initial condition $\Phi_0^i(x) = x^i$, we obtain a uniquely solvable problem (over at least a finite parameter interval of τ .)

EXAMPLE Let $M = \mathbb{R}^n$ and $v_x = Ax$, where $A \in GL(n)$. The flow of this vector field is $\Phi: M \times \mathbb{R} \to M, (x, \tau) \mapsto \Phi_{\tau}(x) = \exp(A\tau)x$.

Lie derivative of forms

Given a vector field and its flux one may ask how a differential form $\phi \in \Lambda^p M$ changes as one moves along its flux lines. The answer to this question is provided by the so-called Lie derivative. The Lie derivative compares $\phi_x \in \Lambda^p(T_x M)^*$ with the pullback of $\phi_{\Phi_\tau(x)} \in \Lambda^p(T_{\Phi_\tau(x)})^*$ under Φ_τ^* (where we interpret $\Phi_\tau : U \to U$ as a diffeomorphism of an open neighbourhood $U \ni x$.) The rate of change of ϕ along the flux lines is described by the differential quotient $\lim_{\tau}^{-1}(\Phi_\tau^*(\phi_{\Phi_\tau(x)}) - \phi_x)$, the Lie derivative of ϕ in the direction of v.

Formally, the Lie derivative is a degree-conserving map,

$$L_{v}: \Lambda^{p}M \rightarrow \Lambda^{p}M,$$

$$\phi \mapsto L_{v}\phi,$$

$$L_{v}\phi = d_{\tau}|_{\tau=0}(\Phi_{\tau}^{*}(\phi_{\Phi_{\tau}(x)}).$$
(2.3)

Properties of the Lie derivative (all immediate consequences of the definition):

- $\triangleright L_v$ is linear, $L_v(w+w') = L_vw + L_vw'$, and
- \triangleright obeys the Leibniz rule, $L_v(\phi \land \psi) = (L_v \phi) \land \psi + \phi \land L_v \psi$.
- $\triangleright\,$ It commutes with the exterior derivative $dL_v=L_vd$ and
- \triangleright L_v is linear in v.

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▷ for $f \in L^0M$ a function, $L_v f = dfv$ reduces to the directional derivative, $L_v f = v(f)$. This can also be written as $L_v f = d_\tau |_{\tau=0} f \circ \Phi_\tau$.

In practice, the computation of Lie derivatives by the formula (2.3) is cumbersome: one first has to compute the flow of the field v, then its pullback map, and finally differentiate w.r.t. time. Fortunately there exists an alternative prescription due to Cartan that is drastically more simple:

$$L_v = i_v \circ d + d \circ i_v. \tag{2.4}$$

In words: to compute the Lie derivative, L_v , of arbitrary forms, one simply has to apply an exterior derivative followed by insertion of v (plus the reverse sequence of operations.) Now, this looks like a manageable operation. Let us sketch the proof of Eq. (2.4). One first checks that the operations on both the left and the right hand side of the equation are derivations. Consequently, it is sufficient to prove the equality for zero forms (functions) and one-forms, dg. (In combination with the Leibniz rule, the expandability of an arbitrary form into wedge products of these building blocks then implies the general identity.)

For functions (cf. the list of properties above), we have $L_v f = df(v) = i_v df = (i_v d + di_v)f$, where we used that $i_v f = 0$ by definition. For one-forms, dg, we obtain

$$L_v dg = d(L_v g) = d(i_v dg) = (di_v + i_v d)dg$$

This proves Eq. (2.4).

Lie derivative of vector fields

A slight variation of the above definitions leads to a derivative operation on vector fields: Let v be a vector field and Φ its flux. Take another vector field w. We may then compare w_x with the image of $w_{\Phi_{\tau}(x)}$ under the tangent map $(T\Phi_{-\tau})_{\Phi_{\tau}(x)}: T_{\Phi_{\tau}(x)}M \to T_xM$. This leads to the definition of the Lie derivative of vector fields,

$$L_v : \operatorname{vect}(M) \to \operatorname{vect}(M),$$

 $w \mapsto L_v w,$

$$(L_v w)_x = d_\tau \Big|_{\tau=0} (T \Phi_{-\tau})_{\Phi_\tau(x)} (w_{\Phi_\tau(x)}).$$
(2.5)

The components of the vector field $L_v w = (L_v w)^i \frac{\partial}{\partial x^i}$ may be evaluated as

$$(L_v w)^i = d_\tau \big|_{\tau=0} (T\Phi_{-\tau})_{\Phi_\tau(x)} (w_{\Phi_\tau(x)})(x^i) = d_\tau \big|_{\tau=0} \frac{\partial \Phi_{-\tau}^i}{\partial x^k} w_{\Phi_\tau(x)}^k$$
$$= \left(d_\tau \big|_{\tau=0} \frac{\partial \Phi_{-\tau}^i}{\partial x^k} \right) w^k + d_\tau \big|_{\tau=0} w_{\Phi_\tau(x)}^i = -\frac{\partial v^i}{\partial x^k} w_k + \frac{\partial w^i}{\partial x^k} v^k.$$

We thus arrive at the identification

$$L_v w = \left(\frac{\partial w^i}{\partial x^k} v^k - \frac{\partial v^i}{\partial x^k} w_k\right) \frac{\partial}{\partial x^i}.$$
(2.6)

Eq. (2.6) implies an alternative interpretation of the Lie derivative: Application of v to a function $f \in \mathcal{C}^{\infty}(M)$ produces another function $v(f) \in \mathcal{C}^{\infty}(M)$. To this function we may apply the vector field w to produce the function $w(v(f)) \equiv (wv)(f)$. Evidently, the formal combination wv acts as a 'derivative' operator on the function f. It is, however, not a derivation. To see this, we choose a coordinate representation and obtain

$$(wv)(f) = w^{i} \frac{\partial}{\partial x^{i}} v^{j} \frac{\partial}{\partial x^{j}} f = w^{i} \frac{\partial v^{j}}{\partial x^{i}} \frac{\partial}{\partial x^{j}} f + w^{i} v^{j} \frac{\partial^{2}}{\partial x^{i} \partial x^{j}} f.$$

The presence of second order derivatives signals that wv is not a linear derivative operation. However, consider now the skew symmetric combination

$$vw - wv \equiv [v, w].$$

Application of this combination to f obtains:

$$(vw - wv) = \left(\frac{\partial w^i}{\partial x^k}v^k - \frac{\partial v^i}{\partial x^k}w_k\right)\frac{\partial}{\partial x^i}f.$$

The second order derivatives have canceled out, which signals that [v, w] is a vector field; the space of vector fields admits a product operation to be explored in more detail below. Second, our result above implies the important identification

$$[v,w] = L_v w. (2.7)$$

EXAMPLE Consider the vector field $v = x^1 \partial_{x^2} - x^2 \partial_{x^1}$. The flow of this field is given by the (linear) map: $\Phi_{\tau}(x) = O_{\tau}x$, where the matrix

$$O_{\tau} = \begin{pmatrix} \cos \tau & -\sin \tau \\ \sin \tau & \cos \tau \end{pmatrix}$$

Now, consider the constant vector field $w = \partial_{x^1}$. With $w_x^i = \delta_{i,1}$, we obtain

$$L_v w = (L_v w)^i \frac{\partial}{\partial x^i} = d_\tau \big|_{\tau=0} \frac{\partial (O_{-\tau} x)^i}{\partial x^j} w^j \frac{\partial}{\partial x^i} = d_\tau \big|_{\tau=0} \frac{\partial (O_{-\tau} x)^i}{\partial x^1} \frac{\partial}{\partial x^i} = -\frac{\partial}{\partial x^2}.$$

Notice that $L_v w \neq 0$, even at the origin where $\Phi_t 0 = 0$ is stationary.

