Understanding quantum measurement from the solution of dynamical models

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Abstract

The quantum measurement problem, that is, understanding why a unique outcome is obtained in each individual run of an experiment, is currently tackled by solving dynamical models. After a general introduction on the subject, an extensive review of the various approaches is presented. Next, a flexible and rather realistic model, describing the dynamics of the measurement of the z-component of a spin through interaction with a magnetic memory, is fully solved within standard quantum mechanics. The latter apparatus is simulated by a Curie–Weiss magnet including $N \gg 1$ spins weakly coupled to a phonon bath. The Hamiltonian evolution exhibits several time scales. The reduction, a rapid decay of the off-diagonal blocks of the system–apparatus density matrix, arises from the many degrees of freedom of the pointer which is the magnetization. The "which basis" paradox does not show up, reduction takes place continuously on the basis set by the interaction between the system and the apparatus. Recurrences are ruled out by two alternative mechanisms. The first one relies on randomenss in the system-pointer coupling. In the second one, which involves the bath, a resonance between phonon and spin frequencies selects the decoherence basis. The apparatus is prepared in a metastable paramagnetic state and would, by itself, go to either its up or down ferromagnetic state. This process is triggered here by the measurement, i.e., by the coupling with the tested system, which breaks the ergodicity and produces a stable final state involving correlations between the system and the indications of the pointer, thus registering the measurement. Conditions on the parameters of the model are given, which ensure that the process satisfies all the features of ideal measurements, including collapse and Born's rule. Violations of some conditions producing various imperfections of the measurement are discussed, as well as attempts of incompatible measurements. As usual, irreversibility is ensured by the macroscopic size of the apparatus, in particular by the large value of N. This study of a measurement process inspires various pedagogical exercises, and suggests lessons for future work on models. The outcome of many dynamical models supports a specified version of the statistical interpretation of quantum mechanics, which is promoted for teaching. Within this interpretation, the solution of the quantum measurement problem relies on the dynamics of the process, on the reduction, on the system–pointer correlations created during the registration and on thermodynamic stability of the final state. Standard quantum statistical mechanics is sufficient to explain the emergence of classicality in the process and the production of a unique answer for the outcome of each run.

Keywords: quantum measurement problem, apparatus, pointer, dynamical models, ideal and imperfect measurements, collapse of the wavefunction, decoherence, registration, statistical interpretation

Dedicated to our teachers and inspirers Nico G. van Kampen and Albert Messiah

Bring vor, was wahr ist;
schreib' so, daß es klar ist
und verficht's, bis es mit dir gar ist[1]
Ludwig Boltzmann

Contents

[1] Put forward what is true, write it such that it is clear, and fight for it till you're done

1. General features of quantum measurements

For this thing is *too heavy for thee,*
thou art not able to perform it thyself alone
Exodus 18.18

In spite of a century of progress and success, quantum mechanics still gives rise to passionate discussions about its interpretation. Understanding quantum measurements is an important issue in this respect, since measurements are a priviledged means to grasp the microscopic physical quantities. Two major steps in this direction were already taken in the early days. In 1926, Born gave the expression of the probabilities of the various possible outcomes of an ideal quantum measurement [1][2]. In 1932, von Neumann worked out the first model of quantum measurement, exhibiting the collapse [2]. Since then, many theorists have worked out models of quantum measurements, with the aim of understanding not merely the dynamics of such processes, but in particular solving the so-called measurement problem. This problem is raised by a conceptual contrast between quantum theory, which is irreducibly probabilistic, and our macroscopic experience. If a measurement is treated as a quantum physical process, in which the tested system interacts with an apparatus, the superposition principle seems to preclude the occurrence of a unique outcome, whereas each single run of a quantum measurement appears to yield a unique result. The challenge has remained to fully explain how this property emerges, ideally without new ingredients, that is, from the mere laws of quantum mechanics alone. Many authors have tackled this deep problem of measurements with the help of models so as to get insight on the interpretation of quantum mechanics. The tasks we undertake in this paper are first to review these works, then to solve in full detail a specific family of dynamical models and to finally draw conclusions from their solutions.

1.1. Measurements and interpretation of quantum mechanics

When you can measure what you are speaking about,
and express it in numbers, you know something about it
Lord Kelvin

Quantum mechanics textbooks dwell neither on questions of interpretation nor on quantum measurements, in spite of their importance in the comprehension of the theory. Generations of students have therefore stumbled over the problem of measurement, before leaving it aside when they pursued research work. Most physicists have never returned to it, considering that it is not worth spending time on a problem which "probably cannot be solved" and which has in practice little implication on physical predictions. Such a fatalistic attitude has emerged after the efforts of the brightest physicists, including Einstein, Bohr, de Broglie, von Neumann and Wigner, failed to lead to a universally accepted solution or even viewpoint; see for reviews [2, 3, 4, 5, 6, 7, 8, 9]. However, the measurement problem has never been forgotten, owing to its intimate connection with the foundations of quantum mechanics, which it may help to formulate more sharply, and owing to its philosophical implications.

In this review we shall focus on the simplest measurements, ideal projective measurements [1], and shall consider non-idealities and unsuccessful processes only occasionally and in section 8. While this is the standard (and only) case taught in standard courses, it is interesting to mention that the first experiment based on a nearly ideal measurement was only carried out recently [10]. An optical analog of a von Neumann measurement has been proposed too [11].

Experimentalists meet the theoretical discussions about quantum measurements with a feeling of speaking different languages. While theorists ponder about the initial pure state of the apparatus, the collapse of its wave packet and the question "when and in which basis does this collapse occur" and "how does this collapse agree with the Schrödinger equation", experimentalists deal with different issues, such as choosing an appropriate apparatus for the desired experiment or stabilizing it before the measurement starts. If an experimentalist were asked to describe one cubic nanometer of his apparatus in theoretical terms, he would surely start with a quantum mechanical approach. But this raises the question whether it is possible to describe the whole apparatus, and also its dynamics, i. e., the dynamics and outcome of the measurement, by quantum mechanics itself. It is this question that we shall answer

[2] Born wrote: "Only one interpretation is possible: $\Phi_{n,m}$ gives the probability* for the electron ...", and the footnote: " * Addition in proof: More careful consideration shows that the probability is proportional to the square of the quantity $\Phi_{n,m}$." (Translation from Wheeler and Zurek [3])

positively in the present work, thus closing the gap between what experimentalists intuitively feel and the formulation of the theory of ideal quantum measurements. To do so, we shall consider models that encompass the points relevant to experimentalists.

For theorists there has remained another unsolved paradox, even deeper than previous ones, the so-called *quantum measurement problem*: How can quantum mechanics, with its superposition principle, be compatible with the fact that *each individual run of a quantum measurement yields a well-defined outcome?* This uniqueness is at variance with the description of the measurement process by means of a pure state, the evolution of which is governed by the Schrödinger equation. Many workers believe that the quantum measurement problem cannot be answered within quantum mechanics. Some of them then hope that a hypothetical "sub-quantum theory", more basic than the current quantum mechanics, might predict what happens in individual systems [12, 13, 14, 15]. Our purpose is, however, to prove that the probabilistic framework of quantum mechanics is sufficient to explain that the outcome of a single measurement is unique although unpredictable within this probabilistic framework. We thus wish to show that quantum theory not only predicts the probabilities for the various possible outcomes of a set of measurements – as a minimalist attitude would state – but also accounts for the uniqueness of the result of each run.

A measurement is the only means through which information may be gained about a physical system S [2, 3, 4, 5, 6, 7, 8, 9, 16, 17]. Both in classical and in quantum physics, it is a dynamical process which couples this system S to another system, the apparatus A. Some correlations are thereby generated between the initial (and possibly final) state of S and the final state of A. Observation of A, in particular the value indicated by its pointer, then allows us to gain by inference some quantitative information about S. A measurement thus involves, in one way or another, the observers[3]. It also has statistical features, through unavoidable uncertainties and, more deeply, through the irreducibly probabilistic nature of our description of quantum systems.

Throughout decades many thoughts were therefore devoted to quantum measurements in relation to the interpretation of quantum theory. Both Einstein [18] and de Broglie [19] spent much time on such questions after their first discovery; the issue of quantum measurements was formulated by Heisenberg [20, 21] and put in a mathematically precise form by von Neumann [2]; the foundations of quantum mechanics were reconsidered in this light by people like Bohm [13, 14] or Everett [22, 23] in the fifties; hidden variables were discussed by Bell in the sixties [24]; the use of a statistical interpretation to analyze quantum measurements was then advocated by Blokhintsev [5, 6] and Ballentine [4] (subtleties of the statistical interpretation are underlined by Home and Whitaker [25]); the most relevant papers were collected by Wheeler and Zurek in 1983 [3]. Earlier reviews on this problem were given by London and Bauer [26] and Wigner [8]. We can presently witness a renewed interest for measurement theory; among many recent contributions we may mention the book of de Muynck [27] and the review articles by Schlosshauer [28] and Zurek [29]. Extensive references are given in the pedagogical article [30] and book [31] by Laloë which review paradoxes and interpretations of quantum mechanics. Indeed, these questions have escaped the realm of speculation owing to progresses in experimental physics which allow to tackle the foundations of quantum mechanics from different angles. Not only Bell's inequalities [24, 30, 32] but also the Greenberger–Horne–Zeilinger (GHZ) logical paradox have been tested experimentally [33]. Moreover, rather than considering cases where quantum interference terms (the infamous "Schrödinger cat problem" [3, 8, 34]) vanish owing to decoherence processes [35], experimentalists have become able to control these very interferences [36], which are essential to describe the physics of quantum superpositions of macroscopic states and to explore the new possibilities offered by quantum information [17, 37]. Examples include left and right going currents in superconducting circuits [10, 38, 39, 40], macroscopic atomic ensembles [36] and entangled mechanical oscillators [41].

1.1.1. Quantum versus classical measurements

When the cat and the mouse agree, the grocer is ruined
Iranian proverb

The quantum measurement problem arises from two major differences between quantum and classical measurements, stressed in most textbooks [2, 21, 42, 43].

(*i*) In classical physics it is legitimate to imagine that quantities such as the position and the momentum of a structureless particle like an electron might in principle be measured with increasingly large precision; this allowed

[3] We shall make the case that observation itself does not influence the outcome of the quantum measurement

us to regard both of them as well-defined physical quantities. This is no longer true in quantum mechanics, where we cannot forget about the random nature of physical quantities. Statistical fluctuations are unavoidable, as exemplified by Heisenberg's inequality [20, 21]: we cannot even think of values that would be taken simultaneously by non-commuting quantities whether or not we measure them. In general both the theory and the measurements provide us only with *probabilities*.

Consider a measurement of an observable $\hat{s}$ of the system S of interest, having eigenvectors $|s_i\rangle$ and eigenvalues s_i. It is an experiment in which S interacts with an apparatus A that has the following property [2, 8, 26, 42]. A physical quantity $\hat{A}$ pertaining to the apparatus A may take at the end of the process one value among a set A_i which are in one-to-one correspondence with s_i. If initially S lies in the state $|s_i\rangle$, the final value A_i will be produced with certainty, and a repeated experiment will always yield the observed result A_i, informing us that S was in $|s_i\rangle$ However, within this scope, S may also lie initially in a state represented by a wave function which is a linear combination,

$$|\psi\rangle = \sum_i \psi_i \, |s_i\rangle \,, \tag{1.1}$$

of the eigenvectors $|s_i\rangle$. *Born's rule* then states that the probability of observing in a given experiment the result A_i equals $|\psi_i|^2$ [1]. An axiomatic derivation of the Born's rule is given in [44]; see [28, 29] for a modern perspective on the rule. Quantum mechanics does not allow us to predict which will be the outcome A_i of an individual measurement, but provides us with the full statistics of repeated measurements of $\hat{s}$ performed on elements of an ensemble described by the state $|\psi\rangle$. The frequency of occurrence of each A_i in repeated experiments informs us about the moduli $|\psi_i|^2$, but not about the phases of these coefficients. In contrast to a classical state, a quantum state $|\psi\rangle$, even pure, always refers to an ensemble, and cannot be determined by means of a unique measurement performed on a single system. It cannot even be fully determined by repeated measurements of the single observable $\hat{s}$, since only the values of the amplitudes $|\psi_i|$ can thus be estimated.

(*ii*) A second qualitative difference between classical and quantum physics lies in the *perturbation of the system* S brought in by a measurement. Classically one may imagine that this perturbation could be made weaker and weaker, so that S is practically left in its initial state while A registers one of its properties. However, a quantum measurement is carried on with an apparatus A much larger than the tested object S; an extreme example is provided by the huge detectors used in particle physics. Such a process may go so far as to destroy S, as for a photon detected in a photomultiplier. It is natural to wonder whether the perturbation of S has a lower bound. Much work has therefore been devoted to the *ideal measurements*, those which preserve at least the statistics of the observable $\hat{s}$ in the final state of S, also referred to as non-demolition experiments or as measurements of the first kind [27]. Such ideal measurements are usually described by assuming that the apparatus A starts in a pure state. Then by writing that, if S lies initially in the state $|s_i\rangle$ and A in the state $|0\rangle$, the measurement leaves S unchanged: the compound system S + A evolves from $|s_i\rangle\,|0\rangle$ to $|s_i\rangle\,|A_i\rangle$, where $|A_i\rangle$ is an eigenvector of $\hat{A}$ associated with A_i. If however, as was first discussed by von Neumann, the initial state of S has the general form (1.1), S + A may reach any possible final state $|s_i\rangle\,|A_i\rangle$ depending on the result A_i observed. In this occurrence the system S is left in $|s_i\rangle$ and A in $|A_i\rangle$, and according to Born's rule, this occurs with the probability $|\psi_i|^2$. As explained in § 1.1.4, this is mathematically expressed by writing that the final density operator of the ensemble S + A has the *reduced* form

$$\sum_i |s_i\rangle\,|A_i\rangle\,|\psi_i|^2\,\langle A_i|\,\langle s_i|\,, \tag{1.2}$$

rather than the full form (1.3) below. Thus, not only is the state of the apparatus modified in a way controlled by the object, as it should in any classical or quantal measurement to provide us with information on S, but the marginal state of the quantum system is also necessarily modified, even by such an ideal measurement (except in the trivial case where (1.1) contains a single term, as it happens when one repeats the measurement).

1.1.2. Reduction

Ashes to ashes, dust to dust
Genesis 3:19

The rules of quantum measurements that we have recalled display a well known contradiction between the principles of quantum mechanics. On the one hand, if the measurement process leads the initial pure state $|s_i\rangle|0\rangle$ into

$|s_i\rangle|A_i\rangle$, the linearity of the wave functions of the compound system S + A and the unitarity of the evolution of the wave functions of S + A governed by the Schrödinger equation imply that the final density operator of S + A issued from (1.1) should be

$$\sum_{ij} |s_i\rangle\,|A_i\rangle\,\psi_i\psi_j^*\langle A_j|\langle s_j|. \tag{1.3}$$

On the other hand, according to Born's rule [1] and von Neumann's analysis [2], an ideal measurement should lead from the initial pure state $|\psi\rangle\,|0\rangle$ to the statistical mixture (1.2). In the orthodox Copenhagen interpretation, two separate postulates of evolution are introduced, one for the hamiltonian motion governed by the Schrödinger equation, the other for measurements which lead the system from $|\psi\rangle$ to one or the other of the states $|s_i\rangle$, depending on the value A_i observed. This lack of consistency is unsatisfactory and other explanations have been searched for (§ 1.3.1).

It should be noted that the loss of the off-diagonal elements takes place in a well-defined basis, the one in which both the tested observable $\hat{s}$ of S and the pointer variable $\hat{A}$ of A are diagonal (such a basis always exists since the joint Hilbert space of S + A is the tensor product of the spaces of S and A). In usual decoherence processes, it is the interaction between the system and some external bath which selects the basis in which off-diagonal elements are chopped off [28, 29]. We have therefore to elucidate this *preferred basis paradox*, and to explain why the reduction takes place in the specific basis selected by the measuring apparatus.

The occurrence in (1.3) of the off-diagonal $i \neq j$ terms is by itself an essential feature of an interaction process between two systems in quantum mechanics. There exist numerous experiments in which a pair of systems is left after interaction in a state of the form (1.3), not only at the microscopic scale, but even for macroscopic objects, involving for instance quantum superpositions of superconducting currents. Such experiments allow us to observe purely quantum coherences represented by off-diagonal terms $i \neq j$, and even to take advantage of the existence of entanglement [45].

However, such off-diagonal "Schrödinger cat" terms, which contradict both Born's rule [1] and von Neumann's reduction [2], must disappear at the end of a measurement process. Their absence is termed as the *reduction*, or the *collapse*, of the wave packet, or of the state. The term "collapse", or "reduction", has not always the same meaning, depending on the authors. It is often used in a *weak sense*, as the transformation of *the state* $|\psi\rangle$ *of* S into a single pure state $|s_i\rangle$, or into the mixture $\sum_i |s_i\rangle|\psi_i|^2\langle s_i|$. In fact, the latter density operator for the system S can be obtained from the non-reduced joint state (1.3) of S + A by merely tracing out the apparatus. When applied to the reduced joint state (1.2), this operation provides the same marginal state for S, so that the question seems to have been eluded. However, such a viewpoint cannot provide an answer to the measurement problem. The very aim of a measurement is to create correlations between S and A and to read the indications of A so as to derive indirectly information about S; but the elimination of the apparatus suppresses both the correlations between S and A and the information gained by reading A. We will therefore always use the term "reduction" in a *strong sense*, referring to the *final joint state of* S + A, which should have the form (1.2) (or, more generally, the form (1.7) below). The difficulty is then to explain how such a reduced state is produced by the dynamics of S + A.

Physically, a set of repeated experiments involving interaction of S and A can be regarded as a measurement only if we observe on A in each run some well defined result A_i, with probability $|\psi_i|^2$. For an ideal measurement we should be able to predict that S is then left in the corresponding state $|s_i\rangle$. We will show in section 12 that these properties are mathematically expressed by the reduced form (1.2), or more generally, (1.7) below, of the final state of S + A, and that the solution of the "measurement problem" amounts to the proof that the dynamics produces such a final state. As stressed by Bohr and Wigner, the reduction, interpreted as expressing the "uniqueness of physical reality", is at variance with the superposition principle which produces the final state (1.3). The challenge is to solve this contradiction, answering Wigner's wish: "*The simplest way that one may try to reduce the two kinds of changes of the state vector to a single kind is to describe the whole process of measurement as an event in time, governed by the quantum mechanical equations of motion*". Our purpose is to show that this is feasible, contrary to Wigner's own negative conclusion [8].

1.1.3. Selection of outcomes

Non-discrimination is a cross-cutting principle
United Nations human rights, 1996

The lack of off-diagonal terms expresses that we expect to consistently find the same answer if we repeat the process on a given system. Reduction thus implies *repeatability* and conversely repeatability implies that the marginal density operator of the system S has lost the elements $i \neq j$ during the first one of the successive measurement processes [46].

A quantum measurement consists in a set of repeated experiments performed on a statistical ensemble of identically prepared systems S, initially in a state $|\psi\rangle$. If we are able to show that, at the end of the process, the resulting ensemble $\mathcal{E}$ of compound systems S + A is described by the density operator (1.2), we can sort the runs according to the indication A_i of the pointer, and split $\mathcal{E}$ into sub-ensembles $\mathcal{E}_i$, each one labelled by i and described by the density operator $|s_i\rangle|A_i\rangle\langle A_i|\langle s_i|$. Selecting the sub-ensemble $\mathcal{E}_i$ by filtering A_i allows us to set S into the given state $|s_i\rangle$ with a view to future experiments on S. An ideal measurement followed by filtering can therefore be used as a *preparation* of the state of S [47]. We will make the argument more precise in § 10.1.4 and section 12.

Note that some authors call "measurement" a repeated experiment in which the occurrence of *some given eigenvalue of* $\hat{s}$ is detected, and in which only the corresponding events are selected for the outcoming system S. Here we use the term "measurement" to designate a repeated experiment performed on an *ensemble of identically prepared systems* which informs us about *all possible values* s_i of the observable $\hat{s}$ of S, and the term "ideal measurement" if the process perturbs S as little as allowed by quantum mechanics, in the sense that it does not affect the statistics of the observables that commute with $\hat{s}$. We do not regard the sorting as part of the measurement, but as a subsequent operation, and prefer to reserve the word "preparation through measurement" to such processes including a selection.

1.2. The need for quantum statistical mechanics

Om een paardendief te vangen heb je een paardendief nodig[4]
Un coupable en cache un autre[5]
Dutch and French proverbs

We wish for consistency to use quantum mechanics for treating the dynamics of the interaction process between the apparatus and the tested system. However, the apparatus must be a macroscopic object in order to allow the outcome to be read off from the final position of its pointer. The natural framework to reconcile these requirements is non-equilibrium quantum statistical mechanics. It will appear that not only the registration process can be addressed in this way, but also the reduction of the wave packet.

1.2.1. Irreversibility of measurement processes

The first time ever I saw your face
I thought the sun rose in your eyes
Written by Ewan MacColl, sung by Roberta Flack

Among the features that we wish to explain, the *reduction* compels us to describe states by means of density operators. The sole use of pure states (quantum states describable by a wave function or a ket), is prohibited by the form of (1.2), which is in general a statistical mixture. Even if we start from a pure state $|\psi\rangle|0\rangle$, we must end up with the reduced state (1.2) through an *irreversible* process. This irreversibility is also exhibited by the fact the same final state (1.2) is reached if one starts from different initial states of the form (1.1) deduced from one another through changes of the phases of the coefficients ψ_i. It is associated with the disappearance of the specifically quantum correlations between S and A described by the off-diagonal terms of (1.3).

Actually, there is a second cause of irreversibility in any effective measurement process. The apparatus A should *register* the result A_i in a robust and permanent way, so that it can be read off by any observer. Such a registration, which is often overlooked in the literature on measurements, is needed for practical reasons especially since S is a microscopic object. Moreover, its very existence allows us to disregard the observers in quantum measurements. Once the measurement has been registered, the result becomes *objective* and can be read off at any time by any observer. It can also be processed automatically, without being read off. Registration requires an *amplification* within the apparatus of a signal produced by interaction with the microscopic system S. For instance, in a bubble chamber, the apparatus

[4] To catch a horse thief, you need a horse thief
[5] One culprit hides another

in its initial state involves a liquid, overheated in a metastable phase. In spite of the weakness of the interaction between the particle to be detected and this liquid, amplification and registration of its track can be achieved owing to local transition towards the stable gaseous phase. This stage of the measurement process thus requires an irreversible phenomenon. It is governed by the kinetics of bubble formation under the influence of the particle and implies a dumping of free energy. Similar remarks hold for photographic plates, photomultipliers or other types of detectors.

Since the amplification and the registration of the measurement results require the apparatus A to be a large object so as to behave irreversibly, we must use quantum statistical mechanics to describe A. In particular, the above assumption that A lay initially in a pure state $|0\rangle$ was unrealistic – nevertheless this assumption is frequent in theoretical works on measurements, see e.g. [28]. Indeed, preparing an object in a pure state requires controlling a complete set of commuting observables, performing their measurement and selecting the outcome (§ 1.1.3). While such operations are feasible for a few variables, they cannot be carried out for a macroscopic apparatus nor even for a mesoscopic apparatus involving, say, 1000 particles. What the experimentalist does in a quantum measurement is quite the opposite [5, 6, 21, 27]: rather than purifying the initial state of A, he lets it stabilize macroscopically by controlling a few collective parameters such as the temperature of the apparatus. The adequate theoretical representation of the initial state of A, which is a *mixed state*, is therefore a *density operator* denoted as $\hat{\mathcal{R}}(0)$. Using pure states in thought experiments or models would require averaging so as to reproduce the actual situation (§§ 10.1.4 and 11.1.4). Moreover the initial state of A should be *metastable*, which requires a sudden change of, e.g., temperature.

Likewise the final possible stable marginal states of A are not pure. As we know from quantum statistical physics, each of them, characterized by the value of the pointer variable A_i that will be observed, should again be described by means of a density operator $\hat{\mathcal{R}}_i$, and not by means of pure states $|A_i\rangle$ as in (1.3). Indeed, the number of state vectors associated with a sharp value of the *macroscopic* pointer variable A_i is huge for any actual measurement: As always for large systems, we must allow for small fluctuations, negligible in relative value, around the mean value $A_i = \mathrm{tr}_\mathrm{A}\hat{A}\hat{\mathcal{R}}_i$. The fact that the possible final states $\hat{\mathcal{R}}_i$ are exclusive is expressed by $\mathrm{tr}_\mathrm{A}\hat{\mathcal{R}}_i\hat{\mathcal{R}}_j \simeq 0$ for $j \neq i$, which implies

$$\hat{\mathcal{R}}_i\hat{\mathcal{R}}_j \to 0 \quad \text{for } N \to \infty \text{ when } i \neq j. \tag{1.4}$$

In words, these macroscopic pointer states are practically orthogonal.

1.2.2. The paradox of irreversibility

La vida es sueño[6]
Calderón de la Barca

If we disregard the system S, the irreversible process leading A from the initial state $\hat{\mathcal{R}}(0)$ to one among the final states $\hat{\mathcal{R}}_i$ is reminiscent of relaxation processes in statistical physics, and the measurement problem raises the same type of puzzle as the paradox of irreversibility. In all problems of statistical mechanics, the evolution is governed at the microscopic level by equations that are invariant under time-reversal: Hamilton or Liouville equations in classical physics, Schrödinger, or Liouville–von Neumann equations in quantum physics. Such equations are reversible and conserve the von Neumann entropy, which measures our missing information. Nevertheless we observe at our scale an irreversibility, measured by an increase of macroscopic entropy. The explanation of this paradox, see, e.g., [48, 49, 50, 51, 52, 53], relies on the large number of microscopic degrees of freedom of thermodynamic systems, on statistical considerations and on plausible assumptions about the dynamics and about the initial state of the system.

Let us illustrate these ideas by recalling the historic example of a classical gas, for which the elucidation of the paradox was initiated by Boltzmann [48, 49, 50]. The microscopic state of a set of N structureless particles enclosed in a vessel is represented at each time by a point $\xi(t)$ in the $6N$-dimensional phase space, the trajectory of which is generated by Hamilton's equations, the energy E being conserved. We have to understand why, starting at the time $t = 0$ from a more or less arbitrary initial state with energy E, we always observe that the gas reaches at the final time t_f a state which macroscopically has the equilibrium properties associated with N and E, to wit, homogeneity and Maxwellian distribution of momenta – whereas a converse transformation is never seen in spite of the reversibility of the dynamics. As we are not interested in a single individual process but in generic features, we can resort to

[6] Life is a dream

statistical considerations. We therefore consider an initial macroscopic state $\mathcal{S}_{\rm init}$ characterized by given values of the (non uniform) densities of particles, of energy, and of momentum in ordinary space. Microscopically, $\mathcal{S}_{\rm init}$ can be realized by any point $\xi_{\rm init}$ lying in some volume $\Omega_{\rm init}$ of phase space. On the other hand, consider the volume Ω_E in phase space characterized by the total energy E. A crucial fact is that the immense majority of points ξ with energy E have macroscopically the equilibrium properties (homogeneity and Maxwellian distribution): the volume $\Omega_{\rm eq}$ of phase space associated with equilibrium occupies nearly the whole volume $\Omega_{\rm eq}/\Omega_E \simeq 1$. Moreover, the volume Ω_E is enormously larger than $\Omega_{\rm init}$. We understand these properties by noting that the phase space volumes characterized by some macroscopic property are proportional to the exponential of the thermodynamic entropy. In particular, the ratio $\Omega_{\rm eq}/\Omega_{\rm init}$ is the exponential of the increase of entropy from $\mathcal{S}_{\rm init}$ to $\mathcal{S}_{\rm eq}$, which is large as N. We note then that Hamiltonian dynamics implies Liouville's theorem. The bunch of trajectories issued from the points $\xi(0)$ in $\Omega_{\rm init}$ therefore reach at the time $t_{\rm f}$ a final volume $\Omega_{\rm f} = \Omega_{\rm init}$ that occupies only a tiny part of Ω_E, but which otherwise is expected to have nothing special owing to the complexity of the dynamics of collisions. Thus most end points $\xi(t_{\rm f})$ of these trajectories are expected to be typical points of Ω_E, that is, to lie in the equilibrium region $\Omega_{\rm eq}$. Conversely, starting from an arbitrary point of Ω_E or of $\Omega_{\rm eq}$, the probability of reaching a point that differs macroscopically from equilibrium is extremely small, since such points occupy a negligible volume in phase space compared to the equilibrium points. The inconceivably large value of Poincaré's recurrence time is also related to this geometry of phase space asociated with the macroscopic size of the system.

The above argument has been made rigorous [48, 49, 50] by merging the dynamics and the statistics, that is, by studying the evolution of the density in phase space, the probability distribution which encompasses the bunch of trajectories evoked above. Indeed, it is easier to control theoretically the Liouville equation than to study the individual Hamiltonian trajectories and their statistics for random initial conditions. The initial state of the gas is now described by a non-equilibrium density in the $6N$-dimensional phase space. Our full information about this initial state, or the full order contained in it, is conserved by the microscopic evolution owing to the Liouville theorem. However, the successive collisions produce correlations between larger and larger numbers of particles. Thus, while after some time the gas reaches at the macroscopic scale the features of thermodynamic equilibrium, the initial order gets hidden into microscopic variables, namely many-particle correlations, that are inaccessible. Because the number of degrees of freedom is large – and it is actually gigantic for any macroscopic object – this order cannot be retrieved (except in some exceptional controlled dynamical phenomena such as spin echoes [54, 55, 56, 57, 58]). In any real situation, it is therefore impossible to recover, for instance, a non-uniform density from the very complicated correlations created during the relaxation process. For all practical purposes, we can safely keep track, even theoretically, only of the correlations between a number of particles small compared to the total number of particles of the system: the exact final density in phase space cannot then be distinguished from a thermodynamic equilibrium distribution. It is this dropping of information about undetectable and ineffective degrees of freedom, impossible to describe even with the largest computers, which raises the macroscopic entropy [48, 49, 50, 51, 52]. Such approximations can be justified mathematically through limiting processes where $N \to \infty$.

Altogether, irreversibility can be derived rigorously for the Boltzmann gas under assumptions of smoothness and approximate factorization of the single particle density. The change of scale modifies qualitatively the properties of the dynamics, for all accessible times and for all accessible physical variables. The emergence of an irreversible relaxation from the reversible microscopic dynamics is a statistical phenomenon which becomes nearly deterministic owing to the large number of particles. We shall encounter similar features in quantum measurement processes.

1.2.3. The measurement problem in the language of statistical physics

Now the whole earth was of one language and of one speech[7]

[7] Metaphorically, the discovery of quantum theory and the lack of agreement about its interpretation may be phrased in the next lines of Genesis 11 [59]: 2. And it came to pass, as they journeyed from the east, that they found a plane in the land of Shinar; and they dwelt there. 3. And they said one to another, Go to, let us make brick, and burn them throughly. And they had brick for stone, and slime had they for mortar. 4. And they said, Go to, let us build a city, and a tower whose top *may reach* unto heaven; and let us make us a name, lest we be scattered abroad upon the face of the whole earth. 5. And the Lord came down to see the city and the tower, which the children of men builded. 6. And the Lord said, Behold, the people *is* one, and they have all one language; and this they begin to do: and now nothing will be restrained from them, which they have imagined to do. 7. Go to, let us go down, and there confound their language, that they may not understand one another's speech. 8. So the Lord scattered them abroad from thence upon the face of all the earth: and they left off to build the city. 9. Therefore is the name of it called Babel; because the Lord did there confound the language of all the earth: and from thence did the Lord scatter them abroad upon the face of all the earth

The theoretical description of a measurement process should be inspired by the same ideas. Actually, a measurement process looks like a relaxation process, but with several complications. On the one hand, the final stable state of A is not unique, and the dynamical process can have *several possible outcomes* for A. In photodetection (the eye, a photomultiplier), one just detects whether an avalanche has or not been created by the arrival of a photon. In a magnetic dot, one detects the direction of the magnetization. The apparatus is therefore comparable to a material which, in statistical physics, has a broken invariance and can relax towards one equilibrium phase or another, starting from a single metastable phase. On the other hand, the evolution of A towards one among the final states $\hat{\mathcal{R}}_i$ characterized by the variable A_i should be triggered by interaction with S, in a way depending on the initial microscopic state of S and, for an ideal measurement, the outcome A_i should be *correlated* to the final microscopic state of S. Thus, contrary to theories of standard relaxation processes in statistical physics, the theory of a measurement process will require a simultaneous control of *microscopic and macroscopic* variables. In the coupled evolution of A and S which involves reduction and registration, coarse graining will be adequate for A, becoming exact in the limit of a large A, but not for S. Moreover the final state of S + A keeps *memory* of the initial state of S, at least partly. The very essence of a measurement lies in this feature, whereas memory effects are rarely considered in standard relaxation processes.

Denoting by $\hat{r}\,(0)$ and $\hat{\mathcal{R}}(0)$ the density operators of the system S and the apparatus A, respectively, before the measurement, the initial state of S + A is characterized in the language of quantum statistical mechanics by the density operator

$$\hat{\mathcal{D}}\,(0) = \hat{r}\,(0) \otimes \hat{\mathcal{R}}\,(0)\,. \tag{1.5}$$

In the Schrödinger picture, where the wave functions evolve according to the Schrödinger equation while observables are time-independent, the density operator $\hat{\mathcal{D}}(t)$ of the compound system S + A evolves according to the Liouville-von Neumann equation of motion

$$i\hbar \frac{\mathrm{d}\hat{\mathcal{D}}}{\mathrm{d}t} = [\hat{H}, \hat{\mathcal{D}}] \equiv \hat{H}\hat{\mathcal{D}} - \hat{\mathcal{D}}\hat{H}, \tag{1.6}$$

where $\hat{H}$ is the Hamiltonian of S + A including the interaction between S and A. By solving (1.6) with the initial condition (1.5), we find the expectation value $\langle \hat{A}(t) \rangle$ of any observable $\hat{A}$ of S + A at the time t as $\mathrm{tr}[\hat{\mathcal{D}}(t)\hat{A}]$ (see subsection 10.1 and Appendix G).

We wish to show that the final state of S + A at the time t_f can be represented, for an ideal measurement, by von Neumann's reduced density operator

$$\hat{\mathcal{D}}\,(t_\mathrm{f}) = \sum_i \left(\hat{\Pi}_i \hat{r}\,(0)\,\hat{\Pi}_i\right) \otimes \hat{\mathcal{R}}_i = \sum_i p_i \hat{r}_i \otimes \hat{\mathcal{R}}_i, \tag{1.7}$$

where $\hat{\Pi}_i$ denotes the projection operator (satisfying $\hat{\Pi}_i \hat{\Pi}_j = \delta_{ij}\hat{\Pi}_i$) on the eigenspace s_i of $\hat{s}$ in the Hilbert space of S, with $\hat{s} = \sum_i s_i \hat{\Pi}_i$ and $\sum_i \hat{\Pi}_i = \hat{\mathrm{I}}$. (If the eigenvalue s_i is non-degenerate, $\hat{\Pi}_i$ is simply equal to $|s_i\rangle\langle s_i|$.) We have denoted by

$$\hat{r}_i = \frac{1}{p_i} \hat{\Pi}_i \hat{r}(0) \hat{\Pi}_i \tag{1.8}$$

the corresponding normalized projected state (which reduces to $|s_i\rangle\langle s_i|$ if s_i is non-degenerate), and by

$$p_i \equiv \mathrm{tr_S} \hat{r}(0) \hat{\Pi}_i \tag{1.9}$$

the normalizing factor (which reduces to $r_{ii}(0)$ if s_i in non-degenerate). Together they characterize the final state expected for the tested system S, when one disregards the apparatus, which is represented in agreement with (1.7) by the density operator

$$\hat{r}\,(t_\mathrm{f}) \equiv \mathrm{tr_A} \hat{\mathcal{D}}(t_\mathrm{f}) = \sum_i p_i \hat{r}_i = \sum_i \hat{\Pi}_i \hat{r}(0) \hat{\Pi}_i = \sum_i p_i |s_i\rangle\langle s_i| = \sum_i r_{ii}(0) |s_i\rangle\langle s_i|. \tag{1.10}$$

The last two expressions in (1.10) hold when the egenvalues s_i of $\hat{s}$ are non-degenerate. The expression (1.7) generalizes (1.2) to arbitrary density operators; it encompasses Born's rule [1], which is obtained by noting that, for the final marginal state

$$\hat{\mathcal{R}}(t_\mathrm{f}) = \mathrm{tr}_\mathrm{S}\, \hat{\mathcal{D}}(t_\mathrm{f}) = \sum_i p_i \hat{\mathcal{R}}_i \tag{1.11}$$

of the apparatus, the probability of observing A_i is expressed by p_i in terms of the initial state of S. The density operator (1.7) is expected to be reached at the end of the process for an ensemble of measurements, whether the registered results are read or not. As indicated in § 1.1.4, reading allows us to select subensembles labelled by the index i. The full expression (1.7) exhibits a correlation between the possible final states of A and those of S, which is encoded in $\langle \hat{\Pi}_i (\hat{A} - A_i)^2 \rangle = 0$ for each i, meaning that in an ideal measurement $\hat{s}$ takes the value s_i when $\hat{A}$ takes the value A_i.

Generalizing the vocabulary of § 1.1.3, we term as *reduction* the process which leads, for the compound system S + A, to a final state of the form (1.7) devoid of off-diagonal blocks in a basis where the measured observable $\hat{s}$ is diagonal. This definition is more stringent than the *weak reduction*, which takes place in some models of measurement, for which it is only the marginal state of the tested system S at the end of the process which has the diagonal form (1.10). That simply tracing out A leads to a diagonal state for S but solves in no way the physics of the measurement process, is well known, see e.g. [5, 6, 27, 60, 61].

1.2.4. Entropy changes in a measurement

Discussions about entropy have produced quite a bit of heat
Anonymous

A density operator $\hat{\mathcal{D}}$ plays in quantum mechanics the role of a set p_i of probabilities, and it refers to a statistical ensemble of physical systems prepared under identical conditions. In this prospect, the von Neumann entropy $S[\hat{\mathcal{D}}] = -\mathrm{tr}\hat{\mathcal{D}} \ln \hat{\mathcal{D}}$ is the quantum analogue of Shannon's entropy $S[p] = -\sum_i p_i \ln p_i$, which is the average amount of information gained when we acknowledge one among the events i with probabilities p_i, or equivalently the information missing before acknowledgement. Although $S[\hat{\mathcal{D}}]$ can be identified with the thermodynamic entropy when $\hat{\mathcal{D}}$ describes a system in equilibrium, it has in general a partly subjective meaning: as Shannon's entropy, it measures the amount of information that is missing when the system is known through $\hat{\mathcal{D}}$, or the uncertainty associated with $\hat{\mathcal{D}}$. One can also identify it with disorder [52, 62, 63, 64, 65],

The equation of motion of S + A is deterministic and reversible, and some manipulations justified by the large size of A are necessary, as in any relaxation problem, to understand how the state of S + A may end as (1.7). Strictly speaking, the Liouville-von Neumann evolution (1.6) conserves the von Neumann entropy $-\mathrm{tr}\hat{\mathcal{D}} \ln \hat{\mathcal{D}}$ associated with the whole set of degrees of freedom of S + A; in principle no information is lost. However, in statistical physics, irreversibility means that information (identified with order) is transferred towards inaccessible degrees of freedom, in the form of many-particle correlations, without possibility of return in a reasonable delay. A measure of this loss of information is provided by the "relevant entropy" [52, 62, 63, 64, 65], which is the von Neumann entropy of the state that results from the elimination of the information about such inaccessible correlations. Here the reduced state $\hat{\mathcal{D}}(t_\mathrm{f})$ should have the latter status: As regards to all accessible degrees of freedom $\hat{\mathcal{D}}(t_\mathrm{f})$ should be equivalent to the state issued from $\hat{\mathcal{D}}(0)$ through the equation of motion (1.6), but we got rid in $\hat{\mathcal{D}}(t_\mathrm{f})$ of the irrelevant correlations involving a very large number of elements of the macroscopic apparatus A; such correlations are irremediably lost.