PHYSICS (M) Consider the cotangent bundle, TM^* of a manifold M of generalized coordinates $\{q^i\}$. The bundle TM^* is parameterized by coordinates $\{q^i, p_i\}$ where the canonical momenta correspond to a Hamiltonian H = H(q, p).

Now, consider the so-called symplectic two-form $\omega = \sum_i dq^i \wedge dp_i$. A the existence of a twoform on the vector spaces $T_{(q,p)}TM^*$. (Don't be afraid of the accumulation of tangent/co-tangent structures! Just think of TM^* as a manifold with coordinates $\{(q^i, p_i)\}$ and $T_{(q,p)}TM^*$ its tangent space at (q, p).) enables us to switch between $T_{(q,p)}TM^*$ and its dual space $(T_{(q,p)}TM^*)^*$, i.e. the space of one-forms on $T_{(q,p)}TM^*$. We apply this correspondence to the particular one-form $dH \in T(TM^*)^*$: define a vector field $X_H \in T(TM^*)$ by the condition

$$\omega(X_H, \cdot) \equiv dH(\cdot). \tag{2.8}$$

In words: substitution of a second vector, $Y \in T(TM^*)$, into $(\omega(X_H, Y)$ gives the same as evaluating dH(Y). The vector field X_H is called the **Hamiltonian vector field** of H.

2.2 Tangent space

The Hamiltonian vector field represents a very useful concept in the description of mechanical systems. We first note that the **flow of the Hamiltonian vector field** represents describes the mechanical trajectories of the system. We may check this by direct evaluation of (2.8) on the vectors $\frac{\partial}{\partial q^i}$ and $\frac{\partial}{\partial p_i}$. Decomposing $X_H = (Q_H, P_H)$ into a 'coordinate' and a 'momentum' sector, we obtain

$$\omega(X_H, \frac{\partial}{\partial q^i}) = dq^j \wedge dp_j(X_H, \frac{\partial}{\partial q^i}) = -(P_H)_i \stackrel{!}{=} dH(\frac{\partial}{\partial q^i}) = \frac{\partial H}{\partial q^i}$$

In a similar manner, we get $(Q_H)^i = \frac{\partial H}{\partial p_i}$. Thus, the components of the Hamiltonian flow $\Phi \equiv (\Phi_Q, \Phi_P)$ obey the equations

$$d_t(\Phi_Q)^i = (Q_H)^i = \frac{\partial H}{\partial p_i},$$

$$d_t(\Phi_P)_i = (P_H)_i = -\frac{\partial H}{\partial q^i},$$
(2.9)

which we recognize as Hamiltons equations of motion.

Many fundamental statements of classical mechanics can be concisely expressed in the language of Hamiltonian flows and the symplectic two form. For example, the time evolution of a function in phase space is described by $f_t(x) \equiv f(\Phi_t(x))$, where we introduced the shorthand notation x = (q, p), and assumed the absence of explicit time dependence in H for simplicity. In incremental form, this assumes the form $d_t f_t(x) = df(d_t \Phi_t(x)) = df(X_H) = X_H(f) = L_{X_H} f$. For example, the statement of the **conservation of energy** assumes the form $d_t H_t(x) = d_H(X_H) = \omega(X_H, X_H) = 0$.

Phase space flow maps regions in phase space onto others and this concept is very powerful in the description of mechanical motion. By way of example, consider a subset $A \subset TM^*$. We interpret A as the set of 'initial conditions' of a large number, N, of point particles, where we assume for simplicity, that these particles populate A at constant density N/vol(A). (Notice that we haven't defined yet, what the 'volume' of A is (cf. the figure.) The flow will transport the volume A to another one, $\Phi_{\tau}(A)$. One may now ask, in which



way the density of particles changes in the process: will the 'evaporate' to fill all of phase space? Or may the distribution 'shrink' down to a very small volume? The answer to these questions is given by **Liouville's theorem**, stating that the density of phase space points remains 'constant'. In other words, the 'volume' of A remains constant under phase space evolution.⁴

To properly formulate Liouville's theorem, we first need to clarify what is meant by the 'volume' of A. This is easy enough, because phase space comes with a **canonical volume form**. Much like a metric induces a volume form, the symplectic two-form ω does so two: define

$$\Omega \equiv \frac{1}{n!} \underbrace{\omega \wedge \cdots \wedge \omega}_{n} = S \sum_{i} dq^{i} \wedge dp_{i},$$

where S is an inessential sign factor. The volume of A is then defined as

$$\operatorname{vol}(A) = \int_A \Omega.$$

⁴ Notice that nothing is said about the *shape* of *A*. If the dynamics is sufficiently wild (chaotic) an initially regular *A* may transform into a ragged object, whose filaments cover all of phase space. In this case, particles do get scattered over phase space. Nonetheless they stay confined in a structure of constant volume.

Liouville's theorem states that

$$\int_A \Omega = \int_{\Phi_\tau(A)} \Omega.$$

Using Eq. 1.40, this is equivalent to $\int_A \Omega = \int_A \Phi_\tau^* \Omega$, or to the condition $\Phi_\tau^* \Omega_{\Phi_\tau(x)} = \Omega_x$. Comparing with the definition (2.3), we may reformulate this as a vanishing Lie derivative, $L_{X_H} \Omega = 0$. Finally, using the Leibniz property (i.e. the fact that the Lie-derivative separately acts on the ω -factors constituting Ω , we conclude that

Liouville's theorem is equivalent to a vanishing of the Lie derivative $L_{X_H}\omega = 0$ of the symplectic form in the direction of the Hamiltonian flow.

The latter statement is proven by straightforward application of Eq. (2.4): $L_{X_H}\omega = i_{X_H}d\omega + di_{X_H}\omega = 0 + 0 = 0$, where the first 0 follows from the $d\omega = 0$ and the second from $di_{X_H}\omega \stackrel{(2.8)}{=} ddH = 0$.

The discussion above heavily relied on the existence of the symplectic form, ω . In general, a manifold equipped with a two-form that is skew symmetric and non-degenerate is called a **symplectic manifold**. Many of the niceties that came with the existence of a scalar product also apply in the symplectic case (think of the existence of a canonical mapping between tangent and cotangent space, or the existence of a volume form.) However, the significance of the symplectic form to the formulation of classical mechanics goes much beyond that, as we exemplified above.

What remains to be show is that phase space actually is a symplectic manifold: our introduction of ω above was ad hoc and tied to a specific system of coordinates. To see, why any cotangent bundle is symplectic, consider the 'projection' $\pi : T(TM^*)$

2.2.6 Orientation

As with the open subsets of \mathbb{R}^n discussed above, an orientation on a manifold may be introduced by defining a no-where vanishing *n*-form ω . However, not for every manifold can such *n*-forms be defined, i.e. not every manifold is orientable (the Moebius strip being a prominent example of a non-orientable manifold.)

Given an orientation (provided by a no-where vanishing *n*-form), a chart (U, α) is called positively oriented, if the corresponding coordinate frame obeys $\omega_x(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}) > 0$, or, equivalently, $\omega = f dx^1 \wedge \cdots \wedge dx^n$ with a positive function f. An atlas containing oriented charts is called oriented. Orientation of an atlas is equivalent to the statement that for any two sets of overlapping coordinate systems $\{x^i\}$ and $\{y^i\}$, $\det(\partial x^i/\partial y^j) > 0$. (Exercise: show that for the Moebius strip no orientable atlas exists.)







Figure 2.3 On the definition of manifolds with boundaries

2.2.7 Manifolds with boundaries

An *n*-dimensional manifold is an object locally looking like an open subset of \mathbb{R}^n . Replacing \mathbb{R}^n by the half space

$$\mathbb{H}^n \equiv \{x = (x^1, \dots, x^n\} \in \mathbb{R}^n | x^n \ge 0\},\$$

we obtain what is called a **manifold with boundary**. The boundary of M, ∂M , is the set of all points mapping onto the boundary of \mathbb{H} , $\partial \mathbb{H}^n = \{x = (x^1, \dots, x^n) \in \mathbb{R}^n | x^n = 0\}$,

$$\partial M = \bigcup_r \alpha_r^{-1}(\alpha_r(M) \cap \partial \mathbb{H}^n),$$

where the index r runs over all charts of an atlas. (One may show that this definition is independent of the chosen atlas.)

EXAMPLE Show that the unit ball $B^n = \{x = (x^1, \dots, x^n) \in \mathbb{R}^n | (x^1)^2 + \dots + (x^n)^2 \le 1\}$ in *n*-dimensions is a manifold whose boundary is the unit sphere S^{n-1} .

With the above definitions, ∂M is (a) a manifold of dimensionality n-1 which (b) is boundaryless, $\partial \partial M = \{\}$. As with the boundary of cells discussed earlier, the boundary ∂M of a manifold inherits an orientation from the bulk, M. To see this, let $x \in \partial M$ be a boundary point and $v = v^i \frac{\partial}{\partial x^i} \in T_x M$ be a tangent vector. If $v^n = 0$, $v \in T_x(\partial M)$ is tangent to the boundary (manifold). If $v^n < 0$, v is called a **outward normal vector**. (Notice, however, that 'normal' does not imply 'orthogonality'; we are not using a metric yet.)

With any outward normal vector n, the (n-1)-form $\tilde{\omega} \equiv i_n \omega \in \Lambda^{n-1} \partial M$ then defines an orientation of the boundary manifold.

2.2.8 Integration

Partition of unity

An atlas $\{(U_{\alpha}, \alpha)\}$ of a manifold M is called locally finite, if every $x \in M$ possesses a neighbourhood such that the number of charts with $U_{\alpha} \cap U \neq \{\}$ is finite. Locally finite atlases always exist.

A partition of unity subordinate to a locally finite covering of M is a family of mappings h_{α} with the following properties:

(i)
$$\forall x \in M, h_{\alpha}(x) \ge 0$$
,
(ii) $\operatorname{supp}(h_{\alpha}) \subset U$

(ii)
$$\operatorname{supp}(h_{\alpha}) \subset U_{\alpha}$$
,

(iii) $\forall x \in M, \sum_{\alpha} h_{\alpha}(x) = 1.$

(Due to the local finiteness of the covering, the sum in 3. contains only a finite number of terms.)

EXAMPLE Let $\{B_{\alpha} | \alpha \in I\}$ be a countable set of unit balls centered at points $x_{\alpha} \in \mathbb{R}^{n}$ covering \mathbb{R}^{n} . Define the functions

$$f_{\alpha} \equiv \begin{cases} \exp(-(1 - |x - x_{\alpha}|^2)^{-1}) & , |x - x_{\alpha}| \le 1, \\ 0 & , \text{ else} \end{cases}$$

Then the functions

$$h_{\alpha}(x) \equiv \frac{f_{\alpha}(x)}{\sum_{\beta} f_{\beta}(x)}$$

define a partition of unity in \mathbb{R}^n .

Integration

Let M be a manifold (with or without boundary), $\{(U_{\alpha}, \alpha)\}$ a locally finite atlas, $\{h_{\alpha}\}$ a partition of unity and $\phi \in \Lambda^n M$ an *n*-form. For an integration domain $U \subset U_{\alpha}$ contained in a single chart domain, the integral over ϕ is defined as

$$\int_U \phi = \int_{\alpha(U)} \alpha^{-1*} \phi,$$

where the second integral is evaluated according to our earlier definition of integrals over open subsets of \mathbb{R}^n . If U is not contained in a single chart, we define

$$\int_U \phi = \sum_\alpha \int_{U \cap U_\alpha} h_\alpha \phi.$$

One may show that the definition does not depend on the reference partition of unity.

Finally, Stokes theorem assumes the form

$$\phi \in \Lambda^{p-1}M: \ \int_M d\phi = \int_{\partial M} \phi.$$

(For a manifold without boundary, the l.h.s. vanishes.)

2.3 Summary and outlook

Metric

A metric on a manifold is a non-degenerate symmetric bilinear form g_x on each tangent space $T_x M$ which depends smoothly on x (i.e. for two vector fields v_1, v_2 the function $g_x(v_1(x), v_2(x))$) depends smoothly on x.) In section 1.3.5 we introduced metric structures on open subsets of $U \subset \mathbb{R}^n$. Most operations relating to the metric, the canonical isomorphism $T_x U \xrightarrow{J} (T_x U)^*$, the Hodge star, the co-derivative, the definition of a volume form, etc. where defined locally. Thanks to the local equivalence of manifolds to open subsets of \mathbb{R}^n , these operations carry over to manifolds without any changes. (To globally define a volume form, the manifold must be orientable.)

There are but a few 'global' aspects where the difference between a manifold and an open subset of \mathbb{R}^n may play a role. While, for example, it is always possible to define a metric of any given signature on an open subset of \mathbb{R}^n , the situation on manifolds is more complex, i.e. a global metric of pre-designated signature need not exist.

2.3 Summary and outlook

In this section, we introduced the concept of manifolds to describe geometric structures that cannot be globally identified with open subsets of \mathbb{R}^n . Conceptually, all we had to do to achieve this generalization was to patch up the local description of a manifold – provided by charts and the ensuing differentiable structures – to a coherent global description. By construction, the transition from one chart to another is mediated by differentiable functions between subsets of \mathbb{R}^n , i.e. objects we know how to handle. In the next section, we will introduce a very important family of differentiable manifolds, viz. manifolds carrying a group structure.

Lie groups

In this chapter we will introduce Lie groups, a class of manifolds of paramount importance in both physics and mathematics. Loosely speaking, a Lie group is a manifold carrying a group structure. Or, changing the perspective, a group that is at the same time a manifold. The linkage of concepts from group theory and the theory of differential manifolds generates a rich mathematical structure. At the same time, Lie groups are the 'natural' mathematical objects to describe symmetries in physics, notably in quantum physics. In section In section 1.4 above, we got a first impression of the importance of Lie groups in quantum theory, when we saw how these objects implement the concept of gauge transformations. In this section, however, the focus will be on the mathematical theory of Lie groups.