We can therefore measure the irreversibility of the measurement process leading from $\hat{\mathcal{D}}(0)$ to $\hat{\mathcal{D}}(t_\mathrm{f})$ by the following entropy balance. The von Neumann entropy of the initial state (1.5) is split into contributions from S and A, respectively, as

$$S\left[\hat{\mathcal{D}}(0)\right] = -\mathrm{tr}\,\hat{\mathcal{D}}(0)\, \ln \hat{\mathcal{D}}(0) = S_\mathrm{S}\left[\hat{r}(0)\right] + S_\mathrm{A}\left[\hat{\mathcal{R}}(0)\right], \tag{1.12}$$

whereas that of the final state (1.7) is

$$S\left[\hat{\mathcal{D}}(t_\mathrm{f})\right] = S_\mathrm{S}\left[\hat{r}(t_\mathrm{f})\right] + \sum_i p_i S_A\left[\hat{\mathcal{R}}_i\right], \tag{1.13}$$

where $\hat{r}\,(t_\mathrm{f})$ is the marginal state (1.10) of S at the final time. The latter expression is found by using the orthogonality $\hat{\mathcal{R}}_i\hat{\mathcal{R}}_j = 0$ for $i \neq j$, so that $-\hat{\mathcal{D}}(t_\mathrm{f}) \ln \hat{\mathcal{D}}(t_\mathrm{f})$ is equal to the sum of its separate blocks, $\sum_i p_i\hat{r}_i \otimes \hat{\mathcal{R}}_i(-\ln p_i - \ln \hat{r}_i - \ln \hat{\mathcal{R}}_i)$, and hence that the entropy of $\hat{\mathcal{D}}(t_\mathrm{f})$ is a sum of contributions arising from each i. The trace over A of the first two terms leads to $\sum_i p_i\hat{r}_i(-\ln p_i - \ln \hat{r}_i)$, the trace over S of which may be identified with the entropy $S_\mathrm{S}[\hat{r}(t_\mathrm{f})]$ of (1.10); the trace of the last term leads to the last sum in (1.13). This equality entails separate contributions from S and A. The increase of entropy from (1.12) to (1.13) clearly arises from the two above-mentioned reasons, reduction and registration. On the one hand, when the density operator $\hat{r}\,(0)$ involves off-diagonal blocks $\hat{\Pi}_i\hat{r}\,(0)\,\hat{\Pi}_j$ $(i \neq j)$, their truncation raises the entropy. On the other hand, a robust registration requires that the possible final states $\hat{\mathcal{R}}_i$ of A are more stable than the initial state $\hat{\mathcal{R}}\,(0)$, so that their entropy is larger. The latter effect dominates because the apparatus is large, typically S_A will be macroscopic and S_S microscopic.

An apparatus is a device which allows us to gain some information on the state of S by reading the outcomes A_i. The price we have to pay for being thus able to determine the probabilities (1.9) is a complete loss of information about the off-diagonal elements $\hat{\Pi}_i\hat{r}\,(0)\,\hat{\Pi}_j$ $(i \neq j)$ of the initial state of S[8], and a rise in the thermodynamic entropy of the apparatus. More generally, in other types of quantum measurements, some information about a system may be gained only at the expense of erasing other information about this system [66] (see subsection 2.5).

The quantitative estimation of the gains and losses of information in the measurement process is provided by an entropic analysis, reviewed in [17, 63, 67]. Applications of entropy for quantifying the uncertainties in quantum measurements are also discussed in [68]. We recall here the properties of the entropy of the marginal state of S and their interpretation in terms of information. We have just noted that $S_\mathrm{S}\,[\hat{r}\,(t_\mathrm{f})] - S_\mathrm{S}\,[\hat{r}\,(0)]$, which is non-negative, measures the increase of entropy of S due to reduction. This means that, in case we know $\hat{r}(0)$, the interaction with A (*without reading the pointer*) lets us loose the amount of information $S_\mathrm{S}\,[\hat{r}\,(t_\mathrm{f})] - S_\mathrm{S}\,[\hat{r}\,(0)]$ about all observables that do not commute with $\hat{s}$ [63, 67]. In fact, this loss is the largest possible among the set of states that preserve the whole information about the observables commuting with $\hat{s}$. Any state of S that provides, for all observables commuting with $\hat{s}$, the same expectation values as $\hat{r}(t_\mathrm{f})$ is less disordered than $\hat{r}(t_\mathrm{f})$, and has an entropy lower than $S_\mathrm{S}[\hat{r}(t_\mathrm{f})]$. In other words, among all the processes that leave the statistics of the observables commuting with $\hat{s}$ unchanged, the ideal measurement of $\hat{s}$ is the one which destroys the largest amount of information (about the other observables of S).

Reading the pointer value A_i, which occurs with probability p_i, allows us to ascertain (for the considered ideal measurement) that S is in the state $\hat{r}_i$ after the measurement. By acknowledging the outcomes of a large sequence of runs of the measurement, we gain therefore on average an amount of information given on the one hand by the Shannon entropy $-\sum_i p_i \ln p_i$, and equal on the other hand to the difference between the entropies of the final state and of its separate components,

$$S_\mathrm{S}\,[\hat{r}(t_\mathrm{f})] - \sum_i p_i S_\mathrm{S}\,[\hat{r}_i] = -\sum_i p_i \ln p_i \geq 0. \tag{1.14}$$

The equality expresses *additivity* of information, or of uncertainty, at the end of the process, when we have not yet read the outcomes A_i: Our uncertainty $S_\mathrm{S}[\hat{r}(t_\mathrm{f})]$, when we know directly that $\hat{r}(t_\mathrm{f})$, the density operator of the final state, encompasses all possible marginal final states $\hat{r}_i$, each with its probability p_i, is given by the left-hand side of (1.14). It is the same as if we proceed in two steps. As we have not yet read A_i, we have a total uncertainty $S_\mathrm{S}\,[\hat{r}(t_\mathrm{f})]$ because we miss the corresponding amount of Shannon information $-\sum_i p_i \ln p_i$ about the outcomes; and we miss also, with the probability p_i for each possible occurrence of A_i, some information on S equal to $S\,[\hat{r}_i]$, the entropy of the state $\hat{r}_i$. As it stands, the equality (1.14) also expresses the *equivalence between negentropy and information* [62, 69, 70]: *sorting* the ensemble of systems S according to the outcome i lowers the entropy by a quantity equal on average to the left-hand side of (1.14), while *reading* the indication A_i of the pointer provides, in Shannon's sense, an additional amount of information $-\ln p_i$, on average equal to the right-hand side.

Two inequalities are satisfied in the whole process, including the sorting of results:

$$-\sum_i p_i \ln p_i \geq S_\mathrm{S}\,[\hat{r}(0)] - \sum_i p_i S_\mathrm{S}\,[\hat{r}_i] \geq 0. \tag{1.15}$$

[8] In the language of section 1.1: Loss of information about the phases of the ψ_i

The first inequality expresses that the additivity of the gained information, exhibited by (1.14) for the final state of S, is spoiled in quantum mechanics when one considers the whole process, due to the quantum perturbation of the initial state of S which eliminates its off diagonal sectors. The second inequality, derived in [71], expresses that measurements yield a positive balance of information about S in spite of the losses resulting from the perturbation of S. Indeed, this inequality means that, on average over many runs of the measurement process, and after sorting of the outcomes, the entropy of S has decreased, i. e., more information on S is available at the time t_f than at the initial time. The equality holds only if all possible final states $\hat{r}_i$ of S have the same entropy.

Note finally that, if we wish to perform repeated quantum measurements in a closed cycle, we must reset the apparatus in its original metastable state. As for a thermal machine, this requires lowering the entropy and costs some supply of energy.

1.3. Towards a solution of the measurement problem?

It is good to be visiting, but it is better to be at home
Russian proverb

The quantum measurement problem arises from the acknowledgement that individual measurements provide well-defined outcomes. Standard quantum mechanics yields only probabilistic results and thus seems unable to explain such a behavior. We have advocated above the use of quantum statistical physics, which seems even less adapted to draw conclusions about individual systems. Most of the present work will be devoted to show how a statistical approach may nevertheless solve the measurement problem as will be discussed in section 12. We begin with a brief survey of the more current approaches.

1.3.1. Various approaches

Ջորին յոթը գետում լողալ գիտի, բայց ջուր տեսնելիս բոլորը մոռանում է: [9]
Armenian proverb

In the early days of quantum mechanics, the apparatus was supposed to behave classically, escaping the realm of quantum theory [72, 73]. A similar idea survives in theoretical or experimental works exploring the possible existence of a border between small or large, or between simple and complex objects, which would separate the domains of validity of quantum and classical physics (Heisenberg's cut [21]).

Another current viewpoint has attributed the reduction in a measurement to the "act of observing the result". Again, the observer himself, who is exterior to the system, is not described in the framework of quantum mechanics. In the many-worlds interpretation, reduction is even denied, and regarded as a delusion due to the limitations of the human mind. From another angle, people who wish to apply quantum theory to the whole universe, even have a non-trivial task in defining what is observation. A more rational attitude is taken within the consistent histories approach, where one is careful with defining when and where the events happen, but then holds that the measurements simply reveal the pre-existing values of events (this approach is discussed below in section 2.9). For interpretations based on entanglement and information, see Peres [17] and Jaeger [74].

A more recent line of thought, going "beyond the quantum" [15] relies on modification of the Schrödinger mechanics by additional non-linear and stochastic terms; see Refs. [12, 75, 76] for review. Such generalizations are based in the belief, emphasized in the standard Copenhagen interpretation of quantum mechanics, that the Schrödinger equation is unable to describe the joint evolution of a system S and an apparatus A, so that a special separate postulate is needed to account for the rules of quantum measurements, in particular reduction. Indeed, a hamiltonian evolution seems to preclude the emergence of a single result in each single realization of a measurement [2, 8, 26].

We will focus below on the most conservative approach where S + A is treated as an isolated quantum object governed by a Hamiltonian, and yet where reduction can be understood. The measurement is not considered on formal and general grounds as in many conceptual works aimed mainly at the interpretation of quantum mechanics, but it is fully analyzed as a dynamical process. Unfortunately the theory of specific experimental measurement processes based on hamiltonian dynamics is made difficult by the complexity of a real measuring apparatus. One can gain full

[9] The mule can swim over seven rivers, but as soon as it sees the water it forgets everything

insight only by solving models that mimic actual measurements. The formal issue is to explain how S + A may reach a final state of the form (1.7) by starting from a state (1.5) and evolving along with (1.6).

The realization of such a program should meet the major challenge raised long ago by Bell [77]: *"So long as the wave packet reduction is an essential component, and so long as we do not know exactly when and how it takes over from the Schrödinger equation, we do not have an exact and unambiguous formulation of our most fundamental physical theory"*. Indeed, a full understanding of quantum mechanics requires knowledge of the time scales involved in measurements. Knowing how the reduction proceeds in time, how long it takes, is a prerequisite for clearing up the meaning of this phenomenon. On the other hand, the registration is part of the measurement; it is important to exhibit the time scale on which it takes place, to determine whether it interferes with the collapse or not, and to know when and how the correlations between S and A are established. These are the tasks we undertake in the body of this work on a specific but flexible model. We resume in section 9 how the solution of this model answers such questions.

1.3.2. Outline

We review in section 2 the works that tackled the program sketched above, and discuss to which extent they satisfy the various features that we stressed in the introduction. For instance, do they explain reduction by relying on a full dynamical solution, or do they only invoke environment-induced decoherence? Do they solve the preferred basis paradox? Do they account for a robust registration? Do they produce the time scales involved in the process?

In section 3 we present the Curie–Weiss model, which encompasses many properties of the previous models and on which we will focus afterwards. It is sufficiently simple to be completely solvable, sufficiently elaborate to account for all characteristics of ideal quantum measurements, and sufficiently realistic to resemble actual experiments: The apparatus simulates a magnetic dot, a standard registering device.

The detailed dynamical solution of this model is worked out in sections 4 to 7, some calculations being given in appendices. After analysing the equations of motion of S + A (section 4), we exhibit several time scales. The reduction rapidly takes place (section 5). It is then made irremediable owing to two alternative mechanisms (section 6). Amplification and registration require much longer delays since they involve a macroscopic change of the apparatus (section 7).

Solving several variants of the Curie–Weiss model allows us to explore various dynamical processes which can be interpreted either as imperfect measurements or as failures (section 8). In particular, we study what happens when the pointer has few degrees of freedom or when one tries to simultaneously measure non-commuting observables. The calculations are less simple than for the original model, but are included in the text for completeness.

The results of the Curie-Weiss model are resumed and analyzed in section 9.

The quantum measurement problem and more generally the interpretation of quantum mechanics are enlightened by models. The statistical interpretation, in a form presented in section 10, appears as the most natural and consistent one in this respect. Several lessons for future work are drawn in section 11. Finally, the unicity of the result of a single measurement, as well as the occurrence of classical probabilities, are seen to emerge from the dynamics of measurement processes (section 12), in the same probabilistic way as irreversibility emerges from Liouvillian dynamics in statistical mechanics.

The reader interested only in the outcomes may skip the technical sections 4 to 8, and focus upon section 9, which gathers all the outcomes of the dynamics in the Curie–Weiss model, and can be regarded as a *self-contained reading guide*.

1.3.3. Terminology employed for the basic concepts

Authors do not always assign the same meaning to some current words. In order to avoid misunderstandings, we gather here the definitions that we are using throughout.

- *Statistical ensemble*: a real or virtual set of systems prepared under identical conditions.
- *Quantum state*: a mathematical object from which all the probabilistic properties of a statistical ensemble of systems can be obtained. It is generally represented by a density operator or a density matrix. Pure states are characterized by an absence of statistical fluctuations for a complete set of commuting observables.

- *Measurement*: a dynamical process which involves an apparatus A coupled to a tested system S and which provides information about one observable $\hat{s}$ of S. The time-dependent state of the compound system S + A describes a statistical ensemble of runs, not individual runs. With this definition, the reading of the outcomes and the selection of the results are not encompassed in the "measurement".

- *Individual run of a measurement*: a single interaction process between tested system and apparatus (prepared in a metastable state), followed by the reading of the outcome.

- *Ideal measurement*: a measurement which does not perturb the observables of S that commute with $\hat{s}$. A non-ideal measurement informs us only about the initial state of S.

- *Reduction; disappearance of Schrödinger cat states*: the disappearance, at the end of the measurement process, of the off-diagonal blocks of the density matrix of S + A, in a basis where $\hat{s}$ is diagonal [10] [11].

- *Decoherence*: in general, a decay of the off-diagonal blocks of a density matrix under the effect of a random environment, such as a thermal bath.

- *Pointer*; *pointer variable*: a part of the apparatus which undergoes a change that can be read off or registered. In general the pointer should be macroscopic and the pointer variable should be collective.

- *Registration*: the creation during a measurement of correlations between S and the macroscopic pointer of A.

- *Selection*: the sorting of the runs of an ideal measurement, after reduction and registration, according to the indication of the pointer. The original ensemble that underwent the process is thus split into subensembles characterized by a well-defined value of $\hat{s}$. Measurement followed by selection may constitute a preparation.

2. The approach based on models

Point n'est besoin d'espérer pour entreprendre,
ni de réussir pour persévérer[12]
Charles le Téméraire and William of Orange

We have briefly surveyed in § 1.3.1 many theoretical ideas intended to elucidate the problem of quantum measurements. We feel that it is more appropriate to think along the lines of an experimentalist who performs measurements in his laboratory. For this reason, it is instructive to formulate and solve models with this scope. We review in this section various models in which S + A is treated as a compound system which evolves during the measurement process according to the standard rules of quantum mechanics. The existing models are roughly divided into related classes. Several models serve to elucidate open problems. Besides concrete models, we shall discuss several more general approaches to quantum measurements (e.g., the decoherence and consistent histories approaches).

2.1. Heisenberg–von Neumann setup

Before you milk a cow, tie it up
South African proverb

A general set-up of quantum measurement was proposed and analysed by Heisenberg [20, 21]. His ideas were formalized by von Neumann who proposed the very first mathematically rigorous model of quantum measurement [2]. An early review on this subject is by London and Bauer [26], in the sixties it was carefully reviewed by Wigner [8]; see [78] for a modern review.

[10] We will refrain from using popular terms as "collapse of the wave function" or "reduction of the wave packet"

[11] We refer to the disappearance of the off-diagonal blocks of the marginal density matrix of S as a "weak reduction". Reduction implies weak reduction, but weak reduction need not imply reduction

[12] It is not necessary to hope for undertaking, neither to succeed for persevering

Von Neumann formulated the measurement process as a coupling between two quantum systems with a specific interaction Hamiltonian that involves the (tensor) product between the measured observable of the tested system and the pointer variable, the observable of the apparatus. This interaction conserves the measured observable and ensures a correlation between the tested quantity and the pointer obervable. In one way or another the von Neumann interaction Hamiltonian is applied in all subsequent models of ideal quantum measurements. However, von Neumannn's model does not account for the differences between the microscopic [system] and macroscopic [apparatus] scales. As a main consequence, it does not have a mechanism to ensure the specific classical correlations (in the final state of the system + apparatus) necessary for the proper interpretation of a quantum measurement. Another drawback of this approach is its requirement for the initial state of the measuring system (the apparatus) to be a pure state (so it is described by a single wave function). Moreover, this should be a specific pure state, where fluctuations of the pointer variable are small. Both of these features are unrealistic.

With all these specific features it is not surprising that the von Neumann model has only one characteristic time driven by the interaction Hamiltonian. Over this time the apparatus variable gets correlated with the initial state of the measured system.

Jauch considers the main problem of the original von Neumann model, i.e. that in the final state it does not ensure specific classical correlations between the apparatus and the system [79]. A solution of this problem is attempted within the lines suggested (using his words) during "the heroic period of quantum mechanics" that is looking for classical features of the apparatus. To this end, Jauch introduces the concept of equivalence between two states (as represented by density matrices): two states are equivalent with respect to a set of observables, if these observables cannot distinguish one of these states from another [79]. Next, he shows that for the von Neumann model there is a natural set of commuting (hence classical) observables, so that with respect to this set the final state of the model is not distinguishable from the one having the needed classical correlations. At the same time Jauch accepts that some other observable of the system and the apparatus can distinguish these states. Next, he makes an attempt to *define* the measurement event via his concept of classical equivalence. In our opinion this attempt is interesting, but not successful.

2.2. Quantum–classical models: an open issue?

Following suggestions of Bohr that the proper quantum measurement should imply a classical apparatus [72, 73], there were several attempts to work out interaction between a quantum and an explicitly classical system [80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92]. (Neither Bohr [72], nor Landau and Lifshitz [73] who present Bohr's opinion in quite detail, consider the proper interaction processes.) This subject is referred to as hybrid (quantum–classical) dynamics. Besides the measurement theory it is supposed to apply in quantum chemistry [80, 81] (where the full modeling of quantum degrees of freedom is difficult) and in quantum gravity [93], where the proper quantum dynamics of the gravitational field is not known. There are several versions of the hybrid dynamics. The situation, where the classical degree of freedom is of a mean-field type is especially well-known [80, 81]. In that case the hybrid dynamics can be derived variationally from a simple combination of quantum and classical Lagrangian. More refined versions of the hybrid dynamics attempt to describe interactions between the classical degree(s) of freedom and quantum fluctuations. Such theories are supposed to be closed and self-consistent, and (if they really exist) they would somehow get the same fundamental status as their limiting cases, i.e., as quantum and classical mechanics. The numerous attempts to formulate such fundamental quantum–classical theories have encountered severe difficulties [83, 84, 85, 86, 87, 88, 89, 90, 91]. There are no–go theorems showing in which specific sense such theories cannot exist [91].

As far as the quantum measurement issues are concerned, the hybrid dynamical models have not received the attention they deserve. This is surprising, because Bohr's insistence on the classicality of the apparatus is widely known and frequently repeated. The existing works are summarized as follows. Diosi and co-authors stated that their scheme for the hybrid dynamics is useful for quantum measurements [83], albeit that they did not come with a more or less explicit analysis. Later on Terno has shown that the problem of a quantum measurement cannot be solved via a certain class of hybrid dynamic systems [94]. His arguments rely on the fact that the majority of hybrid system have pathological features in one way or another. Terno also reviews some earlier attempts, in particular by him in collaboration with Peres [90], to describe quantum measurements via hybrid dynamics; see the book of Peres [17] for preliminary ideas within this approach. However, recently Hall and Reginato [92, 95] suggest a scheme for the hybrid dynamics that seems to be free of pathological features. This scheme is based on coupled quantum and

classical ensembles. A related set-up of hybrid dynamics is proposed by Elze and coworkers based on a path-integral formulation [96], see also [97]. If Hall and Reginato's claim is true that such schemes can circumvent no-go theorems [92, 95], it should be interesting to look again at the features of quantum measurements from the perspective of an explicitly classical apparatus: Bohr's program can still be opened!

In closing this subsection we note that the relation between quantum and classical has yet another, *geometrical* twist, because the pure-state quantum dynamics (described by the Schrödinger equation) can be exactly mapped to a classical Hamiltonian dynamics evolving in a suitable classical symplectic space [98, 99, 100]. Quantum aspects (such as uncertainties and the Planck's constant) are then reflected via a Riemannian metrics in this space [99, 100]; see also [101] for a recent review. This is a geometrical counterpart to the usual algebraic description of quantum mechanics, and is considered to be a potentially rich source for various generalizations of quantum mechanics [101, 102]. A formulation of the quantum measurement problem in this language was attempted in [102]. We note that so far this approach is basically restricted to pure states (see, however, [100] in this context).

Everitt, Munro and Spiller discuss a measurement model which also incorporates explicitly classical features of the apparatus [103]. The model consists of a two-level system (the measured system), the apparatus, which is a one-dimensional quartic oscillator under external driving, and environment whose influence on the system + apparatus is described within the Lindblad master-equation approach. The main point of this work is that the apparatus can display chaotic features. Everitt, Munro and Spiller insist on this point for two model-dependent reasons (neither of which seems to us fundamental): The feature of chaos allows one to distinguish quantum versus classical regimes for the apparatus – to some extent only, because in a limited sense also a quantum system can be chaotic – while transition from regular to chaotic oscillations relates to comparing measurements with different time delays. The model reproduces certain features expected from individual measurement outcomes, but this happens at the cost of *unravelling* the master equation, a relatively arbitrary procedure of going from density matrices to random wave-functions. The authors of Ref. [103] are aware of this arbitrariness and attempt to minimize it.

2.3. Explicitly infinite apparatus: Coleman–Hepp and related models

Several authors argued that once the quantum measurement apparatus is supposed to be a macroscopic system, the most natural framework for describing measurements is to assume that it is explicitly infinite; see the review by Bub [104]. C^*-algebras is the standard tool for dealing with this situation [105]. Its main pecularity is that there are (many) inequivalent unitary representations of the algebra of observables, i.e., certain superpositions between wavefunctions cannot be physical states (in contrast to finite-dimensional Hilbert spaces) [104]. This is supposed to be helpful in constructing measurement models. Hepp proposed first such models [7]. He starts his investigation by clearly stating the goals of quantum measurement models. In particular, he stresses that the essence of the problem is in getting classical correlations between the measured observable and the pointer variable of the apparatus, and that quantum mechanics is a theory that describes probabilities of certain events. Hepp then argues that the quantum measurement problem can be solved, i.e., the required classical correlations can be established dynamically, if one restricts oneself to macroscopic observables. He then moves to concrete models, which are solved in the C^*-algebraic framework. The infinite system approach is also employed in the quantum measurement model proposed by Whitten-Wolfe and Emch [106]. However, working with an infinite measuring apparatus hides the physical meaning of the approach, because some important dynamic scales of the quantum measurement do depend on the number of degrees of freedom of the apparatus [60]. Thus, making the apparatus explicitly infinite (instead of taking it large, but finite) misses an important piece of physics, and does not allow to understand which features of the quantum measurement will survive for a apparatus having a mesoscopic scale.

Hepp also studies several exactly solvable models, which demonstrate various aspects of his proposal. One of them—proposed to Hepp by Coleman and nowadays called the Coleman–Hepp model— describes an ultra-relativistic particle interacting with a linear chain of spins. Hepp analyzes this model in the infinite apparatus situation; this has several drawbacks, e.g., the overall measurement time is obviously infinite. The physical representation of the Coleman–Hepp Hamiltonian is improved by Nakazato and Pascazio [107]. They show that the basic conclusions on the Coleman–Hepp Hamiltonian approach can survive in a more realistic model, where the self-energy of the spin chain is taken into account. Nakazato and Pascazio also discuss subtleties involved in taking the thermodynamic limit for the model [107]. The Coleman–Hepp model with a large but finite number of the apparatus particles is studied by Sewell [108, 109, 110]. He improves on previous treatments by carefully calculating the dependence of

the characteristic times of the model on this number, and discusses possible imperfections of the measurement model arising from a finite number of particles.

Using the example of the Coleman–Hepp model, Bell demonstrates explicitly [77] that the specific features of the quantum measurement hold only for a certain class of observables, including macroscopic observables [61, 108, 109, 110]. It is then possible to construct an observable for which those features do not hold [77]. We recall that the same holds in the irreversibility problem: it is always possible to construct an observable of a macroscopic system (having a large, but finite number of particles) that will not show the signs of irreversible dynamics, i.e., it will not be subject to relaxation. Bell takes this aspect as an essential drawback and states that the quantum measurement was not and cannot be solved within a statistical mechanics approach [77]. Our attitude in the present paper is different. We believe that although concrete models of quantum measurements may have various drawbacks, the resolution of the measurement problem is definitely to be sought along the routes of quantum statistical mechanics. The fact that certain restrictions on the set of observables are needed, simply indicates that, similar to irreversibility, a quantum measurement is an emergent phenomenon of a large system – the tested system combined with the apparatus – over some characteristic time.

2.4. Quantum statistical models

If I have a thousand ideas and only one turns out to be good, I am satisfied
Alfred Bernhard Nobel

Here we describe several models based on quantum statistical mechanics. In contrast to the previous chapter, these models do not invoke anything beyond the standard quantum mechanics of finite though large systems.

Green proposed a realistic model of quantum measurement [111]. He emphasizes the necessity of describing the apparatus via a mixed, quasi-equilibrium state and stresses that the initial state of the apparatus should be macroscopic and metastable. The model studied in [111] includes a spin-$\frac{1}{2}$ particle interacting with two thermal baths at different temperatures. The two-temperature situation serves to simulate metastability. The tested particle switches interaction between the baths. By registering the amount of heat flow through the baths (a macroscopic pointer variable), one can draw certain conclusions about the initial state of the spin. Off-diagonal terms of the spin density matrix are suppressed via a mechanism akin to inhomogeneous broadening. However, an explicit analysis of the dynamic regime and its characteristic times is absent.

Cini studies a simple model for the quantum measurement process which illustrates some of the aspects related to the macroscopic character of the apparatus [112]. The model is exactly solvable and can be boiled down to a spin-$\frac{1}{2}$ particle (tested spin) interacting with a spin-L particle (apparatus). The interaction Hamiltonian is $\propto \sigma_z L_z$, where σ_z and L_z are, respectively, the third components of the spin-$\frac{1}{2}$ and spin L. Cini shows that in the limit $L \gg 1$ and for a sufficiently long interaction time, the off-diagonal terms introduced by an (arbitrary) initial state of the tested spin give negligible contributions to the observed quantities, i.e., to the variables of the tested spin and the collective variables of the apparatus. The characteristic times of this process are analyzed, as well as the situation with a large but finite value of L.

In Refs. [5, 6] Blokhintsev studies, within the statistical interpretation of quantum mechanics, several interesting measurement models with a metastable initial state of the apparatus: an incoming test particle interacting with an apparatus-particle in a metastable potential well, a test neutron triggering a nuclear chain reaction, et cetera. Though the considered models are physically appealing, the involved measurement apparatuses are frequently not really macroscopic. Neither does Blokhintsev pay proper attention to the correlations between the system and the apparatus in the final state.

Requardt studies a quantum measurement model, in which due to collisional interaction with the tested system, the pointer variable of a macroscopic measuring apparatus undergoes a coherent motion, in which the momentum correlates with the values of the measured observable (coordinate) [113]. It is stressed that for the approach to have a proper physical meaning, the apparatus should have a large but finite number of degrees of freedom. However, no detailed account of characteristic measurement times is given. Requardt also assumes that the initial state of the measurement apparatus is described by a wave function, which is merely consistent with the macroscopic information initially available on this apparatus. He focuses on those aspects of the model which will likely survive in a more general theory of quantum measurements; see in this context his later work [61] that is reviewed below.

An interesting statistical mechanical model of quantum measurement was proposed and studied in Ref. [114] by Gaveau and Schulman. The role of apparatus is played by a one-dimensional Ising spin model. Two basic energy parameters of the model are an external field and the spin-spin coupling (exchange coupling). An external field is tuned in such a way that a spontaneous flipping of one spin is energetically not beneficial, while the characteristic time of flipping two spins simultaneously is very large. This requirement of metastability puts an upper limit on the number of spins in the apparatus. The tested spin $\frac{1}{2}$ interacts only with one spin of the apparatus; this is definitely an advantage of this model. The spin-apparatus interaction creates a domino effect bringing the apparatus to a unique ferrromagnetic state. This happens for the tested spin pointing up. For the tested spin pointing down nothing happens, since in this state the tested spin does not interact with the apparatus. Characteristic times of the measurement are not studied in detail, though Gaveau and Schulman calculate the overall relaxation time and the decay time of the metastable state. It is unclear whether this model is supposed to work for an arbitrary initial state of the tested spin.

Ref. [115] by Merlin studies a quantum mechanical model for distinguishing two different types of bosonic particles. The model is inspired by Glaser's chamber device, and has the realistic feature that the bosonic particle to be tested interacts only with one particle of the apparatus (which by itself is made out of bosons). The initial state of the apparatus is described by a pure state and it is formally metastable (formally, because this is not a thermodynamic metastability). The relaxation process is not accounted for explicitly; its consequences are simply postulated. No analysis of characteristic relaxation times is presented. Merlin analyses the relation of measurement processes with the phenomenon of spontaneous symmetry breaking.

2.4.1. Spontaneous symmetry breaking

The role of spontaneous symmetry breaking as an essential ingredient of the quantum measurement process is underlined in papers by Grady [116], Fioroni and Immirzi [117] and Pankovic and Predojevic [118]. They stress that superpositions of vacuum states are not allowed in quantum field theory, since these superpositions do not satisfy the cluster property. All three approaches stay mainly at a qualitative level, though Fioroni and Immirzi go somewhat further in relating ideas on quantum measurement process to specific first-order phase transition scenarios. An earlier discussion on symmetry breaking, quantum measurements and geometrical concepts of quantum field theory is given by Ne'eman [119].

Ref. [120] by Zimanyi and Vladar also emphasizes the relevance of phase transitions and symmetry breaking for quantum measurements. They explicitly adopt the statistical interpretation of quantum mechanics. General statements are illustrated via the Caldeira-Leggett model [121, 122, 123, 124]: a two-level system coupled to a bath of harmonic oscillators. This model undergoes a second-order phase transition with relatively weak decay of off-diganonal terms in the thermodynamic limit, provided that the coupling of the two-level system to the bath is sufficiently strong. The authors speculate about extending their results to first-order phase transitions. A dynamical consideration is basically absent and the physical meaning of the pointer variable is not clear.

Thus the concept of spontaneous symmetry breaking is frequently discussed in the context of quantum measurement models (although it is not anymore strictly spontaneous, but driven by the interaction with the system of which the observable is to be measured). It is also an essential feature of the approach discussed in the present paper. It should however be noted that so far only one scenario of symmetry breaking has been considered in the context of quantum measurements (the one that can be called the classical scenario), where the higher temperature extremum of the free energy becomes unstable (or at least metastable) and the system moves to another, more stable state (with lower free energy). Another scenario is known for certain quantum systems (e.g., quantum antiferromagnets) with a low-temperature spontaneously symmetry broken state; see, e.g., [125]. Here the non-symmetric state is not an eigenstate of the Hamiltonian, and (in general) does not have less energy than the unstable ground state. The consequences of this (quantum) scenario for quantum measurements are so far not explored. However, recently van Wezel, van den Brink and Zaanen studied specific decoherence mechanisms that are induced by this scenario of symmetry breaking. [125].

2.4.2. System-pointer-bath models

Refs. [126] by Haake and Walls and [127] by Haake and Zukowski study a measurement of a discrete-spectrum variable coupled to a single-particle apparatus (the meter). The latter is a harmonic oscillator, and it interacts with a thermal bath, which is modeled via harmonic oscillators. The interaction between the tested system and the meter is impulsive (it lasts a short time) and involves the tensor product of the measured observable and the momentum

of the meter. There are two characteristic times here: on the shorter time, the impulsive interaction correlates the states of the object and of the meter, while on the longer time scale the state of the meter becomes classical under the influence of the thermal bath, and the probability distribution of the meter coordinate is prepared via mixing well-localized probability distributions centered at the eigenvalues of the measured quantity, with the weights satisfying the Born rule [1]. (This sequence of processes roughly corresponds to the ideas of decoherence theory; see below for more detail.) At an even longer time scale the meter will completely thermalize and forget about its interaction with the tested system. The authors of [126, 127] also consider a situation where the meter becomes unstable under the influence of the thermal bath, since it now feels an inverted parabolic potential. Then the selection of the concrete branch of instability can be driven by the interaction with the object. Since the initial state of such an unstable oscillator is not properly metastable, one has to select a special regime where the spontaneous instability decay can be neglected.

The quantum measurement model studied in [128] by Venugopalan is in many aspects similar to models investigated in [126, 127]. The author stresses relations of the studied model to ideas from the decoherence theory.

Ref. [129] by the present authors investigates a model of quantum measurement where the macroscopic measurement apparatus is modeled as an ideal Bose gas, in which the amplitude of the condensate is taken as the pointer variable. The model is essentially based on the properties of irreversibility and of ergodicity breaking, which are inherent in the model apparatus. The measurement process takes place in two steps: First, the reduction of the state of the tested system takes place, this process is governed by the apparatus-system interaction. During the second step classical correlations are established between the apparatus and the tested system over the much longer time scale of equilibration of the apparatus. While the model allows to understand some basic features of the quantum measurement as a driven phase-transition, its dynamical treatment contains definite drawbacks. First, the Markov approximation for the apparatus-bath interaction, though correct for large times, is incorrectly employed for very short times, which greatly overestimates the reduction time. Another drawback is that the model is based on the phase transition in an ideal Bose gas. This transition is known to have certain pathological features (as compared to a more realistic phase-transition in a weakly interacting Bose gas). Though the authors believe that this fact did not influence the qualitative outcomes of the model, it is certainly desirable to have better models, where the phase transition scenario would be generic and robust. Such models will be considered in later chapters of this work.

In Ref. [130, 131] Spehner and Haake present a measurement model that in several aspects improves upon previous models. The model includes the tested system, an oscillator (generally anharmonic), which plays the role of apparatus, and a thermal bath coupled to the oscillator. The time scales of the model are set in such a way that the correlations between the measured observable of the system and the pointer variable of the apparatus (here the momentum of the anharmonic oscillator) and the decay of the off-diagonal terms of the tested system density matrix are established simultaneously. This implies realistically that no macroscopic superpositions are generated. In addition, the initial state of the apparatus and its bath is not assumed to be factorized, which makes it possible to study strong (and also anharmonic) apparatus-bath couplings.

Ref. [132] by Mozyrsky and Privman studies a quantum measurement model, which consists of three parts: the tested system, the apparatus and a thermal bath that directly couples to the system (and not to the apparatus). The initial state of the apparatus is not metastable, it is chosen to be an equilibrium state. The dynamics of the measured observable of the system is neglected in the course of measurement. The authors of [132] show that after some decoherence time their model is able to reproduce specific correlations that are expected for a proper quantum measurement.

Omnès recently studied a model for a quantum measurement [133]. The pointer variable of the apparatus is supposed to be its (collective) coordinate. The introduction of the measurement process is accompanied by a discussion on self-organization. For solving this Omnès partially involves the mean-field method, because the many-body apparatus density matrix is substituted by the tensor product of the partial density matrices. The dynamics of the model involves both decoherence and reduction. These two different processes are analysed together and sometimes in rather common terms, which can obscure important physical differences between them. In the second part Omnès studies fluctuations of the observation probabilities for various measurement results. These fluctuations are said to arise due to a coupling with an external environment modeled as a phonon bath.

Van Kampen stresses the importance of considering a macroscopic and metastable measuring apparatus and proposes a model that is supposed to illustrate the main aspects of the measurement process [9]. The model consists of a single atom interacting with a multi-mode electromagnetic field, which is playing the role of apparatus. The emitted

photon that is generated correlates with the value of the measured observable. The apparatus can be macroscopic (since the vacuum has many modes), but its (thermodynamically) metastable character is questionable. The model is not solved in detail, and its main dynamical consequences are not analyzed. Nevertheless, van Kampen offers a qualitative analysis of this model, which appears to support the common intuition on quantum measurements. The resulting insights are summarized in his "ten theorems" on quantum measurements.

2.4.3. Towards model-independent approaches

Qui se soucie de chaque petite plume ne devrait pas faire le lit[13]
Swiss proverb

Sewell [108, 109, 110] and independently Requardt [61] attempt to put the results obtained from several models into a single model-independent approach, which presumably may pave a way towards a general theory of quantum measurements. The basic starting point of the approach is that the measuring apparatus, being a many-body quantum system, does have a set of macroscopic, mutually commuting observables $\{A_1, \ldots, A_{\mathcal{M}}\}$ with $\mathcal{M}$ a macroscopic integer. The commutation is approximate for a large, but finite number of reservoir particles, but it becomes exact in the thermodynamic limit for the apparatus. Each A_k is typically a normalized sum over a large number of apparatus particles. The set $\{A_1, \ldots, A_{\mathcal{M}}\}$ is now partioned into macroscopic cells; each such cell refers to some subspace in the Hilbert space formed by a common eigenvector. The cells are distinguished from each other by certain combinations of the eigenvalues of $\{A_1, \ldots, A_{\mathcal{M}}\}$. The purpose of partioning into cells is to correlate each eigenvalue of the microscopic observable to be measured with the corresponding cell. In the simplest situation the latter set reduces to just one observable A, while two cells refer to the subspace formed by the eigenvectors of A associated with positive or negative eigenvalues. Further derivations, which so far were carried out on the levels of models only [61, 108, 109, 110], amount to showing that a specific coupling between the system and the apparatus can produce their joint final state, which from the viewpoint of observables $A_k \otimes S$ — where S is any observable of the microscopic measured system — does have several features required for a good (or even ideal) quantum measurement.

2.4.4. Ergodic theory approach

Daneri, Loinger and Prosperi approach quantum measurements via the quantum ergodic theory [16]. Such an approach was anticipated in the late forties by the works of Jordan [134] and Ludwig [135]. Daneri, Loinger and Prosperi model the measuring apparatus as a macroscopic system, which in addition to energy has another conserved quantity, which serves the role of the pointer variable. Under the influence of the system to be measured this conservation is broken, and there is a possibility to correlate different values of the measured observable with the pointer values. Daneri, Loinger and Prosperi invoke the basic assumption of ergodic theory and treat the overall density matrix via time-averaging [16]. The time-averaged density matrix satisfies the necessary requirements for an ideal measurement. However, the use of the time-averaging does not allow to understand the dynamics of the quantum measurement process, because no information about the actual dynamical time scales is retained in the time-averaged density matrix. Also, although the initial state of the measuring apparatus does have some properties of metastability, it is not really metastable in the thermodynamic sense.

The publication of the paper by Daneri, Loinger and Prosperi in early sixties induced a hot debate on the measurement problem; see [136] for a historical outline. We shall not attempt to review this debate here, but only mention one aspect of it: Tausk (see [136] for a description of his unpublished work) and later on Jauch, Wigner and Yanase [137] criticize the approach by Daneri, Loinger and Prosperi via the argument of an interaction free measurement. This type of measurements is first discussed by Renninger [138]. The argument goes as follows: sometimes one can gather information about the measured system even without any macroscopic process generated in the measuring apparatus. This can happen, for instance, in the double-slit experiment when the apparatus measuring the coordinate of the particle is placed only at one slit. Then the non-detection by this apparatus will — ideally — indicate that the particle passed through the other slit. The argument thus intends to demonstrate that quantum measurements need not be related to macroscopic (or irreversible) processes. This argument however does not present any special difficulty within the statistical interpretation of quantum mechanics, where both the wavefunction and the density matrix refer

[13] Who cares about every little feather should not make the bed

to an ensemble of identically prepared system. Although it is true that not every single realization of the apparatus-particle interaction has to be related to a macroscopic process, the probabilities of getting various measurement results do rely on macroscopic processes in the measuring apparatus.

2.5. No-go theorems and small measuring apparatuses

Non ho l'età, per amarti[14]
Lyrics by Mario Penzeri, sung by Gigliola Cinquetti

The quantum measurement process is regarded as a fundamental problem, also because over the years several no–go theorems were established showing that the proper conditions for quantum measurement cannot be satisfied if they are demanded as exact features of the final state of the apparatus [8, 139, 140, 141]. The first such theorem was established by Wigner [8]. Then several extensions of this theorem were elaborated by Fine [142] and Shimony with co-authors [139, 140, 141]. The presentation by Fine is especially clear, as it starts from the minimal conditions required from a quantum measurement [142]. After stating the no-go theorem, Fine proceeds to discuss in which sense one should look for approximate schemes that satisfy the measurement conditions, a general program motivating also the present study. The results of Refs. [139, 140, 141] show that even when allowing certain imperfections in the apparatus functioning, the quantum measurement problem remains unsolvable in the sense that the existence of specific classical correlations in the final state of the system + apparatus cannnot be ensured; see also in this context the recent review by Bassi and Ghirardi [12]. In our viewpoint, the no-go theorems do not preclude approximate satisfaction of the quantum measurement requirements – owing to a macroscopic size of the apparatus.