3.1 Generalities

A (finite dimensional, real) Lie group, G, is a differentiable manifold carrying a group structure. One requires that the group multiplication, $G \times G \to G$, $(g,g') \mapsto gg'$ and the group inversion, $G \to G$, $g \mapsto g^{-1}$ be differentiable maps.

Notice that there are lots of mathematical features that may be attributed to a Lie group (manifold):

- ▷ manifold: dimensionality, compactness, conectedness, etc.
- ▷ group: abelian, simplicity, nilpotency, etc.

The joint group/manifold structure entails two immediate further definitions: a **Lie subgroup** $H \subset G$ is a subgroup of G which is also a sub-manifold. We denote by e the unit element of G and by G^e the **connected component** of G. The set G^e is a Lie subgroup of the same dimensionality as G (think why!)

A few elementary (yet important) examples of Lie groups:

- ▷ There are but two different connected one–dimensional Lie groups: the **real numbers** \mathbb{R} with its additive group structure is a simply connected non–compact abelian Lie group. The **unit** circle $\{z \in \mathbb{C} | |z| = 1\}$ with complex multiplication as group operation is a compact abelian non–simply connected Lie group (designated by U(1) or SO(2) depending on whether one identifies \mathbb{C} with \mathbb{R}^2 or not.)
- ▷ The general linear group, GL(n), i.e. the set of all real $(n \times n)$ -matrices with non-vanishing determinants is a Lie group. Embedded into \mathbb{R}^n it contains two connected components, $GL^+(n)$ and $GL^-(n)$, the set of matrices of positive and negative determinant, respectively

3.2 Lie group actions

(the former containing the group identity, $\mathbf{1}_n$.) $\operatorname{GL}(n)$ is non-compact, non-connected, and non-abelian.

- ▷ The orthogonal group, O(n) is the group of all real orthogonal matrices, i.e. $O(n) = \{O \in \operatorname{GL}(n) | O^T O = \mathbf{1}_n\}$. It is the maximal compact subgroup of $\operatorname{GL}(n)$. It is of dimension n(n-1)/2 and non-connected. Similarly, the special orthogonal group, $\operatorname{SO}(n) = \{O \in O(n) | \det O = 1\}$ is the maximal compact subgroup of $\operatorname{GL}^+(n)$. It is connected yet not simply connected. (Think of $\operatorname{SO}(2)$.)
- ▷ The special unitary group, SU(n) is the group of all *complex valued* $(n \times n)$ -matrices, U, obeying the conditions $U^{\dagger}U = \mathbf{1}_n$ and det U = 1. Alternatively, one may think of SU(n) as a *real* manifold viz. as a real subgroup of Gl(2n). It is of dimension $n^2 1$, compact, and simply connected.

By way of example, consider SU(2). We are going to show that SU(2) is isomorphic to the real manifold S^3 , the three sphere. To see this, write an element $U \in SU(2)$ (in complex representation) as

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The conditions $U^{\dagger}U = \mathbf{1}_2$ and $\det U = 1$ translate to $a\bar{a} + c\bar{c} = 1, b\bar{b} + d\bar{d} = 1, a\bar{b} + c\bar{d} = 0$, and ad - cb = 1. Defining $a = x^1 + ix^2$ and $c = x^3 + ix^4$, $x^1, \ldots, x^4 \in \mathbb{R}$, these conditions are resolved by

$$U = \begin{pmatrix} x^1 + ix^2 & -x^3 + ix^4 \\ x^3 + ix^4 & x^1 - ix^2 \end{pmatrix}, \qquad \sum_{i=1}^4 (x^i)^2 = 1.$$

This representation establishes an diffeomorphism between SU(2) and the three sphere, $S^3 = \{(x^1, x^2, x^3, x^4) \in \mathbb{R}^4 | \sum_{i=1}^4 (x^i)^2 = 1\}.$

3.2 Lie group actions

3.2.1 Generalities

Let G be a Lie group and M an arbitrary manifold. A (left) action¹ of G on M is a differentiable mapping,

$$\begin{array}{rcl} \rho: G \times M & \to & M, \\ (g, x) & \mapsto & \rho(g, x) \equiv \rho_a(x), \end{array}$$

assigning to each group element a smooth map $\rho_g: M \to M$. The composition of these maps must be compatible with the group structure multiplication in the sense that

$$\rho_{gg'} = \rho_g \circ \rho_{g'}, \qquad \rho_e = \mathrm{id}_M.$$

¹ Sometimes group actions are also called 'group representations'. However, we prefer to reserve that terminology for the linear group actions to be defined below.

Lie groups

EXAMPLE Let G = SO(3) be the three dimensional rotation group and $M = S^2$ the two-sphere. The group G acts on M by rotating the unit-vectors constituting S^2 . More generally, the isometries of a Riemannian manifold define a group acting on the manifold.

A **right action** of G on M is defined in the same manner, only that the compatibility relation reads

$$\rho_{gg'} = \rho_{g'} \circ \rho_g.$$

Notation: Instead of ρ_g we will occasionally write $\rho(g)$ or just g. For brevity, the left/right action of g on $x \in M$ is often designated as gx/xg

If ρ is a left action, then $g\mapsto \rho_{g^{-1}}$ defines a right action. For a fixed $x\in M,$ we define the map,

$$\operatorname{bit}_x : G \to M,$$

 $g \mapsto \operatorname{bit}_x g \equiv \rho_{g^{-1}} x.$ (3.1)

The **orbit** of x is the image of bit_x ,

$$\operatorname{orbit}(x) \equiv \operatorname{bit}_x(G).$$

If $\operatorname{orbit}(x) = M$, the action of the group is called **transitive**. (Exercise: why is it sufficient to prove transitivity for an arbitrary reference point?) For a transitive group action, two arbitrary points, $x, y \in M$ are connected by a group transformation, $y = \rho_g x$. An action is called **faithful** if there are no actions other than ρ_e acting as the identity transform: $(\forall x \in M : \rho_g x = x) \Rightarrow g = e$. It is called **free** iff bit_x is injective for all $x \in M$. (A free action is faithful.) The **isotropy group** of an element $x \in M$, is defined as

$$I(x) \equiv \{g \in G | \rho_g x = x\}.$$

The action is free iff the isotropy group of all $x \in M$ contains only the unit element, $I(x) = \{e\}$.

EXAMPLE The action of the rotation group SO(3) on the two–sphere S^2 is transitive and faithful, but not free. The isotropy group is SO(2).

3.2.2 Action of a Lie group on itself

A Lie group acts on itself, M = G, in a number of different and important ways. The action by **left translation** is defined by

$$\begin{array}{rccc} L_g:G & \to & G, \\ & h & \mapsto & gh, \end{array} \tag{3.2}$$

i.e. g acts by left multiplication. This representation is transitive and free. Second, it acts on itself by the **inner automorphism**,

$$\operatorname{aut}_g: G \to G,$$

$$h \mapsto ghg^{-1}.$$
 (3.3)

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In general, this representation is neither transitive nor faithful. The stability group, I(e) = G. Finally, the **right translation** is a right action defined by

$$\begin{array}{rccc} R_g:G & \to & G, \\ & h & \mapsto & hg. \end{array} \tag{3.4}$$

Right and left translation commute, and we have $\operatorname{aut}_q = L_q \circ R_{q^{-1}}$.