Turning this point over, one may ask which features of proper quantum measurements (as displayed by successful models of this phenomenon) would survive for an apparatus that is not macroscopically large. There are several different ways to pose this question, e.g., below we shall study the measuring apparatus (that already performs well in the macroscopic limit) for a large but finite number of particles. Another approach was recently worked out by Allahverdyan and Hovhannisyan [66]. They assume that the measuring apparatus is a finite system, and study system-apparatus interaction setups that lead to transferring certain matrix elements of the unknown density matrix λ of the system into those of the final state $\widetilde{r}$ of the apparatus. Such a transfer process represents one essential aspect of the quantum measurement with a macroscopic apparatus. No further limitations on the interaction are introduced, because the purpose is to understand the implications of the transfer on the final state of the system. It is shown that the transfer process eliminates from the final state of the system the memory about the transferred matrix elements (or certain other ones) [66]. In particular, if one diagonal matrix element is transferred, $\widetilde{r}_{aa} = \lambda_{aa}$, the memory on all non-diagonal elements $\lambda_{a\neq b}$ or $\lambda_{b\neq a}$ related to this diagonal element is completely eliminated from the final density operator of the system (the memory on other non-diagonal elements λ_{cd}, where $c \neq a$ and $d \neq a$ may be preserved). Thus, the general aspect of state disturbance in quantum measurements is the loss of memory about off-diagonal elements, rather than diagonalization (which means the vanishing of the off-diagonal elements).

2.6. An open problem: A model for a non-statistical interpretation of the measurement process.

We can't go on forever, with suspicious minds
Written by Mark James, sung by Elvis Presley

The statistical interpretation together with supporting models does provide a consistent view on measurements within the standard quantum mechanics. However, it should be important to understand whether there are other consistent approaches *from within* the standard quantum formalism that can provide an alternative view on quantum measurements. Indeed, it cannot be excluded that the real quantum measurement is a wide notion, which combines instances of different interpretations. In the present review we will not cover approaches that introduce additional ingredients to the standard quantum theory, and will only mention them in subsection 2.8.

We focus only on one alternative to the statistical interpretation, which is essentially close to the Copenhagen interpretation [72, 73, 143] and is based on effectively non-linear Schrödinger equation. We should however stress that so far the approach did not yet provide a fully consistent and unifying picture of quantum measurements even for one model.

[14] I do not have the age to love you

Recently Brox, Olaussen and Nguyen approached quantum measurements via a non-linear Schrödinger equation [144]. The authors explicitly adhere to a version of the Copenhagen interpretation, where the wave function (the pure quantum state) refers to a single system. They present a model which is able to account for single measurement events. The model consists of a spin-$\frac{1}{2}$ (the system to be measured), a ferromagnet (the measuring apparatus), and the apparatus environment. The overall system is described by a pure wavefunction. The ferromagnetic apparatus is prepared in an (unbiased) initial state with zero magnetization. The two ground states of the ferromagnet have a lower energy and, respectively, positive and negative magnetization. Moving towards one of these states under influence of the tested system is supposed to amplify the weak signal coming from this tested system. (The latter features will also play an important role in the models to be considered in detail later on.) The environment is modeled as a spin-glass: environmental spins interact with random (positive or negative) coupling constants. So far all these factors are more or less standard, and — as stressed by the authors — these factors alone cannot account for a solution of the measurement problem within an interpretation that ascribes the wavefunction to a single system. The new point introduced by Brox, Olaussen and Nguyen is that the effective interaction between the apparatus and the measured system is non-linear in the wavefunction: it contains an analogue of a self-induced magnetic field [144]. In contrast to the existing approaches, where non-linearity in the Schrödinger equation are introduced axiomatically, Brox, Olaussen and Nguyen state that their non-linearity can in fact emerge from the Hartree-Fock approach: it is known that in certain situations (the Vlasov limit) the many-body Schrödinger equation can be reduced to a non-linear equation for the single-particle wave function [145]. Examples of this are the Gross-Pitaevskii equation for Bose condensates [145] or the non-linear equation arising during quantum feedback control [146]. However, the statement by Brox, Olaussen and Nguyen on the emergent non-linearity is not really proven, which is an essential drawback. Leaving this problem aside, these authors show numerically that the specific nonlinearity in the system-apparatus interaction may lead to a definite, albeit random, measurement result. The statistics of this randomness approximately satisfies the Born rule [1], which emerges due to the macroscopic size of the apparatus. The cause of this randomness is the classical randomness related to the choice of the spin-glass interaction constants in the environment [144], i.e., for different such choices (each one still ensuring the proper relaxation of the apparatus) one gets different single-measurement results. Thus in this approach the cause of the randomness in measurement results is not the irreducible quantum randomness, but rather the usual classical randomness, which is practically unavoidable in the preparation of a macroscopic environment. Brox, Olaussen and Nguyen argue that the nonlinearity in the system-bath interaction — which is crucial for obtaining all the above effects — need not be large, since the amplification may be ensured by a large size of the ferromagnet [144]. Their actual numerical calculations are however carried out only for moderate-size spin systems.

2.7. Decoherence theory

Presently it is often believed that decoherence theory solves the quantum measurement problem. So let us introduce this concept. Decoherence refers to a process, where due to coupling with an external environment, off-diagonal elements of the system density matrix decay in time; see [28, 29, 35, 147, 148, 149, 150] for reviews. The basis where this decay happens is selected by the structure of the system-environment coupling. In this way the system acquires some classical features.

Decoherence is well known since the late 40's [151]. One celebrated example is spin relaxation in NMR experiments. The decay of the transverse polarization, perpendicular to the permanently applied field, is in general characterized by the relaxation time $\mathcal{T}_2$; it can be viewed as a decoherence of the spin system, since it exhibits the decay of the off-diagonal contributions to the spin density matrix in the representation where the applied Hamiltonian is diagonal [152, 153]. Another standard example is related to the Pauli equation for an open quantum system weakly coupled to an external thermal bath [154]. This equation can be visualized as a classical stochastic process during which the system transits from one energy level to another.

More recently decoherence has attracted attention as a mechanism of quantum-to-classical transition, and was applied to the quantum measurement problem [28, 29, 35, 147, 148, 149, 150]. The standard pattern of such an application relies on an initial impulsive interaction of the von Neumann type which correlates (entangles) the measuring apparatus with the system to be measured. Generally, this step is rather unrealistic, since it realizes macroscopic superpositions, which were never seen in any realistic measurement or any measurement model. Next, one assumes a specific environment for the apparatus, with the environment-apparatus interaction Hamiltonian directly related to the variable to be measured. Moreover, within the decoherence approach it is stressed – e.g., by Zurek in [29] and

by Milburn and Walls in [149] – that the observable to-be-measured is determined during the process generated by the apparatus-environment interaction. The latter is supposed to diagonalize the density matrix of the system plus the apparatus in a suitable basis. This second step is again unrealistic, since it assumes that the variable to be measured, which is normally under control of the experimentalist, must somehow correlate with the structure of the system's environment, which – by its very definition – is out of direct control. To put it in metaphoric terms, decoherence theory asserts that *the surrounding air measures a person's size*. But without explicit pointer variable that can be read off, this is not what one normally understands under *measuring a person's size*; we consider *measurement without a readable pointer variable* merely as a linguistic redefinition of the concept, that obscures the real issue. These criticisms of the decoherence theory approach agree with the recent analysis by Requardt [61].

One even notes that, as far as the problem of quantum-to-classical transition is concerned, the decoherence cannot be regarded as the only – or even as the basic – mechanism of this transition. As convincingly argued by Wiebe and Ballentine [155] and Ballentine [156], realistic macroscopic Hamiltonian systems can – and sometimes even should – achieve the classical limit without invoking any decoherence effect. This concerns both chaotic and regular Hamiltonian systems, although the concrete scenarios of approaching the classical limit differ for the two cases.

In spite of these caveats that prevent decoherence theory to provide *the* solution, it has been valuable in shaping the ideas on quantum measurement models, In particular, this concerns a recent attempt by Omnès to develop a general theory of decoherence via ideas and methods of non-equilibrium statistical mechanics [157] (see also [133] that we reviewed above). Among the issues addressed in [157] is the generality of the system-environment structure that leads to decoherence, the physical meaning of separating the system from the environment, and the relation of the decoherence theory to the hydrodynamic description.

2.8. Seeking the solution outside quantum mechanics

No, no, you're not thinking; you're just being logical
Niels Bohr

Though this review will restrict itself to approaches to quantum measurements within the standard quantum mechanics, we briefly list for completeness a number of attempts to seek the solution for the quantum measurement problem beyond it. The de Broglie–Bohm approach [13, 14, 19] is currently one of the most popular alternatives to the standard quantum mechanics. It introduces an additional set of variables (coordinates of the physical particles) and represents the Schrödinger equation as an equation of motion for those particles, *in addition* to the motion of the wavefunction, which keeps the physical meaning of a separate entity (guiding field). Hence in this picture there are two fundamental and separate entities: particles and fields. Recently Smolin attempted to construct a version of the de Broglie–Bohm approach, where the wavefunction is substituted by certain phase-variables, which, together with coordinates, are supposed to be features of particles [158]. In this context see also a related contribution by Schmelzer, where the fundamental character of the wavefunction is likewise negated [159]. The approach by Smolin is coined in terms of a real ensemble, which—in contrast to ensembles of non-interacting objects invoked for validation of any probabilistic theory—does contain highly-nonlocal (distance independent) interactions between its constituents. It is presently unclear to which extent this substitution of the wavefunction by phase-variables will increase the eligibility of the de Broglie–Bohm approach, while Smolin does not discuss the issues of measurement that are known to be non-trivial within the approach [13, 14, 160].

Another popular alternative is the spontaneous localization approach by Ghirardi, Rimini and Weber [161]. This approach is based on a non-linear and stochastic generalization of the Schrödinger equation such that the collapse of the wavefunction happens spontaneously (i.e., without any measurement) with a certain rate governed by classical white noise. Bassi and Ghirardi recently reviewed this and related approaches in full detail [12]; other useful sources are the book by Adler [162] and the review paper by Pearle [163]. Non-linear modifications of the Schrödinger equation have by now a long history [75, 76, 164, 165, 166, 167, 168]. All of them in one way or another combine non-linearities with classical randomness. The first such model was introduced by Bohm and Bub [164] starting from certain hidden-variables assumption. The approaches that followed were either oriented towards resolving the quantum measurement problem [75, 166, 167] or trying to obtain fundamentally non-linear generalizations of the Schrödinger equation and quantum mechanics [168]. Several approaches of the former type were unified within a formalism proposed by Grigorenko [76]. Recently Svetlichny presented a resource letter on the latter type of

approaches [169]. Some of those approaches based on non-linear generalizations of the Schrödinger equation were confronted to experiments, see e.g. Refs. [170, 171], but so far with negative result.

A very different approach was taken by De Raedt and Michielsen, who simulate the measurement process by specifying a set of simple rules that mimic the various components of the measurement setup, such as beam splitters, polarizers and detectors. They perform numerical simulations using algorithms that the mimic the underlying events, and are able to reproduce the statistical distributions given by quantum mechanics [172, 173].

2.9. A short review on consistent histories

The consistent histories approach negates the fundamental need of measurements for understanding quantum measurements (quantum mechanics without measurements). It was proposed by Griffiths [174] based on earlier ideas of Aharonov, Bergmann and Lebowitz [175]. The approach is reviewed, e.g., by Griffiths [176], Gell-Mann [177], Hohenberg [178], and Omnès [179]. It aims to develop an interpretation of quantum mechanics valid for a closed system of any size and any number of particles, the evolution of which is governed by the Liouville–von Neumann (or Heisenberg) equation. Within this approach the notion of an event—together with its probability—is defined from the outset and measurements simply reveal the pre-existing values of physical quantities. In particular, it is not necessary to invoke either the micro-macro separation or statistical assumptions on the initial states needed to derive the irreversibility aspect of quantum measurements. All of these may still be needed to describe concrete measurements, but the fundamental need for understanding quantum measurements from the viewpoint of statistical mechanics would be gone, if the consistent histories approach is viable or, at least, will turn out to be really viable in the end.

However, as it stands presently the approach produces results at variance with predictions of the measurement-based quantum mechanics [180] (then it is not important which specific interpretation one prescribes). Hence, within its present status, the consistent histories approach cannot be a substitute for the statistical mechanics-based theory of quantum measurements. Some authors by-pass problems of the consistent histories approach and state that it is useful as a paradox-free reformulation of the standard mecanics; see e.g. the recent review by Hohenberg [178] and the book by Griffiths [176]. In fact the opposite is true: as we explain below, the consistent histories approach adds paradoxes that do not exist within the statistical interpretation of quantum mechanics.

2.9.1. Deeper into consistent histories

The easiest method of introducing the consistent histories approach is to highlight as soon as possible its differences with respect to the standard measurement-based approach. Let us start with the quantum mechanics formula for the probability of two consecutive measurements $\mathcal{M}(t_1)$ and $\mathcal{M}(t_2)$ carried at times t_1 and t_2 ($t_2 > t_1$):

$$p_{t_1,t_2}\,[i,\,j|\mathcal{M}(t_1),\mathcal{M}(t_2)] = \mathrm{tr}\left[\Pi_j(t_2)\Pi_i(t_1)\rho\Pi_i(t_1)\Pi_j(t_2)\right], \tag{2.1}$$

where ρ is the initial state of the system, $\Pi_i(t_1)$ with $\sum_i \Pi_i(t_1) = 1$ and $\Pi_j(t_2)$ with sum $\sum_i \Pi_i(t_2) = 1$ are the projectors for the physical quantities (represented by Hermitean operators) $A(t_1)$ and $B(t_2)$ measured at the times t_1 and t_2, respectively. For simplicity we assume the Heisenberg representation, and do not write in (2.1) explicit indices for A and B. What is however *necessary* to do is to indicate that the joint probability in (2.1) is explicitly conditional on the two measurements $\mathcal{M}(t_1)$ and $\mathcal{M}(t_2)$. As expected, the meaning of (2.1) is that the measurement at time t_1 (with probabilities given by Born's rule) is accompanied by selection of the sub-ensemble referring to the result i. The members of this subensemble are then measured at the time t_2. Generalization to n-time measurements is obvious.

What now the consistent histories approach does is to skip the context-dependence in (2.1) and regard the resulting probabilities $p[i,\,j]$ as a description of events taking place spontaneously, i.e. *without any measurement and without any selection of outcome*. The cost to pay is that the initial state ρ and the projectors $\Pi_i(t_1)$ and $\Pi_j(t_2)$ have to satisfy a special *consistency* condition (without this condition the events are not defined):

$$\mathrm{tr}\left[\Pi_j(t_2)\Pi_i(t_1)\rho\Pi_{i'}(t_1)\Pi_{j'}(t_2)\right] = \delta_{ii'}\delta_{jj'}\,p_{t_1,t_2}[i,\,j], \tag{2.2}$$

where $\delta_{ii'}$ is the Kronecker delta. As a consequence of (2.2), one can sum out the first (i. e., the earlier) random variable and get the probability for the second event only:

$$p_{t_2}[j] = \sum_i p_{t_1,t_2}[i,\,j] = \mathrm{tr}\left[\Pi_j(t_2)\rho\Pi_j(t_2)\right]. \tag{2.3}$$

Note that without condition (2.2), i.e. just staying within the standard approach (2.1), (2.3) would not hold, e.g. generally $\sum_i p_{t_1,t_2}[i,j|\mathcal{M}(t_1),\mathcal{M}(t_2)]$ still depends on $\mathcal{M}(t_1)$ and is not equal to $p_{t_2}[j|\mathcal{M}(t_2)]$ (probability of the outcome j in the second measurement provided that no first measurement was done). This is natural, since quantum measurements generally do perturb the state of the measured system. Hence (2.2) selects only those situations, where this perturbation is seemingly absent.

Any time-ordered sequence of events defines a history. A set of histories satisfying (2.2) is called a *consistent histories* set. Due to (2.2), the overall probability of the consistent histories sums to one.

In effect (2.2) forbids superpositions; hence, it is called decoherence condition [176, 177, 178]. One notes that (2.2) is sufficient, but not necessary for deriving (2.3). Hence, certain weaker conditions instead of (2.2) were also studied [174], but generally they do not satisfy the straightforward statistical independence conditions (independently evolving systems have independent probabilities) [181].

It was however noted that the consistent histories approach can produce predictions at variance with the measurement based quantum mechanics [180]. The simplest example of such a situation is given in [182]. Consider a quantum system with zero Hamiltonian in the pure initial state

$$\rho = |\phi\rangle\langle\phi|, \quad |\phi\rangle = \frac{1}{\sqrt{3}}[|a_1\rangle + |a_2\rangle + |a_3\rangle], \tag{2.4}$$

where the vectors $\{|a_k\rangle\}_{k=1}^3$ are orthonormal: $\langle a_k|a_{k'}\rangle = \delta_{kk'}$. Define a two-event history with projectors

$$\{\Pi_1(t_1) = |a_1\rangle\langle a_1|, \Pi_2(t_1) = 1 - |a_1\rangle\langle a_1|\} \text{ and } \{\Pi_1(t_2) = |\psi\rangle\langle\psi|, \Pi_2(t_2) = 1 - |\psi\rangle\langle\psi|\}, \quad t_2 > t_1, \tag{2.5}$$

where

$$|\psi\rangle = \frac{1}{\sqrt{3}}[|a_1\rangle + |a_2\rangle - |a_3\rangle]. \tag{2.6}$$

This history is consistent, since conditions (2.2) hold due to $\langle\phi|\psi\rangle = \langle\phi|a_1\rangle\langle a_1|\psi\rangle$. One now calculates

$$p_{t_1,t_2}[a_1,\psi] = \mathrm{tr}\left[\Pi_1(t_2)\Pi_1(t_1)\rho\Pi_1(t_1)\Pi_1(t_2)\right] = \langle\psi|a_1\rangle\langle a_1|\phi\rangle\langle\phi|a_1\rangle\langle a_1|\psi\rangle = \frac{1}{9}, \tag{2.7}$$

$$p_{t_2}[\psi] = \mathrm{tr}\left[\Pi_1(t_2)\rho\Pi_1(t_2)\right] = |\langle\psi|\phi\rangle|^2 = \frac{1}{9}. \tag{2.8}$$

Given two probabilities (2.7) and (2.8) one can calculate the following conditional probability:

$$p_{t_1|t_2}[a_1|\psi] = \frac{p_{t_1,t_2}[a_1,\psi]}{p_{t_2}[\psi]} = 1. \tag{2.9}$$

Yet another two-event consistent history is defined with projectors

$$\{\widetilde{\Pi}_1(t_1) = |a_2\rangle\langle a_2|, \widetilde{\Pi}_2(t_1) = 1 - |a_2\rangle\langle a_2|\} \text{ and } \{\Pi_1(t_2) = |\psi\rangle\langle\psi|, \Pi_2(t_2) = 1 - |\psi\rangle\langle\psi|\}, \quad t_2 > t_1. \tag{2.10}$$

Comparing (2.10) with (2.5) we note that the first measurement at t_1 is different, i.e. it refers to measuring a different physical observable. Analogously to (2.9) we calculate for the second consistent history

$$p_{t_1|t_2}[a_2|\psi] = 1. \tag{2.11}$$

The consistent histories (2.5) and (2.10) share one event, ψ, at the later time. On the base of this event (2.9) retrodicts with probability one (i.e., with certainty) that a_1 happened. Likewise, (2.11) retrodicts with certainty that a_2 happened. But the events a_1 and a_2 are mutually incompatible, since their projectors are orthogonal, $\langle a_1|a_2\rangle = 0$: if one happened, the other one could not happen.

Note that such an inconsistency is excluded within the measurement-based approach. There the analogues of (2.9) and (2.11) refer to different contexts [different measurements]: they read, respectively, $p[a_1|\psi,\mathcal{M}(t_1),\mathcal{M}(t_2)] = 1$ and

$p[a_2|\psi, \widetilde{\mathcal{M}}(t_1), \mathcal{M}(t_2)] = 1$. It is not surprising that different contexts, $\mathcal{M}(t_1) \neq \widetilde{\mathcal{M}}(t_1)$, force conditional probabilities to retrodict incompatible events. Naturally, if within the standard approach one makes the same measurements the incompatible events cannot happen, e.g. $p[a_1|\psi, \mathcal{M}(t_1), \mathcal{M}(t_2)] \times p[a_2|\psi, \mathcal{M}(t_1), \mathcal{M}(t_2)] = 0$, because the second probability is zero.

Following Kent [180] we interpret this effect as a disagreement between the predictions (or more precisely: the retrodictions) of the consistent history approach versus those of the measurement-based quantum mechanics. In response to Kent, Griffiths and Hartle suggested that for avoiding above paradoxes, predictions and retrodictions of the approach are to be restricted to a single consistent history [182, 183]. Conceptually, this seems to betray the very point of the approach, because in effect it brings back the necessity of fixing the context within which a given consistent history takes place. And what fixes this context, once measurements are absent?

Another possible opinion is that condition (2.2) is not strong enough to prevent a disagreement with the measurement based approach, and one should look for a better condition for defining events [184, 185]. To our knowledge, such a condition is so far not found. Bassi and Ghirardi [186] pointed out another logical problem with the consistent histories approach. This produced another debate on the logical consistency of the approach [187, 188], which we will not discuss here.

We hold the opinion that in spite of being certainly thought-provoking and interesting, the consistent histories approach, as it presently stands, cannot be a substitute for the theory of quantum measurements: both conceptually and practically we still need to understand what is going on in realistic measurements, with their imperfections.

2.10. What we learned from these models

Ձկնորսը ձկանը ասեց. "Ի՞նչ կա-չկա ծովում":[15]
"Ասելիք շատ ունեմ, բայց բերանս լիքը ջուր է":
Armenian proverb

Each one of the above models enlightens one or another among the many aspects of quantum measurements. However, none of them reproduces the whole set of desired features: reduction of S + A, Born's rule, complete scenario of the joint evolution of S + A, with an evaluation of its characteristic times, metastability of the initial state of A, amplification within A of the signal, unbiased and robust registration by A in the final state, accurate establishment between S and the pointer variable of A of the correlations that characterize an ideal measurement, influence of the parameters of the model on possible imperfections of the measurement. In particular, permanent registration requires the pointer to be macroscopic. In the following we study in detail a model, introduced in Refs. [60, 189, 190, 191, 192, 193, 194], which encompasses these various requirements.

3. A Curie–Weiss model for quantum measurements

What I cannot build, I cannot understand
Richard Feynman

3.1. General features

We take for S, the system to be measured, the simplest quantum system, namely a spin $\frac{1}{2}$ (or any two-level system). The observable $\hat{s}$ to be measured is its third Pauli matrix $\hat{s}_z$, with eigenvalues s_i equal to ± 1. The statistics of this observable should not change significantly during the measurement process [2, 8, 195]. Hence $\hat{s}_z$ should commute with the Hamiltonian of S + A, at least nearly.

We have stressed at the end of § 1.2.1 that the apparatus A should lie initially in a metastable state [196, 197], and at the end of the process in either one of several possible stable states (see section 2 for other models of this type). This suggests to take for A, as in several models described in section 2, a quantum system that may undergo a phase transition with broken invariance. The initial state $\hat{\mathcal{R}}(0)$ of A is the metastable phase with unbroken invariance. The

[15] Fisherman: "What's the news from the sea?" Fish: "I have a lot to say, but my mouth is full of water"

states $\hat{\mathcal{R}}_i$ represent the stable phases with broken invariance, in each of which registration can be permanent. The symmetry between the outcomes prevents any bias.

Here we need two such stable states, in one–to–one correspondence with the two eigenvalues s_i of $\hat{s}_z$. The simplest system which satisfies these properties is an Ising model [197]. Conciliating mathematical tractability and realistic features, we thus take as apparatus A = M + B, a model that simulates a *magnetic dot*: The magnetic degrees of freedom M consist of $N \gg 1$ spins with Pauli operators $\hat{\sigma}_a^{(n)}$, $(n = 1, 2, \cdots, N;\ a = x, y, z)$, which read for each n

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \hat{\sigma}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{3.1}$$

where $\hat{\sigma}_0$ is the corresponding identity matrix. The non-magnetic degrees of freedom such as phonons behave as a thermal bath B (Fig. 3.1). Anisotropic interactions between these spins can generate Ising ferromagnetism below the Curie temperature T_c. As pointer variable $\hat{A}$ we take the order parameter, which is the magnetization in the z-direction (within normalization), as represented by the quantum observable

$$\hat{m} = \frac{1}{N} \sum_{n=1}^{N} \hat{\sigma}_z^{(n)}. \tag{3.2}$$

We let N remain finite, which will allow us to keep control of the equations of motion. It should however be sufficiently large so as to ensure the required properties of phase transitions: The relaxation from $\hat{\mathcal{R}}(0)$ to either one of the two states $\hat{\mathcal{R}}_i$, at the temperature T (below T_c) imposed by the bath B, should be irreversible, the fluctuations of the order parameter $\hat{m}$ in each equilibrium state $\hat{\mathcal{R}}_i$ should be weak (as $1/\sqrt{N}$), and the transition between these two states $\hat{\mathcal{R}}_i$ should be nearly forbidden.

The initial state $\hat{\mathcal{R}}(0)$ of A is the metastable paramagnetic state. We expect the final state (1.7) of S + A to involve for A the two stable ferromagnetic states $\hat{\mathcal{R}}_i$, $i = \uparrow$ or $\downarrow$, that we denote as $\hat{\mathcal{R}}_{\Uparrow}$ or $\hat{\mathcal{R}}_{\Downarrow}$, respectively[16]. The equilibrium temperature T will be imposed to M by the phonon bath [145, 199] through weak coupling between the magnetic and non-magnetic degrees of freedom. Within small fluctuations, the order parameter (3.2) vanishes in $\hat{\mathcal{R}}(0)$ and takes two opposite values in the states $\hat{\mathcal{R}}_{\Uparrow}$ and $\hat{\mathcal{R}}_{\Downarrow}$, $A_i \equiv \langle \hat{m} \rangle_i$ equal to $+m_F$ for $i = \uparrow$ and to $-m_F$ for $i = \downarrow$[17]. As in real magnetic registration devices [198], information will be stored by A in the form of the sign of the magnetization.

3.2. The Hamiltonian

I ask not for a lighter burden, but for broader shoulders
Jewish proverb

The full Hamiltonian can be decomposed into terms associated with the system, with the apparatus and with their coupling:

$$\hat{H} = \hat{H}_S + \hat{H}_{SA} + \hat{H}_A. \tag{3.3}$$

3.2.1. The system

As indicated above, for an ideal measurement the observable $\hat{s}$ should commute with $\hat{H}$ [8, 129, 195]. The simplest self-Hamiltonian that ensures this property (no evolution of S without coupling to A), is the trivial one,

$$\hat{H}_S = 0. \tag{3.4}$$

More generally, in order to describe an imperfect measurement where $\hat{s}$ may move noticeably during the measurement (subsection 8.2), we shall introduce there a magnetic field acting on the tested spin.

[16] Here and in the following, single arrows $\uparrow, \downarrow$ will denote the spin S, while double arrows $\Uparrow, \Downarrow$ denote the magnet M

[17] Note that the values $A_i = \pm m_F$, which we wish to come out associated with the eigenvalues $s_i = \pm 1$, are determined from equilibrium statistical mechanics; they are not the eigenvalues of $\hat{A} \equiv \hat{m}$, which range from -1 to $+1$ with spacing $2/N$, but thermodynamic expectation values around which small fluctuations of order $1/\sqrt{N}$ occur

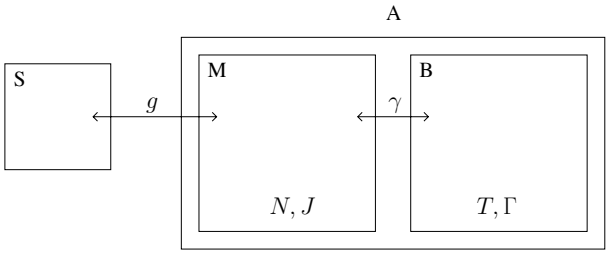

Figure 3.1: The Curie-Weiss measurement model and its parameters. The system S is a spin $\hat{\mathbf{s}}$. The apparatus A includes a magnet M and a bath B. The magnet, which acts as a pointer, consists of N spins coupled to one another through an Ising interaction J. The phonon bath B is characterized by its temperature T and a Debye cutoff Γ. It interacts with M through a spin-boson coupling γ. The process is triggered by the interaction g between the measured observable $\hat{s}_z$ and the magnetization per spin, $\hat{m}$, of the pointer.

The coupling between the tested system and the apparatus,

$$\hat{H}_{\mathrm{SA}} = -g\hat{s}_z \sum_{n=1}^{N} \hat{\sigma}_z^{(n)} = -Ng\hat{s}_z\hat{m}, \tag{3.5}$$

has the usual form of a spin-spin coupling in the z-direction [197], and the constant $g > 0$ characterizes its strength. As wished, it commutes with $\hat{s}_z$. We have assumed that the range of the interaction between the spin S and the N spins of M is large compared to the size of the magnetic dot, so that we can disregard the space-dependence of the coupling. The occurrence of the factor N in (3.5) should not worry us, since we will not take the thermodynamic limit $N \to \infty$. Although sufficiently large to ensure the existence of a clear phase transition, N is finite. We shall resume in § 9.4 the conditions that N should satisfy. In a realistic setting, the interaction between S and M would first be turned on, then turned off continuously, while the tested spin approaches the dot then moves away. For simplicity we assume $\hat{H}_{\mathrm{SA}}$ to be turned on suddenly at the initial time $t = 0$, and it will be turned off at a final time t_{f}, as we discuss below[18].

3.2.2. The magnet

The apparatus A consists, as indicated above, of a magnet M and a phonon bath B (Fig. 3.1), and its Hamiltonian can be decomposed into

$$\hat{H}_{\mathrm{A}} = \hat{H}_{\mathrm{M}} + \hat{H}_{\mathrm{B}} + \hat{H}_{\mathrm{MB}}. \tag{3.6}$$

The magnetic part is chosen as [198]

$$\hat{H}_{\mathrm{M}} = -N \sum_{q=2,4} J_q \frac{\hat{m}^q}{q} = -NJ_2\frac{\hat{m}^2}{2} - NJ_4\frac{\hat{m}^4}{4}, \tag{3.7}$$

where the magnetization operator $\hat{m}$ was defined by (3.2). It couples all q-plets of spins $\hat{\boldsymbol{\sigma}}^{(n)}$ symmetrically, with the same coupling constant $J_q N^{1-q}$ for each q-plet. (The factor N^{1-q} is introduced only for convenience.) The space-independence of this coupling is fairly realistic for a small magnetic dot, as in (3.5). The interaction is fully anisotropic, involving only the z-components, and ferromagnetic ($J > 0$). The exponent q is even in order to ensure the up-down symmetry of the apparatus. The term $q = 4$ describes so-called super-exchange interactions as realized for metamagnets [142]. We shall only consider ferromagnetic interactions ($J_2 > 0$ or $J_4 > 0$ or both).

We will see in § 3.3.4 that the Hamiltonian (3.6) produces a paramagnetic equilibrium state at high temperature and two ferromagnetic states at low temperature, with a transition of second order for $J_2 > 3J_4$, of first order for $3J_4 > J_2$. The former case is exemplified by the Curie–Weiss Ising model for an anisotropic magnetic dot [197], with pairwise interactions in $\hat{\sigma}_z^{(n)}\hat{\sigma}_z^{(p)}$, recovered here for $J_4 = 0$,

[18] Contrary to the switching on, this switching off need not be performed suddenly since m_{F} is close to $m_{\Uparrow}$

$$\hat{H}_{\rm M} = -\frac{J_2 N}{2}\hat{m}^2 = -\frac{J_2}{2N}\sum_{i,j=1}^{N}\hat{\sigma}_z^{(i)}\hat{\sigma}_z^{(j)}, \qquad (q=2). \tag{3.8}$$

Likewise, the first-order case is exemplified by letting $J_2 = 0$, keeping in (3.6) only the quartic "super-exchange" term:

$$\hat{H}_{\rm M} = -\frac{J_4 N}{4}\hat{m}^4 = -\frac{J_4}{4N^3}\sum_{i,j,k,l=1}^{N}\hat{\sigma}_z^{(i)}\hat{\sigma}_z^{(j)}\hat{\sigma}_z^{(k)}\hat{\sigma}_z^{(l)}, \qquad (q=4). \tag{3.9}$$

The more physical case (3.6) of mixtures of $q = 2$ and $q = 4$ terms will not differ qualitatively from either one of the two pure cases $q = 2$ or $q = 4$. It will therefore be sufficient for our purpose, in section 7, to illustrate the two situations $J_2 > 3J_4$ and $J_2 < 3J_4$ by working out the Hamiltonians (3.8) and (3.9), respectively. We may summarize these two cases by $H_{\rm M} = -(NJ/q)\hat{m}^q$ with $q = 2$ and 4, respectively.

Using A as a measurement apparatus requires this lifetime of the initial state to be larger than the overall measurement time. An advantage of a first-order transition is the local stability of the paramagnetic state, even below the transition temperature, which ensures a much larger lifetime as in the case of a second order transition. We shall see, however (§ 7.3.2), that the required condition can be satisfied even for $q = 2$.

3.2.3. The bath

The interaction between the magnet and the bath, which drives the apparatus to equilibrium, is taken as a standard spin-boson Hamiltonian [145, 199, 200]

$$\hat{H}_{\rm MB} = \sqrt{\gamma}\sum_{n=1}^{N}\left(\hat{\sigma}_x^{(n)}\hat{B}_x^{(n)} + \hat{\sigma}_y^{(n)}\hat{B}_y^{(n)} + \hat{\sigma}_z^{(n)}\hat{B}_z^{(n)}\right) \equiv \sqrt{\gamma}\sum_{n=1}^{N}\sum_{a=x,y,z}\hat{\sigma}_a^{(n)}\hat{B}_a^{(n)}, \tag{3.10}$$

which couples each component $a = x, y, z$ of each spin $\hat{\sigma}^{(n)}$ with some hermitean linear combination $\hat{B}_a^{(n)}$ of phonon operators. The dimensionless constant $\gamma \ll 1$ characterizes the strength of the thermal coupling between M and B, which is weak.

For simplicity, we require that the bath acts independently for each spin degree of freedom n, a. (The so-called independent baths approximation.) This can be achieved (i) by introducing Debye phonon modes labelled by the pair of indices k, l, with eigenfrequencies ω_k depending only on k, so that the bath Hamiltonian is

$$\hat{H}_{\rm B} = \sum_{k,l}\hbar\omega_k\hat{b}_{k,l}^{\dagger}\hat{b}_{k,l}, \tag{3.11}$$

and (ii) by assuming that the coefficients C in

$$\hat{B}_a^{(n)} = \sum_{k,l}\left[C\,(n,a;k,l)\,\hat{b}_{k,l} + C^*\,(n,a;k,l)\,\hat{b}_{k,l}^{\dagger}\right] \tag{3.12}$$

are such that

$$\sum_{l} C\,(n,a;k,l)\,C^*\,(m,b;k,l) = \delta_{n,m}\delta_{a,b}\,c\,(\omega_k)\,. \tag{3.13}$$

This requires the number of values of the index l to be at least equal to $3N$. For instance, we may associate with each component a of each spin $\hat{\sigma}^{(n)}$ a different set of phonon modes, labelled by k, n, a, identifying l as (n, a), and thus define $\hat{H}_{\rm B}$ and $\hat{B}_a^{(n)}$ as

$$\hat{H}_{\rm B} = \sum_{n=1}^{N}\sum_{a=x,y,z}\sum_{k}\hbar\omega_k\hat{b}_{k,a}^{\dagger(n)}\hat{b}_{k,a}^{(n)}, \tag{3.14}$$

$$\hat{B}_a^{(n)} = \sum_{k}\sqrt{c\,(\omega_k)}\left(\hat{b}_{k,a}^{(n)} + \hat{b}_{k,a}^{\dagger(n)}\right). \tag{3.15}$$

We shall see in § 3.3.2 that the various choices of the phonon set, of the spectrum (3.11) and of the operators (3.12) coupled to the spins are equivalent, in the sense that the joint dynamics of S + M will depend only on the spectrum ω_k and on the coefficients $c\,(\omega_k)$.

$$\hat{\mathcal{D}} \left\{ \hat{D} \left\{ \begin{matrix} \mathrm{S} & \hat{r} \\ \mathrm{M} & \hat{R}_{\mathrm{M}} \end{matrix} \right\} \hat{\mathcal{R}} \right. \qquad \hat{\mathcal{D}} \left\{ \hat{D} \left\{ \begin{matrix} \mathrm{S} & \hat{r} \\ \mathrm{M} & \hat{R}_{\mathrm{M}} \\ \mathrm{B} & \hat{R}_{\mathrm{B}} \end{matrix} \right\} \hat{\mathcal{R}} \right.$$

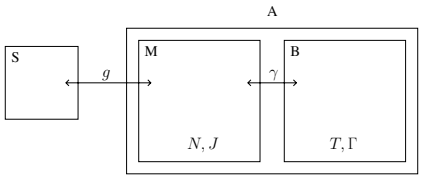

Figure 3.2: Notations for the density operators of the system S + A and the subsystems M and B of A. The full density matrix $\hat{\mathcal{D}}$ is parametrized by its submatrices $\hat{\mathcal{R}}_{ij}$ (with $i, j = \pm 1$ or $\uparrow, \downarrow$), the density matrix $\hat{D}$ of S + M by its submatrices $\hat{R}_{ij}$. The marginal density operator of S is denoted as $\hat{r}$ and the one of A as $\hat{\mathcal{R}}$. The marginal density operator of M itself is denoted as $\hat{R}_{\mathrm{M}}$ and the one of B as $\hat{R}_{\mathrm{B}}$.

3.3. Structure of the states

If you do not enter the tiger's cave, you will not catch its cub
Japanese proverb

3.3.1. Notations

The full state $\hat{\mathcal{D}}$ of the system evolves according to the Liouville–von Neumann equation (1.6), which we have to solve. It will be convenient to define through partial traces, at any instant t, the following marginal density operators: $\hat{r}$ for the tested system S, $\hat{\mathcal{R}}$ for the apparatus A, $\hat{R}_{\mathrm{M}}$ for the magnet M, $\hat{R}_{\mathrm{B}}$ for the bath, and $\hat{D}$ for S + M after elimination of the bath (as depicted schematically in Fig. 3.2), according to

$$\hat{r} = \mathrm{tr}_{\mathrm{A}}\hat{\mathcal{D}}, \qquad \hat{\mathcal{R}} = \mathrm{tr}_{\mathrm{S}}\hat{\mathcal{D}}, \qquad \hat{R}_{\mathrm{M}} = \mathrm{tr}_{\mathrm{B}}\hat{\mathcal{R}} = \mathrm{tr}_{\mathrm{S,B}}\hat{\mathcal{D}}, \qquad \hat{R}_{\mathrm{B}} = \mathrm{tr}_{\mathrm{S,M}}\hat{\mathcal{D}}, \qquad \hat{D} = \mathrm{tr}_{\mathrm{B}}\hat{\mathcal{D}}. \tag{3.16}$$

The expectation value of any observable $\hat{A}$ pertaining, for instance, to the subsystem S + M of S + A (including products of spin operators $\hat{s}_a$ and $\hat{\sigma}_a^{(n)}$) can equivalently be evaluated as $\langle\hat{A}\rangle = \mathrm{tr}_{S+A}\hat{\mathcal{D}}\hat{A}$ or as $\langle\hat{A}\rangle = \mathrm{tr}_{S+M}\hat{D}\hat{A}$.

As indicated in subsection 1.2, the apparatus A is a large system, treated by methods of statistical mechanics, while we need to follow in detail the microscopic degrees of freedom of the system S and their correlations with A. To this aim, we shall analyze the full state $\hat{\mathcal{D}}$ of the system into several sectors, characterized by the eigenvalues of $\hat{s}_z$. Namely, in the two-dimensional eigenbasis of $\hat{s}_z$ for S, $|\uparrow\rangle, |\downarrow\rangle$, with eigenvalues $s_i = +1$ for $i = \uparrow$ and $s_i = -1$ for $i = \downarrow$, $\hat{\mathcal{D}}$ can be decomposed into the four blocks

$$\hat{\mathcal{D}} = \begin{pmatrix} \hat{\mathcal{R}}_{\uparrow\uparrow} & \hat{\mathcal{R}}_{\uparrow\downarrow} \\ \hat{\mathcal{R}}_{\downarrow\uparrow} & \hat{\mathcal{R}}_{\downarrow\downarrow} \end{pmatrix}, \tag{3.17}$$

where each $\hat{\mathcal{R}}_{ij}$ is an operator in the space of the apparatus. We shall also use the partial traces (see again Fig. 3.2)

$$\hat{R}_{ij} = \mathrm{tr}_{\mathrm{B}}\hat{\mathcal{R}}_{ij}, \qquad \hat{D} = \mathrm{tr}_{\mathrm{B}}\hat{\mathcal{D}} = \begin{pmatrix} \hat{R}_{\uparrow\uparrow} & \hat{R}_{\uparrow\downarrow} \\ \hat{R}_{\downarrow\uparrow} & \hat{R}_{\downarrow\downarrow} \end{pmatrix} \tag{3.18}$$

over the bath; each $\hat{R}_{ij}$ is an operator in the 2^N-dimensional space of the magnet. Indeed, we are not interested in the evolution of the bath variables, and we shall eliminate B by relying on the weakness of its coupling (3.10) with M. The operators $\hat{R}_{ij}$ code our full statistical information about S and M. We shall use the notation $\hat{R}_{ij}$ whenever we refer to S + M and $\hat{R}_{\mathrm{M}}$ when referring to M alone. The magnet M is thus described by $\hat{R}_{\mathrm{M}} = \hat{R}_{\uparrow\uparrow} + \hat{R}_{\downarrow\downarrow}$, the system S alone by the matrix elements $r_{ij} = \mathrm{tr}_{\mathrm{M}}\hat{R}_{ij}$ of $\hat{r}$. The correlations of $\hat{s}_z$, $\hat{s}_x$ or $\hat{s}_y$ with and any function of the observables $\hat{\sigma}_a^{(n)}$ ($a = x, y, z$, $n = 1$, $\ldots N$) are represented by $\hat{R}_{\uparrow\uparrow} - \hat{R}_{\downarrow\downarrow}$, $\hat{R}_{\uparrow\downarrow} + \hat{R}_{\downarrow\uparrow}$, $i\hat{R}_{\uparrow\downarrow} - i\hat{R}_{\downarrow\uparrow}$, respectively. The operators $\hat{R}_{\uparrow\uparrow}$ and $\hat{R}_{\downarrow\downarrow}$ are hermitean positive, but not normalized, whereas $\hat{R}_{\downarrow\uparrow} = \hat{R}_{\uparrow\downarrow}^{\dagger}$. Notice that we now have from (3.16) – (3.18)

$$\hat{r}_{ij} = \mathrm{tr}_{\mathrm{A}}\hat{\mathcal{R}}_{ij} = \mathrm{tr}_{\mathrm{M}}\hat{R}_{ij}, \qquad \hat{\mathcal{R}} = \hat{\mathcal{R}}_{\uparrow\uparrow} + \hat{\mathcal{R}}_{\downarrow\downarrow}, \qquad \hat{R}_{\mathrm{M}} = \hat{R}_{\uparrow\uparrow} + \hat{R}_{\downarrow\downarrow}, \qquad \hat{R}_{\mathrm{B}} = \mathrm{tr}_{\mathrm{M}}(\hat{\mathcal{R}}_{\uparrow\uparrow} + \hat{\mathcal{R}}_{\downarrow\downarrow}). \tag{3.19}$$

So far all elements are functions of time t that elapses from the beginning of the measurement $t = 0$ until its final time t_{f}. To introduce further notation, we mention that the combined system S + A = S + M + B should end up in

$$\hat{\mathcal{D}}(t_{\mathrm{f}}) = \begin{pmatrix} p_{\uparrow}\hat{\mathcal{R}}_{\Uparrow} & 0 \\ 0 & p_{\downarrow}\hat{\mathcal{R}}_{\Downarrow} \end{pmatrix} = p_{\uparrow}\hat{\mathcal{R}}_{\Uparrow} \otimes |\uparrow\rangle\langle\uparrow| + p_{\downarrow}\hat{\mathcal{R}}_{\Downarrow} \otimes |\downarrow\rangle\langle\downarrow|, \tag{3.20}$$

where $\hat{\mathcal{R}}_{\Uparrow}$ ($\hat{\mathcal{R}}_{\Downarrow}$) is density matrix of the thermodynamically stable state of the magnet and bath, after the measurement, in which the magnetization is up, taking the value $m_{\Uparrow}(g)$ (down, taking the value $m_{\Downarrow}(g)$); these events occur with probabilities $p_{\uparrow}$ and $p_{\downarrow}$, respectively[19]. The Born rule then predicts that $p_{\uparrow} = \mathrm{tr}_{\mathrm{S}}\hat{r}(0)\Pi_{\uparrow} = r_{\uparrow\uparrow}(0)$ and $p_{\downarrow} = r_{\downarrow\downarrow}(0)$.

Since no off-diagonal terms occur in (3.20), a point that we wish to explain, and since we expect B to remain nearly in its intial equilibrium state, we may trace out the bath, as is standard in classical and quantum thermal physics, without losing significant information. It will therefore be sufficient for our purpose to show that the final state is

$$\hat{D}(t_{\mathrm{f}}) = \begin{pmatrix} p_{\uparrow}\hat{R}_{\mathrm{M}\Uparrow} & 0 \\ 0 & p_{\downarrow}\hat{R}_{\mathrm{M}\Downarrow} \end{pmatrix} = p_{\uparrow}\hat{R}_{\mathrm{M}\Uparrow} \otimes |\uparrow\rangle\langle\uparrow| + p_{\downarrow}\hat{R}_{\mathrm{M}\Downarrow} \otimes |\downarrow\rangle\langle\downarrow|, \tag{3.21}$$

now referring to the magnet M and tested spin S alone.

Returning to Eq. (3.19), we note that from any density operator $\hat{R}$ of the magnet we can derive the probabilities $P_{\mathrm{M}}^{\mathrm{dis}}(m)$ for $\hat{m}$ to take the eigenvalues m, where "dis" denotes their discreteness. These $N+1$ eigenvalues,

$$m = -1, \qquad -1 + \frac{2}{N}, \qquad \ldots, \qquad 1 - \frac{2}{N}, \qquad 1, \tag{3.22}$$

have equal spacings $\delta m = 2/N$ and multiplicities

$$G(m) = \frac{N!}{\left[\frac{1}{2}N(1+m)\right]!\left[\frac{1}{2}N(1-m)\right]!} \simeq \sqrt{\frac{2}{\pi N(1-m^2)}}\exp\left[\frac{N}{2}\left(\ln\frac{4}{1-m^2} - m\ln\frac{1+m}{1-m}\right) + O\left(\frac{1}{N}\right)\right]. \tag{3.23}$$

Denoting by $\delta_{\hat{m},m}$ the projection operator on the subspace m of $\hat{m}$, the dimension of which is $G(m)$, we have

$$P_{\mathrm{M}}^{\mathrm{dis}}(m,t) = \mathrm{tr}_{\mathrm{M}}\hat{R}_{\mathrm{M}}(t)\delta_{\hat{m},m}. \tag{3.24}$$

In the limit $N \gg 1$, where m becomes basically a continuous variable, we shall later work with the functions $P_{\mathrm{M}}(m,t)$

$$P_{\mathrm{M}}(m,t) = \frac{N}{2}P_{\mathrm{M}}^{\mathrm{dis}}(m,t), \qquad \int_{-1}^{1}\mathrm{d}m\, P_{\mathrm{M}}(m,t) = \sum_{m} P_{\mathrm{M}}^{\mathrm{dis}}(m,t) = 1, \tag{3.25}$$

that have a finite and smooth limit for $N \to \infty$, and use similar relations between the functions P_{ij} and P_{ij}^{dis}, and C_a and C_a^{dis}, introduced next.

In what follows, the density operators $\hat{R}_{\mathrm{M}}$ will most often depend only on the observables $\hat{\sigma}_z^{(n)}$ and be symmetric functions of these observables. Hence, $\hat{R}_{\mathrm{M}}$ will reduce to a mere function of the operator $\hat{m}$ defined by (3.2). In such a circumstance, eq. (3.24) can be inverted: the knowledge of $P_{\mathrm{M}}(m)$ is then sufficient to determine the density operator $\hat{R}_{\mathrm{M}}$, through a simple replacement of the scalar m by the operator $\hat{m}$ in

$$\hat{R}_{\mathrm{M}}(t) = \frac{1}{G(\hat{m})}P_{\mathrm{M}}^{\mathrm{dis}}(\hat{m},t)\,. \tag{3.26}$$

The expectation value of any product of operators $\hat{\sigma}_a^{(n)}$ of the magnet can then be expressed in terms of $P_{\mathrm{M}}^{\mathrm{dis}}(m)$. For instance, the two-spin correlations ($n \neq p$) are related to the second moment of $P_{\mathrm{M}}^{\mathrm{dis}}(m)$ by

$$\mathrm{tr}_{\mathrm{S,A}}\hat{\mathcal{D}}\hat{\sigma}_a^{(n)}\hat{\sigma}_b^{(p)} = \mathrm{tr}_{\mathrm{M}}\hat{R}_{\mathrm{M}}\hat{\sigma}_a^{(n)}\hat{\sigma}_b^{(p)} = \frac{\delta_{a,z}\delta_{b,z}}{N-1}\left[N\sum_{m}P_{\mathrm{M}}^{\mathrm{dis}}(m)\,m^2 - 1\right]. \tag{3.27}$$

[19]Notice that in the final state we denote properties of the tested system by $\uparrow,\downarrow$ and of the apparatus by $\Uparrow,\Downarrow$. In sums like (1.7) we will also use $i = \pm 1$

Likewise, when the operators $\hat{R}_{ij}$ in (3.18) depend only on $\hat{m}$, we can parameterize them at each time, according to

$$\hat{R}_{ij}(t) = \frac{1}{G(\hat{m})} P_{ij}^{\rm dis}(\hat{m}, t), \tag{3.28}$$

by functions $P_{ij}^{\rm dis}(m)$ defined on the set (3.22) of values of m, with $[P_{ij}^{\rm dis}(m)]^* = P_{ji}^{\rm dis}(m)$. (For the moment we refrain from denoting the explicit t dependence.) All statistical properties of S + M at the considered time can then be expressed in terms of these functions $P_{ij}^{\rm dis}(m)$. Indeed the combinations

$$C_x^{\rm dis}(m) = P_{\uparrow\downarrow}^{\rm dis}(m) + P_{\downarrow\uparrow}^{\rm dis}(m), \qquad C_y^{\rm dis} = iP_{\uparrow\downarrow}^{\rm dis} - iP_{\downarrow\uparrow}^{\rm dis}, \qquad C_z^{\rm dis} = P_{\uparrow\uparrow}^{\rm dis} - P_{\downarrow\downarrow}^{\rm dis} \tag{3.29}$$

encompass all the correlations between $\hat{s}_x$, $\hat{s}_y$ or $\hat{s}_z$ and any number of spins of the apparatus. In particular, the expectation values of the components of $\hat{\mathbf{s}}$ are given by

$$\mathrm{tr}\hat{\mathcal{D}}\hat{s}_a = \sum_m C_a^{\rm dis}(m) = \int_{-1}^{1} \mathrm{d}m\, C_a(m), \tag{3.30}$$

with the continuous functions $C_a(m) = \frac{1}{2} N C_a^{\rm dis}(m)$ as in (3.25), while the correlations between $\hat{\mathbf{s}}$ and one spin of M are

$$\mathrm{tr}\hat{\mathcal{D}}\hat{s}_a\hat{\sigma}_b^{(n)} = \delta_{b,z} \sum_m C_a^{\rm dis}(m)\, m = \delta_{b,z} \int_{-1}^{1} \mathrm{d}m\, C_a(m)\, m. \tag{3.31}$$

Correlations of $\hat{\mathbf{s}}$ with many spins of M involve higher moments of $C_a^{\rm dis}(m)$ as in eq. (3.27). We can interpret $P_{\uparrow\uparrow}^{\rm dis}(m)$ as the joint probability to find S in $|\uparrow\rangle$ and $\hat{m}$ equal to m, so that $P_{\uparrow\uparrow}^{\rm dis}(m) + P_{\downarrow\downarrow}^{\rm dis}(m) = P_{\rm M}^{\rm dis}(m)$ reduces to the probability $P_{\rm M}^{\rm dis}(m)$ which characterizes through (3.26) the marginal state of M.

3.3.2. Equilibrium state of the bath

At the initial time, the bath is set into equilibrium at the temperature $T = 1/\beta$[20]. The density operator of the bath,

$$\hat{R}_{\rm B}(0) = \frac{1}{Z_{\rm B}} e^{-\beta \hat{H}_{\rm B}}, \tag{3.32}$$

when $\hat{H}_{\rm B}$ is given by (3.11), describes the set of phonons at equilibrium in independent modes.

As shown in section 4.2 the bath will be involved in our problem only through its *autocorrelation function* in the equilibrium state (3.32), defined in the Heisenberg picture (see § 10.1.2) by

$$\mathrm{tr}_{\rm B}\left[\hat{R}_{\rm B}(0)\, \hat{B}_a^{(n)}(t)\, \hat{B}_b^{(p)}(t')\right] = \delta_{n,p}\delta_{a,b}\, K(t - t'), \tag{3.33}$$

$$\hat{B}_a^{(n)}(t) \equiv \hat{U}_{\rm B}^{\dagger}(t)\, \hat{B}_a^{(n)} \hat{U}_{\rm B}(t), \tag{3.34}$$

$$\hat{U}_{\rm B}(t) = e^{-i\hat{H}_{\rm B}t/\hbar}, \tag{3.35}$$

in terms of the evolution operator $\hat{U}_{\rm B}(t)$ of B alone. The bath operators (3.12) have been defined in such a way that the equilibrium expectation value of $B_a^{(n)}(t)$ vanishes [145, 199, 200]. Moreover, the condition (3.13) ensures that the equilibrium correlations between different operators $\hat{B}_a^{(n)}(t)$ and $\hat{B}_b^{(p)}(s)$ vanish, unless $a = b$ and $n = p$, and that the autocorrelations for $n = p$, $a = b$ are all the same, thus defining a unique function $K(t)$ in (3.33). We introduce the Fourier transform and its inverse,

$$\tilde{K}(\omega) = \int_{-\infty}^{+\infty} \mathrm{d}t\, e^{-i\omega t} K(t), \qquad K(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathrm{d}\omega\, e^{i\omega t} \tilde{K}(\omega) \tag{3.36}$$

[20] We use units where Boltzmann's constant is equal to one [196]; otherwise, T and $\beta = 1/T$ should be replaced throughout by $k_B T$ and $1/k_B T$, respectively

and choose for $\tilde{K}(\omega)$ the simplest expression having the required properties, namely the quasi-Ohmic form [122, 123, 145, 199, 200]

$$\tilde{K}(\omega) = \frac{\hbar^2}{4}\frac{\omega e^{-|\omega|/\Gamma}}{e^{\beta\hbar\omega}-1}. \tag{3.37}$$

The temperature dependence accounts for the quantum bosonic nature of the phonons [145, 199, 200]. The Debye cutoff Γ characterizes the largest frequencies of the bath, and is assumed to be larger than all other frequencies entering our problem. The normalization is fixed so as to let the constant γ entering (3.10) be dimensionless.

The form (3.37) of $\tilde{K}(\omega)$ describes the spectral function of the Nyquist-noise correlator, which is the quantum generalization of the classical white noise. It can be obtained directly through general reasonings based on the detailed balance and the approach to equilibrium [145, 199]. We can also derive it from the expressions (3.11) for $\hat{H}_{\rm B}$, (3.12) and (3.34) for $\hat{B}_a^{(n)}(t)$, and (3.32) for $\hat{R}_{\rm B}(0)$, which under general conditions provide a universal model for the bath [145, 199, 200]. Indeed, by inserting these expressions into the left-hand side of (3.33), we recover the diagonal form of the right-hand side owing to (3.13), with $K(t)$ given by

$$\begin{aligned} K(t) &= \sum_k c(\omega_k)\left(\frac{e^{i\omega_k t}}{e^{\beta\hbar\omega_k}-1}+\frac{e^{-i\omega_k t}}{1-e^{-\beta\hbar\omega_k}}\right) \\ &= \int_0^\infty \mathrm{d}\omega\rho(\omega)c(\omega)\left(\frac{e^{i\omega t}}{e^{\beta\hbar\omega}-1}+\frac{e^{-i\omega t}}{1-e^{-\beta\hbar\omega}}\right). \end{aligned} \tag{3.38}$$

We have expressed above $K(t)$ in terms of the density of modes

$$\rho(\omega) = \sum_k \delta(\omega-\omega_k), \tag{3.39}$$

and this provides

$$\tilde{K}(\omega) = 2\pi\rho(|\omega|)c(|\omega|)\frac{\operatorname{sgn}\omega}{e^{\beta\hbar\omega}-1}. \tag{3.40}$$

In agreement with Kubo's relation, we also find for the dissipative response

$$\int_{-\infty}^{+\infty}\mathrm{d}t e^{-i\omega t}\mathrm{tr}_{\rm B}\left\{\hat{R}_{\rm B}(0)\left[\hat{B}_a^{(n)}(t),\hat{B}_b^{(p)}(0)\right]\right\} = -2\pi\rho(|\omega|)c(|\omega|)\operatorname{sgn}\omega. \tag{3.41}$$

In the limit of a spectrum ω_k of the phonon modes sufficiently dense so that the relevant values of $t/\hbar$ and β are small compared to the inverse level spacing of the phonon modes, we can regard $\rho(\omega)c(\omega)$ as a continuous function. In the quasi-Ohmic regime [122, 123, 124, 145, 199, 200], the dissipative response at low frequencies is proportional to ω, as obvious for a friction-dominated harmonic oscillator. We thus take (for $\omega > 0$)

$$\rho(\omega)c(\omega) = \frac{\hbar^2}{8\pi}\omega e^{-\omega/\Gamma}, \tag{3.42}$$

where ω is called the Ohmic factor, and where we include a Debye cutoff Γ on the phonon spectrum and a proper normalization. Then (3.40) reduces to the assumed expression (3.37).

3.3.3. Initial state

In the initial state $\hat{\mathcal{D}}(0) = \hat{r}(0)\otimes\hat{\mathcal{R}}(0)$ where S and A are statistically independent, the 2×2 density matrix $\hat{r}(0)$ of S is arbitrary, with elements $r_{\uparrow\uparrow}(0)$, $r_{\uparrow\downarrow}(0)$, $r_{\downarrow\uparrow}(0)$ and $r_{\downarrow\downarrow}(0)$ satisfying

$$r_{\uparrow\uparrow}(0)+r_{\downarrow\downarrow}(0)=1, \qquad r_{\uparrow\downarrow}(0)=r_{\downarrow\uparrow}^*(0), \qquad r_{\uparrow\uparrow}(0)\,r_{\downarrow\downarrow}(0)\geq r_{\uparrow\downarrow}(0)\,r_{\downarrow\uparrow}(0). \tag{3.43}$$

According to the discussion of the section 3.1, the initial density operator $\hat{\mathcal{R}}(0)$ of the apparatus describes the magnetic dot in a metastable paramagnetic state. As justified below, we take for it the factorized form

$$\hat{\mathcal{R}}(0) = \hat{R}_{\rm M}(0)\otimes\hat{R}_{\rm B}(0), \tag{3.44}$$

where the bath is in the equilibrium state (3.32), at the temperature $T = 1/\beta$ lower than the transition temperature of M, while the magnet with Hamiltonian (3.6) is in a paramagnetic equilibrium state at a temperature $T_0 = 1/\beta_0$ higher than its transition temperature:

$$\hat{R}_{\rm M}(0) = \frac{1}{Z_{\rm M}} e^{-\beta_0 \hat{H}_{\rm M}}. \tag{3.45}$$

How can the apparatus be actually initialized in the non-equilibrium state (3.44) at the time $t = 0$? This *initialization* takes place during the time interval $-\tau_{\rm init} < t < 0$. The apparatus is first set at earlier times into equilibrium at the temperature T_0. Due to the smallness of γ, its density operator is then factorized and proportional to $\exp[-\beta_0(\hat{H}_{\rm M} + \hat{H}_{\rm B})]$. At the time $-\tau_{\rm init}$ the phonon bath is suddenly cooled down to T. We shall evaluate in § 7.3.2 the relaxation time of M towards its equilibrium ferromagnetic states under the effect of B at the temperature T. Due to the weakness of the coupling γ, this time turns out to be much longer than the duration of the experiment. We can safely assume $\tau_{\rm init}$ to be much shorter than this relaxation time so that M remains unaffected by the cooling. On the other hand, the quasi continuous nature of the spectrum of B can allow the phonon-phonon interactions (which we have disregarded when writing (3.11)) to establish the equilibrium of B at the temperature T within a time shorter than $\tau_{\rm init}$. It is thus realistic to imagine an initial state of the form (3.44).

An alternative method of initialization consists in applying to the magnetic dot a strong radiofrequency field, which acts on M but not on B. The bath can thus be thermalized at the required temperature, lower than the transition temperature of M, while the populations of spins of M oriented in either direction are equalized. The magnet is then in a paramagnetic state, as if it were thermalized at an *infinite* temperature T_0 in spite of the presence of a cold bath. In that case we have the initial state (see Eq. (3.1))

$$\hat{R}_{\rm M}(0) = \frac{1}{2^N} \prod_{n=1}^{N} \hat{\sigma}_0^{(n)}. \tag{3.46}$$

The initial density operator (3.45) of M being simply a function of the operator $\hat{m}$, we can characterize it as in (3.24) by the probabilities $P_{\rm M}^{\rm dis}(m, 0)$ for $\hat{m}$ to take the values (3.22). Those probabilities are the normalized product of the degeneracy (3.23) and the Boltzmann factor,

$$P_{\rm M}^{\rm dis}(m,0) = \frac{1}{Z_0} G(m) \exp\left[\frac{N}{T_0}\left(\frac{J_2}{2}m^2 + \frac{J_4}{4}m^4\right)\right], \qquad Z_0 = \sum_m G(m) \exp\left[\frac{N}{T_0}\left(\frac{J_2}{2}m^2 + \frac{J_4}{4}m^4\right)\right]. \tag{3.47}$$

For sufficiently large N, the distribution $P_{\rm M}(m, 0) = \frac{1}{2} N P_{\rm M}^{\rm dis}(m, 0)$ is peaked around $m = 0$, with the Gaussian shape

$$P_{\rm M}(m,0) \simeq \frac{1}{\sqrt{2\pi}\,\Delta m} e^{-m^2/2\Delta m^2} = \sqrt{\frac{N}{2\pi\delta_0^2}} e^{-Nm^2/2\delta_0^2}. \tag{3.48}$$

This peak, which has a narrow width of the form

$$\Delta m = \sqrt{\langle m^2 \rangle} = \frac{\delta_0}{\sqrt{N}}, \tag{3.49}$$

involves a large number, of order $\sqrt{N}$, of eigenvalues (3.22), so that the spectrum can be treated as a continuum (except in sections 5.3 and 6). For the Hamiltonian (3.9) with $q = 4$, only the multiplicity (3.23) contributes to Δm, so that the paramagnetic initial state (3.45) is characterized at any initial temperature T_0 by the distribution $P_{\rm M}(m, 0)$ equal to

$$P_{\rm M}(m,0) = P_{\rm M0}(m) = \frac{1}{2^N} G(m) \equiv \sqrt{\frac{N}{2\pi}} e^{-Nm^2/2}. \tag{3.50}$$

For the general Hamiltonian (3.7), the width is larger, due to correlations between spins, and given by

$$\delta_0 = \sqrt{\frac{T_0}{T_0 - J_2}}, \qquad \Delta m = \sqrt{\frac{T_0}{N(T_0 - J_2)}}. \tag{3.51}$$

In the pure $q = 2$ case with Hamiltonian (3.8), and in general in case $J_2 > 0$, the temperature T_0 of initialization should be sufficiently higher than the Curie temperature so that $\delta_0^2 \ll N$, which ensures the narrowness of the peak. For an initialization caused by a radiofrequency, the initial distribution is again (3.50).

3.3.4. Ferromagnetic equilibrium states of M

We expect the final state (1.7) of S + A after measurement to involve the two ferromagnetic equilibrium states $\hat{\mathcal{R}}_i$, $i = \Uparrow$ or $\Downarrow$. As above these states $\hat{\mathcal{R}}_i$ of the apparatus factorize, in the weak coupling limit ($\gamma \ll 1$), into the product of (3.32) for the bath and a ferromagnetic equilibrium state $\hat{R}_{\mathrm{M}i}$ for the magnet M. It is tempting to tackle broken invariance by means of the mean-field approximation, which becomes exact at equilibrium for infinite N owing to the long range of the interactions [197, 198]. However, we are interested in a finite, though large, value of N, and the probability distribution $P_{\mathrm{M}i}(m)$ associated with $\hat{R}_{\mathrm{M}i}$ has a significant width around the mean-field value for m. Moreover, we shall see in subsection 7.3 that, in spite of the constancy of the interaction between all spins, the time-dependent mean-field approximation may fail even for large N. We will study there the dynamics of the whole distribution $P_{\mathrm{M}}(m, t)$ including the final regime where it is expected to tend to $P_{\mathrm{M}\Uparrow}(m)$ or $P_{\mathrm{M}\Downarrow}(m)$, and will determine in particular the lifetime of the metastable state (3.44). We focus here on equilibrium only. For later convenience we include an external field h acting on the spins of the apparatus, so as to arrive from (3.7) at[21]

$$\hat{H}_{\mathrm{M}} = -Nh\hat{m} - NJ_2\frac{\hat{m}^2}{2} - NJ_4\frac{\hat{m}^4}{4}. \tag{3.52}$$

As in (3.26) we characterize the canonical equilibrium density operator of the magnet $\hat{R}_{\mathrm{M}} = (1/Z_{\mathrm{M}})\exp[-\beta\hat{H}_{\mathrm{M}}]$, which depends only on the operator $\hat{m}$, by the probability distribution

$$P_{\mathrm{M}}(m) = \frac{\sqrt{N}}{Z_{\mathrm{M}}\sqrt{8\pi}}e^{-\beta F(m)}, \tag{3.53}$$

where m takes the discrete values m_i given by (3.22), and where the *free energy function*

$$F(m) = -NJ_2\frac{m^2}{2} - NJ_4\frac{m^4}{4} - Nhm + \frac{NT}{2}m\ln\frac{1+m}{1-m} + \frac{(N+1)T}{2}\ln\frac{1-m^2}{4} + O\left(\frac{1}{N}\right), \tag{3.54}$$

arises from $\hat{H}_{\mathrm{M}}$ (Eq. 3.6) and from (3.23). The distribution (3.53) displays narrow peaks at the minima of $F(m)$, and the *equilibrium free energy* $-T\ln Z_{\mathrm{M}}$ is equal for large N to the absolute minimum of (3.54). The function $F(m)$ reaches its extrema at values of m given by the self-consistent equation

$$m\left(1 - \frac{1}{N}\right) = \tanh\left[\beta\left(h + J_2 m + J_4 m^3\right)\right], \tag{3.55}$$

which as expected reduces to the mean-field result for large N. In the vicinity of a minimum of $F(m)$ at $m = m_i$, the probability $P_{\mathrm{M}}(m)$ presents around each m_i a nearly Gaussian peak, given within normalization by

$$P_{\mathrm{M}i}(m) \propto \exp\left\{-\frac{N}{2}\left[\frac{1}{1-m_i^2} - \beta J_2 - 3\beta J_4 m_i^2\right](m-m_i)^2 - \frac{N}{3}\left[\frac{m_i}{(1-m_i^2)^2} - 3\beta J_4 m_i\right](m-m_i)^3\right\}. \tag{3.56}$$

This peak is located at a distance of order $1/N$ from the mean-field value, it has a width of order $1/\sqrt{N}$ and a weak asymmetry. The possible values of m are dense within the peak, with equal spacing $\delta m = 2/N$. With each such peak $P_{\mathrm{M}i}(m)$ is associated through (3.25), (3.26) a density operator $\hat{R}_i$ of the magnet M which may describe a locally stable equilibrium. Depending on the values of J_2 and J_4 and on the temperature, there may exist one, two or three such locally stable states. We note the corresponding average magnetizations m_i, for arbitrary h, as m_{P} for a paramagnetic state and as $m_\Uparrow$ and $m_\Downarrow$ for the ferromagnetic states, with $m_\Uparrow > 0$, $m_\Downarrow < 0$. We also note as $\pm m_{\mathrm{F}}$ the ferromagnetic

[21] In section 7 we shall identify h with $+g$ in the sector $\hat{R}_{\uparrow\uparrow}$ of $\hat{D}$, or with $-g$ in its sector $\hat{R}_{\downarrow\downarrow}$, where g is the coupling between S and A, while a true field in the y-direction will be introduced in section 8.2 and denoted by b, see Eq. (8.46)

magnetizations for $h = 0$. When h tends to 0 (as happens at the end of the measurement where we set $g \to 0$), $m_{\rm P}$ tends to 0, $m_{\Uparrow}$ to $+m_{\rm F}$ and $m_{\Downarrow}$ to $-m_{\rm F}$.

For $h = 0$, the system M is invariant under change of sign of m [197]. This invariance is spontaneously broken below some temperature [197]. In the case $q = 2$ of the Ising interaction (3.8), there is above the Curie temperature $T_{\rm c} = J_2$ a single paramagnetic peak $P_{\rm M0}(m)$ at $m_{\rm P} = 0$, given by (3.48), (3.51), and for $T < J_2$ two symmetric ferromagnetic peaks (3.56), $i = \Uparrow$ or $\Downarrow$, at the points $m_{\Uparrow} = m_{\rm F}$ and $m_{\Downarrow} = -m_{\rm F}$, given by $m_{\rm F} = \tanh \beta J_2 m_{\rm F}$. These peaks are well separated provided

$$\frac{N}{2}\left(\frac{1}{1-m_{\rm F}^2} - \beta J_2\right) m_{\rm F}^2 \gg 1, \tag{3.57}$$

in which case they characterize two different equilibrium ferromagnetic states. This condition is satisfied for large N and $\beta J_2 - 1$ finite; near $\beta J_2 = 1$, where $m_{\rm F}^2 \sim 3(\beta J_2 - 1)$, the two states $\hat{R}_{\Uparrow}$ and $\hat{R}_{\Downarrow}$ still have no overlap as soon as the temperature differs significantly from the critical temperature, as

$$\frac{J_2 - T}{T} \gg \frac{1}{\sqrt{3N}}. \tag{3.58}$$

This property is needed to ensure a faithful registration by M of the measurement. Little is changed for the Hamiltonian (3.7) with $J_4 > 0$ but still $J_2 > 3J_4$.

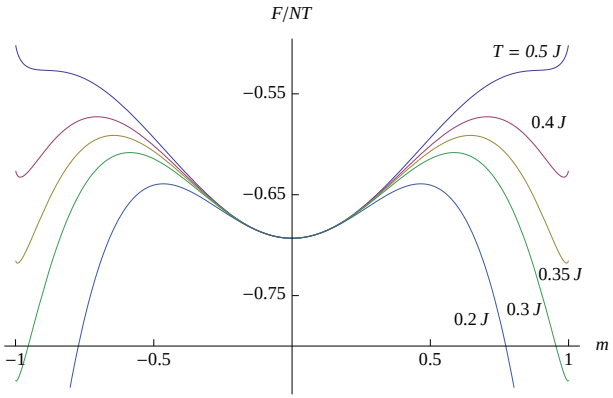

Figure 3.3: The free energy F in units of NT for a pure quartic interaction (eq. (3.9), evaluated from Eq. (3.54) with $h = 0$, as function of the magnetization m at various temperatures. There is always a local paramagnetic minimum at $m = 0$. A first-order transition occurs at $T_{\rm c} = 0.363 J_4$, below which the ferromagnetic states associated with the minima at $\pm m_{\rm F}$ near ± 1 become the most stable.

Still for $h = 0$, but in the case $3J_4 > J_2$ of a first-order transition, $F(m)$ has a minimum at $m = 0$ if $T > J_2$ and hence (3.53) has there a peak as (3.50) at $m = 0$ whatever the temperature, see Fig. 3.3. For the pure quartic interaction of Eq. (3.9), the two additional ferromagnetic peaks $P_{\rm M\Uparrow}(m)$ and $P_{\rm M\Downarrow}(m)$ appear around $m_{\Uparrow} = m_{\rm F} = 0.889$ and $m_{\Downarrow} = -m_{\rm F}$ when the temperature T is lower than $0.496 J_4$. As T decreases, $m_{\rm F}$ given by $m_{\rm F} = \tanh \beta J_4 m_{\rm F}^3$ increases and the value of the minimum $F(m_{\rm F})$ decreases; the weight (3.53) is transferred from $P_{\rm M0}(m)$ to $P_{\rm M\Uparrow}(m)$ and $P_{\rm M\Downarrow}(m)$. A first-order transition occurs when $F(m_{\rm F}) = F(0)$, for $T_{\rm c} = 0.363 J_4$ and $m_{\rm F} = 0.9906$, from the paramagnetic to the two ferromagnetic states, although the paramagnetic state remains locally stable. The spontaneous magnetization $m_{\rm F}$ is always very close to 1, behaving as $1 - m_{\rm F} \sim 2e^{-2J_4/T}$.

For the general Hamiltonian (3.7), it is a simple exercise to study the cross-over between first and second-order transitions, which takes place for $m_i \ll 1$. To this aim, the free energy (3.54) is expanded as

$$\frac{F(m) - F(0)}{N} \approx (T - J_2)\frac{m^2}{2} + (T - 3J_4)\frac{m^4}{12} + T\frac{m^6}{30}, \tag{3.59}$$

and its shape and minima are studied as function of J_2, J_4 and T. This approximation holds for $|T - J_2| \ll J_2$, $|3J_4 - J_2| \ll J_2$. For $J_2 > 3J_4$, the second-order transition takes place at $T_{\rm c} = J_2$ whatever J_4. For $3J_4 > J_2$, the

first-order transition temperature T_c is given by $T_c - J_2 \sim 5(3J_4 - J_2)^2/48J_2$, and the equilibrium magnetization jumps from 0 to $\pm m_F$, with $m_F^2 \sim 5(3J_4 - J_2)/4J_2$. The paramagnetic state is locally stable down to $T > J_2$, the ferromagnetic states up to $T - J_2 < (4/3)(T_c - J_2)$. When $3J_4 > J_2$, a metastability with a long lifetime of the paramagnetic state is thus ensured if the bath temperature satisfies $T_c > T > J_2$.

Strictly speaking, the canonical equilibrium state of M below the transition temperature, characterized by (3.53), has for $h = 0$ and finite N the form $\hat{R}_{Meq} = \frac{1}{2}(\hat{R}_{M\Uparrow} + \hat{R}_{M\Downarrow})$. However this state is not necessarily the one reached at the end of a relaxation process governed by the bath B, when a field h, even weak, is present: this field acts as a source which breaks the invariance. The determination of the state $\hat{R}_M(t_f)$ reached at the end of a relaxation process involving the thermal bath B and a weak field h requires a dynamical study which will be worked out in subsection 7.3. In our model of measurement, the situation is similar, though slightly more complicated. The system-apparatus coupling (3.5) plays the rôle of an operator-valued source, with eigenvalues behaving as a field $h = g$ or $h = -g$. We shall determine in section 7 towards which state M is driven under the conjugate action of the bath B and of the system S, depending on the parameters of the model.

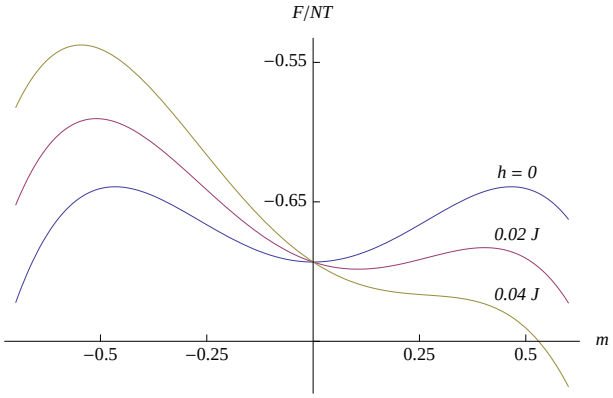

Figure 3.4: The effect of a positive field h on $F(m)$ for $q = 4$ at temperature $T = 0.2J_4$. As h increases the paramagnetic minimum m_P shifts towards positive m. At the critical field $h_c = 0.0357J_4$ this local minimum disappears, and the curve has an inflexion point with vanishing slope at $m = m_c = 0.268$. For larger fields, like in the displayed case $g = 0.04J_4$, the locally stable paramagnetic state disappears, and there remain only the two ferromagnetic states, the most stable one with positive magnetization $m_{\Uparrow} \simeq 1$ and the metastable one with negative magnetization $m_{\Downarrow} \simeq -1$.

As a preliminary step, let us examine here the effect on the free energy (3.54) of a small positive field h. Consider first the minima of $F(m)$ [196, 197]. The two ferromagnetic minima $m_{\Uparrow}$ and $m_{\Downarrow}$ given by (3.55) are slightly shifted away from m_F and $-m_F$, and $F(m_{\Uparrow}) - F(m_F)$ behaves as $-Nhm_F$. Hence, as soon as $\exp\{-\beta[F(m_{\Uparrow}) - F(m_{\Downarrow})]\} \sim \exp(2\beta Nhm_F) \gg 1$, only the single peak $P_{M\Uparrow}(m)$ around $m_{\Uparrow} \simeq m_F$ contributes to (3.53), so that the canonical equilibrium state of M has the form $\hat{R}_{Meq} = \hat{R}_{M\Uparrow}$. The shape of $F(m)$ will also be relevant for the dynamics. For a second order transition, although $F(m)$ has when $h = 0$ a maximum at $m = 0$, its stationarity allows the state $\hat{R}_M(m, 0) \propto P_M(\hat{m}, 0)$ given by (3.48) to have a long lifetime for $N \gg 1$. The introduction of h produces a negative slope $-Nh$ at $m = 0$, which suggests that the dynamics will let $\langle m \rangle$ increase. For a first order transition, the situation is different (Fig. 3.4). If h is sufficiently small, $F(m)$ retains its paramagnetic minimum, the position of which is shifted as $m_P \sim h/T$; the paramagnetic state $\hat{R}_M(0)$ remains locally stable. It may decay towards a stable ferromagnetic state only through mechanisms of thermal activation or quantum tunnelling, processes with very large characteristic times, of exponential order in N. However, there is a threshold h_c above which this paramagnetic minimum of $F(m)$, which then lies at $m = m_c$, disappears. The value of h_c is found by eliminating $m = m_c$ between the equations $d^2F/dm^2 = 0$ and $dF/dm = 0$. In the pure $q = 4$ case ($J_2 = 0$) on which we focus as an illustration for first order transitions, we find $2m_c^2 = 1 - \sqrt{1 - 4T/3J_4}$, $h_c = \frac{1}{2}T \ln[1 + m_c)/(1 - m_c)] - J_4 m_c^3$. At the transition temperature $T_c = 0.363J_4$, we have $m_c = 0.375$ and $h_c = 0.0904J_4$; for $T = 0.2J_4$, we obtain $m_c = 0.268$ and $h_c = 0.036J_4$; for $T \ll J_4$, m_c behaves as $\sqrt{T/3J_4}$ and h_c as $\sqrt{4T^3/27J_4}$. Provided $h > h_c$, $F(m)$ has now a negative slope in the whole interval $0 < m < m_F$. We can thus expect, in our measurement problem, that the registration will take place in a reasonable delay, either for for a first order transition if the coupling g is larger than h_c, or for a second order transition. In the latter case, it will be necessary to check, however, that the lifetime of the initial state is larger than the duration of the measurement.

This will be done in § 7.3.2.

4. Equations of motion

Τὰ πάντα ῥεῖ[22]
Quoted from Heraklitos by Plato and Simplicius

In this technical section, we rewrite the dynamical equations for our model in a form which will help us, in the continuation, to discuss the physical features of the solution. We will make no other approximation than the weak spin-phonon coupling, $\gamma \ll 1$, and will derive the equations up to first order in γ. In subsection 4.5, we take advantage of the large size of the apparatus, $N \gg 1$, to reduce the equations of motion into a pair of partial differential equations.

4.1. A conserved quantity, the measured component of the spin, and the Born rule

All the world's Great Journeys begin with the first step
A 1000 miles journey starts with a single step
Tibetan and Aboriginal Australian proverbs

With $\hat{H}_S = 0$ to express absence of dynamics during the measurement process and, according to the von Neumann prescription, only the measured observable $\hat{s}_z$ (and not $\hat{s}_x$ or $\hat{s}_y$) occurring in the coupling (3.5) between S and A, we can already conclude that $\hat{s}_z$ is conserved during the measurement, viz. $i\hbar \mathrm{d}\hat{s}_z/\mathrm{d}t = [\hat{s}_z, \hat{H}] = 0$. This implies that the diagonal elements of the density matrix of the spin are conserved, viz. $r_{\uparrow\uparrow}(t_f) = r_{\uparrow\uparrow}(t) = r_{\uparrow\uparrow}(0)$ and $r_{\downarrow\downarrow}(t_f) = r_{\downarrow\downarrow}(0)$. The result embodies *the Born rule*: the probabilities for the possible *outcomes* of an ideal measurement are given by the diagonal elements of the *initial* density matrix of S. But $r_{\uparrow\downarrow}$ and $r_{\downarrow\uparrow}$, on the other hand, are not conserved (viz. $[\hat{s}_a, \hat{H}] \neq 0$ for $a = x, y$), and they will evolve and ultimately vanish[23].

4.2. Eliminating the bath variables

Don't call the alligator a big-mouth till you have crossed the river
Brazilian proverb

A complete description of the measurement process requires the solution, in the Hilbert space of S + A, of the Liouville–von Neumann equation of motion [196]

$$i\hbar \frac{\mathrm{d}\hat{\mathcal{D}}}{\mathrm{d}t} = \left[\hat{H}, \hat{\mathcal{D}}\right], \tag{4.1}$$

with the initial condition

$$\hat{\mathcal{D}}(0) = \hat{r}(0) \otimes \hat{R}_M(0) \otimes \hat{R}_B(0) = \hat{D}(0) \otimes \hat{R}_B(0). \tag{4.2}$$

We are not interested, however, in the bath variables, and the knowledge of $\hat{D}(t) = \mathrm{tr}_B \hat{\mathcal{D}}(t)$ is sufficient for our purpose. As usual in non-equilibrium statistical mechanics [145, 199, 200, 201], we rely on the weakness of the coupling $\hat{H}_{MB}$ between the magnet and the bath, so as to treat perturbatively the dissipative effect of the bath.