3.2.3 Linear representations

An action is called a **linear representation** or just representation if the manifold it acts upon is a vector space, M = V, and if all diffeomorphisms ρ_g are linear. Put differently, a linear representation is a group homeomorphism $G \to \operatorname{GL}(V)$. Linear representations are not transitive (think of the zero vector.) They are called **irreducible representations** if V does not possess ρ_q invariant subspaces other than itself (and space spanned by the zero vector.)

EXAMPLE The 'fundamental' representation of SO(3) on \mathbb{R}^3 is irreducible.

A group action is called an affine representation if it acts on an affine space and if all ρ_g are affine maps.²

Two more remarks on representations,

- $\triangleright \text{ Given a linear (or affine) representation, } \rho, \text{ a general left action may be constructed as } \rho'_g \equiv F \circ \rho_g \circ F^{-1}, \text{ where } F : V \to V \text{ is some diffeomorphism; } \rho'_g \text{ need no longer be linear.} \\ \text{Conversely, given a left action } \rho_g \text{ it is not always straightforward to tell whether } \rho_g \text{ is a linear representation } \rho'_g \text{ disguised by some diffeomorphism, } \rho_g = F \circ \rho_g \circ F^{-1}. \\ \end{cases}$
- ▷ Depending on the dimensionality of the vector space V, one speaks of a finite or an **infinite dimensional representation**. For example, given an open subset of $U \subset \mathbb{R}^n$ the Lie group GL(n) acts on the frame bundle of U — the infinite dimensional vector space formed by all frames — by left multiplication. This is an infinite dimensional representation of GL(n).

3.3 Lie algebras

3.3.1 Definition

Recall that a (finite or infinite) **algebra** is a vector space V equipped with a product operation, $V \times V \rightarrow V$. A **Lie algebra** is an algebra whose product (the bracket notation is standard for Lie algebras)

$$\begin{bmatrix} \ , \ \end{bmatrix} : V \times V \quad \to \quad V$$
$$(v, w) \quad \mapsto \quad [v, w]$$

satisfies certain additional properties:

 \triangleright [,] is bilinear,

 $^{^{2}}$ A map is called affine if it is the sum of a linear map and a constant map.

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 \triangleright skew-symmetric: $\forall v, w \in V, [v, w] = -[w, v]$, and satisfies the

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▷ Jacobi identity, $\forall u, v, w \in V, [u, [v, w]] + [w, [u, v]] + [v, [w, u]] = 0.$

EXAMPLE In section 2.2.5 we have introduced the Lie derivative Eq. (2.5) as a derivative operation on the infinite dimensional space of vector fields on a manifold, vect(M). Alternatively, we may think of the Lie derivative as a product,

$$(v, w) \mapsto L_v w = [v, w]$$

$$(3.5)$$

assigning to two vector fields a new one, [v, w]. The above product operation is called the Lie bracket of vector fields. Both, skew symmetry and the Jacobi identity are immediate consequences of Eq. (2.6). We thus conclude that vect(M) is an infinite dimensional Lie algebra, the Lie algebra of vector fields on a manifold.

3.3.2 Lie algebra of a Lie group

Let $A \in \text{vect}(G)$ be a vector field on a Lie group G. A is a **left invariant vector field** if it is invariant under all left translations,

$$\forall g, h \in G: \ (TL_q)_h A_h = A_{qh}$$

EXAMPLE The left invariant vector fields on the abelian Lie group \mathbb{R}^n are the constant vector fields.

Due to the linearity of TL_g , linear combinations of left invariant vector fields are again left invariant, i.e. the set of left invariant vector fields forms a linear space, here denoted by g.

However, as we are going to show below, \mathfrak{g} carries much more mathematical structure than just linearity. To see this, a bit of preparatory work is required: Let M be a manifold and $F: M \to M$ a smooth map. A vector field v on M is called **invariant** under the map F, if $\forall x \in M: TF_x v_x = v_{F(x)}$. Recalling that $TF_x v_x(f) = v_x(f \circ F)$, and that $v_{F(x)}(f) = (v(f) \circ F)(x)$ this condition may be rewritten as

$$\forall f \in \mathcal{C}^{\infty}(M) : \qquad v(f \circ F) = v(f) \circ F.$$

Now, consider two vector fields $A, B \in \mathfrak{g}$. Applying the above invariance criterion to $F = L_g$ and considering the two cases $v = A, f = B(\tilde{f})$, and $v = B, f = A(\tilde{f})$, where $\tilde{f} \in \mathcal{C}^{\infty}(G)$, we obtain

$$(AB)(f) \circ L_g = A(B(f)) \circ L_g = A(B(f) \circ L_g) = A(B(f \circ L_g)) = (AB)(f \circ L_g),$$

$$(BA)(\tilde{f}) \circ L_g = B(A(\tilde{f})) \circ L_g = B(A(\tilde{f}) \circ L_g) = B(A(\tilde{f} \circ L_g)) = (BA)(\tilde{f} \circ L_g).$$

Subtraction of these formulas gives

$$(AB - BA)(\tilde{f}) \circ L_q = (AB - AB)(\tilde{f} \circ L_q),$$

which shows that

$$A, B \in \mathfrak{g} \Rightarrow [A, B] \in \mathfrak{g},$$
i.e. that the space of left invariant vector fields, \mathfrak{g} , forms a Lie subalgebra of the space of vector fields. \mathfrak{g} is called the **Lie algebra** of G.

The left action of a Lie group on itself is transitive. Specifically, each element g may be reached by left multiplication of the unit element, $g = L_g e = ge$. As we shall see, this implies an isomorphism of the tangent space T_eG onto the Lie algebra of the group. Indeed, the left-invariance criterion implies that

$$A_g = (TL_g)_e A_e, (3.6)$$

i.e. the value of the vector field at arbitrary g is determined by its value in the tangent space at unity, T_eG . As a corollary we conclude that

- \triangleright The dimension of the Lie algebra, $\dim(\mathfrak{g}) = \dim(G)$, and
- \triangleright On g there exists a global frame, $\{B_i\}, i = 1, \dots, \dim(\mathfrak{g})$, where $(B_i)_g = (TL_g)_e E^i$ and $\{E^i\}$ is a basis of T_eG ; Lie group manifolds are parallelizable.