Let us therefore split the Hamiltonian (3.3) into $\hat{H} = \hat{H}_0 + \hat{H}_{MB} + \hat{H}_B$ with $\hat{H}_0 = \hat{H}_S + \hat{H}_{SA} + \hat{H}_M$. Regarding the coupling $\hat{H}_{MB}$ as a perturbation, we introduce the unperturbed evolution operators, namely (3.35) for the bath, and

$$\hat{U}_0(t) = e^{-i\hat{H}_0 t/\hbar} \tag{4.3}$$

for S + M. We can then expand the full evolution operator in powers of the coupling $\sqrt{\gamma}$, in the interaction picture, and take the trace over B of eq. (4.1) so as to generate finally an equation of motion for the density operator $\hat{D}(t)$ of S + M. This calculation is worked out in Appendix A.

[22] Everything flows
[23] This has the popular name "decay of Schrödinger cat terms"

The result involves the autocorrelation function $K(t)$ of the bath, defined by (3.32) – (3.35) and expressed in our model by (3.36), (3.37). It also involves the operators $\hat{\sigma}_a^{(n)}(u)$ in the space of S + M, defined in terms of the memory time $u = t - t'$ by

$$\hat{\sigma}_a^{(n)}(u) \equiv \hat{U}_0(t)\,\hat{U}_0^\dagger(t')\,\hat{\sigma}_a^{(n)}\hat{U}_0(t')\,\hat{U}_0^\dagger(t) = \hat{U}_0(u)\,\hat{\sigma}_a^{(n)}\hat{U}_0^\dagger(u)\,. \tag{4.4}$$

Altogether we obtain a differential equation for $\hat{D}(t)$, the kernel of which involves times earlier than t through $K(u)$ and $\hat{\sigma}_a^{(n)}(u)$ [145, 199, 200]:

$$\frac{\mathrm{d}\hat{D}}{\mathrm{d}t} - \frac{1}{i\hbar}\left[\hat{H}_0, \hat{D}\right] = \frac{\gamma}{\hbar^2}\int_0^t \mathrm{d}u \sum_{n,a}\left\{K(u)\left[\hat{\sigma}_a^{(n)}(u)\,\hat{D}, \hat{\sigma}_a^{(n)}\right] + K(-u)\left[\hat{\sigma}_a^{(n)}, \hat{D}\hat{\sigma}_a^{(n)}(u)\right]\right\} + O\left(\gamma^2\right). \tag{4.5}$$

As anticipated in § 3.3.2, the phonon bath occurs in this equation, which governs the dynamics of S + M, only through the function $K(t)$, the memory time being the time-range $\hbar/2\pi T$ of $K(t)$ [145, 199, 200].

4.3. Decoupled equations of motion

In our model, the Hamiltonian commutes with the measured observable $\hat{s}_z$, hence with the projection operators $\hat{\Pi}_i$ onto the states $|\uparrow\rangle$ and $|\downarrow\rangle$ of S. The equations for the operators $\hat{\Pi}_i\hat{D}\hat{\Pi}_j$ are therefore decoupled. We can replace the equation (4.5) for $\hat{D}$ in the Hilbert space of S + M by a set of four equations for the operators $\hat{R}_{ij}$ defined by (3.18) in the Hilbert space of M. We shall later see (section 8.2) that this simplification underlies the ideality of the measurement process.

The Hamiltonian $\hat{H}_0$ in the space S + M gives rise to two Hamiltonians $\hat{H}_\uparrow$ and $\hat{H}_\downarrow$ in the space M, which according to (3.5) and (3.7) are simply two functions of the observable $\hat{m}$, given by

$$\hat{H}_i = H_i(\hat{m}) = -gNs_i\hat{m} - N\sum_{q=2,4}\frac{J_q}{q}\hat{m}^q, \qquad (i = \uparrow, \downarrow) \tag{4.6}$$

with $s_i = +1$ (or -1) for $i = \uparrow$ (or $\downarrow$). These Hamiltonians $\hat{H}_i$, which describe interacting spins $\hat{\sigma}^{(n)}$ in an external field gs_i, occur in (4.5) both directly and through the operators

$$\hat{\sigma}_a^{(n)}(u, i) = e^{-i\hat{H}_i u/\hbar}\hat{\sigma}_a^{(n)}e^{i\hat{H}_i u/\hbar}, \tag{4.7}$$

obtained by projection of (4.4) with $\hat{\Pi}_i$ and reduction to the Hilbert space of M.

The equation (4.5) for $\hat{D}(t)$ which governs the joint dynamics of S+M thus reduces to the four differential equations in the Hilbert space of M (we recall that $i, j = \uparrow, \downarrow$ or ± 1):

$$\frac{\mathrm{d}\hat{R}_{ij}(t)}{\mathrm{d}t} - \frac{\hat{H}_i\hat{R}_{ij}(t) - \hat{R}_{ij}(t)\hat{H}_j}{i\hbar} = \frac{\gamma}{\hbar^2}\int_0^t \mathrm{d}u \sum_{n,a}\left\{K(u)\left[\hat{\sigma}_a^{(n)}(u, i)\,\hat{R}_{ij}(t), \hat{\sigma}_a^{(n)}\right] + K(-u)\left[\hat{\sigma}_a^{(n)}, \hat{R}_{ij}(t)\hat{\sigma}_a^{(n)}(u, j)\right]\right\}. \tag{4.8}$$

4.4. Reduction to scalar equations

4.4.1. Representing the pointer by a scalar variable

Even a small star shines in the darkness
Finnish proverb

For each operator $\hat{R}_{ij}$, the initial conditions are given according to (3.43) and (3.44) by

$$\hat{R}_{ij}(0) = r_{ij}(0)\,\hat{R}_\mathrm{M}(0)\,, \tag{4.9}$$

and $\hat{R}_\mathrm{M}(0)$ expressed by the Gibbs state (3.45) is simply a function of the operator $\hat{m}$. We show in Appendix B that this property is preserved for $\hat{R}_{ij}(t)$ by the evolution (4.8), owing to the form (4.6) of $\hat{H}_i$ and in spite of the occurrence of the operators $\hat{\sigma}_a^{(n)}$ in the right-hand side.

We can therefore parametrize, as anticipated at the end of § 3.3.1, the operators $\hat{R}_{ij}$ in the form $\hat{R}_{ij} = P_{ij}^\mathrm{dis}(\hat{m})/G(\hat{m})$. Their equations of motion (4.8) are then diagonal in the eigenspace of $\hat{m}$, and are therefore equivalent to scalar equations which govern the functions $P_{ij}(m) = (N/2)P_{ij}^\mathrm{dis}(m)$ of the variable m taking the discrete values (3.22).

4.4.2. Equations of motion for $P_{ij}(m,t)$

The equations resulting from this parametrization are derived in Appendix B. The integrals over u entering in (4.8) yield the functions

$$\tilde{K}_{t>}(\omega) = \int_0^t \mathrm{d}u e^{-i\omega u} K(u) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \mathrm{d}\omega' \tilde{K}(\omega') \frac{e^{i(\omega'-\omega)t}-1}{\omega'-\omega}, \tag{4.10}$$

and

$$\tilde{\mathrm{K}}_{t<}(\omega) = \int_0^t \mathrm{d}u e^{i\omega u} K(-u) = \int_{-t}^0 \mathrm{d}u e^{-i\omega u} K(u) = \left[\tilde{K}_{t>}(\omega)\right]^* = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \mathrm{d}\omega' \tilde{K}(\omega') \frac{1-e^{i(\omega-\omega')t}}{\omega'-\omega}, \tag{4.11}$$

where ω takes, depending on the considered term, the values $\Omega_\uparrow^+, \Omega_\uparrow^-, \Omega_\downarrow^+, \Omega_\downarrow^-$, given by

$$\hbar\Omega_i^\pm(m) = H_i(m\pm\delta m) - H_i(m), \qquad (i=\uparrow,\downarrow), \tag{4.12}$$

in terms of the Hamiltonians (4.6) and of the level spacing $\delta m = 2/N$. They satisfy the relations

$$\Omega_i^\pm(m\mp\delta m) = -\Omega_i^\mp(m). \tag{4.13}$$

The quantities (4.12) are interpreted as excitation energies of the magnet M arising from the flip of one of its spins in the presence of the tested spin S (with value s_i); the sign + (−) refers to a down-up (up-down) spin flip. Their explicit values are:

$$\hbar\Omega_i^\pm(m) = \mp 2gs_i + 2J_2(\mp m - \frac{1}{N}) + 2J_4(\mp m^3 - \frac{3m^2}{N} \mp \frac{4m}{N^2} - \frac{2}{N^3}), \tag{4.14}$$

with $s_\uparrow = 1$, $s_\downarrow = -1$.

The operators $\hat{\sigma}_a^{(n)}$ which enter (4.8) are shown in Appendix B to produce a flip of the spin $\hat{\boldsymbol{\sigma}}^{(n)}$, that is, a shift of the operator $\hat{m}$ into $\hat{m}\pm\delta m$. (More precisely, the $\hat{\sigma}_{x,y}^{(n)}$ induce a shift, while the $\hat{\sigma}_z^{(n)}$ have no effect.) We introduce the notations

$$\Delta_\pm f(m) = f(m_\pm) - f(m), \qquad m_\pm = m\pm\delta m, \qquad \delta m = \frac{2}{N}. \tag{4.15}$$

The resulting dynamical equations for $P_{ij}(m,t)$ take different forms for the diagonal and for the off-diagonal components. On the one hand, the first *diagonal block* of $\hat{D}$ is parameterized by the *joint probabilities* $P_{\uparrow\uparrow}(m,t)$ to find S in $|\uparrow\rangle$ and $\hat{m}$ equal to m at the time t. These probabilities evolve according to

$$\frac{\mathrm{d}P_{\uparrow\uparrow}(m,t)}{\mathrm{d}t} = \frac{\gamma N}{\hbar^2}\left\{\Delta_+\left[(1+m)\tilde{K}_t\left(\Omega_\uparrow^-(m)\right)P_{\uparrow\uparrow}(m,t)\right] + \Delta_-\left[(1-m)\tilde{K}_t\left(\Omega_\uparrow^+(m)\right)P_{\uparrow\uparrow}(m,t)\right]\right\}, \tag{4.16}$$

with initial condition $P_{\uparrow\uparrow}(m,0) = r_{\uparrow\uparrow}(0)P_{\mathrm{M}}(m,0)$ given by (3.48); likewise for $P_{\downarrow\downarrow}(m)$, which involves the frequencies $\Omega_\downarrow^\mp(m)$. The factor $\tilde{K}_t(\omega)$ is expressed by the combination of two terms,

$$\tilde{K}_t(\omega) \equiv \tilde{K}_{t>}(\omega) + \tilde{K}_{t<}(\omega) = \int_{-t}^{+t} \mathrm{d}u e^{-i\omega u} K(u) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega'}{\pi} \frac{\sin(\omega'-\omega)t}{\omega'-\omega}\tilde{K}(\omega'). \tag{4.17}$$

It is real and tends to $\tilde{K}(\omega)$, given in Eq. (3.37), at times t larger than the range $\hbar/2\pi T$ of $K(t)$ [145, 199, 200]. This may be anticipated from the relation $\sin[(\omega'-\omega)t]/\pi(\omega'-\omega) \to \delta(\omega'-\omega)$ for $t\to\infty$ and it may be demonstrated with help of the contour integration techniques of Appendix D, which we leave as a student exercise, see section 9.6.

On the other hand, the sets $P_{\uparrow\downarrow}(m,t)$ and $P_{\downarrow\uparrow} = P^{\star}_{\uparrow\downarrow}$ which parameterize the *off-diagonal blocks* of $\hat{D}$, and which are related through (3.29) to the correlations between $\hat{s}_x$ or $\hat{s}_y$ and any number of spins of M, evolve according to

$$\frac{\mathrm{d}}{\mathrm{d}t}P_{\uparrow\downarrow}(m,t) - \frac{2iNgm}{\hbar}P_{\uparrow\downarrow}(m,t) = \frac{\gamma N}{\hbar^2}\left\{\Delta_+\left[(1+m)\,\tilde{K}_-(m,t)P_{\uparrow\downarrow}(m,t)\right] + \Delta_-\left[(1-m)\,\tilde{K}_+(m,t)P_{\uparrow\downarrow}(m,t)\right]\right\}, \quad (4.18)$$

with initial condition $P_{\uparrow\downarrow}(m,0) = r_{\uparrow\downarrow}(0)\,P_{\mathrm{M}}(m,0)$. Here $\tilde{K}_{t>}$ and $\tilde{K}_{t<}$ enter in the combinations

$$\tilde{K}_{\pm}(m,t) \equiv \tilde{K}_{t>}\left[\Omega^{\pm}_{\uparrow}(m)\right] + \tilde{K}_{t<}\left[\Omega^{\pm}_{\downarrow}(m)\right]. \quad (4.19)$$

4.4.3. Interpretation as quantum balance equations

Je moet je evenwicht bewaren[24]
Dutch expression

Our basic equations (4.16) and (4.18) fully describe the dynamics of the measurement. The diagonal equation (4.16) can be interpreted as a balance equation [145, 199, 200]. Its first term represents elementary processes in which one among the spins $\sigma^{(n)}$ flips from $\sigma_z^{(n)} = +1$ to $\sigma_z^{(n)} = -1$. For the value m of the magnetization, a value taken with probability $P_{\uparrow\uparrow}(m,t)$ at the time t, there are $\frac{1}{2}N(1+m)$ spins pointing upwards, and the probability for one of these spins to flip down between the times t and $t+\mathrm{d}t$ under the effect of the phonon bath can be read off from (4.18) to be equal to $2\gamma\hbar^{-2}\tilde{K}_t\left(\Omega^{-}_{\uparrow}\right)\mathrm{d}t$. This process produces a decrease of $P_{\uparrow\uparrow}(m)$ and it is accounted for by the negative contribution (which arises from the second part of Δ_+ and is proportional to $-P_{\uparrow\uparrow}(m,t)$) to the first term in the right-hand side of (4.16). The coefficient $\tilde{K}_t(\omega)$ depends on the temperature T of the bath B, on the duration t of its interaction with M, and on the energy $\hbar\omega$ that it has transferred to M; this energy is evaluated for $P_{\uparrow\uparrow}$ (or $P_{\downarrow\downarrow}$) as if the spins of M were submitted to an external field $+g$ (or $-g$). The first term in (4.16) also contains a positive contribution arising from the same process, for which the magnetization decreases from $m+\delta m$ to m, thus raising $P_{\uparrow\uparrow}(m)$ by a term proportional to $P_{\uparrow\uparrow}(m+\delta m)$. Likewise, the second term in the right-hand side of (4.16) describes the negative and positive changes of $P_{\uparrow\uparrow}(m)$ arising from the flip of a single spin from $\sigma_z^{(n)} = -1$ to $\sigma_z^{(n)} = +1$. Quantum mechanics occurs in (4.16) through the expression (3.37) of $\tilde{K}(\omega)$; though in the flipping probabilities the factors $\hbar$ cancel, owing to the factor $\hbar^2$ that enters $\tilde{K}(\omega)$, their quantum nature is still expressed by the Bose-Einstein occupation numbers.

The equation (4.18) for $P_{\uparrow\downarrow}$ has additional quantum features. Dealing with an off-diagonal block, it involves simultaneously the two Hamiltonians $\hat{H}_{\uparrow}$ and $\hat{H}_{\downarrow}$ of Eq. (4.6) in the Hilbert space of M through the $\Omega^{\pm}_{\uparrow,\downarrow}$. The quantities $P_{\uparrow\downarrow}$ and $P_{\downarrow\uparrow}$ are complex and cannot be interpreted as probabilities, although we recognize in the right-hand side the same type of balance as in Eq. (4.16). In fact, while $\sum_m P^{\mathrm{dis}}_{\uparrow\uparrow}(m) = 1 - \sum_m P^{\mathrm{dis}}_{\downarrow\downarrow}(m)$, or in the $N \gg 1$ limit $\int \mathrm{d}m\, P_{\uparrow\uparrow}(m) = 1 - \int \mathrm{d}m\, P_{\downarrow\downarrow}(m)$, remains constant in time because the sum over m of the right-hand side of (4.16) vanishes, the term in the left-hand side of (4.18), which arises from $H_i - H_j$, prevents $\sum_m P^{\mathrm{dis}}_{\uparrow\downarrow}(m)$ from being constant; It will, actually, lead to the disappearance of these "Schrödinger cat" terms.

Comparison of the right-hand sides of (4.16) and (4.18) shows moreover that the bath acts in different ways on the diagonal and off-diagonal blocks of the density operator $\hat{D}$ of S + M.

4.5. Large N expansion

Except in subsection 8.1 we shall deal with a magnetic dot sufficiently large so that $N \gg 1$. The set of values (3.22) on which the distributions $P_{ij}(m,t)$ are defined then become dense on the interval $-1 \le m \le +1$. At the initial time, $P_{ij}(m,0)$, proportional to (3.48), extends over a range of order $1/\sqrt{N}$ while the spacing of the discrete values of m is $\delta m = 2/N$. The initial distributions P_{ij} are thus smooth on the scale δm, and $P_{\uparrow\uparrow}$ and $P_{\downarrow\downarrow}$ will remain smooth at later times. It is therefore legitimate to *interpolate* the set of values of the diagonal quantities $P_{ii}(m,t)$ defined at the discrete points (3.22) into a continuous function of m. If we assume the two resulting functions P_{ii} to be several times differentiable with respect to m, the discrete equation (4.16) satisfied by the original distributions will give rise

[24] You have to keep your balance

to continuous equations, derived below, involving an asymptotic expansion in powers of $1/N$. Within exponentially small corrections, the characteristic functions associated with $P_{ii}(m,t)$ then reduce to integrals:

$$\Psi_{ii}(\lambda,t) \equiv \sum_m P_{ii}^{\rm dis}(m,t)\, e^{\lambda m} = \int \mathrm{d}m P_{ii}(m,t)\, e^{\lambda m}, \tag{4.20}$$

provided $\lambda \ll N$. The moments of $P_{ii}(m)$ of order less than N can also be evaluated as integrals.

However, the left-hand side of Eq. (4.18) generates for finite times rapid variations of $P_{\uparrow\downarrow}(m,t)$ and $P_{\downarrow\uparrow}(m,t)$ as functions of m, and it will be necessary in sections 5 and 6 to account for the discrete nature of m. When writing below the equations of motion for these quantities in the large N limit, we will take care of this difficulty.

The differences $\Delta_\pm$ defined by (4.15) satisfy

$$\Delta_\pm[f(m)g(m)] = [\Delta_\pm f(m)]\, g(m) + f(m)\,[\Delta_\pm g(m)] + [\Delta_\pm f(m)]\,[\Delta_\pm g(m)], \tag{4.21}$$

and give rise to derivatives with respect to m according to

$$\Delta_\pm f(m) \approx \pm\frac{2}{N}\frac{\partial f(m)}{\partial m} + \frac{2}{N^2}\frac{\partial^2 f(m)}{\partial m^2} \pm \frac{4}{3N^3}\frac{\partial^3 f(m)}{\partial m^3}. \tag{4.22}$$

We can also expand the excitation energies $\hbar\Omega_i^\pm$, defined by (4.12) and (4.6), for large N as

$$\Omega_i^\pm(m) \approx \mp 2\omega_i - \frac{2}{N}\frac{\mathrm{d}\omega_i}{\mathrm{d}m} = \left(1 \pm \frac{1}{N}\frac{\mathrm{d}}{\mathrm{d}m}\right)(\mp 2\omega_i), \tag{4.23}$$

where we introduced the quantity

$$\hbar\omega_i = -\frac{1}{N}\frac{\mathrm{d}H_i}{\mathrm{d}m} = g s_i + J_2 m + J_4 m^3, \qquad (s_i = \pm 1), \tag{4.24}$$

interpreted as the effective energy of a single spin of M coupled to the other spins of M and to the tested spin S.

The above expansions will allow us to transform, for large N, the equations of motion for P_{ij} into partial differential equations. In case $\partial P_{ij}/\partial m$ is finite for large N, we can simply replace in (4.16) and (4.18) $N\Delta_\pm$ by $\pm 2\partial/\partial m$ and $\Omega_i^\pm$ by $\mp 2\omega_i$. However, such a situation is exceptional; we shall encounter it only in § 7.3.2. In general P_{ij} will behave for large N as $A \exp NB$. This property, exhibited at $t = 0$ in §§ 3.3.3 and 3.3.4, is preserved by the dynamics. As $\partial P_{ij}/\partial t$ involves leading contributions of orders N and 1, we need to include in the right-hand sides of (4.16) and (4.18) contributions of the same two orders. Let us therefore introduce the functions

$$X_{ij}(m,t) \equiv \frac{1}{N}\frac{\partial \ln P_{ij}}{\partial m} = \frac{1}{NP_{ij}}\frac{\partial P_{ij}}{\partial m}, \tag{4.25}$$

which contain parts of order 1 and $1/N$, and their derivatives

$$X'_{ij} \equiv \frac{1}{N}\frac{\partial^2 \ln P_{ij}}{\partial m^2} = \frac{\partial X_{ij}}{\partial m}, \tag{4.26}$$

which can be truncated at finite order in N. The discrete increments of P_{ij} are thus expanded as

$$\begin{aligned}\Delta_\pm P_{ij} &= P_{ij}\left[\exp(\Delta_\pm \ln P_{ij}) - 1\right] \approx P_{ij}\left[\exp\left(\pm 2X_{ij} + \frac{2}{N}X'_{ij}\right) - 1 + O\left(\frac{1}{N^2}\right)\right] \\ &\approx P_{ij}\left[\exp\left(\pm 2X_{ij}\right) - 1 + \frac{2X'_{ij}}{N}\exp\left(\pm 2X_{ij}\right) + O\left(\frac{1}{N^2}\right)\right].\end{aligned} \tag{4.27}$$

Using the full relation (4.21) with $f = P_{\uparrow\uparrow}$ and $g = (1 \pm m)\tilde{K}_t(\Omega_\uparrow^\mp)$ with $\Delta_\pm f$ from (4.27) and $\Delta_\pm g = \pm(2/N)\partial g/\partial m$, Eq. (4.16) then becomes

$$\frac{\partial P_{\uparrow\uparrow}}{\partial t} \approx \frac{2\gamma}{\hbar^2} P_{\uparrow\uparrow} \left\{ N \sinh X_{\uparrow\uparrow} \left[(1+m)\tilde{K}_t(\Omega_\uparrow^-) e^{X_{\uparrow\uparrow}} - (1-m)\tilde{K}_t(\Omega_\uparrow^+) e^{-X_{\uparrow\uparrow}} \right] \right.$$
$$\left. + e^{X_{\uparrow\uparrow}} \frac{\partial}{\partial m} \left[(1+m)\tilde{K}_t(2\omega_\uparrow) e^{X_{\uparrow\uparrow}} \right] - e^{-X_{\uparrow\uparrow}} \frac{\partial}{\partial m} \left[(1-m)\tilde{K}_t(-2\omega_\uparrow) e^{-X_{\uparrow\uparrow}} \right] + O\left(\frac{1}{N}\right) \right\}. \tag{4.28}$$

The first term on the right-hand side determines the evolution of the exponent of $P_{\uparrow\uparrow}$, which contains parts of order N, but contains also contributions of order 1 arising from the terms of order $1/N$ from (4.27) and from X_{ij} and their derivatives. The remaining terms in $1/N$ determine the evolution of the amplitude of P_{ij}. The bath term of the equation (4.18) for $P_{\uparrow\downarrow}(m,t)$ (and for $P_{\downarrow\uparrow} = P_{\uparrow\downarrow}^*$) has a similar form, again obtained from all the terms in (4.21) and (4.27), namely, using the notation (4.19):

$$\frac{\partial P_{\uparrow\downarrow}}{\partial t} - \frac{2iNgm}{\hbar} P_{\uparrow\downarrow} \approx \frac{2\gamma}{\hbar^2} P_{\uparrow\downarrow} \left\{ N \sinh X_{\uparrow\downarrow} \left[(1+m)\tilde{K}_-(m,t) e^{X_{\uparrow\downarrow}} - (1-m)\tilde{K}_+(m,t) e^{-X_{\uparrow\downarrow}} \right] \right.$$
$$\left. + e^{X_{\uparrow\downarrow}} \frac{\partial}{\partial m} \left[(1+m)\tilde{K}_-(m,t) e^{X_{\uparrow\downarrow}} \right] - e^{-X_{\uparrow\downarrow}} \frac{\partial}{\partial m} \left[(1-m)\tilde{K}_+(m,t) e^{-X_{\uparrow\downarrow}} \right] + O\left(\frac{1}{N}\right) \right\}. \tag{4.29}$$

A further simplification occurs for large N in the diagonal sector. Then $P_{\uparrow\uparrow}$, which is real, takes significant values only in the vicinity of the maximum of $\ln P_{\uparrow\uparrow}$. This maximum is reached at a point $m = \mu(t)$, and $P_{\uparrow\uparrow}$ is concentrated in a range for $|m - \mu(t)|$ of order $1/\sqrt{N}$ [25]. In this range, $X_{\uparrow\uparrow}$ is proportional to $\mu(t) - m$, and it is therefore of order $1/\sqrt{N}$[26]. We can therefore expand (4.28) in powers of $X_{\uparrow\uparrow}$, noting also that $X'_{\uparrow\uparrow}$ is finite, and collect the $X_{\uparrow\uparrow}$, $X_{\uparrow\uparrow}^2$, $X'_{\uparrow\uparrow}$ and $X_{\uparrow\uparrow}X'_{\uparrow\uparrow}$ terms. Thus, if we disregard the exponentially small tails of the distribution $P_{\uparrow\uparrow}$, which do not contribute to physical quantities, we find at the considered order, using (4.25) and (4.26),

$$\frac{\partial P_{\uparrow\uparrow}}{\partial t} \approx \frac{\partial}{\partial m} \left[-v\,(m,t)\, P_{\uparrow\uparrow} \right] + \frac{1}{N} \frac{\partial^2}{\partial m^2} \left[w\,(m,t)\, P_{\uparrow\uparrow} \right], \tag{4.30}$$

where

$$v\,(m,t) = \frac{2\gamma}{\hbar^2} \left[(1-m)\,\tilde{K}_t\,(-2\omega_\uparrow) - (1+m)\,\tilde{K}_t\,(2\omega_\uparrow) \right] + O\left(\frac{1}{N}\right), \tag{4.31}$$

$$w\,(m,t) = \frac{2\gamma}{\hbar^2} \left[(1-m)\,\tilde{K}_t\,(-2\omega_\uparrow) + (1+m)\,\tilde{K}_t\,(2\omega_\uparrow) \right] + O\left(\frac{1}{N}\right). \tag{4.32}$$

The next contribution to the right hand side of (4.30) would be $-2vX_{\uparrow\uparrow}X'_{\uparrow\uparrow}P_{\uparrow\uparrow}$, of order $1/\sqrt{N}$. We have replaced in v and w the frequencies $\Omega_\uparrow^\pm$ by $\mp 2\omega_\uparrow$, which has the sole effect of shifting the position and width of the distribution $P_{\uparrow\uparrow}$ by a quantity of order $1/N$. As shown by the original equation (4.28), the two terms of (4.30) have the same order of magnitude (in spite of the presence of the factor $1/N$ in the second one) when $P_{\uparrow\uparrow}$ has an exponential form in N. Only the first one contributes if $P_{\uparrow\uparrow}$ becomes smooth (§ 7.3.2). The equation for $P_{\downarrow\downarrow}$ is obtained from (4.30) by changing g into $-g$.

In the regime where the registration will take place (§ 7.1.2), we shall be allowed to replace $\tilde{K}_t(\pm 2\omega_i)$ by $\tilde{K}(\pm 2\omega_i)$, which according to (3.37) is equal to

$$\tilde{K}(\pm 2\omega_i) = \frac{\hbar^2 \omega_i}{4} \left[\coth(\beta\hbar\omega_i) \mp 1 \right] \exp\left(-\frac{2|\omega_i|}{\Gamma} \right), \qquad (i = \uparrow, \downarrow). \tag{4.33}$$

[25] Numerically we find for $N = 1000$ extended distributions, see Figs. 7.5 and 7.6, since the typical peak width $1/\sqrt{N}$ is still sizable

[26] This property does not hold for $P_{\uparrow\downarrow}$, since $X_{\uparrow\downarrow}$ contains a term $2igt/\hbar$ arising from the left hand side of (4.29)

Eqs. (4.31) and (4.32) will thereby be simplified.

The final equations (4.29) and (4.30), with the initial conditions $P_{ij}(m,0) = r_{ij}P_{\rm M}(m,0)$ expressed by (3.48), describe the evolution of S + M during the measurement process. We will work them out in sections 5 to 7. The various quantities entering them were defined by (4.25) and (4.26) for X_{ij} and X'_{ij}, by (4.31) and (4.32) for v and w, by (3.37), (4.10), (4.11), (4.17) and (4.19) for $\tilde{K}_{t>}$, $\tilde{K}_{t<}$, $\tilde{K}_t$ and $\tilde{K}_\pm$, respectively, and by (4.24) for ω_i.

The dynamics of $P_{\uparrow\downarrow}$ has a purely quantum nature. The left-hand side of (4.29) governs the evolution of the normalization $\int {\rm d}m P_{\uparrow\downarrow}(m,t)$, equal to the off-diagonal element $r_{\uparrow\downarrow}(t)$ of the marginal state $\hat{r}(t)$ of S. The bath gives rise on the right-hand side to a non-linear partial differential structure, which arises from the discrete nature of the spectrum of $\hat{m}$.

The final equation of motion (4.30) for $P_{\uparrow\uparrow}$ has the form of a Fokker–Planck equation [201, 202], which describes a stochastic motion of the variable m. Its coefficient v, which depends on m and t, can be interpreted as a drift velocity, while its coefficient w characterizes a diffusion process. This analogy with a classical diffusion process, should not, however, hide the quantum origin of the diffusion term, which is as sizeable for large N as the drift term. While the drift term comes out by bluntly taking the continuous limit of (4.16), the diffusion term originates, as shown by the above derivation, from the conjugate effect of two features: (i) the smallness of the fluctuations of m, and (ii) the discreteness of the spectrum of the pointer observable $\hat{m}$. Although the pointer is macroscopic, its quantum nature is essential, not only in the off-diagonal sector, but also in the diagonal sector which accounts for the registration of the result.

5. Very short times: von Neumann's reduction

Alea iacta est[27]
Julius Caesar

Since the coupling γ of the magnet M with the bath B is weak, some time is required before B acts significantly on M. In the present section, we therefore study the behavior of S + M at times sufficiently short so that we can neglect the right-hand sides of (4.16) and (4.18). We shall justify this approximation in section 6.

5.1. Mechanism of the reduction

5.1.1. The reduction time

When their right-hand sides are dropped, the equations (4.16) and (4.18) with the appropriate boundary conditions are readily solved as

$$P_{\uparrow\uparrow}(m,t) = r_{\uparrow\uparrow}(0)\,P_{\rm M}(m,0)\,, \qquad P_{\downarrow\downarrow}(m,t) = r_{\downarrow\downarrow}(0)\,P_{\rm M}(m,0)\,, \tag{5.1}$$

$$P_{\uparrow\downarrow}(m,t) = [P_{\downarrow\uparrow}(m,t)]^* = r_{\uparrow\downarrow}(0)\,P_{\rm M}(m,0)\,e^{2iNgmt/\hbar}. \tag{5.2}$$

From the viewpoint of the tested spin S, these equations describe a Larmor precession around the z-axis [54], under the action of an effective magnetic field Ngm which depends on the state of M. From the viewpoint of the magnet M, we shall see in § 5.1.3 that the phase occurring in (5.2) generates time-dependent correlations between M and the transverse components of $\mathbf{s}$.

The expectation values $\langle \hat{s}_a(t)\rangle$ of the components of $\mathbf{s}$ are found from (3.29) by summing (5.1) and (5.2) over m. These equations are valid for arbitrary N and arbitrary time t as long as the bath is inactive. If N is sufficiently large and t sufficiently small so that the summand is a smooth function on the scale $\delta m = 2/N$, that is, if $N \gg 1$ and $t \ll \hbar/g$, we can use (4.20) to replace the summation over m by an integration. These conditions will be fulfilled in subsections 5.1 and 5.2; we shall relax the second one in subsection 5.3 where we study the effects of the discreteness of m. Using the expression (3.48), (3.49) of $P_{\rm M}(m,0)$, we find by integrating (5.2) over m:

$$r_{\uparrow\downarrow}(t) = r_{\uparrow\downarrow}(0)\,e^{-(t/\tau_{\rm red})^2}, \tag{5.3}$$

[27] The die is cast

or equivalently

$$\langle \hat{s}_a(t) \rangle = \langle \hat{s}_a(0) \rangle e^{-(t/\tau_{\rm red})^2}, \qquad (a = x,y), \tag{5.4}$$

$$\langle \hat{s}_z(t) \rangle = \langle \hat{s}_z(0) \rangle, \tag{5.5}$$

where we introduced the reduction time

$$\tau_{\rm red} \equiv \frac{\hbar}{\sqrt{2}\, Ng\Delta m} = \frac{\hbar}{\sqrt{2N}\, \delta_0 g}. \tag{5.6}$$

Although $P_{\uparrow\downarrow}(m,t)$ is merely an oscillating function of t for each value of m, the summation over m has given rise to a damping. This property arises from the dephasing that exists between the oscillations for different values of m.

In the case $T_0 = \infty$ of a fully disordered initial state, we may solve directly (4.8) (without right-hand side) from the initial condition (4.9). We obtain, for arbitrary N, $\hat{R}_{\uparrow\downarrow}(t) = r_{\uparrow\downarrow}(0) 2^{-N} \exp(2iNg\hat{m}t/\hbar)$, whence by using the definition (3.2) of $\hat{m}$ and taking the trace over M, we find the exact result[28]

$$r_{\uparrow\downarrow}(t) = r_{\uparrow\downarrow}(0) \left(\cos \frac{2gt}{\hbar} \right)^N, \tag{5.7}$$

which reduces to (5.3) for times of order $\tau_{\rm red}$.

Thus, over a time scale of order $\tau_{\rm red}$, the transverse components of the spin S decay and vanish while the z-component is unaltered: the off-diagonal elements $r_{\uparrow\downarrow} = r^*_{\downarrow\uparrow}$ of the marginal density matrix of S disappear during the very first stage of the measurement process. It was to be expected that the apparatus, which is a large object, has a rapid and strong effect on the much smaller system S. In the present model, this rapidity arises from the large number N of spins of the magnet, which shows up through the factor $1/\sqrt{N}$ in the expression (5.6) of $\tau_{\rm red}$.

As we shall see in § 5.1.3, the off-diagonal block $\hat{\mathcal{R}}_{\uparrow\downarrow} = \hat{\mathcal{R}}^\dagger_{\downarrow\uparrow}$ of the full density matrix $\hat{\mathcal{D}}$ of S + A is proportional to $\hat{r}_{\uparrow\downarrow}(t)$ and its elements also decrease as $\exp[-(t/\tau_{\rm red})^2]$, at least those elements which determine correlations involving a number of spins of M small compared to N. Von Neumann's reduction therefore takes place for the overall system S + A over the brief initial time lapse $\tau_{\rm red}$.

The quantum nature of the reduction process manifests itself through the occurrence of two different Hamiltonians $\hat{H}_\uparrow$ and $\hat{H}_\downarrow$ in the Hilbert space of M. Both of them occur in the dynamical equation (4.18) for $P_{\uparrow\downarrow}$, whereas only $\hat{H}_\uparrow$ occurs in (4.16) for $P_{\uparrow\uparrow}$ through $\Omega^\pm_\uparrow$, and likewise only $\hat{H}_\downarrow$ for $P_{\downarrow\downarrow}$, through $\Omega^\pm_\downarrow$.

The reduction time $\tau_{\rm red}$ is inversely proportional to the coupling g between $\hat{s}_z$ and each spin $\hat{\sigma}^{(n)}_z$ of the magnet. It does not depend directly on the couplings J_q ($q = 2, 4$) between the spins $\hat{\sigma}^{(n)}_z$. Indeed, the dynamical equations (4.16), (4.18) without bath-magnet coupling involve only $H_\uparrow(m) - H_\downarrow(m)$, so that the interactions $\hat{H}_{\rm M}$ which are responsible for ferromagnetism cancel out therein. These interactions occur only through the right-hand side which describes the effect of the bath. They also appear indirectly in $\tau_{\rm red}$ through the factor δ_0 of Δm given by (3.51), in the case $q = 2$ of an Ising magnet M. When $J_2 \neq 0$, the occurrence of $\delta_0 > 1$ thus contributes to accelerate the reduction process.

5.1.2. Reduction versus decoherence: a general phenomenon

It is often said [28, 29, 35, 147, 148, 149, 150] that von Neumann's reduction "is a decoherence effect". As is well known, decoherence is the rapid destruction of coherent superpositions of distinct pure states induced by a random environment, such as a thermal bath. In the latter seminal case, the characteristic decoherence time has the form of $\hbar/T$ divided by some power of the number of degrees of freedom and by a coupling constant between the system and the bath (see also our discussion of the decoherence approach in section 2). Here, things are different. As we have just seen and as will be studied below in detail, the reduction process involves only the magnet. Although the bath is part of the apparatus, it has no effect here and the characteristic reduction time $\tau_{\rm red}$ does not depend on the bath temperature. Indeed the dimensional factor of (5.6) is $\hbar/g$, and not $\hbar/T$. The thermal fluctuations are replaced by the

[28] An equivalent way to derive this result is to employ (3.28) for making the identification $P^{\rm dis}_{\uparrow\downarrow}(m,t) = G(m) \times r_{\uparrow\downarrow}(0) 2^{-N} \exp(2iNgmt/\hbar)$, and to sum over the values (3.22) of m

fluctuation Δm of the pointer variable, which does not depend on T for $q = 4$ and which decreases with T_0 as (3.51) for $q = 2$.

The fact that the reduction is controlled only by the coupling of the pointer variable with S is exhibited by the occurrence, in (5.6), of its number N of degrees of freedom. Registration of s_z requires this variable to be *collective*, so that $N \gg 1$. However, long before registration begins to take place in A through the influence on $\hat{\mathcal{D}}$ of $r_{\uparrow\uparrow}(0)$ and $r_{\downarrow\downarrow}(0)$, the large size of the detector entails the loss of $r_{\uparrow\downarrow}(0)$ and $r_{\downarrow\uparrow}(0)$.

Moreover, the *basis* in which the reduction takes place is *selected* by the very design of the apparatus. It depends on the observable which is being measured. Had we proceeded to measure $\hat{s}_x$ instead of $\hat{s}_z$, we would have changed the orientation of the magnetic dot; the part of the initial state $\hat{r}(0)$ of S that gets lost would have been different. Contrary to standard decoherence, reduction is here a controlled effect.

Altogether, it is only the pointer degrees of freedom *directly coupled to* S that are responsible for the rapid reduction. As such, it is a *dephasing*. The effects of the bath are important (sections 6.2 and 7), but do not infer on the initial reduction process, on the time scale $\tau_{\rm red}$. We consider it therefore confusing to use the term "decoherence" for the decay of the off-diagonal blocks in a quantum measurement, since this mechanism of reduction or collapse can be fundamentally different from a standard environment-induced decoherence. Here the reduction is a consequence of dephasing between oscillating terms which should be summed to generate the physical quantities[29].

The above considerations hold for the *class of models* of quantum measurements for which *the pointer has many degrees of freedom* directly coupled to S [60, 130, 131] (see also [150] in this context). We have already found for the reduction time a behavior analogous to (5.6) in a model where the detector is a Bose gas [129], with a scaling in $N^{-1/4}$ instead of $N^{-1/2}$. More generally, suppose we wish to measure an arbitrary observable $\hat{s}$ of a microscopic system S, with discrete eigenvalues s_i and corresponding projections $\hat{\Pi}_i$. The result should be registered by some pointer variable $\hat{m}$ of an apparatus A coupled to $\hat{s}$. The full Hamiltonian has still the form (3.3), and it is natural to assume that the system–apparatus coupling has the same form

$$\hat{H}_{\rm SA} = -Ng\hat{s}\hat{m}, \qquad \text{(general operators } \hat{s}, \hat{m}) \tag{5.8}$$

as (3.5). The coupling constant g refers to each one of the N elements of the collective pointer, so that a factor N appears in (5.8) as in (3.5), if $\hat{m}$ is dimensionless and normalized in such a way that the range of its relevant eigenvalues is finite when N becomes large. The reduced density matrix $\hat{r}(t)$ is made of blocks $\langle i\alpha|\,\hat{r}\,|j\beta\rangle$ where α takes as many values as the dimension of $\hat{\Pi}_i$. It can be obtained as

$$\langle i\alpha|\,\hat{r}(t)\,|j\beta\rangle = \sum_m \langle i\alpha|\,\mathcal{P}(m,t)\,|j\beta\rangle\,, \tag{5.9}$$

where $\langle i\alpha|\,\mathcal{P}(m,t)\,|j\beta\rangle$, which generalizes $P_{\uparrow\downarrow}(m,t)$, is defined by

$$\langle i\alpha|\,\mathcal{P}(m,t)\,|j\beta\rangle = \langle i\alpha|\,{\rm tr}_{\rm A}\left(\delta_{\hat{m},m}\hat{\mathcal{D}}\right)|j\beta\rangle\,. \tag{5.10}$$

We have denoted by m the eigenvalues of $\hat{m}$, and by $\delta_{\hat{m},m}$ the projection operator on m in the Hilbert space of A. The quantity (5.10) satisfies an equation of motion dominated by (5.8):

$$\left[i\hbar\frac{\rm d}{{\rm d}t} + Ng(s_i - s_j)m\right]\langle i\alpha|\,\mathcal{P}(m,t)\,|j\beta\rangle \simeq 0. \tag{5.11}$$

In fact, the terms arising from $\hat{H}_{\rm S}$ (which no longer vanishes but commutes with $\hat{s}$) and from $\hat{H}_{\rm A}$ (which commutes with the initial density operator $\hat{\mathcal{R}}(0)$) are small during the initial instants compared to the term arising from the coupling $\hat{H}_{\rm SA}$. We therefore find for short times

$$\langle i\alpha|\,\hat{r}(t)\,|j\beta\rangle = \langle i\alpha|\,\hat{r}(0)\,|j\beta\rangle\,{\rm tr}_{\rm A}\hat{\mathcal{R}}(0)\,e^{iNg(s_i-s_j)\hat{m}t/\hbar}. \tag{5.12}$$

The rapidly oscillating terms in the right-hand side interfere destructively as in (5.3) on a short time, if $\hat{m}$ has a dense spectrum and an initial distribution involving many eigenvalues. Each contribution is merely oscillating, but

[29] In section 6.2 we shall discuss the effects of *decoherence* by the bath, which does take place, but long after the reduction time scale

the summation over eigenvalues produces a relaxation. This decrease takes place on a time scale of order $\hbar/Ng\delta s\Delta m$, where δs is the level spacing of the measured observable $\hat{s}$ and Δm is the width of the distribution of eigenvalues of $\hat{m}$ in the initial state of the apparatus. Leaving aside the later stages of the measurement process, we thus acknowledge the generality of the present reduction mechanism, and that of the expression (5.6) for the reduction time in the spin $\frac{1}{2}$ situation where $\delta s = 2$.

5.1.3. Establishment and disappearance of correlations

Let us now examine how the apparatus evolves during this first stage of the measurement process, described by Eqs. (5.1) and (5.2). The first equation implies that the marginal density operator $\hat{R}_{\rm M}(t) = \hat{R}_{\uparrow\uparrow}(t) + \hat{R}_{\downarrow\downarrow}(t)$ of M remains unchanged. This property agrees with the idea that M, a large object, has a strong influence on S, a small object, but that conversely a long time is required before M is affected by its interaction with S. Eqs. (5.1) also imply that no correlation is created between $\hat{s}_z$ and M.

However, although $\hat{R}_{\rm M}(t) = \hat{R}(0)$, correlations are created between M and the transverse component $\hat{s}_x$ (or $\hat{s}_y$) of S. These correlations are described by the quantities $C_x = P_{\uparrow\downarrow} + P_{\downarrow\uparrow}$ and $C_y = i(P_{\uparrow\downarrow} - P_{\downarrow\uparrow})$ introduced in (3.29). Since $\hat{R}_{\uparrow\downarrow}$ is a function of $\hat{m}$ only, the components $\hat{\sigma}_x^{(n)}$ and $\hat{\sigma}_y^{(n)}$ of the spins of M remain statistically independent, with $\langle\hat{\sigma}_x^{(n)}\rangle = \langle\hat{\sigma}_y^{(n)}\rangle = 0$ and with the quantum fluctuations $\langle\hat{\sigma}_x^{(n)2}\rangle = \langle\hat{\sigma}_y^{(n)2}\rangle = 1$. The correlations between M and S involve only the z-component of the spins $\hat{\sigma}^{(n)}$ of the magnet and the x- or y-component of the tested spin **s**. We can derive them as functions of time from the generating function

$$\Psi_{\uparrow\downarrow}(\lambda,t) \equiv \sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\langle\hat{s}_-\hat{m}^k(t)\rangle = \sum_m P^{\rm dis}_{\uparrow\downarrow}(m,t)e^{\lambda m} = r_{\uparrow\downarrow}(0)\sum_m P^{\rm dis}_{\rm M}(m,0)e^{2iNgmt/\hbar+\lambda m}, \tag{5.13}$$

where $\hat{s}_- = \frac{1}{2}(\hat{s}_x - i\hat{s}_y)$. In fact, whereas $\Psi_{\uparrow\downarrow}(\lambda,t)$ generates the expectation values $\langle\hat{s}_-\hat{m}^k\rangle$, the correlations $\langle\hat{s}_-\hat{m}_k\rangle_{\rm c}$ are defined by the cumulant expansion

$$\Psi_{\uparrow\downarrow}(\lambda,t) = \sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\langle\hat{s}_-\hat{m}^k\rangle_{\rm c}\left(\sum_{k'=0}^{\infty}\frac{\lambda^{k'}}{k'!}\langle\hat{m}^{k'}\rangle\right) = \sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\langle\hat{s}_-\hat{m}^k\rangle_{\rm c}\exp\left(\sum_{k'=1}^{\infty}\frac{\lambda^{k'}}{k'!}\langle\hat{m}^{k'}\rangle_{\rm c}\right), \tag{5.14}$$

which factors out the correlations $\langle\hat{m}^{k'}\rangle_{\rm c}$ within M. The latter correlations are the same as at the initial time, so that we shall derive the correlations between S and M from

$$\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\langle\hat{s}_-\hat{m}^k(t)\rangle_{\rm c} = r_{\uparrow\downarrow}(0)\frac{\Psi_{\uparrow\downarrow}(\lambda,t)}{\Psi_{\uparrow\downarrow}(\lambda,0)}. \tag{5.15}$$

For correlations involving not too many spins (we will discuss this point in § 5.3.2), we can again replace the summation over m in (5.13) by an integral. Since $P_{\rm M}(m,0)$ is a Gaussian, the sole non-trivial cumulant $\langle\hat{m}^k\rangle_{\rm c}$ is $\langle\hat{m}^2\rangle = \Delta m^2$, given by (3.48), (3.49), and we get from (5.13) and (5.15)

$$\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\langle\hat{s}_-\hat{m}^k(t)\rangle_{\rm c} = r_{\uparrow\downarrow}(0)\exp\left(-\frac{t^2}{\tau_{\rm red}^2} + i\sqrt{2}\frac{t}{\tau_{\rm red}}\lambda\Delta m\right) = r_{\uparrow\downarrow}(t)\exp\left(i\sqrt{2}\frac{t}{\tau_{\rm red}}\lambda\Delta m\right). \tag{5.16}$$

At first order in λ, the correlations between S and any single spin of M are thus expressed by

$$\begin{aligned}
\langle\hat{s}_x\hat{\sigma}_z^{(n)}(t)\rangle &= \langle\hat{s}_x\hat{m}(t)\rangle_{\rm c} = \sum_m C_x^{\rm dis}(m,t)\,m = \sqrt{2}\frac{t}{\tau_{\rm red}}\langle\hat{s}_y(t)\rangle\Delta m = \sqrt{2}\frac{t}{\tau_{\rm red}}\langle\hat{s}_y(0)\rangle e^{-(t/\tau_{\rm red})^2}\Delta m,\\
\langle\hat{s}_y\hat{\sigma}_z^{(n)}(t)\rangle &= \langle\hat{s}_y\hat{m}(t)\rangle_{\rm c} = \sum_m C_y^{\rm dis}(m,t)\,m = -\sqrt{2}\frac{t}{\tau_{\rm red}}\langle\hat{s}_x(t)\rangle\Delta m,
\end{aligned} \tag{5.17}$$

where we used (5.4). These correlations first increase, reach a maximum for $t = \tau_{\rm red}/\sqrt{2}$, then decrease along with $\langle\hat{s}_x(t)\rangle$ and $\langle\hat{s}_y(t)\rangle$ (Fig. 5.1). At this maximum, their values satisfy

$$\frac{\langle\hat{s}_x\hat{m}(t)\rangle}{\Delta m} = \langle\hat{s}_y(t)\rangle = \frac{\langle\hat{s}_y(0)\rangle}{\sqrt{e}}, \qquad \frac{\langle\hat{s}_y\hat{m}(t)\rangle}{\Delta m} = -\frac{\langle\hat{s}_x(0)\rangle}{\sqrt{e}}. \tag{5.18}$$

They do not lie far below the bound yielded by Heisenberg's inequality

$$|\langle \hat{s}_x \hat{m} \rangle|^2 = \left| \frac{1}{2i} \langle [\hat{s}_y - \langle \hat{s}_y \rangle, \hat{s}_z \hat{m}] \rangle \right|^2 \leq \left(1 - \langle \hat{s}_y \rangle^2\right) \Delta m^2, \tag{5.19}$$

which implies at all times

$$\left(\frac{2t^2}{\tau_{\rm red}^2} + 1 \right) \langle \hat{s}_y(t) \rangle^2 \leq 1, \tag{5.20}$$

since the left-hand side of (5.20) is $2/e$ at the maximum of (5.17).

The next order correlations are obtained from (5.16) as ($a = x, y$)

$$\langle \hat{s}_a \hat{m}^2 (t) \rangle_{\rm c} \equiv \langle \hat{s}_a \hat{m}^2 (t) \rangle - \langle \hat{s}_a (t) \rangle \langle \hat{m}^2 \rangle = -\frac{2t^2}{\tau_{\rm red}^2} \langle \hat{s}_a (t) \rangle \Delta m^2. \tag{5.21}$$

These correlations again increase, but more slowly than (5.17), reach (in absolute value) a maximum later, at $t = \tau_{\rm red}$, equal to $(-2/e)\langle \hat{s}_a (0) \rangle \Delta m^2$, then decrease together with $\langle \hat{s}_a (t) \rangle$. Accordingly, the correlations between $\hat{s}_x$ and two spins of M, evaluated as in (3.27), are given by

$$\langle \hat{s}_x \hat{\sigma}_a^{(n)} \hat{\sigma}_b^{(p)} (t) \rangle_{\rm c} = \langle \hat{s}_x (t) \rangle \frac{\delta_{a,z} \delta_{b,z}}{N-1} \left(-\frac{2t^2}{\tau_{\rm red}^2} N \Delta m^2 - 1 \right), \tag{5.22}$$

which for large N behaves as (5.21).

Likewise, (5.16) together with (5.3) provides the hierarchy of correlations through the real and imaginary parts of

$$\langle (\hat{s}_x - i\hat{s}_y) \hat{m}^k (t) \rangle_{\rm c} = \langle (\hat{s}_x - i\hat{s}_y) (0) \rangle \left(i\sqrt{2} \frac{t}{\tau_{\rm red}} \Delta m \right)^k e^{-(t/\tau_{\rm red})^2}, \tag{5.23}$$

with Δm from (3.51). This expression also holds for more detailed correlations such as $\langle \hat{s}_a \hat{\sigma}_z^{(1)} \hat{\sigma}_z^{(2)} \cdots \hat{\sigma}_z^{(k)} (t) \rangle_{\rm c}$ within corrections of order $1/N$ as in eq. (5.22), provided k/N is small.

Altogether (Fig. 5.1) the correlations (5.23) scale as $\Delta m^k = \left(\delta_0 / \sqrt{N} \right)^k$. If the rank k is odd, $\langle \hat{s}_x \hat{m}^k (t) \rangle_{\rm c}$ is proportional to $\langle \hat{s}_y (0) \rangle$, if k is even, it is proportional to $\langle \hat{s}_x (0) \rangle$, with alternating signs. The correlations of rank k depend on time as $(t/\tau_{\rm red})^k \exp[-(t/\tau_{\rm red})^2]$. Hence, correlations of higher and higher rank begin to grow later and later, in agreement with the factor t^k, and they reach a maximum later and later, at the time $t = \tau_{\rm red} \sqrt{k/2}$. For even k, the maximum of $\left| \langle \hat{s}_x \hat{m}^k (t) \rangle_{\rm c} \right|$ is given by

$$\max \left| \frac{\langle \hat{s}_x \hat{m}^k (t) \rangle_{\rm c}}{\langle \hat{s}_x (0) \rangle \Delta m^k} \right| = \frac{1}{k!} \left(\frac{2k}{e} \right)^{k/2} \left(\frac{k}{2} \right)! \simeq \frac{1}{\sqrt{2}}, \tag{5.24}$$

which is nearly independent of k.

5.1.4. The reduction, a cascade process

Het viel in gruzelementen[30]
Dutch saying

The mechanism of reduction in the present model is therefore comparable to a current mechanism of irreversibility in statistical mechanics (§ 1.2.2). In a classical Boltzmann gas, initially off-equilibrium with a non-uniform density, the relaxation toward uniform density takes place through the establishment of correlations between a larger and larger number of particles, under the effect of successive collisions [48, 49, 50]. Here, similar features occur although quantum dynamics is essential. The relaxation (5.4) of the off-diagonal elements $r_{\uparrow\downarrow} = r_{\downarrow\uparrow}^*$ of the marginal state $\hat{r}$ of S is accompanied by the generation, owing to the coupling $\hat{H}_{\rm SA}$, of correlations between S and M.

[30] It fell and broke into tiny pieces

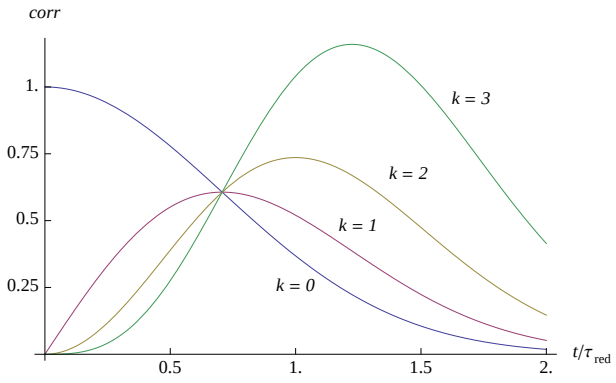

Figure 5.1: The relative correlations $corr = \langle(\hat{s}_x - i\hat{s}_y)\hat{m}^k(t)\rangle_c / \langle(\hat{s}_x - i\hat{s}_y)(0)\rangle(i\sqrt{2}\Delta m)^k$ from Eq. (5.23), as function of $t/\tau_{\rm red}$. For $k = 0$ $\langle\hat{s}_x(t)\rangle$ decreases as a Gaussian. The curves for $k = 1, 2$ and 3 show that the correlations develop, reach a maximum, then disappear later and later.

Such correlations, absent at the initial time, are built up in a *cascade*, as shown by eq. (5.23) and Fig. 5.1. Let us characterize the state $\hat{R}$ of S + M by the expectation values and correlations of the operators $\hat{s}_a$ and $\hat{\sigma}_a^{(n)}$. The order of S, initially embedded in the expectation values of the transverse components $r_{\uparrow\downarrow}(0)$ of the spin $\hat{\mathbf{s}}$, is progressively transferred to correlations (5.17) between these components and one spin of M, then in turn to correlations (5.22) with two spins, with three spins, and so on. The larger the rank k of the correlations, the smaller they are, as $\Delta m^k \sim 1/N^{k/2}$ (Eq. (5.23)); but the larger their number is, as $N!/k!\,(N-k)!$. Their time-dependence, in $t^k \exp[-(t/\tau_{\rm red})^2]$, shows how they blow up and blow out successively.

As a specific feature of our model of quantum measurement, the interaction process does not affect the marginal statistical state of M. All the multiple correlations produced by the coupling $\hat{H}_{\rm SA}$ lie astride S and M.

Von Neumann's reduction [2], defined as the disappearance of the off-diagonal blocks $\hat{\mathcal{R}}_{\uparrow\downarrow} = \hat{\mathcal{R}}_{\downarrow\uparrow}$ of the full density matrix $\hat{\mathcal{D}}$ of S + A, or equivalently of the expectation values of all operators involving $\hat{s}_x$ or $\hat{s}_y$, results from the proportionality of $\hat{\mathcal{R}}_{\uparrow\downarrow}(t)$ to $r_{\uparrow\downarrow}(t)$, within a polynomial coefficient in t associated with the factor t^k in the k-th rank correlations. Initially, only few among the $2^N \times 2^N$ elements of the matrix $\hat{R}_{\uparrow\downarrow}(0)$ do not vanish, those which correspond to $2r_{\uparrow\downarrow}(0) = \langle\hat{s}_x(0)\rangle - i\langle\hat{s}_y(0)\rangle$ and to $P_{\rm M}(m, 0)$ given by (3.48). The very many elements of $\hat{R}_{\uparrow\downarrow}(0)$ which describe correlations between $\hat{s}_x$ or $\hat{s}_y$ and the spins of M, absent at the initial time, grow, while an overall factor $\exp[-(t/\tau_{\rm red})^2]$ damps $\hat{R}_{\uparrow\downarrow}(t)$. At times $\tau_{\rm red} \ll t \ll \hbar/g$, all elements of $\hat{R}_{\uparrow\downarrow}(t)$ and hence of $\hat{\mathcal{R}}_{\uparrow\downarrow}(t)$ have become negligibly small[31]. In principle, no information is lost since the equations of motion are reversible, but the initial datum $r_{\uparrow\downarrow}(0)$ gets spread among very many matrix elements of $\hat{\mathcal{R}}_{\uparrow\downarrow}$ which nearly vanish, exactly as in the irreversibility paradox (§ 1.2.2).

If N could be made infinite, the progressive creation of correlations would provide a rigorous mathematical characterization of the irreversibility of the reduction process, as for relaxation processes in statistical mechanics. Consider, for some fixed value of K, the set of correlations (5.23) of ranks k such that $0 \leq k \leq K$, including $\langle\hat{s}_x\rangle$ and $\langle\hat{s}_y\rangle$ for $k = 0$. All correlations of this set vanish in the limit $N \to \infty$ for fixed t, since $\tau_{\rm red}$ then tends to 0. (The coupling constant g may depend on N, in which case it should satisfy $Ng^2 \to \infty$.) This property holds even for infinite K, *provided* $K \to \infty$ *after* $N \to \infty$, a limit which characterizes the irreversibility. However, such a limit is not uniform: the reversibility of the underlying dynamics manifests itself through the finiteness of high-order correlations for sufficiently large t (§ 5.3.2).

Anyhow N is not allowed in physics to go to infinity, since the time $\tau_{\rm red}$ would unrealistically vanish. For large but finite N, there is no rigorous qualitative characterization of irreversibility, neither in this model of measurement nor in statistical mechanics, but the above discussion remains relevant. In fact, physically, it is legitimate to regard as equal to zero a quantity which is less than some small bound, and to regard as unobservable and irrelevant all correlations which involve a number k of spins exceeding some bound K much smaller than N. We shall return to this issue in § 11.2.3.

[31] The latter implication follows because the bath contributions cannot raise the S + A correlations

5.2. Randomness of the initial state of M

Initial states $\hat{\mathcal{R}}(0)$ that can actually be prepared at least in a thought experiment, such as the paramagnetic canonical equilibrium distribution of § 3.3.3, involve large randomness. In particular, if the initialization temperature T_0 is sufficiently large, the state (3.46), i. e., $\hat{R}_{\rm M}(0) = 2^{-N}\prod_n \hat{\sigma}_0^{(n)}$, is the most disordered statistical state of M; in such a case, $P_{\rm M}(m,0)$ is given by (3.50). We explore in this subsection how the reduction process is modified for other, less random, initial states of M.

5.2.1. Arbitrary initial states

The derivations of the equations of motion in subsections 4.1 and 4.2 were general, irrespective of the initial state. However, in subsections 4.3 and 5.1 we have relied on the fact that $\hat{R}_{\rm M}(0)$ depends only on $\hat{m}$. In order to deal with an arbitrary initial state $\hat{R}_{\rm M}(0)$, we return to eq. (4.8), where we can as above neglect for very short times the coupling with the bath. The operators $\hat{R}_{ij}(t)$ and $\hat{H}_i$ in the Hilbert space of M no longer commute because $\hat{R}_{ij}$ now involves spin operators other than $\hat{m}$. However, the probabilities and correlations $P_{ij}(m,t)$ defined by (3.24) still satisfy Eqs. (B.13) of Appendix B without right-hand side. Hence the expressions (5.1) and (5.2) for $P_{ij}(m)$ at short times hold *for any initial state* $\hat{R}_{\rm M}(0)$, with $P_{\rm M}(m,0)$ given by $\mathrm{tr}_{\rm M}\hat{R}_{\rm M}(0)\,\delta_{\hat{m},m}$.

The various expressions (5.4), (5.5), (5.6), (5.17), (5.21), (5.23) relied only on the Gaussian shape of the probability distribution $P_{\rm M}(0,m)$ associated with the initial state. They will therefore remain valid for any initial state $\hat{R}_{\rm M}(0)$ that provides a narrow distribution $P_{\rm M}(m,0)$, centered at $m=0$ and having a width Δm small ($\Delta m \ll 1$) though large compared to the level spacing, viz. $\Delta m \gg 2/N$. Indeed, within corrections of relative order $1/N$, such distributions are equivalent to a Gaussian. The second condition ($\Delta m \gg 2/N$) ensures that $\tau_{\rm red}$ is much shorter than $\hbar/g$, another characteristic time that we shall introduce in § 5.3.1.

In fact, the behavior in $1/\sqrt{N}$ for Δm is generic, so that the reduction time has in general the same expression (5.6) as for a paramagnetic canonical equilibrium state, with δ_0 defined by $\delta_0^2 = N\mathrm{tr}_{\rm M}\hat{R}_{\rm M}(0)\,\hat{m}^2$. The dynamics of the reduction process described above holds for most possible initial states of the apparatus: decay of $\langle\hat{s}_x(t)\rangle$ and $\langle\hat{s}_y(t)\rangle$; generation of a cascade of correlations $\langle\hat{s}_a\hat{m}^k(t)\rangle$ of order Δm^k between the transverse components of the spin S and the pointer variable $\hat{m}$; increase, then decay of the very many matrix elements of $\hat{R}_{\uparrow\downarrow}(t)$, which are small as $\left(\sqrt{2}\,\Delta m\, t/\tau_{\rm red}\right)^k \exp[-(t/\tau_{\rm red})^2]$ for $t \ll \hbar/g$.

In case the initial density operator $\hat{R}_{\rm M}(0)$ is a symmetric function of the N spins, the correlations between $\hat{s}_x$ or $\hat{s}_y$ and the z-components of the individual spins of M are still given by expressions such as (5.22). However, in general, $\hat{R}_{\rm M}(0)$ no longer depends on the operator $\hat{m}$ only; it involves transverse components $\hat{\sigma}_x^{(n)}$ or $\hat{\sigma}_y^{(n)}$, and so does $\hat{R}_{\uparrow\downarrow}(t)$, which now includes correlations of $\hat{s}_x$ or $\hat{s}_y$ with x- or y-components of the spins $\hat{\sigma}^{(n)}$. The knowledge of $P_{\uparrow\downarrow}(m,t)$ is in this case not sufficient to fully determine $\hat{R}_{\uparrow\downarrow}(t)$, since (3.24) holds but not (3.26).

The proportionality of the reduction time $\tau_{\rm red} = \hbar/\sqrt{2}\,Ng\Delta m$ to the inverse of the fluctuation Δm shows that the reduction is a disorder effect, since Δm measures the randomness of the pointer variable in the initial state. This is easy to understand: S sees an effective magnetic field Ngm which is random through m, and it is this very randomness which causes the relaxation. The existence of such a randomness in the initial state, even though it is small as $1/\sqrt{N}$, is necessary to ensure the transfer of the initial order embodied in $r_{\uparrow\downarrow}(0)$ into the cascade of correlations between S and M and to entail a brief reduction time $\tau_{\rm red}$. Boltzmann's elucidation of the irreversibility paradox also relied on statistical considerations about the initial state of a classical gas which will relax to equilibrium.

5.2.2. Pure versus mixed initial state

It is therefore natural to wonder whether the reduction of the state would still take place for pure initial states of M, which are the least random ones in quantum physics, in contrast to the paramagnetic state (3.46) or (3.50) which is the most random one. To answer this question, we first consider the pure state with density operator

$$\hat{R}_{\rm M}(0) = \prod_{n=1}^{N} \frac{1}{2}\left(1+\hat{\sigma}_x^{(n)}\right), \tag{5.25}$$

in which all spins $\hat{\sigma}^{(n)}$ point in the x-direction. This initialization may be achieved by submitting M to a strong field in the x-direction and letting it thermalize with a cold bath B for a long duration before the beginning of the measurement. The fluctuation of $\hat{m}$ in the state (5.25) is $1/\sqrt{N}$. Hence, for this pure initial state of M, the reduction takes place

exactly as for the fully disordered initial paramagnetic state, since both yield the same probability distribution (3.50) for m.

A similar conclusion holds for the most general factorized pure state, with density operator

$$\hat{R}_{\rm M}(0) = \prod_{n=1}^{N} \frac{1}{2}\left(1 + \mathbf{u}^{(n)} \cdot \hat{\boldsymbol{\sigma}}^{(n)}\right), \tag{5.26}$$

where the $\mathbf{u}^{(n)}$ are arbitrary unit vectors[32]. The fluctuation Δm, then given by

$$\delta_0^2 = N\Delta m^2 = \frac{1}{N}\sum_{n=1}^{N}\left[1 - \left(u_z^{(n)}\right)^2\right], \tag{5.27}$$

is in general sufficiently large to ensure again the properties of subsection 5.1, which depend on $\hat{R}_{\rm M}(0)$ only through Δm.

Incoherent or coherent superpositions of such pure states will yield the same effects. We will return to this point in subsection 11.1.4, noting conversely that an irreversibility which occurs for a mixed state is also statistically present in most of the pure states that underlie it.

Quantum mechanics brings in another feature: a given mixed state can be regarded as a superposition of pure states in many different ways. For instance, the completely disordered paramagnetic state (3.46), $\hat{R}_{\rm M}(0) = 2^{-N}\prod_n \hat{\sigma}_0^{(n)}$, can be described by saying that each spin points at random in the $+z$ or in the $-z$-direction; it can also be described as an incoherent superposition of the pure states (5.26) with randomly oriented vectors $\mathbf{u}^{(n)}$. This ambiguity makes the analysis into pure components of a quantum mixed state unphysical (§§ 10.1.5 and 11.1.4).

Let us stress that the *statistical or quantum nature of the fluctuations* Δm of the pointer variable in the initial state *is irrelevant* as regards the reduction process. In the most random state (3.46) this fluctuation $1/\sqrt{N}$ appears as purely statistical; it would be just the same for "classical spins" having only a z-component with random values ± 1. In the pure state (5.25), it is merely quantal; indeed, its value $1/\sqrt{N}$ is the lower bound provided by Heisenberg's inequality

$$\Delta m_y^2 \Delta m_z^2 \geq \frac{1}{4}\left|\left\langle\left[\hat{m}_y, \hat{m}_z\right]\right\rangle\right|^2 = \frac{1}{N^2}\langle \hat{m}_x\rangle^2 \tag{5.28}$$

for the operators $\hat{m}_a = N^{-1}\sum_n \hat{\sigma}_a^{(n)}$ ($a = x, y, z$), with here $\Delta m_y = \Delta m_z = 1/\sqrt{N}$, $\langle \hat{m}_x\rangle = 1$. Differences between these two situations arise only at later times, through the coupling $\hat{H}_{\rm MB}$ with the bath.

5.2.3. Pathological initial states

Zo moet je niet beginnen[33]
Dutch expression

There exists states $\hat{R}_{\rm M}(0)$ for which the fluctuation Δm is not large compared to the spacing $\delta m = 2/N$. An extreme case is, for even N, a pure state in which $N/2$ spins point in the $+z$-direction, $N/2$ in the $-z$-direction; then $P_{\rm M}(m, 0) = \delta_{m,0}$. Coherent or incoherent superpositions of such states yield the same distribution $P_{\rm M}(m, 0) = \delta_{m,0}$, in particular the microcanonical paramagnetic state $\hat{R}_{\rm M}(0) = \delta_{\hat{m},0}[(N/2)!]^2/N!$. In all such cases, m and Δm exactly vanish so that the Hamiltonian and the initial state of S + M satisfy $(\hat{H}_{\rm SA} + \hat{H}_{\rm M})\hat{D}(0) = 0$, $\hat{D}(0)(\hat{H}_{\rm SA} + \hat{H}_{\rm M}) = 0$. According to Eq. (4.8), nothing will happen, both in the diagonal and off-diagonal sectors, until the bath begins to act through the weak terms of the right-hand side. The above mechanism of reduction based on the coupling between S and M thus fails for the states $\hat{D}(0)$ such that $P_{\rm M}(m, 0) = \delta_{m,0}$, whether these states are pure or not.

The situation is similar for all states in which Δm is of order $\delta m = 2/N$, with about half of the spins oriented in nearly the $+z$-direction and half in the $-z$-direction. When the bath B is disregarded, the off-diagonal block $\hat{R}_{\uparrow\downarrow}(t)$ then evolves, as shown by (5.2), on a time scale of order $\hbar/2g$ instead of the much smaller reduction time (5.6), of order $1/\sqrt{N}$.

[32] The consideration of such a state is academic since it would be impossible, even in a thought experiment, to set M in it

[33] You should not begin in this way

In such cases the reduction will appear (contrary to our discussion of § 5.1.2) as a phenomenon of the decoherence type, governed indirectly by B through $\hat{H}_{\rm SA}$ and $\hat{H}_{\rm MB}$, and taking place on a time scale much longer than $\tau_{\rm red}$. This circumstance occurs in many models of measurement, see section 2, in particular those for which S is not coupled with many degrees of freedom of the pointer. It is clearly the large size of M which is responsible here for the reduction. We return to this point in § 8.1.4.

However, these considerations are formal and have little physical relevance since Δm is of order $1/\sqrt{N}$ in nearly all states of M. The states for which it is of order δm are extremely scarce. Moreover, such states cannot be produced by physical (*and macroscopic*) means since they involve correlations of a large number of spins, of order $N/2$, difficult to control. No reasonable interaction with an external device can give rise to such correlations. In particular, while M can easily be set into a canonical paramagnetic state, by preparation at a large temperature T_0 or by a strong RF field (§ 3.3.3), it is difficult to imagine how it might be set into a microcanonical state.

Here again, we recover ideas that were introduced to elucidate the irreversibility paradox. In a Boltzmann gas, one can theoretically imagine initial states with a uniform density which would give rise after some time to a macroscopic inhomogeneity [49, 50]. But such states are extremely scarce and involve subtle specific correlations. Producing one of them would involve the impossible task of handling the particles one by one.

5.3. Consequences of discreteness

Hij keek of hij water zag branden[34]
Dutch proverb

Somewhat surprisingly, it appears that the discreteness of the pointer variable m has specific implications in the off-diagonal blocks of the density matrix. We shall later see that such effects do not occur in the diagonal sectors related to registration.

5.3.1. The recurrence time

De klok is rond gegaan[35]
Dutch expression

Although we have displayed the reduction of the state as an irreversible process on the time scale $\tau_{\rm red}$, the dynamics of our model without the bath is so simple that we expect the reversibility of the equations of motion to manifest itself for finite N. As a matter of fact, the irreversibility arises as usual (§ 1.2.2) from an approximate treatment, justified only under the conditions considered above: large N, short time, correlations of finite order. This approximation, which underlined the results (5.4) and (5.16) of subsections 5.1 and 5.2, consisted in treating m as a continuous variable. We now go beyond it by returning to the expression (5.13), which is exact if the bath is inactive ($\gamma = 0$), and by taking into account the discreteness of the spectrum of $\hat{m}$.

For $N \gg 1$, we can still use for $P_{\rm M}(m, 0)$ the Gaussian form (3.48) based on (3.23). The generating function then reads

$$\Psi_{\uparrow\downarrow}(\lambda, t) = r_{\uparrow\downarrow}(0)\,\sqrt{\frac{2}{\pi}}\,\frac{1}{N\Delta m}\sum_m \exp\left[-\frac{m^2}{2\Delta m^2} + i\pi N m\frac{t}{\tau_{\rm recur}} + \lambda m\right], \tag{5.29}$$

where we have introduced the recurrence time

$$\tau_{\rm recur} \equiv \frac{\pi\hbar}{2g} = \pi\sqrt{2}\frac{\Delta m}{\delta m}\tau_{\rm red}. \tag{5.30}$$

The values (3.22) of m that contribute to the sum (5.28) are equally spaced, at distances $\delta m = 2/N$. When the time t increases and begins to approach $\tau_{\rm recur}$ within a delay of order $\tau_{\rm red}$, the correlations undergo an *inverse cascade*: Simpler and simpler correlations are gradually generated from correlations involving a huge number of spins of M. This process is the time-reversed of the one described in § 5.1.3. When t reaches $\tau_{\rm recur}$, or a multiple of it, the various

[34] He looked as if he saw water burn, i.e., he was very surprised
[35] The clock has made a turn

terms of (5.29) add up, instead of interfering destructively as when t is of order of $\tau_{\rm red}$. In fact, the generating function (5.29) satisfies

$$\Psi_{\uparrow\downarrow}(\lambda, t+\tau_{\rm recur}) = (-1)^N \Psi_{\uparrow\downarrow}(\lambda, t), \tag{5.31}$$

so that without the bath the state $\hat{D}(t)$ of S + M *evolves periodically*, returning to its initial expression $\hat{r}(0)\otimes\hat{R}(0)$ at equally spaced times: the Schrödinger cat terms revive.

This recurrence is a quantum phenomenon [49, 50]. It arises from the discreteness and regularity of the spectrum of the pointer variable operator $\hat{m}$, and from the oversimplified nature of the model solved in the present section, which includes only the part (3.5) of the Hamiltonian. We will exhibit in section 6 two mechanisms which, in less crude models, modify the dynamics on time scales larger than $\tau_{\rm red}$ and prevent recurrences to occur.

The recurrence time (5.30) is much longer than the reduction time, since $\Delta m/\delta m = \frac{1}{2}\delta_0\sqrt{N}$. Thus, long after the initial order carried by the transverse components $\langle\hat{s}_x\rangle$ and $\langle\hat{s}_y\rangle$ of the spin S has dissolved into numerous and weak correlations, *this order revives* through an inverse cascade. At the time $\tau_{\rm recur}$, S gets decorrelated from M, with $r_{\uparrow\downarrow}(\tau_{\rm recur}) = (-)^N r_{\uparrow\downarrow}(0)$. The memory of the off-diagonal elements, which was hidden in correlations, was dephased, it was not lost for good, and it emerges back. Such a behavior of the transverse components of the spin S is reminiscent of the behavior of the transverse magnetization in spin echo experiments [54, 55, 56, 57, 58]. By itself it is a dephasing which can cohere again, and will do so unless other mechanisms (see section 6) prevent this.

5.3.2. High-order correlations

We can write $\Psi_{\uparrow\downarrow}(\lambda, t)$ given by (5.29) more explicitly, for large N, by formally extending the summation over m beyond -1 and $+1$, which is innocuous, and by using Poisson's summation formula, which reads

$$\sum_m f(m) = \frac{N}{2}\sum_{p=-\infty}^{+\infty}(-1)^{pN}\int {\rm d}m\, e^{-i\pi Nmp} f(m). \tag{5.32}$$

As a result, we get

$$\Psi_{\uparrow\downarrow}(\lambda, t) = r_{\uparrow\downarrow}(0)\sum_{p=-\infty}^{+\infty}(-1)^{pN}\exp\left(\frac{\lambda\Delta m}{\sqrt{2}} + i\frac{t-p\tau_{\rm recur}}{\tau_{\rm red}}\right)^2, \tag{5.33}$$

which is nothing but a sum of contributions deduced from (5.15), (5.16) and (5.3) by repeated shifts of t (with alternating signs for odd N). This obviously periodic expression exhibits the recurrences and the corrections to the results of subsections 5.1 and 5.2 due to the discreteness of m.

In fact, $\Psi_{\uparrow\downarrow}(\lambda, t)$ is related to the elliptic function θ_3 [203] through

$$\begin{aligned}\frac{\Psi_{\uparrow\downarrow}(\lambda,t)}{r_{\uparrow\downarrow}(0)} &= \exp\left(\frac{\lambda\Delta m}{\sqrt{2}}+\frac{it}{\tau_{\rm red}}\right)^2\theta_3\left[\frac{1}{2}\left(\lambda\delta_0^2+\eta+i\pi N^2\Delta m^2\frac{t}{\tau_{\rm recur}}\right),\frac{N^2\Delta m^2}{2}\right]\\ &= \sqrt{\frac{2}{\pi}}\frac{1}{N\Delta m}\exp\left[-\eta\left(\frac{i\pi t}{\tau_{\rm recur}}+\frac{\lambda}{N}+\frac{1}{2N\delta_0^2}\right)\right]\theta_3\left[\frac{t}{\tau_{\rm recur}}-\frac{i}{N\pi}\left(\lambda+\frac{\eta}{\delta_0^2}\right),\frac{2}{\pi^2N^2\Delta m^2}\right],\end{aligned} \tag{5.34}$$

with $\eta = 0$ for even N, $\eta = 1$ for odd N. It satisfies the two periodicity properties (5.31) and

$$\Psi_{\uparrow\downarrow}\left(\lambda+\frac{2}{\delta_0^2}, t\right) = \exp\left(\frac{2\pi i t}{\tau_{\rm recur}}+\frac{2\lambda}{N}+\frac{2}{N\delta_0^2}\right)\Psi_{\uparrow\downarrow}(\lambda,t). \tag{5.35}$$

According to (5.15) and (5.33) the dominant corrections to the results of § 5.1.3 are given for $t \ll \tau_{\rm recur}$ by the terms $p = \pm 1$ in $\Psi_{\uparrow\downarrow}(\lambda, t)$ and $\Psi_{\uparrow\downarrow}(\lambda, 0)$, that is,

$$\begin{aligned}\langle\hat{s}_-\hat{m}^k\rangle_{\rm c} &= r_{\uparrow\downarrow}(0)\exp\left(-\frac{t^2}{\tau_{\rm red}^2}\right)\left(i\sqrt{2}\Delta m\right)^k\left[\left(\frac{t}{\tau_{\rm red}}\right)^k+(-1)^N A_k(t)\exp\left(-\frac{\tau_{\rm recur}^2}{\tau_{\rm red}^2}\right)\right],\\ A_k(t) &\equiv \left(\frac{t-\tau_{\rm recur}}{\tau_{\rm red}}\right)^k\exp\left(\frac{2t\tau_{\rm recur}}{\tau_{\rm red}^2}\right)+\left(\frac{t+\tau_{\rm recur}}{\tau_{\rm red}}\right)^k\exp\left(\frac{-2t\tau_{\rm recur}}{\tau_{\rm red}^2}\right)+\left[(-1)^{k+1}-1\right]\left(\frac{\tau_{\rm recur}}{\tau_{\rm red}}\right)^k.\end{aligned} \tag{5.36}$$

For $t \to 0$, the correction behaves as t^2 or t depending on whether k is even or odd, whereas the main contribution behaves as t^k. However the coefficient is so small that this correction is negligible as soon as $t > \tau_{\rm red} \exp(-\pi^2 N\delta_0^2/2k)$, an extremely short time for $k \ll N$.

We expected the expression (5.23) for the correlations to become invalid for large k. In fact, the values of interest for t are of order $\tau_{\rm red}$, or of $\tau_{\rm red}\sqrt{k}$ for large k, since the correlations reach their maximum at $t = \tau_{\rm red}\sqrt{k/2}$. In this range, the correction in (5.36) is dominated by the first term of $A_k(t)$, which is negligibly small provided

$$\left(\frac{t}{\tau_{\rm red}}\right)^k \gg \left(\frac{\tau_{\rm recur}}{\tau_{\rm red}}\right)^k \exp\left[-\frac{\tau_{\rm recur}(\tau_{\rm recur}-2t)}{\tau_{\rm red}^2}\right]. \tag{5.37}$$

Hence, in the relevant range $t \sim \tau_{\rm red}\sqrt{k}$, the expression (5.23) for the correlations of rank k is valid provided

$$k \ll \frac{\pi^2 N\delta_0^2}{2\ln(\tau_{\rm recur}/t)}, \tag{5.38}$$

but that simple shape does not hold for correlations between very many particles.

In fact, when t becomes sizeable compared to $\tau_{\rm recur}$, the generating function (5.33) is dominated by the terms $p = 0$ and $p = 1$. The correlations take, for arbitrary k, the form

$$\langle \hat{s}_- \hat{m}^k\rangle_{\rm c} = r_{\uparrow\downarrow}(0)\left(i\pi\delta_0^2\right)^k \left\{\left(\frac{t}{\tau_{\rm recur}}\right)^k \exp\left(-\frac{t^2}{\tau_{\rm red}^2}\right) + \left(\frac{\tau_{\rm recur}-t}{\tau_{\rm recur}}\right)^k \exp\left[-\frac{(\tau_{\rm recur}-t)^2}{\tau_{\rm red}^2}\right]\right\}. \tag{5.39}$$

They are all exponentially small for $N \gg 1$ since $\tau_{\rm recur}^2/\tau_{\rm red}^2$ is large as N. The large rank correlations dominate. If for instance t is half the recurrence time, both terms of (5.39) have the same size, and the correlations increase with k by the factor $(\pi\delta_0^2/2)^k$, where $\delta_0 \geq 1$.

6. Irreversibility of the reduction

Quare fremuerunt gentes, et populi meditati sunt inania?[36]
Psalm 2

The sole consideration of the interaction between the tested spin S and the pointer M has been sufficient to explain and analyze the reduction of the state, which takes place on the time scale $\tau_{\rm red}$, at the very early stage of the measurement process. However this Hamiltonian (Eq. (3.5)) is so simple that if it were alone it would give rise to recurrences around the times $\tau_{\rm recur}$, $2\tau_{\rm recur}$, ... In fact the evolution is modified by other processes, which as we shall see hinder the possibility of recurrence and render the reduction irreversible on any reachable time scale.