EXAMPLE By way of example, let us consider the Lie group GL(n). (Many of the structures discussed below instantly carry over to other classical matrix groups.) $GL(n) \subset \mathbb{R}^{n^2}$ is open in \mathbb{R}^{n^2} and can be represented in terms of a global set of coordinates. The standard coordinates are, of course, $x^{ij}(g)$, where x^{ij} is the *i*th row and *j*th column of the matrix representing *g*. A tangent vector at *e* can be represented as (remember: summation convention)

$$A_e = a^{ij} \frac{\partial}{\partial x^{ij}}$$

In order to compute the corresponding invariant vector field A_g , we first note that the L_g acts by left matrix–multiplication,

$$x^{ij}(L_gh) = x^{ik}(g)x^{kj}(h).$$

Using Eq. (1.14), we then obtain

$$\left((TL_g)_e A_e\right)^{ij} = \frac{\partial x^{ij}(L_g h)}{\partial x^{kl}(h)}\Big|_{h=e} a^{kl} = x^{ik}(g)a^{kl},$$

Or $A_g = gA_e$, where $g(A_e)$ is identified with the $n \times n$ matrices $\{x^{ij}(g)\}$ $(\{a^{ij}\})$ and matrix multiplication is implied. Another representation reads

$$A_g = x(g)^{ik} a^{kj} \frac{\partial}{\partial x^{ij}} \equiv x^{ik} a^{kj} \frac{\partial}{\partial x^{ij}}.$$

This latter expression may be used to calculate the Lie bracket of two left invariant vector fields,

$$[A,B] = x^{ij} a^{kj} \frac{\partial}{\partial x^{ij}} x^{lm} b^{mn} \frac{\partial}{\partial x^{ln}} - (a \to b) = x^{ik} (ab - ba)^{kj} \frac{\partial}{\partial x^{ij}}$$

where $\{b^{ij}\}$ are the matrix indices identifying B and ab is shorthand for standard matrix multiplication. This result (a) makes the left invariance of [A, B] manifest, and (b) shows that the commutator [A, B] simply obtains by taking the *matrix* commutator of the coordinate matrices [a, b], (more formally, the identification $A \rightarrow a = \{a^{ij}\}$ is a homomorphism of the Lie algebras '(left invariant vector fields, Lie bracket)' and '(ordinary matrices, matrix commutator)'.

When working with (left invariant) vector fields on Lie groups, it is often convenient to employ the 'equivalence classes of curves' definition of vector fields. For a given $A \in T_eG$, there are

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many curves $\gamma_A(t)$ tangent to A at t = 0: $\gamma_A(0) = e$, $d_t\Big|_{t=0} \gamma_A(t) = A$. Presently, however, it will be convenient to consider a distinguished curve, viz. $g_{A,t} \equiv \Phi_{A,t}(e)$, where $\Phi_t(e)$ is the flow of the left invariant vector field A_g corresponding to A. We note that the vector field A evaluated at $g_{A,s}$, $A_{g_{A,s}}$, affords two alternative representations. On the one hand, $A_{g_{A,s}} = T(L_{g_{A,s}})_e A_e = d_t\Big|_{t=0} g_{A,s} g_{A,t}$ and on the other hand, $A_{g_{A,s}} = d_s g_{A,s} = d_t\Big|_{t=0} g_{A,s+t}$. This implies that

$$g_{A,s}g_{A,t} = g_{A,s+t},$$

i.e. $\{g_{A,t}\}$ is a **one parameter subgroup** of G. Later on, we shall see that these subgroups play a decisive role in establishing the connection between the Lie algebra and the global structure of the group. Presently, we only note that the left invariant vector field A_g may be represented as (cf. Eq. (3.6))

$$A_g = d_t \big|_{t=0} gg_{A,t}$$

Put differently, the flow of the left invariant vector field³, $\Phi_{A,t}$ acts as

$$\begin{array}{rccc} \Phi_{A,t}:G & \to & G, \\ g & \mapsto & gg_{A,t} \end{array}$$

for $\Phi_{A,0} = \mathrm{id}_G$ and $d_t|_{t=0} \Phi_{A,t}(g) = A_g$, as required. This latter representation may be used to compute the Lie derivative of two left invariant vector fields A and B, $L_A B$. With $B_g = d_s|_{s=0} gg_{B,s}$, we have

$$(L_AB)_e = d_t \Big|_{t=0} (T\Phi_{-t})_{\Phi_{A,t}(e)} (B_{\Phi_{A,t}(e)}) = d_{s,t}^2 \Big|_{s,t=0} (T\Phi_{-t})_{g_{A,t}} g_{A,t} g_{B,s} = = d_{s,t}^2 \Big|_{s,t=0} g_{A,t} g_{B,s} g_{A,-t}.$$

Using that $g_t g_{-t} = g_{t-t} = g_0 = e$, i.e. that $g_{-t} = g_t^{-1}$, we conclude that the Lie derivative of two left invariant vector fields is given by

$$(L_A B)_e = d_{s,t}^2 \Big|_{s,t=0} g_{A,t} g_{B,s} g_{A,t}^{-1}.$$
(3.7)

3.4 Lie algebra actions

3.4.1 From the action of a Lie group to that of its algebra

Let $F_a: M \to M$, $a \in \mathbb{R}$ be a one parameter family of diffeomorphisms, smoothly depending on the parameter a. Assume that $F_0 = \mathrm{id}_M$. For a infinitesimal, we may write,

$$F_a(x) \simeq x + a \frac{\partial}{\partial a} \Big|_{a=0} F_a(x) + \mathcal{O}(a^2).$$

³ Notice that $g_{A,t}$ was constructed from the flow through the origin whilst we here define the global flow.

This shows that, asymptotically for small a, F_a may be identified with a vector field (whose components are given by $\partial_a|_{a=0}x^i(F_a(x))$). Two more things we know are that (a) the infinitesimal variant of Lie group elements (the elements of the tangent space at unity) constitute the Lie algebra, and (b) Lie group actions map Lie group elements into diff(M). Summarizing

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$$\begin{array}{ccc} G & \stackrel{\text{repr.}}{\longrightarrow} & \text{diff}(M), \\ \\ \text{infinit.} & \downarrow & \downarrow & \text{infinit.} \\ \\ \mathfrak{g} & \stackrel{?}{\longrightarrow} & \text{vect}(M). \end{array}$$

The diagram suggests that there should exist an 'infinitesimal' variant of Lie group representations mapping Lie algebra elements onto vector fields. Also, one may expect that this mapping is a Lie algebra homomorphism, i.e. is compatible with the Lie bracket.

Identifying $\mathfrak{g} = T_e G$, the representation of the Lie algebra, i.e. an assignment $\mathfrak{g} \ni A \mapsto v \in$ vect(M) may be constructed as follows: consider $x \in M$. The group G acts on x as $x \mapsto \rho_g(x)$. We may think of $\rho_{\bullet}(x) : G \to M, g \mapsto \rho_g(x)$ as a smooth map from G to M. Specifically, the unit element e maps onto x. Thus, the tangent mapping $T\rho_{\bullet}(x)$ maps \mathfrak{g} into $T_x M$. For a given Lie group representation ρ we thus define the induced **representation of the Lie algebra** as

$$\tilde{\rho} : \mathfrak{g} \to \operatorname{vect}(M),$$
$$A \mapsto \tilde{\rho}_A,$$
$$(\tilde{\rho}_A)_x = (T\rho_{\bullet}(x))_e(A).$$