6.1. Destructive interferences

Bis repetita (non) placent[37]
diverted from Horace

We still neglect in this subsection the effects of the phonon bath (keeping $\gamma = 0$), but will show that the recurrent behavior exhibited in § 5.3.1 is suppressed by a small change in the model, which makes it a little less idealized.

[36] Why do the heathen rage, and the people imagine a vain thing?
[37] Repetitions are (not) appreciated

6.1.1. Spread of the coupling constants

When we introduced the interaction (3.5) between S and A, we assumed that the coupling constants between the tested spin $\hat{\mathbf{s}}$ and each of the spins $\hat{\sigma}^{(n)}$ of the apparatus were all the same. However, even though the range of the forces is long compared to the size of the magnetic dot, these forces can be different, at least slightly. This is similar to the inhomogeneous broadening effect well known in NMR physics [54, 55, 56, 57, 58]. We thus replace here $\hat{H}_{\mathrm{SA}}$ by the more general interaction

$$\hat{H}'_{\mathrm{SA}} = -\hat{s}_z \sum_{n=1}^{N} (g + \delta g_n)\, \hat{\sigma}_z^{(n)}, \tag{6.1}$$

where the couplings $g + \delta g_n$ are constant in time and have the small dispersion

$$\delta g^2 = \frac{1}{N} \sum_{n=1}^{N} \delta g_n^2, \qquad\qquad \sum_{n=1}^{N} \delta g_n = 0. \tag{6.2}$$

The equations of motion (4.8) for $\hat{D}$, the right-hand side of which we disregard, remain valid, with the effective Hamiltonian

$$\hat{H}_i = -s_i \sum_{n} (g + \delta g_n)\, \hat{\sigma}_z^{(n)} - \sum_{q} \frac{NJ_q}{q} \hat{m}^q \tag{6.3}$$

instead of (4.6). This Hamiltonian, as well as the initial conditions $\hat{R}_{ij}(0) = r_{ij}(0)\, \hat{R}_{\mathrm{M}}(0)$, depends only on the commuting observables $\hat{\sigma}_z^{(n)}$. Hence the latter property is also satisfied by the operators $\hat{R}_{ij}(t)$ at all times. Accordingly, $\hat{R}_{\uparrow\uparrow}(t)$ and $\hat{R}_{\downarrow\downarrow}(t)$ remain constant, and the part $\hat{H}_{\mathrm{M}}$ of $\hat{H}_i$ does not contribute to the equation for $\hat{R}_{\uparrow\downarrow}(t)$, which is readily solved as

$$\hat{R}_{\uparrow\downarrow}(t) = r_{\uparrow\downarrow}(0)\, \hat{R}_{\mathrm{M}}(0) \exp \frac{2i}{\hbar} \left(Ng\hat{m}t + \sum_{n=1}^{N} \delta g_n \hat{\sigma}_z^{(n)} t \right), \tag{6.4}$$

with $\hat{R}_{\mathrm{M}}(0)$ given in terms of $\hat{m}$ by (3.45). Notice that here the operator $\hat{R}_{\uparrow\downarrow}$ does not depend only on $\hat{m}$.

If $\hat{R}_{\mathrm{M}}(0)$ is the most random paramagnetic state (3.46), produced for $q = 2$ by initializing the apparatus with $T_0 \gg J$ or with a strong RF field, or for $q = 4$ with any temperature higher than the transition, (6.4) takes the form

$$\hat{R}_{\uparrow\downarrow}(t) = r_{\uparrow\downarrow}(0) \prod_{n=1}^{N} \frac{1}{2} \left[\hat{\sigma}_0^{(n)} \cos \frac{2(g + \delta g_n)\, t}{\hbar} + i\hat{\sigma}_z^{(n)} \sin \frac{2(g + \delta g_n)\, t}{\hbar} \right]. \tag{6.5}$$

The off-diagonal elements of the state of S thus evolve according to

$$r_{\uparrow\downarrow}(t) = r_{\uparrow\downarrow}(0) \prod_{n=1}^{N} \cos \frac{2(g + \delta g_n)\, t}{\hbar}. \tag{6.6}$$

The right-hand side behaves as (5.4) for $\delta g \ll g$ as long as t is of order τ_{red}. However, it is expected to remain extremely small at later times since the factors of (6.6) interfere destructively unless t is close to a multiple of $\pi\hbar/2(g + \delta g_n)$ for most n. In particular, the successive recurrences which occurred in § 4.4.1 at the times $\tau_{\mathrm{recur}}, 2\tau_{\mathrm{recur}}, \ldots$ for $\delta g = 0$ and $\gamma = 0$ are now absent provided the deviations δg_n are sufficiently large. We thus obtain a permanent reduction if we have at the time $t = \tau_{\mathrm{recur}}$

$$1 \gg \prod_{n=1}^{N} \cos \frac{\pi \delta g_n}{g} \approx \prod_{n=1}^{N} e^{-\pi^2 \delta g_n^2 / 2g^2} = e^{-\pi^2 \sum_n \delta g_n^2 / 2g^2} = e^{-N\pi^2 \delta g^2 / 2g^2}, \tag{6.7}$$

that is,

$$\frac{\delta g}{g} \gg \frac{1}{\pi} \sqrt{\frac{2}{N}}. \tag{6.8}$$

Provided this condition is satisfied, all results of subsections 5.1 and 5.2 hold, even for large times. The whole set of correlations $\langle \hat{s}_- \hat{m}^k \rangle_{\mathrm{c}}$, first created by the coupling (6.1), disappear for not too large k after a time of order $\tau_{\mathrm{red}} \sqrt{k}$, and

do not revive as t becomes larger. As in usual irreversible processes of statistical mechanics, it is mathematically not excluded that (6.6) takes significant values around some values of t, if N is not too large and if many deviations δg_n are arithmetically related to one another; but this can occur only for extremely large times, physically out of reach (§ 6.1.2).

These conclusions hold for an arbitrary initial state (3.45). The expression (6.4) is the product of $\hat{R}_{\uparrow\downarrow}(t)$, as evaluated in section 5 for $\delta g = 0$, by the phase factor

$$\prod_{n=1}^{N} \exp\left(\frac{2i\delta g_n \sigma_z^{(n)} t}{\hbar}\right). \tag{6.9}$$

A generic set of coupling constants satisfying (6.2) provides the same results as if they were chosen at random, with a narrow gaussian distribution of width δg. Replacing then (6.9) by its expectation value, we find that the whole statistics of S + M (without the bath) is governed by the product of the generating function (5.33) by [54]

$$\prod_{n=1}^{N} \overline{\exp\left(2i\delta g_n \sigma_z^{(n)} t/\hbar\right)} = e^{-\left(t/\tau_{\text{irrev}}^{\text{M}}\right)^2}, \tag{6.10}$$

which introduces a characteristic decay time

$$\tau_{\text{irrev}}^{\text{M}} = \frac{\hbar}{\sqrt{2N}\delta g}. \tag{6.11}$$

This damping factor suppresses all the recurrent terms with $p \neq 0$ in (5.33) if δg satisfies the condition (6.8). Since the exponent of (6.10) is $(\delta g/g\delta_0)^2 (t/\tau_{\text{red}})^2$, the first correlations $\langle \hat{s}_- \hat{m}^k(t)\rangle_{\text{c}}$ are left unchanged if $\delta g \ll g$, while those of higher order are overdamped for large k since $(t/\tau_{\text{red}})^2 = k/2$ at their maximum.

Thus, the reduction of the state produced on the time scale τ_{red} by the coupling $\hat{H}'_{\text{SA}}$ of eq. (6.1), characterized by the decay (5.4) of $\langle \hat{s}_x(t)\rangle$ and $\langle \hat{s}_y(t)\rangle$ and by the time dependence (5.23) of $\langle \hat{s}_- \hat{m}^k(t)\rangle_{\text{c}}$, is fully irreversible. The time $\tau_{\text{irrev}}^{\text{M}}$ characterizes this *irreversibility induced by the magnet* M alone, caused by the dispersion of the constants $g + \delta g_n$ which couple $\hat{\mathbf{s}}$ with the elements $\hat{\sigma}_z^{(n)}$ of the pointer variable. If $\tau_{\text{irrev}}^{\text{M}}$ is such that $\tau_{\text{red}} \ll \tau_{\text{irrev}}^{\text{M}} \ll \tau_{\text{recur}}$, that is, when (6.8) is satisfied, the off-diagonal blocks $\hat{R}_{\uparrow\downarrow}(t)$ of $\hat{D}(t)$ remain negligible on time scales of order τ_{recur}. We will show in § 6.1.2 that recurrences might still occur, but at inaccessibly large times.

6.1.2. Generality of the direct damping mechanism

We have just seen that a modification of the direct coupling between the tested spin S and the magnet M, without any intervention of the bath, is sufficient to prevent recurrences to exist after the initial damping of the off-diagonal blocks of $\hat{D}$. In fact, recurrences took place in § 5.3.1 only because our original model was peculiar, involving a complete symmetry between the N spins which constitute the pointer. We will now show that the mechanism of irreversibility of § 6.1.1, based merely on the direct coupling between the tested system and the pointer of the apparatus, is quite general: it occurs as soon as the pointer presents no regularity.

Let us therefore return to the wide class of models introduced in § 5.1.2, characterized by a coupling

$$\hat{H}_{\text{SA}} = -Ng\hat{s}\hat{m}, \qquad \text{(general operators } \hat{s}, \hat{m}) \tag{6.12}$$

between the measured observable $\hat{s}$ of the system S and the pointer observable $\hat{m}$ of the apparatus A. We assume that the pointer, which has N degrees of freedom, has no symmetry feature, so that the spectrum of $\hat{m}$ displays neither systematic degeneracies nor arithmetic properties. We disregard the other degrees of freedom of A, in particular the indirect coupling with the bath. The model considered above in § 6.1.1 enters this general frame, since its Hamiltonian (6.1) takes the form (6.12) if we identify our $\hat{s}_z$ with the general $\hat{s}$ and if we redefine $\hat{m}$ as

$$\hat{m} = \frac{1}{N}\sum_{n=1}^{N}\left(1 + \frac{\delta g_n}{g}\right)\hat{\sigma}_z^{(n)}. \tag{6.13}$$

Indeed, provided the condition (6.8) is satisfied, the 2^N eigenvalues of (6.13) are randomly distributed over the interval $(-1, 1)$ instead of occurring at the values (3.22) with the huge multiplicities (3.23).

In all such models governed by the Hamiltonian (6.12), the off-diagonal elements of $\hat{r}$ behave as (5.12) so that their time-dependence, and more generally that of the off-diagonal blocks of $\hat{R}$, has the form

$$F(t) = \frac{1}{Q} \sum_{q=1}^{Q} e^{i\omega_q t}. \tag{6.14}$$

Indeed, the matrix element (5.11) is a sum of exponentials involving the eigenfrequencies

$$\omega_q \equiv \frac{Ng(s_i - s_j)m_q}{\hbar}, \tag{6.15}$$

where m_q are the eigenvalues of $\hat{m}$. The number Q of these eigenfrequencies is large as an exponential of the number N of microscopic degrees of freedom of the pointer, for instance $Q = 2^N$ for (6.13). To study a generic situation, we can regard the eigenvalues m_q or the set ω_q as independent random variables. Their distribution is governed by the density of eigenvalues of $\hat{m}$ and by the initial density operator $\hat{R}(0)$ of the apparatus which enters (5.12) and which describes a metastable equilibrium. For sufficiently large N, we can take for each dimensionless m_q a narrow symmetric gaussian distribution, with width of relative order $1/\sqrt{N}$. The statistics of $F(t)$ that we will study then follows from the probability distribution for the frequencies ω_q,

$$p\left(\omega_q\right) = \frac{1}{\sqrt{2\pi}\Delta\omega} \exp\left(-\frac{\omega_q^2}{2\Delta\omega^2}\right), \tag{6.16}$$

where $\Delta\omega$ is of order $\sqrt{N}$ due to the factor N entering the definition (6.15) of ω_q. This problem has been tackled long ago by Kac [204].

We first note that the expectation value of $F(t)$ for this random distribution of frequencies,

$$\overline{F(t)} = e^{-\Delta\omega^2 t^2/2}, \tag{6.17}$$

decays exactly, for all times, as the Gaussian (5.3) with a reduction time $\tau_{\rm red} = \sqrt{2}/\Delta\omega$, encompassing the expression (5.6) that we found for short times in our original model. This result holds for most sets ω_q, since the statistical fluctuations and correlations of $F(t)$, given by

$$\overline{F(t)\,F(t')} - \overline{F(t)}\;\overline{F(t')} = \tfrac{1}{Q}\left(e^{-\Delta\omega^2(t+t')^2/2} - e^{-\Delta\omega^2\left(t^2+t'^2\right)/2}\right), \tag{6.18}$$

$$\overline{F(t)\,F^*(t')} - \overline{F(t)}\;\overline{F^*(t')} = \tfrac{1}{Q}\left(e^{-\Delta\omega^2(t-t')^2/2} - e^{-\Delta\omega^2\left(t^2+t'^2\right)/2}\right), \tag{6.19}$$

are small for large Q.

Nevertheless, for any specific choice of the set ω_q, nothing prevents the real part of $F(t)$ to reach significant values at some times t large as $t \gg \Delta\omega$, due to the tail of its probability distribution. Given some positive number f (less than 1), say $f = 0.2$, we define the *recurrence time* $\tau_{\rm recur}$ as the typical delay we have to wait on average before $\Re F(t)$ rises back up to f. We evaluate this time in Appendix C. For f sufficiently small so that $\ln I_0(2f) \simeq f^2$, a property which holds for $f = 0.2$, we find

$$\tau_{\rm recur} = \frac{2\pi}{\Delta\omega} \exp\left(Qf^2\right) = \pi\sqrt{2}\tau_{\rm red} \exp\left(Qf^2\right). \tag{6.20}$$

As Q behaves as an exponential of N, this generic recurrence time is *inaccessibly large*. Even for a pointer involving only $N = 10$ spins, in which case $Q = 2^N = 2^{10}$, and for $f = 0.2$, we have $\tau_{\rm recur}/\tau_{\rm red} = 2.7 \cdot 10^{18}$. The destructive interferences taking place between the various terms of (5.12) explain not only the reduction of the state (§ 5.1.2) but also, owing to the randomness of the coupling, the irretrievable nature of this decay process over any reasonable time lapse, in spite of the unitarity of the evolution.

Although we expect the eigenfrequencies ω_q associated with a large pointer to be distributed irregularly, the distribution (6.16) chosen above, for which they are completely random and uncorrelated, is not generic. Indeed, according to (6.15), these eigenfrequencies are quantum objects, directly related to the eigenvalues m_q of the operator

$\hat{m}$. A more realistic model should therefore rely on the idea that $\hat{m}$ is a complicated operator, which is reasonably represented by a random matrix. As well known, the eigenvalues of a random matrix are correlated: they repell according to Wigner's law. The above study should therefore be extended to *random matrices* $\hat{m}$ instead of random uncorrelated frequencies ω_q, using the techniques of the random matrix theory [205]. We expect the recurrence time thus obtained to be shorter than above, due to the correlations among the set ω_q, but still to remain considerably longer than with the regular specrum of § 5.3.1.

6.2. Effect of the bath on the reduction

You can't fight City Hall
Saying

Returning to our original model of subsection 3.2 with a uniform coupling g between S and the spins of M, we now take into account the effect, on the off-diagonal blocks of $\mathcal{D}$, of the coupling γ between M and B. We thus start from eq. (4.29), to be solved for times of the order of the recurrence time. We will show that the damping due to the bath can prevent $P_{\uparrow\downarrow}$ and hence $\hat{R}_{\uparrow\downarrow}$ from becoming significant at all times t larger than $\tau_{\rm red}$, in spite of the regularity of the spectrum of $\hat{m}$ which leads to the anomalously short recurrence time $\pi\hbar/2g$ of (5.30)[38].

Readers interested mainly in the physics of the reduction may jump to § 9.6.1, where the mathematics is simplified using insights gained about the behavior of the equation of motion for $t \gg \hbar/T$ through the rigorous approach of § 6.2.1 and of appendix D.

6.2.1. Determination of $P_{\uparrow\downarrow}(t)$

We have found recurrences in $P_{\uparrow\downarrow}(m,t)$ by solving (4.18) without right-hand side and by taking into account the discreteness of m (§ 5.3.1). The terms arising from the bath will modify for each m the modulus and the phase of $P^{\rm dis}_{\uparrow\downarrow}(m,t) = (2/N)P_{\uparrow\downarrow}(m,t)$.

In order to study these changes, we rely on the equation of motion (4.18), the right-hand side of which has been obtained in the large N limit while keeping however the values of m discrete as in § 5.3.1. Note first that the functions $\tilde{K}_{t>}(\omega)$ and $\tilde{K}_{t<}(\omega)$ defined by Eqs. (4.10) and (4.11), respectively, are complex conjugate for the same value of ω. It then results from Eq. (4.18) together with its initial condition that [39]

$$P_{\uparrow\downarrow}(-m,t) = P^*_{\uparrow\downarrow}(m,t) = P_{\downarrow\uparrow}(m,t). \tag{6.21}$$

For $\gamma = 0$, the solution of (4.18) with the initial condition (3.46) is given by (5.2). Starting from this expression, we parametrize $P_{\uparrow\downarrow}(m,t)$ as

$$P_{\uparrow\downarrow}(m,t) = r_{\uparrow\downarrow}(0)\sqrt{\frac{N}{2\pi\delta_0^2}}\,\exp\left[-\frac{Nm^2}{2\delta_0^2} + \frac{2iNgmt}{\hbar} - NA(m,t)\right], \tag{6.22}$$

in terms of the function $A(m,t)$, to be determined at first order in γ from Eq. (4.29) with the initial condition $A(m,0) = 0$. For large N, $A(m,t)$ contains contributions of orders 1 and $1/N$. Its complete expression is exhibited in Appendix D in terms of the autocorrelation function $K(t)$ of the bath (Eq. (D.3)).

The distribution $P_{\uparrow\downarrow}(m,t)$ takes significant values only within a sharp peak centered at $m = 0$ with a width of order $1/\sqrt{N}$. We can therefore consistently expand $A(m,t)$ in powers of m up to second order, according to

$$A(m,t) \approx B(t) - i\Theta(t)m + \frac{1}{2}D(t)m^2, \tag{6.23}$$

so that we can write from (6.22) and (6.23) the expression for $P^{\rm dis}_{\uparrow\downarrow} = (2/N)P_{\uparrow\downarrow}$ in the form

[38] For the related, effective decay of $\mathcal{R}_{\uparrow\downarrow}(t)$ and $\mathcal{R}_{\downarrow\uparrow}(t)$, see section 11.2.3

[39] Changing g into $-g$ would also change $P_{\uparrow\downarrow}(m,t)$ into $P^*_{\uparrow\downarrow}(m,t)$, but we shall stick to the ferromagnetic interaction $g > 0$

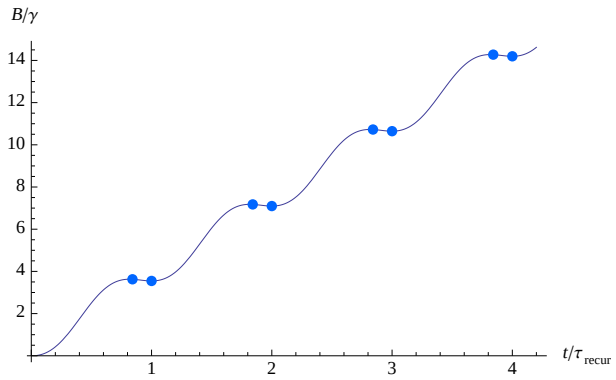

Figure 6.1: The damping function $B(t)$ issued from the interaction of the magnet with the bath. This function is measured in units of the dimensionless magnet-bath coupling constant γ, and the time is measured in units of the recurrence time $\tau_{\rm recur} = \pi\hbar/2g$. The parameters are $T = 0.2J$ and $g = 0.045J$ and $\hbar\Gamma = 50\sqrt{\pi/2}\,J$. After an initial t^4 growth, the curve is quasi linear with periodic oscillations. "Anti-damping" with $\mathrm{d}B/\mathrm{d}t < 0$ occurs during the delay $\alpha\tau_{\rm recur}$ before each recurrence (Eq.(6.33)). The condition $NB(\tau_{\rm recur}) \gg 1$ entails the irreversible suppression of all the recurrences. Bullets denote the local maxima (see (6.36)) and the local minima at integer values of $t/\tau_{\rm recur}$.

$$P^{\rm dis}_{\uparrow\downarrow}(m,t) = r_{\uparrow\downarrow}(0)\sqrt{\frac{2}{\pi N\delta_0^2}}\exp\left\{-NB(t) + iN\left[\frac{2gt}{\hbar} + \Theta(t)\right]m - N\left[\frac{1}{\delta_0^2} + D(t)\right]\frac{m^2}{2}\right\}. \tag{6.24}$$

The functions $B(t)$, $\Theta(t)$ and $D(t)$, proportional to γ, describe the effect of the bath on the off-diagonal blocks of the density matrix of S + M. They are real on account of (6.21). The overall factor $\exp[-NB(t)]$ governs the amplitude of $P^{\rm dis}_{\uparrow\downarrow}$. The term $\Theta(t)$ modifies the oscillations which arose from the coupling between S and M. The term $D(t)$ modifies the width of the peak of $P^{\rm dis}_{\uparrow\downarrow}$. The explicit expressions of these functions, given by (D.14) for $B(t)$, (D.26) for $\Theta(t)$ and (D.29) for $D(t)$, are derived in appendix D from the equation of motion (D.3) for $A(m,t)$, which itself results directly from Eq. (4.29) for $P^{\rm dis}_{\uparrow\downarrow}$. We analyze them below.

6.2.2. The damping function

The main effect of the bath is the introduction in (6.23) of the overall factor $\exp[-NB(t)]$, which produces a damping of the off-diagonal blocks $\hat{R}_{\uparrow\downarrow}$ and $\hat{R}_{\downarrow\uparrow}$ of the density matrix $\hat{D}$ of S + M. The expression for $B(t)$ derives from Eq. (D.8) and is given explicitly by

$$B(t) = \gamma\int_0^\infty \frac{\mathrm{d}\omega\,\omega}{\pi}\coth\frac{\hbar\omega}{2T}\exp\left(-\frac{\omega}{\Gamma}\right)\left\{\frac{\sin^2\Omega t}{2(\omega^2-\Omega^2)} + \frac{\Omega^2(1-\cos\omega t\,\cos\Omega t) - \omega\Omega\sin\omega t\,\sin\Omega t}{(\omega^2-\Omega^2)^2}\right\}, \tag{6.25}$$

with $\Omega = 2g/\hbar$. The ω-integral can be carried out analytically if one replaces in the spectrum of phonon modes (3.37) the Debye cutoff by a quasi Lorentzian one, see Eq. (D.10) and the connection (D.11) between the cutoff parameters; the result for B is given in (D.14). The function $B(t)$ of Eq. (D.14), or, nearly equivalently, Eq. (6.25), is illustrated by fig. 6.1. We discuss here its main features in the limiting cases of interest.

Consider first the short times $t \ll 1/\Gamma$. This range covers the delay $\tau_{\rm red}$ during which the reduction takes place, but it is much shorter than the recurrence time. We have shown in Appendix D that $B(t)$ behaves for $t \ll 1/\Gamma$ as

$$B(t) \sim \frac{\gamma\Gamma^2 g^2}{2\pi\hbar^2}\,t^4, \tag{6.26}$$

increasing slowly as shown by fig. 6.1. If $NB(t)$ remains sufficiently small during the whole reduction process so that $\exp[-NB(t)]$ remains close to 1, the bath is ineffective over the delay $\tau_{\rm red}$. This takes place under the condition

$$NB(\tau_{\rm red}) = N\frac{\gamma\Gamma^2 g^2}{2\pi\hbar^2}\tau_{\rm red}^4 = \frac{\gamma\hbar^2\Gamma^2}{8\pi N\delta_0^4 g^2} \ll 1, \tag{6.27}$$

which is easily satisfied in spite of the large value of $\hbar\Gamma/g$, since $\gamma \ll 1$ and $N \gg 1$. Then the coupling with the bath does not interfere with the reduction by the magnet studied in section 5. Otherwise, if $NB(\tau_{\rm red})$ is finite, the damping by B, which behaves as an exponential of $-t^4$, enhances the reduction effect in $\exp[-(t/\tau_{\rm red})^2]$ of M, and reduces the tails of the curves of fig. 5.1.

Consider now the times t larger than $\hbar/2\pi T$, which is the memory time of the kernel $K(t)$. We are then in the Markovian regime. This range of times encompasses the recurrences which in the absence of the bath occur periodically at the times $t = p\tau_{\rm recur}$, with $\tau_{\rm recur} = \pi\hbar/2g$. Under the condition $t \gg \hbar/2\pi T$, we show in Appendix D Eq. (D.18)), that $B(t)$ has the form

$$B(t) = \frac{\gamma\pi}{4}\coth\frac{g}{T}\left(\frac{t}{\tau_{\rm recur}} - \frac{1}{2\pi}\sin\frac{2\pi t}{\tau_{\rm recur}}\right) + \frac{\gamma}{4\pi}\ln\frac{\hbar\Gamma}{2\pi T}\left(1 - \cos\frac{2\pi t}{\tau_{\rm recur}}\right). \tag{6.28}$$

On average, $B(t)$ thus increases linearly along with the first term of (6.28), as exhibited by fig. 6.1. Hence, the bath generates in this region $t \gg \hbar/2\pi T$ *the exponential damping*

$$\exp[-NB(t)] \sim \exp\left(-\frac{t}{\tau_{\rm irrev}^{\rm B}}\right), \tag{6.29}$$

where the decay is characterized by the bath-induced irreversibility time

$$\tau_{\rm irrev}^{\rm B} = \frac{2\hbar\tanh g/T}{N\gamma g}. \tag{6.30}$$

The recurrences, at $t = p\tau_{\rm recur}$, are therefore attenuated by the factor

$$\exp\left(-\frac{p\tau_{\rm recur}}{\tau_{\rm irrev}^{\rm B}}\right) = \exp\left(-\frac{p\pi N\gamma}{4\tanh g/T}\right). \tag{6.31}$$

Thus, *all recurrences are irreversibly suppressed*, so that the initial reduction becomes definitive, provided the coupling between M and B is sufficiently strong so as to satisfy $NB(\tau_{\rm recur}) \gg 1$, or equivalently $\tau_{\rm irrev}^{\rm B} \ll \tau_{\rm recur}$, that is:

$$\gamma \gg \frac{4\tanh g/T}{\pi N}. \tag{6.32}$$

In case $T \gg g$, the irreversibility time

$$\tau_{\rm irrev}^{\rm B} \sim \frac{2\hbar}{N\gamma T} \tag{6.33}$$

depends only on the temperature of the bath, on the number of spins of the magnet, and on the magnet-bath coupling, irrespective of the system-magnet coupling.

In spite of the smallness of γ, the large value of N makes the condition (6.32) easy to satisfy. In fact, if the hardly more stringent condition $NB(\hbar/2\pi T) \gg 1$, that is, $N\gamma \gg 4\pi$, is satisfied, we have $NB(t) \gg 1$ in the region $t \gg \hbar/2\pi T$ where the approximation (6.28) holds. Thus, although $B(t)$ is quasi linear in this region, the exponential shape of the decay (6.29), with its characteristic time $\tau_{\rm irrev}^{\rm B}$, loses physical relevance since $\exp[-NB(t)]$ is there practically zero.

In this same region $t \gg \hbar/2\pi T$, the expression (6.28) of $B(t)$ involves oscillatory contributions superimposed to the linear increase considered above (fig. 6.1). In fact, the time derivative

$$\frac{\tau_{\rm recur}}{\gamma}\,\frac{{\rm d}B}{{\rm d}t} = \left(\frac{\pi}{2}\coth\frac{g}{T}\,\sin\frac{\pi t}{\tau_{\rm recur}} + \ln\frac{\hbar\Gamma}{2\pi T}\,\cos\frac{\pi t}{\tau_{\rm recur}}\right)\sin\frac{\pi t}{\tau_{\rm recur}}. \tag{6.34}$$

of $B(t)$ is periodic, with period $\tau_{\rm recur}$, and it vanishes at the times t such that

$$\sin\frac{\pi t}{\tau_{\rm recur}} = 0 \quad \text{or} \quad \tan\frac{\pi t}{\tau_{\rm recur}} = -\frac{2}{\pi}\ln\frac{\hbar\Gamma}{2\pi T}\,\tanh\frac{g}{T}. \tag{6.35}$$

The first set of zeros occur at the recurrence times $p\tau_{\rm recur}$, which are local minima of $B(t)$. The second set provide local maxima, which occur somewhat earlier than the recurences (fig. 6.1), at the times

$$t = (p-\alpha)\tau_{\rm recur}, \qquad \alpha = \frac{1}{\pi}\arctan\left(\frac{2}{\pi}\ln\frac{\hbar\Gamma}{2\pi T}\,\tanh\frac{g}{T}\right). \tag{6.36}$$

An unexpected quantum effect thus takes place in the off-diagonal blocks of the density matrix of S + M. Usually, a bath produces a monotonous relaxation. Here, the damping factor $\exp[-NB(t)]$, which results from the coupling of M with the bath, *increases* between the times $(p-\alpha)\tau_{\rm recur}$ and $p\tau_{\rm recur}$. During these periods, the system S + M undergoes an "*anti-damping*". This has no incidence on our measurement process, since the recurrences are anyhow killed under the condition (6.29) and since their duration, $\tau_{\rm red}$, is short compared to the delay $\alpha\tau_{\rm recur}$. One may imagine, however, other processes that would exhibit a similar effect.

6.2.3. Time-dependence of physical quantities

All the off-diagonal physical quantities, to wit, the expectation values $\langle\hat{s}_x(t)\rangle$, $\langle\hat{s}_y(t)\rangle$, and the correlations between $\hat{s}_x$ or $\hat{s}_y$ and any number of spins of the apparatus are embedded in the generating function $\Psi_{\uparrow\downarrow}(\lambda, t)$ defined as in (5.12). As we recalled in § 6.2.1, we must sum over the *discrete values* (3.22) of m, rather than integrate over m; the distinction between summation and integration becomes crucial when the time t reaches $\tau_{\rm recur}$, since then the period in m of the oscillations of $P^{\rm dis}_{\uparrow\downarrow}(m,t)$ becomes as small as the level spacing. From (6.23), we see that the characteristic function, modified by the bath terms, has the same form as in § 5.3.2 within multiplication by $\exp[-NB(t)]$ and within modification of the phase and of the width of $P^{\rm dis}_{\uparrow\downarrow}(m,t)$.

Let us first consider the effect of $\Theta(t)$. Its introduction changes the phase of $P_{\uparrow\downarrow}$ according to

$$\frac{2iNgmt}{\hbar} \mapsto \frac{2iNgmt}{\hbar} + iN\Theta(t)m. \tag{6.37}$$

Hence, the occurrence of the term $\Theta(t)$ might shift the recurrences, which take place when

$$\frac{2gt}{\hbar} + \Theta(t) = p\pi. \tag{6.38}$$

However, the expression of $\Theta(t)$ derived in the appendix D, Eq. (D.26),

$$\Theta(t) \sim -\frac{\gamma}{8g}\left[\left(\frac{2}{\delta_0^2}-1\right)T + J_2\right]\left[1-\cos\frac{2\pi t}{\tau_{\rm recur}}\right]. \tag{6.39}$$

vanishes for $t = p\tau_{\rm recur} = p\pi\hbar/2g$, so that the replacement (6.34) does not affect the values of the recurrence times. Between these recurrence times, the reduction makes all correlations of finite rank negligible even in the absence of the bath, as if $P^{\rm dis}_{\uparrow\downarrow}$ did vanish; then, the phase of $P^{\rm dis}_{\uparrow\downarrow}$ is irrelevant. Altogether, $\Theta(t)$ is completely ineffective.

Likewise, the term $D(t)$ is relevant only at the recurrence times. We evaluate it in Eq. (D.29) as

$$D(p\tau_{\rm recur}) \simeq p\eta, \qquad \eta = \frac{\pi\gamma}{2}\frac{J_2}{g}\left(\frac{J_2}{3T} - 1\right). \tag{6.40}$$

This term changes the width of the distribution $P^{\rm dis}_{\uparrow\downarrow}(m,t)$ by a small relative amount of order $\gamma \ll 1$, according to

$$\Delta m = \frac{\delta_0}{\sqrt{N}} \mapsto \Delta m_p = \frac{\delta_0}{\sqrt{N(1+p\eta\delta_0^2)}} = \Delta m(1 - \frac{1}{2}p\eta\delta_0^2). \tag{6.41}$$

The width therefore increases if $J_2 < 3T$, or decreases if $J_2 > 3T$, but this effect is signficant only if the recurrences are still visible, that is, if the condition (6.32) is not satisfied.

The expression (5.32) of the generating function is thus modified into

$$\Psi_{\uparrow\downarrow}(\lambda,t) = r_{\uparrow\downarrow}(0)e^{-NB(t)}\sum_{p=-\infty}^{\infty}(-1)^{pN}\exp\left(\frac{\lambda\Delta m_p}{\sqrt{2}} + i\frac{t-p\tau_{\rm recur}}{\tau_{\rm red}}\right)^2. \tag{6.42}$$

The crucial change is the presence of the damping factor which invalidates the periodicity (5.30) of $\Psi_{\uparrow\downarrow}(\lambda,t)$ and which inhibits the recurrences. Moreover, for any $t > 0$, the terms $p < 0$ in (6.42) are negligible, since they involve (for $t = 0$) the factor $\exp[-(p\tau_{\rm recur}/\tau_{\rm red})^2]$. Thus, under the conditions (6.28) and (6.32), the sum (6.42) reduces at all times to its term $p = 0$. Accordingly, it is legitimate to express for arbitrary times $P_{\uparrow\downarrow}$ as

$$P_{\uparrow\downarrow}(m,t) = P_{\uparrow\downarrow}(m,0)\exp\left[\frac{2iNgmt}{\hbar} - NB(t)\right], \tag{6.43}$$

and to treat m as a continuous variable. As a consequence, the full density matrix of S + M, which results from (3.26), has off-diagonal blocks given by

$$\hat{R}_{\uparrow\downarrow}(t) = r_{\uparrow\downarrow}(0)\hat{R}_{\rm M}(0)\exp\left[\frac{2iNgmt}{\hbar} - NB(t)\right], \tag{6.44}$$

where we recall the expressions (D.14), (6.26) and (6.28) for $B(t)$.

Altogether, as regards the evolution of the physical quantities $\langle \hat{s}_a \hat{m}^k(t)\rangle$ ($a = x$ or y), nothing is changed in the results of § 5.1.3 on the scale $t \ll \tau_{\rm recur}$; these results are summarized by Eq. (5.22) and illustrated by fig. 5.1. For $t \gg \tau^{\rm B}_{\rm irrev}$, the factor $\exp[-NB(t)]$ makes all these off-diagonal quantities *vanish irremediably*, including the high-rank correlations of § 5.3.2.

In spite of the simplicity of this result, our derivation was heavy because we wanted to produce a rigorous proof. It turned out that the interaction between the spins of M occurs both through δ_0 in the initial state of M and through J_2 in the dynamics generated by the bath, but that it has little effect. Taking this properties for granted, treating M as a set of independent spins and admitting that for $t \gg \hbar/2\pi T$ the autocorrelation function of the bath enters the dynamical equation through (D.21), we present in § 9.6.1 a simpler derivation, which may be used for tutorial purposes and which has an intuitive interpretation: Both the precession of $\hat{s}$ and the damping of $\hat{R}_{\uparrow\downarrow}(t)$ by the bath arise from a dynamical process in which each spin of M is independently driven by its interaction with S and independently relaxes under the effect of the bath B.

6.2.4. The off-diagonal bath effect, an ongoing decoherence process regulated by the tested observable

Het houdt maar niet op[40]

[40] It keeps going on

The damping described above has two unusual features: on the one hand (fig. 6.1), its coefficient does not monotonically decrease; on the other hand, it is governed by a resonance effect. However, it has also clearly the features of a standard decoherence [28, 29, 35, 147, 148, 149, 150]. It takes place in the compound system S + M under the influence of B which plays the role of an environment. The decay (6.29) is quasi-exponential, apart from non-essential oscillations. The expression (6.33) of the irreversibility time $\tau^{\rm B}_{\rm irrev} = 2\hbar/N\gamma T$ (for $T \gg g$) is typical of a bath-induced decoherence: It is inversely proportional to the *temperature* T of B, to the number N which characterizes the *size* of the system S + M, and to the *coupling* γ of this system with its environment, which is here the bath.

Nevertheless, we have stressed (§ 5.1.2) that the fundamental mechanism of the initial reduction of the state $\hat{D}(t)$ of S + M has not such a status of decoherence. It takes place in the brief delay $\tau_{\rm red} = \hbar/\sqrt{N}\delta_0 g$, during which the bath does not yet have any effect. Contrary to *decoherence*, this *dephasing* process is internal to the system S + M, and does not involve its environment B. It is governed by the direct coupling g between S and the pointer M, as shown by the expression of the reduction time. It is during delays of order $\tau_{\rm red}$ that the phenomena described in section 5 occur – decay of the average transverse components of the spin S, creation then disappearance of correlations with higher and higher rank (§ 5.1.3 and fig. 5.1). The bath has no effect on this reduction proper.

When the bath begins to act, that is, when $NB(t)$ becomes significant, the reduction can be considered as *practically achieved* since Eq (6.27) is easily satisfied. The only tracks that remain from the original blocks $\hat{R}_{\uparrow\downarrow}(0)$ and $\hat{R}_{\downarrow\uparrow}(0)$ of $\hat{D}(0)$ are correlations of very high rank (§ 5.3.2), so that the state $\hat{D}(t)$ cannot be distinguished at such times from a state without off-diagonal blocks. However, if the Hamiltonian did reduce solely to $\hat{H}_{\rm SA}$ (Eq. (3.5)), the simplicity of the dynamics would produce, from these hidden correlations, a revival of the initial state $\hat{D}(0)$, taking place just before $\tau_{\rm recur}$, during a delay of order $\tau_{\rm red}$. The weak interaction γ with the bath *wipes out the high rank correlations*, at times t such that $\tau_{\rm red} \ll t \ll \tau_{\rm recur}$ for which they are the only remainder of $r_{\uparrow\downarrow}(0)$. Their destruction prevents the inverse cascade from taking place and thus suppresses all recurrences.

The interaction between S and M does not only produce the initial reduction of $\hat{D}$ described in section 5. It is also an essential ingredient in the very mechanism of decoherence by the bath B. Indeed, the interaction (3.10) between M and B is isotropic, so that it is the coupling between S and M which should govern the selection of the basis in which the suppression of recurrences will occur after the initial reduction. To understand how this ongoing *preferred basis problem* is solved, let us return to the derivation of the expression (6.25) for the damping term $B(t)$, valid in the time range of the bath-induced irreversibility. This expression arose from the integral (D.8), to wit,

$$\frac{{\rm d}B}{{\rm d}t} = \frac{4\gamma \sin\Omega t}{\pi\hbar^2} \int_{-\infty}^{\infty} {\rm d}\omega\, \tilde{K}(\omega)\, \frac{\Omega(\cos\Omega t - \cos\omega t)}{\omega^2 - \Omega^2} \tag{6.45}$$

which analyzes the influence, on the damping, of the various frequencies ω of the autocorrelation function $\tilde{K}(\omega)$ of the phonon bath. The effect of the system-magnet interaction g is embedded in the frequency $\Omega = 2g/\hbar = \pi/\tau_{\rm recur}$, directly related to the period of the recurrences. In appendix D, we have shown that the quasi-linear behaviour of $B(t)$ results from the approximation (D.20) for the last factor of (6.45): This factor is peaked around $\omega = \pm\Omega$ for $t \gg \hbar/2\pi T$. The integral (6.45) then reduces to

$$\frac{{\rm d}B(t)}{{\rm d}t} = \frac{\gamma}{\hbar^2}\left[\tilde{K}(\Omega) + \tilde{K}(-\Omega)\right](1 - \cos 2\Omega t), \tag{6.46}$$

the constant part of which produces the dominant, linear term $B \propto t$ of (6.28). In the autocorrelation function $\tilde{K}(\omega)$ which controls the damping by B in the equation of motion of S + M, $\hbar\omega$ is the energy of the phonon that is created or annihilated by interaction with a spin of the magnet (§ 3.2.2). Thus, through a resonance effect arising from the peak of the integrand in (6.44), the frequency ω of the phonons that contribute to the damping adjusts itself onto the frequency $\Omega = 2g/\hbar$ associated with the precession of the spins of the magnet under the influence of the tested spin. Owing to this *resonance effect*, the bath acts mainly through the frequency of the recurrences. Accordingly, phonons with energy $\hbar\omega$ close to the energy $\hbar\Omega = 2g$ of a spin flip in M (see Eq. (3.5)) are continuously absorbed and emitted, and this produces the shrinking of the off-diagonal blocks $\hat{R}_{\uparrow\downarrow}$ and $\hat{R}_{\downarrow\uparrow}$. The effect is cumulative, since $B \propto t$. *The decoherence by the bath is thus continuously piloted by the coupling of the magnet with* S.