INFO It is an instructive exercise to show that $\tilde{\rho}$ is a Lie algebra (anti) homomorphism. Temporarily denoting $\tilde{\rho}_A \equiv \tilde{A}$, what we need to check is that $\widetilde{[A,B]} = [\tilde{A},\tilde{B}]$. Again, it will be convenient to work in the curve representation of vector fields. With $A = d_t|_{t=0}g_{A,t}$ and $B = d_s|_{s=0}g_{B,s}$, we have $\tilde{A}_x = d_t|_{t=0}g_{A,t}x$, where we denote the group action on M by $\rho_g(x) \equiv gx$. Similarly, the flow of the vector field \tilde{A} is given by $\Phi_{\tilde{A},t}(x) = g_{A,t}x$. We may now evaluate the Lie bracket of the image vector fields as

$$\begin{split} [\tilde{A}, \tilde{B}]_x &= (L_{\tilde{A}}\tilde{B})(x) = d_t \big|_{t=0} (T\Phi_{\tilde{A}, -t})_{\Phi_{\tilde{A}, t}(x)} \tilde{B}_{\Phi_{\tilde{A}, t}(x)} = d_{s, t}^2 \big|_{s, t=0} g_{A, -t} g_{B, s} g_{A, t}(x) = \\ &= -d_{s, t}^2 \big|_{s, t=0} g_{A, t} g_{B, s} g_{A, -t} x. \end{split}$$

Comparison with (3.7) shows that $[\tilde{A}, \tilde{B}]_x = -\widetilde{L_AB}_x = -\widetilde{[A, B]}_x$, i.e. $\mathfrak{g} \to \operatorname{vect}(M)$ is a Lie algebra anti (the sign) homomorphism.

3.4.2 Linear representations

Let $\rho : G \to \operatorname{GL}(V) \subset \operatorname{diff}(V)$ be a linear representation of a Lie group. To understand what vector fields describe the Lie algebra, let $A \in T_eG$ be a Lie algebra element and $g_{A,t}$ be a representing curve. The representation $\rho_g(v) \equiv M_g v$ maps group elements g onto linear transformations $M_g \in \operatorname{GL}(V)$. In a given basis, M_g is represented by an $(n \times n)$ -matrix $\{M_{qj}^{ij}\}$.

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Specifically, the generating curve $g_{A,t}\mapsto M_{g_{A,t}}$ is represented by a matrix valued curve. We thus find

$$(\tilde{\rho}_A)_v = d_t \big|_{t=0} M_{g_{A,t}} v.$$

The vector field $(\tilde{\rho}_A)_v$ depends linearly on v. Defining $X_A \equiv d_t |_{t=0} M_{g_{A,t}} \in \mathrm{gl}(V)$ (X_A is an element of the Lie algebra, $\mathrm{gl}(V)$ of the group $\mathrm{GL}(V)$), we may write have $(\tilde{\rho}_A)_v \equiv X_A v$.

Adjoint representation

In section 3.2.2 we have seen that a Lie group acts on itself by the inner automorphism, aut, where $\operatorname{aut}_g(h) = ghg^{-1}$. Associated to this action we have a linear representation of G on \mathfrak{g} , the **adjoint representation**, Ad, of the group on its Lie algebra:

$$\begin{array}{rcl} \mathrm{Ad}:G\times\mathfrak{g}&\to&\mathfrak{g},\\ (g,A)&\mapsto&\mathrm{Ad}_g(A)\equiv(T\mathrm{aut}_g)_eA, \end{array}$$

where we identify $A \in T_e G$ as an element of the tangent vector space. Since $\operatorname{aut}_g(e) = e$, the image of A, $(\operatorname{Taut}_g)_e A \in T_e G$ is again in the Lie algebra. With $g_{A,t}$ a curve representing A, we have $\operatorname{Ad}_g(A) = d_t \big|_{t=0} gg_{A,t}g^{-1}$.

The corresponding linear representation of the Lie algebra is denoted the **adjoint representation of the Lie algebra**, $\tilde{Ad} \equiv ad$. The adjoint representation is a representation of the Lie algebra on itself. According to our previous discussion, we have

$$\operatorname{ad}_{A}(B) = d_{t,s}^{2} \Big|_{t,s=0} g_{A,t} g_{B,s} g_{A,t}^{-1} = [A, B].$$

The result

$$ad_A(B) = [A, B]$$
(3.8)

plays a pivotal role in the representation theory of Lie groups. Let $\{T_a\}$ be a basis of the Lie algebra. The expansion coefficients f_{abc} of $[T_a, T_b]$,

$$[T_a, T_b] \equiv f_{abc} T_c \tag{3.9}$$

are called the **structure constants** of the Lie algebra. Two Lie algebras are isomorphic, if they share the same structure constants.

3.5 From Lie algebras to Lie groups

Above, we have seen how plenty of structure information is encoded in the Lie algebra \mathfrak{g} . In this final section, we will show that this information actually suffices to recover the structure of the whole group G, at least in some vicinity of the unit element. The section contains the mathematics behind the physicists' strategy to generate a global transformation (an element of the Lie group or one of its actions) out of infinitesimal transformations, or transformation 'generators' (an element of the Lie algebra or one of its actions.)

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3.5.1 The exponential mapping

In section 3.3.2 we have introduced the flow $g_{A,t}$ of a left invariant vector field A through the origin. We now define the map

$$\exp: T_e G \quad \to \quad G,$$

$$A \quad \mapsto \quad \exp(\mathbf{A}) \equiv g_{A,1}.$$
(3.10)

Let us try to understand the background of the denotation 'exp'. We first note that for $s \in \mathbb{R}$, $g_{sA,t} = g_{A,st}$. Indeed, $g_{sA,t}$ solves the differential equation $d_tg_{sA,t} = sAg_{sA,t}$ with initial condition $g_{sA,0} = e$. However, by the chain rule, $d_tg_{A,st} = sd_{st}g_{A,st} = sAg_{A,st}$ (with initial condition $g_{A,s0} = e$.) Thus, $g_{A,st}$ and $g_{sA,t}$ solve the same first order initial value problem which implies their equality. Using the homogeneity relation $g_{sA,t} = g_{A,st}$, we find that

$$\exp(sA)\exp(tA) = g_{sA,1}g_{tA,1} = g_{A,s}g_{A,t} = g_{A,s+t} = g_{(s+t)A,1} = \exp((s+t)A),$$

i.e. the function 'exp' satisfies the fundamental relation of exponential functions which explains its name. The denotation exp hints at another important point. Defining monomials of Lie algebra elements A^n in the obvious manner, i.e. through the *n*-fold application of the vector A, we may tentatively try the power series representation

$$\exp(A) = \sum_{n=0}^{\infty} \frac{1}{n!} A^n.$$
 (3.11)

To see that the r.h.s. of this equation indeed does the job, we use that $\exp(tA) = g_{t,A}$ must satisfy the differential equation $d_t \exp(tA) = \exp(tA)A = A_{\exp(tA)}$. It is straightforward to verify that the r.h.s. of Eq. (3.11) solves this differential equation and, therefore, appears to faithfully represent the exponential function.⁴

INFO We are using the cautious attribute 'appears to' because the interpretation of the r.h.s. of the power series representation is not entirely obvious. A priori, monomials A^n neither lie in the Lie algebra, nor in the group, i.e. the actual meaning of the series requires interpretation. In cases, where $G \subset GL(n)$ is (subset of) the matrix group GL(n) no difficulties arise: certainly, A^n is a matrix and det $exp(A) = exp(\ln det exp(A)) = exp tr \ln exp(A) = exp tr(A) \neq 0$ is non-vanishing, i.e. $exp(A) \in GL(n)$ as required. For the general interpretation of the power series interpretation, we refer to the literature.

3.5.2 Normal coordinates

The exponential map provides the key to 'extrapolating' from local structures (Lie algebra) to global ones (Lie group). Let us quote a few relevant facts:

▷ In general, the exponential map is neither injective, nor surjective. However, in some open neighbourhood of the origin, exp defines a diffeomorphism. This feature may be used to define

⁴ Equivalently, one might have argued that the fundamental relation $\exp(x) \exp(y) = \exp(x + y)$ implies the power series representation.

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a specific set of local coordinates, the so-called **normal coordinates**: Assume that a basis of T_eG has been chosen. The normal coordinates of $\exp(tA)$ are then defined by $\exp(tA)^i \equiv tA^i$, where A^i are the components of $A \in T_eG$ in the chosen basis. (For a proof of the faithfulness of this representation, see the info block below.)

▷ There is one and only one simply connected simply connected Lie group G with $\text{Lie}(G) = \mathfrak{g}$. For every other connected Lie group H with $\text{Lie}(H) = \mathfrak{g}$, there is a group homomorphism $G \to H$ whose kernel is a discrete subgroup of H. (Example: $\mathfrak{g} = \mathbb{R}$, $G = \mathbb{R}$, $H = S^1$, with kernel $\mathbb{Z} \times (2\pi)$.) G is called the **universal covering group** of H.

INFO We wish to prove that, locally, exp defines a diffeomorphism. To this end, let $g^i(g)$ be an arbitrary coordinate system and $\eta^i(g)$ be the normal coordinates. The (local) faithfulness of the latter is proven, once we have shown that the Jacobi matrix $\frac{\partial g^i}{\partial \eta^j}$ is non-singular at the origin. Without loss of generality, we assume that the basis spanning T_eG is the basis of coordinate vectors $\frac{\partial}{\partial g^i}$ of the reference system.

Thus, let $\eta^i \equiv tA^i$ be the normal coordinates of some group element g. Now consider the specific tangent vector $B \equiv \frac{\partial}{\partial t}\Big|_{t=0}g \in T_eG$. Its normal coordinates are given by $B_n^i = \frac{\partial}{\partial t}\eta^i = A^i$. By definition, the components of the group element g in the original system of normal coordinates will be given by $g^i(g) = \exp^i(tA)$. The components of the vector B are given by $B^i = d_t\Big|_{t=0} \exp^i(tA) = d_t\Big|_{t=0}g^i_{t,tA} = d_t\Big|_{t=0}g^i_{t,tA} = A^i$. Thus, $B^i = B_n^i$ coincide. At the same time, by definition, $B^i = \frac{\partial g^i}{\partial \eta^j}B_n^j$, implying that $\frac{\partial g^i}{\partial \eta^j} = id$ has maximal rank, at least at the origin. By continuity, the coordinate transformation will be non-singular in at least an open neighbourhood of g = e.

EXAMPLE The group SU(2) as the universal covering group of SO(3). The groups SU(2) and SO(3) have isomorphic Lie algebras. The three dimensional algebra so(3) consists of all three dimensional antisymmetric real matrices. It may be conveniently spanned by the three matrices

$$T_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \qquad T_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \qquad T_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

generating rotations around the 1,2 and 3 axis, respectively. The structure constants in this basis, $[T_i, T_j] = \epsilon_{ijk}T_k$ coincide with the fully antisymmetric tensor ϵ_{ijk} . In contrast, the Lie algebra su(2) of SU(2) consists of all anti–Hermitean traceless two dimensional complex matrices. It is three dimensional and may be spanned by the matrices $\tau_i \equiv -\frac{i}{2}\sigma_i$, where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are the familiar Pauli matrices. As with so(3), we have $[\tau_i, \tau_j] = \epsilon_{ijk}\tau_k$, i.e. the two algebras are isomorphic to each other.

Above, we have seen that the group manifold $SU(2) \simeq S^3$ is isomorphic to the three-sphere, i.e. it is simply connected. In contrast, the manifold SO(3) is connected yet not simply connected.

To see this, we perform a gedanken experiment: consider a long two-dimensional flexible strip embedded in three dimensional space. Let the (unit)-length of the strip be parameterized by $\tau \in [0, 1]$ and let $v(\tau)$ be the vector pointing in the 'narrow' direction of the strip. Assuming the width of the strip to be uniform, the mapping $v(0) \mapsto v(\tau)$ is mediated by an SO(3) transformation $O(\tau)$. Further, $\tau \mapsto O(\tau)$ defines a curve in SO(3) Assuming that $v(0) \parallel v(1)$, this curve is closed. As we will see, however, it cannot be contracted to a trivial (constant) curve if v(1) obtains from v(0) by

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a 2π rotation (in which case our strip looks like a Moebius strip.) However, it *can* be contracted, if v(1) was obtained from v(0) by a 4π rotation.

To explicate the connection $SU(2) \leftrightarrow SO(3)$, we introduce the auxiliary function

$$f: \mathbb{R}^3 \quad \to \quad \mathrm{su}(2),$$
$$v = v^i e_i \quad \mapsto \quad v^i \tau_i.$$

For an arbitrary matrix $U \in SU(2)$, we have $Uf(v)U^{-1} \in su(2)$ ($Uf(v)U^{-1}$ is anti-Hermitean and traceless.) Further, the map f is trivially bijective, i.e. it has an inverse. We may thus define an action of SU(2) in \mathbb{R}^3 as

$$\rho_U v \equiv f^{-1}(Uf(v)U^{-1}).$$

Due to the linearity of f, ρ_U actually is a linear representation. Furthermore $\rho_U : \mathbb{R}^3 \to \mathbb{R}^3 \in \mathrm{SO}(3)$, i.e. $\rho : \mathrm{SU}(2) \to \mathrm{SO}(3), U \mapsto \rho_U$ defines a map between the two groups $\mathrm{SU}(2)$ and $\mathrm{SO}(3)$. (To see that $\rho_U \in \mathrm{SO}(3)$, we compute the norm $\rho_U v$. First note that for $A \in \mathrm{su}(2)$, $|f^{-1}(A)|^2 = 4 \det(A)$. However, $\det(Uf(v)U^{-1}) = \det(f(v))$, from where follows the norm-preserving of ρ_U . One may also check that ρ_U preserves the orientation, i.e. ρ_U is a norm- and orientation preserving map, $\rho_U \in \mathrm{SO}(3)$.)

It can be checked that the mapping $\rho : \mathrm{SU}(2) \to \mathrm{SO}(3)$ is surjective. However, it is *not* injective. To see this, we need to identify a group element $g \in \mathrm{SU}(2)$ such that $\forall v : g^{-1}f(v)g = f(v)$, or, equivalently, $\forall h \in \mathrm{SU}(2) : g^{-1}hg = h$. The two group elements satisfying this requirement are g = e and g = -e. We have thus found hat the universal covering group of $\mathrm{SO}(3)$ is $\mathrm{SU}(2)$ and that the discrete kernel of the group homomorphism $\mathrm{SU}(2) \to \mathrm{SO}(3)$ is $\{e, -e\}$.