In conclusion, the initial reduction and its further consolidation are in the present model the results of an interplay between the three interacting objects, S, M and B. The main effect, on the time scale $\tau_{\rm red}$, arises from the coupling between S and the many degrees of freedom of M, and it should not be regarded as decoherence. Rather, it is a dephasing effect as known in nuclear magnetic resonance. Viewing the magnet M as "some kind of bath or of environment", as is often done, disregards the essential role of M: to act as the pointer that indicates the outcome of the quantum measurement. Such an idea also confers too much extension to the concept of bath or environment. Decoherence usually requires some randomness of the environment, and we have seen (§ 5.2.2) that reduction may occur even if the initial state of M is pure.

The mechanisms that warrant, on a longer time scale $\tau^{\rm M}_{\rm irrev}$ or $\tau^{\rm B}_{\rm irrev}$, the permanence of the reduction can be regarded as adjuvants of the main initial reduction process, since they become active after all accessible off-diagonal expectation values and correlations have (provisionally) disappeared. We saw in subsection 6.1 that the intervention of B is not necessary to entail this irreversibility, which can result from a dispersion of the coupling constants g_n. For the more efficient mechanism of suppression of recurrences of subsection 6.2, we have just stressed that it is a decoherence process arising from the phonon thermal bath but steered by the spin-magnet coupling.

In section 7, we turn to the most essential role of the bath B in the measurement, to allow the registration of the outcome by the pointer.

7. Registration

Wie schrijft, die blijft[41]
Les paroles s'envolent, les écrits restent[42]
Dutch and French proverbs

The main issue in a measurement process is the establishment of correlations between S and A, which will allow us to gain information on S through observation of A [5, 6, 27, 43, 73]. As shown in § 5.1.3, the process creates correlations in the off-diagonal blocks $\hat{\mathcal{R}}_{\uparrow\downarrow}(t)$ and $\hat{\mathcal{R}}_{\downarrow\uparrow}(t)$ of the density matrix $\hat{\mathcal{D}}(t)$ of S + A, but those which survive after the brief reduction time $\tau_{\rm red}$ involve a large number of spins $\hat{\sigma}^{(n)}$ of M and are inobservable. The considered quantum measurement thus cannot provide information on the off-diagonal elements $r_{\uparrow\downarrow}(0)$ of the density matrix $\hat{r}(0)$ of S. We now show, by studying the dynamics of the diagonal blocks of $\hat{\mathcal{D}}(t)$, how M can register the statistical information embedded in $r_{\uparrow\uparrow}(0)$ and $r_{\downarrow\downarrow}(0)$ through creation of system-apparatus correlations. We shall also see how the process can be used as a preparation of S in the pure state $|\uparrow\rangle$ or $|\downarrow\rangle$.

The registration process presents two different qualitative behaviours, depending on the nature of the phase transition of the magnet, of second order if the parameters of its Hamiltonian (3.7) satisfy $J_2 > 3J_4$, of first order if they satisfy $3J_4 > J_2$. Recalling our discussion below Eq. (3.9), we will exemplify below these two situations with the two pure cases $q = 2$ and $q = 4$. In the former case, for $J_2 \equiv J$ and $J_4 = 0$, the Hamiltonian is expressed by (3.8); in the latter case, for $J_4 \equiv J$ and $J_2 = 0$, it is expressed by (3.9). We summarize these two cases by $H_{\rm M} = -(NJ/q)\hat{m}^q$ with $q = 2$ and 4, respectively.

7.1. Properties of the dynamical equations

7.1.1. Initial regime

The dynamics of the diagonal blocks $\hat{\mathcal{R}}_{\uparrow\uparrow}(t)$ of $\hat{\mathcal{D}}(t)$ results for large N from the equation (4.30) for the scalar function $P_{\uparrow\uparrow}(t)$, with initial condition $P_{\uparrow\uparrow}(0) = r_{\uparrow\uparrow}(0)P_{\rm M}(m, 0)$. The initial distribution $P_{\rm M}(m, 0)$ for the magnetization of M, given by (3.48), is a Gaussian, peaked around $m = 0$ with the small width $\delta_0/\sqrt{N}$. We have noted (subsection 4.4) the analogy of the equation of motion (4.30) with a Fokker-Planck equation [202] for the random variable m submitted to the effects of the thermal bath B. In this equation, which reads

$$\frac{\partial P_{\uparrow\uparrow}}{\partial t} = \frac{\partial}{\partial m}\left(-vP_{\uparrow\uparrow}\right) + \frac{1}{N}\frac{\partial^2}{\partial m^2}\left(wP_{\uparrow\uparrow}\right), \tag{7.1}$$

[41] Who writes, stays
[42] Words fly away, writings stay

the first term describes a *drift*, the second one a *diffusion* [202]. The drift velocity $v(m,t)$ is a function of m and t defined by (4.31), whereas the diffusion coefficient $w_{\uparrow\uparrow}(m,t)$ is defined by (4.32). The normalization of $P_{\uparrow\uparrow}$ remains unchanged in time:

$$\int \mathrm{d}m P_{\uparrow\uparrow}(m,t) = \int \mathrm{d}m P_{\uparrow\uparrow}(m,0) = r_{\uparrow\uparrow}(0), \tag{7.2}$$

so that the ratio $P(m,t) = P_{\uparrow\uparrow}(m,t)/r_{\uparrow\uparrow}(0)$ can be interpreted as a conditional probability of m if $s_z = 1$.

For very *short times* such that $t \ll 1/\Gamma$, we have

$$\tilde{K}_t(\omega) \sim 2tK(0) = \frac{\hbar^2}{4\pi}\Gamma^2 t, \tag{7.3}$$

and hence

$$v \sim -\frac{\gamma}{\pi}\Gamma^2 mt, \qquad w \sim \frac{\gamma}{\pi}\Gamma^2 t\,. \tag{7.4}$$

The solution of (7.1) then provides a Gaussian which remains centered at $m = 0$. Its width $\sqrt{D/N}$ decays for $q = 2$, $\delta_0 > 1$ as

$$D(t) = \delta_0^2 - (\delta_0^2 - 1)\left[1 - \exp\left(-\frac{\gamma}{\pi}\Gamma^2 t^2\right)\right], \tag{7.5}$$

and is constant ($D = \delta_0^2 = 1$) for $q = 4$. Anyhow, on the considered time scale, the change in $P_{\uparrow\uparrow}(m,t)$ is not perceptible since $\gamma \ll 1$. The registration may begin to take place only for larger times.

7.1.2. Markovian regime

The weakness of the magnet-bath coupling γ implies that the time scale of the registration is larger than the memory time $\hbar/2\pi T$ of $K(t)$. Then $\tilde{K}_t(\omega)$ defined by (4.17) reduces to $\tilde{K}(\omega)$, that is, to (3.37). The equation of motion (7.1) for $P_{\uparrow\uparrow}$ becomes Markovian [145, 199, 200], with v and w depending only on m and not on t. As soon as $t \gg \hbar/2\pi T$, $P_{\uparrow\uparrow}$ thus evolves in a short-memory regime. Its equation of motion is invariant under time translation.

The explicit expressions (4.31) and (4.32) of $v_{\uparrow\uparrow}$ and w become in this regime

$$v(m) \quad = \gamma\omega_\uparrow(1 - m\coth\beta\hbar\omega_\uparrow), \tag{7.6}$$

$$w(m) \quad = \gamma\omega_\uparrow(\coth\beta\hbar\omega_\uparrow - m), \tag{7.7}$$

where $\hbar\omega_\uparrow = g + Jm^{q-1}$ from definition (4.24). These functions contain in fact an extra factor $\exp(-2|\omega_\uparrow|/\Gamma)$, which we disregard since the Debye cutoff is large:

$$\hbar\Gamma \gg g, \qquad \hbar\Gamma \gg J. \tag{7.8}$$

While the diffusion coefficient $w(m)$ is everywhere positive, the drift velocity $v(m)$ changes sign at the values $m = m_i$ that are solutions of (3.55). We illustrate the behavior of $v(m)$ in Figs. 7.1 for $q = 2$ and 7.2 for $q = 4$.

7.1.3. Classical features

We have stressed (subsection 4.4) that the drift term in (7.1) is "classical", in the sense that it comes out for large N by taking the continuous limit of the spectrum of $\hat{m}$, and that the diffusion term, although relevant in this large N limit, results from the discreteness of the spectrum of $\hat{m}$ and has therefore a quantum origin. We can, however, forget this origin and regard this diffusion term as a "classical" stochastic effect. As a preliminary exercise, we show below that an empirical classical approach of the registration provides us at least with a drift, similar to the one occurring in eq. (7.1).

For times $t \gg \tau_{\mathrm{red}}$ it is legitimate to disregard the off-diagonal blocks $\hat{\mathcal{R}}_{\uparrow\downarrow}$ and $\hat{\mathcal{R}}_{\downarrow\uparrow}$ of $\hat{\mathcal{D}}$, and the process that takes place later on involves only $P_{\uparrow\uparrow}$ and $P_{\downarrow\downarrow}$. (In our present model the blocks evolve independently anyhow). This process looks like the measurement of a "classical discrete spin" which would take only two values $+1$ and -1 with respective probabilities $r_{\uparrow\uparrow}(0)$ and $r_{\downarrow\downarrow}(0)$; the x- and y-components play no role. The magnet M also behaves, in the present diagonal sectors, as a collection of N classical spins $\sigma_z^{(n)}$, the x- and y-components of which can be disregarded. The

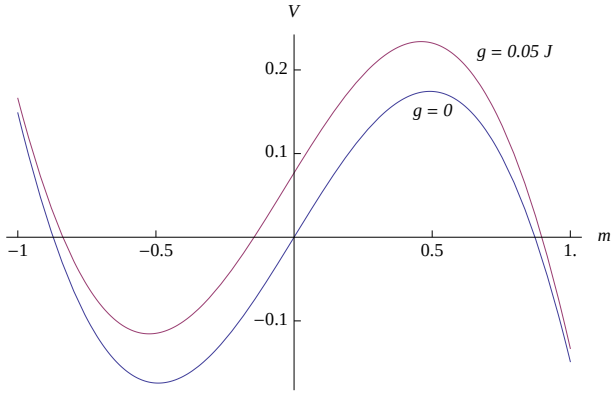

Figure 7.1: The drift velocity field $V(m) = \hbar\nu(m)/\gamma T = \beta(Jm+g)[1-m\coth\beta(Jm+g)]$ for second-order transitions ($q = 2$, i. e., $J_2 = J$, $J_4 = 0$), at the temperature $T = 0.65J$. The fixed points, the zeroes of $V(m)$, are the extrema of the free energy $F(m)$. For g=0, the attractive fixed points lie at $\pm m_{\mathrm{F}} = \pm 0.87$. For g=0.05J, the two attractive fixed points lie at $m_{\Uparrow} = 0.90$ and $m_{\Downarrow} = 0.84$, and the repulsive bifurcation lies at $m = -m_{\mathrm{B}} = -0.14$. For $g = 0$ the attractive fixed points lie at $\pm m_{\mathrm{F}} = \pm 0.91$.

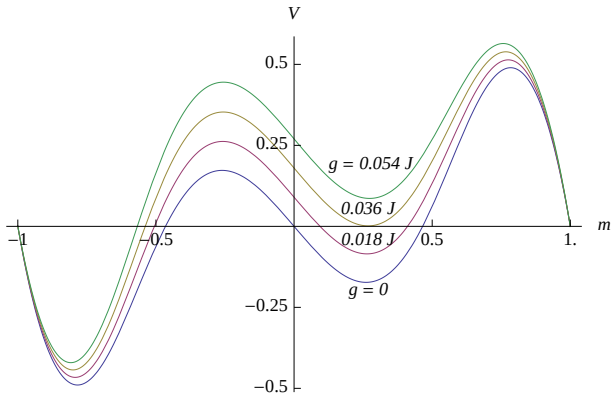

Figure 7.2: The drift velocity field $V(m) = \hbar\nu(m)/\gamma T = \beta(Jm^3+g)[1-m\coth\beta(Jm^3+g)]$ for first-order transitions ($q = 4$, i. e., $J_2 = 0$, $J_4 = J$) at $T = 0.2J$ and for various couplings g. The zeroes of $V(m)$ are the extrema of the free energy $F(m)$ (see Figs. 3.3 and 3.4). For $g = 0$ there are three attractive fixed points, $m_{\mathrm{P}} = 0$ and $\pm m_{\mathrm{F}}$ with $m_{\mathrm{F}} = 1 - 9.1\cdot 10^{-5}$ and two repulsive fixed points, at ± 0.465, close to $\pm\sqrt{T/J} = \pm 0.447$. For increasing g, m_{P} increases up to $m_{\mathrm{c}} = 0.268$ until g reaches $h_{\mathrm{c}} = 0.0357\,J$. For larger g, the paramagnetic fixed point m_{P} disappears together with the positive repulsive point, and, since V is positive for all $m > 0$, the distribution of m can easily move from values near 0 to values near m_{F}, "rolling down the hill" of $F(m)$. If g is too small, $V(m)$ vanishes with a negative slope at the attractive paramagnetic fixed point m_{P} near the origin; the distrbution of m then ends up around m_{P} and the apparatus returns to its paramagnetic state when g is switched off so that the registration fails.

dynamics of M is governed by its coupling with the thermal bath B. If this coupling is treated classically, we recover a standard problem in classical statistical mechanics [54, 206, 207, 208, 209]. Indeed, the dynamics of

$$P(m,t) = \frac{P_{\uparrow\uparrow}(m,t)}{r_{\uparrow\uparrow}(0)} \tag{7.9}$$

is the same as the relaxation of the random order parameter m of an Ising magnet, submitted to a magnetic field $h = g$ and weakly coupled to the bath B at a temperature lower than the transition temperature. Likewise, $P_{\downarrow\downarrow}(m,t)/r_{\downarrow\downarrow}(0)$ behaves as the time-dependent probability distribution for m in a magnetic field $h = -g$.

Such dynamics have been considered long since, see e.g. [54, 55, 56, 57, 58, 206, 207, 208, 209]. The variables $\sigma_z^{(n)}$ are regarded as c-numbers, which can take the two values ± 1. Due to the presence of transverse spin components at the quantum level they may flip with a transition rate imposed by the bath. Since N is large, it seems natural to assume that the variance of m remains weak at all times, as $D(t)/N$. (In fact, this property fails in circumstances that we shall discuss in subsection 7.3.) The probability distribution P is then equivalent to a Gaussian,

$$P(m,t) = \sqrt{\frac{N}{2\pi D(t)}} \exp\left\{-\frac{N[m-\mu(t)]^2}{2D(t)}\right\}, \tag{7.10}$$

In the present classical approximation we neglect D, assuming that m is nearly equal to the expectation value $\mu(t)$. This quantity is expected to evolve according to an equation of the form

$$\frac{\mathrm{d}\mu(t)}{\mathrm{d}t} = v(\mu(t)). \tag{7.11}$$

In order to establish this law and to determine the form of the function v, most authors start from a balance equation governing the probability that each spin $\sigma_z^{(n)}$ takes the values $\sigma_i = \pm 1$ (with $i = \uparrow$ or $\downarrow$). The bath induces a transition probability $W_i(m)$ per unit time, which governs the possible flip of each spin from σ_i to $-\sigma_i$, in a configuration where the total spin is $\sum_i \sigma_i = Nm$. A detailed balance property must be satisfied, relating two inverse processes, that is, relating W_i and W_{-i}; it ensures that the Boltzmann-Gibbs distribution for the magnet at the temperature of the bath is stationary, to wit,

$$\frac{W_{-i}[m - (2/N)\sigma_i]}{W_i(m)} = \exp[-\beta \Delta E_i(m)], \tag{7.12}$$

where $\Delta E_i\,(m)$ is the energy brought in by one spin flip from σ_i to $-\sigma_i$. For large N, we have $\Delta E_i = 2\sigma_i(h + Jm^{q-1})$ (which reads for general couplings $\Delta E_i = 2\sigma_i(h + J_2 m + J_4 m^3)$), so that $W_i(m)$ depends on σ_i as

$$W_i(m) = \frac{1}{2\theta(m)}[1 + \tanh \beta \sigma_i (h + Jm^{q-1})], \tag{7.13}$$

including a transition time $\theta(m)$ which may depend on m and on the temperature $T = \beta^{-1}$ of B. (Indeed, $W_{-i}(m)$, obtained from W_i by changing σ_i into $\sigma_{-i} = -\sigma_i$, satisfies (7.12).)

$$v(m) = \frac{1}{\theta(m)}[\tanh \beta(h + Jm^{q-1}) - m]. \tag{7.14}$$

Various forms for $\theta(m)$ can be found in the works devoted to this subject; they are based either on phenomenology or on an approximate solution of models [206, 207, 208, 209]. In all cases the stable fixed points of the motion (7.11), at which $v(m)$ vanishes, are the values m_i given for large N by (3.55), where the free energy (3.54) is minimal. However, the time-dependence of $\mu(t) = \langle m \rangle$ as well as the behavior of higher order cumulants of m depend on the coefficient $\theta(m)$. For instance, while θ is a constant in [206], it is proportional to $\tanh \beta(h + Jm^{q-1})$ in [208] and [209]; it still has another form if $v(m)$ is taken to be proportional to $-\mathrm{d}F/\mathrm{d}m$.

In the present, fully quantum approach, which relies on the Hamiltonian introduced in subsection 3.2, the drift velocity $v(m)$ has been found to take the specific form (7.6) in the Markovian regime $t \gg \hbar/2\pi T$. We can then identify the coefficient $\theta(m)$ of (7.14) with

$$\theta(m) = \frac{\hbar \tanh \beta(h + Jm^{q-1})}{\gamma(h + Jm^{q-1})}. \tag{7.15}$$

With this form of $\theta(m)$, which arises from a quantum microscopic theory, the dynamical equation (7.11) keeps a satisfactory behavior when h or m becomes negative, contrary to the ad hoc choice $\theta(m) \propto \tanh\beta(h + Jm^{q-1})$. It provides, for $q = 2$, as shown in § 7.3.2, a long lifetime for the paramagnetic state, and better low temperature features than for $\theta(m)$ =constant.

Altogether, our final equations for the evolution of the diagonal blocks of $\hat{\mathcal{D}}$ are, at least in the Markovian regime, similar to equations readily found from a classical phenomenology. However, the quantum starting point and the rather realistic features of our model provide us unambiguously with the form (7.6) for the drift velocity, which meets several natural requirements in limiting cases. The occurrence of Planck's constant in (7.15) reveals the quantum origin of our classical-like equation. Moreover, quantum mechanics is also at the origin of the diffusion term and it provides the explicit form (7.7) for w. Finally, by varying the parameters of the model, we can discuss the validity of this equation and explore other regimes.

7.1.4. H-theorem and dissipation

In order to exhibit the dissipative nature of our quantum equations of motion for $P_{\uparrow\uparrow}$ and $P_{\downarrow\downarrow}$ in the Markovian regime, we establish here an associated H-theorem [202]. This theorem holds for any Markovian dynamics, with or without detailed balance. We start from the general, discrete equation (4.16), valid even for small N, where $\tilde{K}_t(\omega)$ is replaced by $\tilde{K}(\omega)$. We consider the normalized probability $P^{\rm dis}(m,t) = (2/N)P(m,t)$, which encompasses $P^{\rm dis}_{\uparrow\uparrow}(m,t)/r_{\uparrow\uparrow}(0)$ for $h = g > 0$ and $P^{\rm dis}_{\downarrow\downarrow}(m,t)/r_{\downarrow\downarrow}(0)$ for $h = -g < 0$, and denote as $E(m) = -hNm - JNq^{-1}m^q$ the Hamiltonian (4.6) with $h = \pm g$. We associate with $P^{\rm dis}(m,t)$ the time-dependent entropy

$$S(t) = -\sum_m P^{\rm dis}(m,t) \ln \frac{P^{\rm dis}(m,t)}{G(m)}, \tag{7.16}$$

where the denominator $G(m)$ accounts for the multiplicity (3.23) of m, and the average energy

$$U(t) = \sum_m P^{\rm dis}(m,t) E(m). \tag{7.17}$$

The time-dependence of the *dynamical free energy* $F_{\rm dyn}(t) = U(t) - TS(t)$ is found by inserting the equations of motion (4.16) for the set $P(m,t)$ into

$$\frac{{\rm d}F_{\rm dyn}}{{\rm d}t} = \sum_m \frac{{\rm d}P^{\rm dis}(m)}{{\rm d}t} \left[E(m) + T \ln \frac{P^{\rm dis}(m,t)}{G(m)} \right]. \tag{7.18}$$

The resulting expression is simplified through summation by parts, using

$$\sum_m \left[\Delta_+ f_1(m)\right] f_2(m) = \sum_m f_1(m)[\Delta_- f_2(m)] = -\sum_m f_1(m_+)[\Delta_+ f_2(m)], \tag{7.19}$$

with the notations (4.15). (No boundary term arises here.) This yields with $P(m) = (N/2)P^{\rm dis}(m)$

$$\frac{{\rm d}F_{\rm dyn}(t)}{{\rm d}t} = -\frac{2\gamma}{\beta\hbar^2} \sum_m \left[(1+m_+) e^{\beta\Delta_+ E(m)} P(m_+,t) - (1-m)P(m) \right] \tilde{K}[\hbar^{-1}\Delta_+ E(m)]\, \Delta_+ \left[\ln \frac{P(m,t)e^{\beta E(m)}}{G(m)} \right], \tag{7.20}$$

where we used $\tilde{K}(-\omega) = \tilde{K}(\omega) \exp\beta\hbar\omega$. Noting that $(1-m)G(m) = (1+m_+)G(m_+)$, we find

$$\frac{{\rm d}F_{\rm dyn}(t)}{{\rm d}t} = -\frac{\gamma}{2\beta\hbar} \sum_m (1-m)G(m) \frac{\Delta_+ E(m)}{\Delta_+ \exp\beta E(m)} e^{-|\Delta_+ E(m)|/\hbar\Gamma} \Delta_+ \left[\frac{P(m,t)e^{\beta E(m)}}{G(m)} \right] \Delta_+ \left[\ln \frac{P(m,t)e^{\beta E(m)}}{G(m)} \right]. \tag{7.21}$$

The last two factors in (7.21) have the same sign, while the previous ones are positive, so that each term in the sum is negative. Thus the dynamical free energy is a decreasing function of time. The quantity $-\beta{\rm d}F_{\rm dyn}/{\rm d}t$ can be interpreted as the *dissipation rate* (or the entropy production) of the compound system M+B, that is, the increase per unit time of the entropy (7.16) of the magnet plus the increase $-\beta U/{\rm d}t$ of the entropy of the bath. In fact the entropy

of M is lower in the final state than in the initial state, but the increase of entropy of B associated with the energy dumping dominates the balance. The negativity of (7.21) characterizes the irreversibility of the registration.

The right-hand side of (7.21) vanishes only if all its terms vanish, that is, if $P(m,t)\exp[\beta E(m)]/G(m)$ does not depend on m. This takes place for large times, when the *dynamical* free energy $F(t)$ has decreased down to the minimum allowed by the definitions (7.16), (7.17). We then reach the limit $P(m) \propto G(m)\exp[-\beta E(m)]$, which is the distribution associated with the canonical equilibrium of M for the Hamiltonian $E(\hat{m})$, that is, with the *static free energy*[43]. We have thus proven for our model the following property, often encountered in statistical physics [145, 202]. The same probability distribution for m arises in two different circumstances. (*i*) In equilibrium statistical mechanics, (§ 3.3.4), $P^{\rm dis}(m)$ follows from the Boltzmann-Gibbs distribution $\hat{R}_{\rm M} \propto \exp[-\beta \hat{H}_{\rm M}]$ for the magnet alone. (*ii*) In non-equilibrium statistical mechanics, it comes out as the asymptotic distribution reached in the long time limit when M is weakly coupled to the bath.

It is only in the Markovian regime that the dynamical free energy is ensured to decrease. Consider in particular, for the quadratic coupling $q = 2$, the evolution of $P^{\rm dis}(m,t)$ on very short times, which involves the narrowing (7.5) of the initial peak. The free energy associated with a Gaussian distribution centered at $m = 0$, with a time-dependent variance $D(t)/N$, is

$$F_{\rm dyn}(t) = \sum_m P^{\rm dis}(m,t)\left[-gNm - \frac{1}{2}JNm^2 + T\ln\frac{P^{\rm dis}(m,t)}{G(m)}\right] = -\frac{1}{2}(JD + T - TD + T\ln D). \tag{7.22}$$

The time-dependence of D is expressed for short times $t \ll \Gamma^{-1}$ by (7.5). The initial value δ_0^2 of $D(t)$ being given by (3.51), we find

$$\frac{{\rm d}F_{\rm dyn}}{{\rm d}t} = \frac{\gamma\Gamma^2 t}{\pi}\,\frac{J^2(T_0 - T)}{T_0(T_0 - J)}. \tag{7.23}$$

Thus at the very beginning of the evolution, $F_{\rm dyn}$ slightly increases, whereas for $t \gg \hbar/2\pi T$ it steadily decreases according to (7.21). In fact, the negative sign of v in the initial non-Markovian regime (7.4) indicates that, for very short times, the fixed point near $m = 0$ is stable although the bath temperature is lower than J.

7.1.5. Approach to quasi-equilibrium

Անձրևոտ օրը շատերը կասեն. "Ջուր տար, քո հավերին լողացրու":[44]
Armenian proverb

The above proof that the system eventually reaches the canonical equilibrium state $\hat{R}_{\rm M} \propto \exp(-\beta \hat{H}_{\rm M})$ is mathematically correct for finite N and $t \to \infty$. However, this result is not completely relevant physically if N is large, when the temperature of the bath is lower than the transition temperature of M. Indeed, the times that we consider should be attainable in practice, and "large times" does not mean "infinite times" in the mathematical sense [49, 50].

In order to analyze this situation, we note that the summand of (7.21) contains a factor $P^{\rm dis}(m,t)$; thus the ranges of m over which $P^{\rm dis}(m,t)$ is not sizeable should be disregarded. When the time has become sufficiently large so that the rate of decrease of $F(t)$ has slowed down, a regime is reached where $P^{\rm dis}(m,t)\exp[\beta E(m)]/G(m)$ is nearly time-independent and nearly constant (as function of m) in any interval where $P^{\rm dis}(m,t)$ is not small. Within a multiplicative factor, $P^{\rm dis}(m,t)$ is then locally close to $\exp[-\beta F(m)]$ where $F(m) = U(m) - T\ln G(m)$ is given by (3.54). It is thus concentrated in peaks, narrow as $1/\sqrt{N}$ and located in the vicinity of points m_i where $F(m)$ has a local minimum. Above the transition temperature, or when the field $h = \pm g$ is sufficiently large, there is only one such peak, and the asymptotic form of $P^{\rm dis}(m,t)$ is unique. However, below the critical temperature, two separate peaks may occur for $q = 2$, and two or three peaks for $q = 4$, depending on the size of h.

[43] The notions of dynamical (moderate time) and static (infinite time) free energy are well known in the theory of glasses and spin glasses, see e.g. [210, 211, 212]. In corresponding mean field models, they differ strongly; here, however, the dynamical free energy simply refers to processes close to equilibrium and decreases down to the static equilibrium free energy in agreement with the macroscopic Clausius–Duhem inequality [50, 65]

[44] On a rainy day many people offer to bathe your chickens

In such a case, $P^{\rm dis}(m,t)$ can be split into a sum of non-overlapping contributions $P^{\rm dis}_{{\rm M}i}(m,t)$, located respectively near m_i and expected to evolve towards the equilibrium distributions $P^{\rm dis}_{{\rm M}i}(m)$ expressed by (3.56). Since $P^{\rm dis}(m,t)$ decays rapidly as a Gaussian (3.56) astray from its maxima m_i, its equation of motion (7.1) does not allow for transfers from one peak to another over any reasonable delay. (Delays exponentially large with N are physically inaccessible.) Once such a regime has been attained, each term $P^{\rm dis}_{{\rm M}i}(m,t)$ *evolves independently* according to (7.1). Its normalization remains constant, and its shape tends asymptotically to (3.56). Hence, below the transition temperature, *ergodicity is broken* in the physical sense. (A breaking of ergodicity may occur in a mathematically rigorous sense only for infinite N or zero noise.) If the system starts from a configuration close to some m_i, it explores, during a physically large time, only the configurations for which m lies around m_i. Configurations with the same energy but with values of m around other minima of $F(m)$ remain out of reach. This phenomenon is essential if we want to use M as the pointer of a measurement apparatus. If the spin S lies upwards, its interaction with A should lead to values of m that fluctuate weakly around $+m_{\rm F}$, not around $-m_{\rm F}$. Ergodicity would imply that A spends the same average time in all configurations having the same energy, whatever the sign of m [49, 50], once the interaction $\hat{H}_{\rm SA}$ is turned off. The breaking of invariance is thus implemented through the dynamics: unphysical times, exponentially large with N, would be needed to reach the symmetric state $\exp(-\beta\hat{H}_{\rm M})$.

In analogy with what happens in glasses and spin glasses [210, 211, 212], for physical large times t, the asymptotic value of $F_{\rm dyn}(t)$ is not necessarily the absolute minimum of $F(m)$. It is a weighted average of the free energies of the stable and metastable states, with magnetizations m_i. The weights, that is, the normalizations of the contributions $P^{\rm dis}_{{\rm M}i}(m,t)$ to $P^{\rm dis}(m,t)$ are determined by the initial distribution $P^{\rm dis}(m,0)$, and they depend on N and on the couplings g and J which enter the equations of motion. For an ideal measurement, we require the process to end up at a single peak, $+m_{\rm F}$ for $P^{\rm dis}_{\uparrow\uparrow}$, $-m_{\rm F}$ for $P^{\rm dis}_{\downarrow\downarrow}$ (subsection 7.2). Otherwise, if M may reach either one of the ferromagnetic states $\pm m_{\rm F}$, the measurement is not faithful; we will determine in § 7.3.3 its probability of failure.

In the present regime where the variations with m of $Pe^{\beta E}/G$ are slow, we can safely write the continuous limit of the H-theorem (7.21) by expressing the discrete variations Δ_+ over the interval $\delta m = 2/N$ as derivatives. We then find the dissipation rate as

$$\begin{aligned} -\frac{1}{T}\,\frac{{\rm d}F_{\rm dyn}}{{\rm d}t} &= \frac{\gamma NT}{\hbar}\int {\rm d}m\, P(m,t)\phi(m)[\coth\phi(m)-m] \\ &\times \left[\frac{1}{NP}\frac{\partial P}{\partial m}-\frac{\tanh\phi(m)-m}{1-m\tanh\phi(m)}\right]\left[\frac{1}{NP}\frac{\partial P}{\partial m}-\phi(m)+\frac{1}{2}\ln\frac{1+m}{1-m}\right], \end{aligned} \tag{7.24}$$

where we use the notation

$$\phi(m) = \beta(h + Jm^{q-1}), \qquad h = \pm g. \tag{7.25}$$

For large N, the term $(1/NP){\rm d}P/{\rm d}m$ is not negligible in case $\ln P$ is proportional to N, that is, in the vicinity of a narrow peak with width $1/\sqrt{N}$. The expression (7.25) is not obviously positive. However, once $P(m,t) = \sum_{i=\pm1} P_{{\rm M}i}(m,t)$ has evolved into a sum of separate terms represented by peaks around the values m_i, we can write the dissipation as a sum of contributions, each of which we expand around m_i. The last two brackets of (7.24) differ only at order $(m-m_i)^3$, and we get

$$-\frac{1}{T}\,\frac{{\rm d}F_{\rm dyn}}{{\rm d}t} = \frac{\gamma NT}{\hbar}\sum_{i=\pm1}\int {\rm d}m\, P_{{\rm M}i}(m,t)\phi(m)[\coth\phi(m)-m] \tag{7.26}$$

$$\times \left\{\frac{1}{NP_{{\rm M}i}}\frac{\partial P_{{\rm M}i}}{\partial m}+\left[\frac{1}{1-m_i^2}-(q-1)\beta Jm_i^{q-2}\right](m-m_i)+\left[\frac{m_i}{(1-m_i^2)^2}-\frac{(q-1)(q-2)}{2}\beta Jm_i^{q-3}\right](m-m_i)^2\right\}^2 .$$

We thus check that $F_{\rm dyn}(t)$ decreases, down to the weighted sum of free energies associated with the stable or metastable equilibrium distributions (3.56). In fact, among the stationary solutions of (7.1), those which satisfy

$$vP - \frac{1}{N}\frac{{\rm d}(wP)}{{\rm d}m} = 0, \tag{7.27}$$

with v and w given by (7.6) and (7.7), coincide with (3.56) around the values of m_i given by (3.55), not only in the mean-field approximation but also including the corrections that we retained in those formulae.

7.2. Registration times

What then is time? If no one asks me, I know what it is.
If I wish to explain it to him who asks, I do not know
Saint Augustine

7.2.1. Motion of a single narrow peak

In the present subsection, we study the evolution of the distribution $P_{\uparrow\uparrow}(m,t)$, which is such that $(1/N)\ln P_{\uparrow\uparrow}$ is finite for large N. This property holds at $t = 0$ and hence at all times. As a consequence, $P_{\uparrow\uparrow}$ presents a narrow peak with width of order $1/\sqrt{N}$, and it is equivalent to a Gaussian. We first note that the evolution (7.1) conserves its normalization $r_{\uparrow\uparrow}(0)$. The ratio (7.9) can then be parametrized as in (7.10) by the position $\mu(t)$ of the peak and by its width parameter $D(t)$, which are both finite for large N.

The equations of motion for $\mu(t)$ and $D(t)$ are derived by taking the first moments of the equation (7.1) for $P_{\uparrow\uparrow}(m,t)$. Integration over m of (7.1) first entails the conservation in time of the normalization $r_{\uparrow\uparrow}(0)$ of $\int \mathrm{d}m P_{\uparrow\uparrow}(m,t)$. We then integrate (7.1) over m after multiplication, first by $m-\mu(t)$, second by $N[m-\mu(t)]^2 - D(t)$, using on the right-hand side an integration by parts and the steepest descents method. To wit, expanding $v(m,t)$ and $w(m,t)$ in powers of $m-\mu(t)$, we rely on the vanishing of the integrals of $m-\mu(t)$ and of $N[m-\mu(t)]^2 - D(t)$ when weighted by $P_{\uparrow\downarrow}(m,t)$, and we neglect for $k>1$ the integrals of $[m-\mu(t)]^{2k}$, which are small as N^{-k}. This yields for sufficiently large N

$$\frac{\mathrm{d}\mu(t)}{\mathrm{d}t} = v[\mu(t),t], \tag{7.28}$$

$$\frac{1}{2}\frac{\mathrm{d}D(t)}{\mathrm{d}t} = \frac{\partial v[\mu(t),t]}{\partial \mu} D(t) + w[\mu(t),t]. \tag{7.29}$$

At the very beginning of the evolution, when t is not yet large compared to $\hbar/2\pi T$, Eq. (7.28) should be solved self-consistently, using the expressions (4.31) for v and (4.32) for w. However, if the coupling γ is weak, the Markovian regime is reached before the shape of $P_{\uparrow\uparrow}$ is significantly changed. We can thus solve (7.28) and (7.29) with the time-independent forms (7.6) and (7.7) for v and w, the initial conditions being $\mu(0) = 0$, $D(0) = \delta_0^2$.

The solution of (7.28) is then, for $t \gg \hbar/2\pi T$,

$$t = \int_0^\mu \frac{\mathrm{d}\mu'}{v(\mu')} = \frac{\hbar}{\gamma T}\int_0^\mu \frac{\mathrm{d}\mu'}{\phi(\mu')[1-\mu'\coth\phi(\mu')]}, \tag{7.30}$$

where the function ϕ is defined by (7.25) with $h = +g$. Inversion of (7.30) provides the motion $\mu(t)$ of the peak of $P_{\uparrow\uparrow}(m,t)$. For $P_{\downarrow\downarrow}$, we have to change g into $-g$ in (7.25), and $\mu\,(t)$ expressed by (7.30) is then negative.

If N is very large, the probabilistic nature of the registration process fades out and the magnetization is located at $\mu(t)$ with near certainty. The evaluation of the time dependence of $\mu(t)$ may be proposed to students as an exercise (§ 9.6.2). Results for quadratic coupling ($q = 2$) and for quartic coupling ($q = 4$), which exemplify second and first-order transitions, respectively, are illustrated by Fig. 7.3 and by Fig. 7.4, respectively. The evolution from the initial paramagnetic state to the final ferromagnetic state exhibits several stages, which will be studied in § 7.2.3 for $q = 2$ and in § 7.2.4 for $q = 4$.

The width of the peak is obtained by regarding D as a function of $\mu(t)$ and by solving the equation for $\mathrm{d}D/\mathrm{d}\mu$ that results from (7.28) and (7.29). This yields

$$D(\mu) = v^2(\mu)\left[\frac{\delta_0^2}{v^2(0)} + \int_0^\mu \frac{\mathrm{d}\mu'\,2w(\mu')}{v^3(\mu')}\right] = \phi^2(\mu)[1-\mu\,\coth\phi(\mu)]^2\left\{\frac{\delta_0^2}{\beta^2 g^2} + \int_0^\mu \frac{\mathrm{d}\mu'\,2[\coth\phi(\mu')-\mu']}{\phi^2(\mu')[1-\mu'\coth\phi(\mu')]^3}\right\}. \tag{7.31}$$

To analyze this evolution of $D(t)$, we first drop the term in w from the equation of motion (7.1) of $P_{\mathrm{M}}(m,t)$. This simplified equation describes a deterministic flow in the space of m, with a local drift velocity $v(m)$. For any initial condition, its solution is the mapping

$$P_{\mathrm{M}}(m,t) = \frac{1}{v(m)}\int \mathrm{d}m' P_{\mathrm{M}}(m',0)\delta\left(t - \int_{m'}^{m}\frac{\mathrm{d}m''}{v(m'')}\right), \tag{7.32}$$

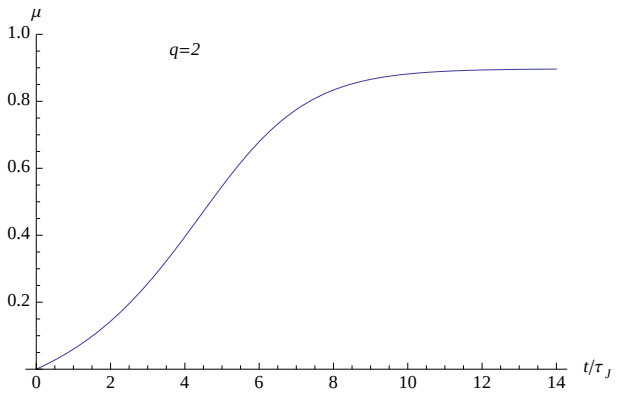

Figure 7.3: The average magnetization $\mu(t)$ for a quadratic interaction ($q = 2$). The time dependence, given by (7.30), results from the local velocity of Fig. 7.1. The parameters are $T = 0.65J$ and $g = 0.05J$, while the time scale is $\tau_J = \hbar/\gamma J$. One can distinguish the three stages of §7.2.3, characterized by the first registration time $\tau_{\rm reg} = [J/(J-T)]\,\tau_J = 2.86\,\tau_J$ (eq. (7.44)) and the second registration time $\tau'_{\rm reg} = 8.4\tau_J$ (eq. (7.48)): (i) Increase, first linearly as $(g/J)(t/\tau_J) = 0.05t/\tau_J$, then exponentially according to (7.42), with a coefficient $m_{\rm B} = g/(J-T) = 0.143$ and a time scale $\tau_{\rm reg}$. After a delay of a few $\tau_{\rm reg}$, the coupling may be switched off without spoiling the registration. (ii) Rise, according to (7.47), up to $m_{\rm F} - \frac{1}{2}m_{\rm B} = 0.80$ reached at the second registration time $\tau'_{\rm reg}$. (iii) Exponential relaxation towards $m_{\Uparrow} = 0.90$ (or $m_{\rm F} = 0.87$ if g is switched off) according to (7.49) with the time scale $1.6\tau_J$.

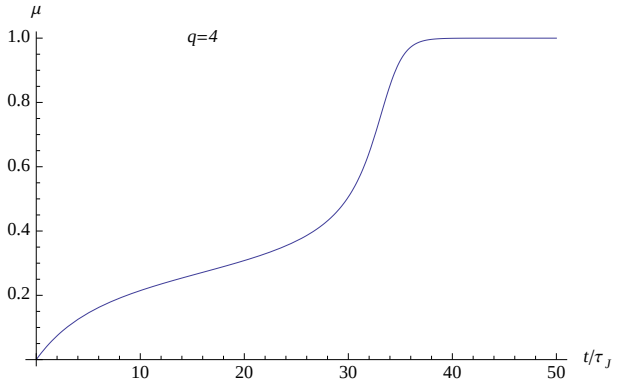

Figure 7.4: The average magnetization $\mu(t)$ for a quartic interaction ($q = 4$). The time dependence, given by (7.30), results from the local velocity of Fig. 7.2. The parameters are $T = 0.2J$ and $g = 0.045J$ while the time scale is $\tau_J = \hbar/\gamma J$. The characteristic registration time $\tau_{\rm reg} = 38\tau_J$ is now given by (7.52). (Note that it is much larger than for a quadratic interaction.) The initial increase of $\mu(t)$ takes place, first linearly as $(g/J)\,t/\tau_J = 0.045t/\tau_J$, then slows down according to (7.51), with a coefficient $g/J = 0.045$ and a time scale $\tau_1 = (g/J)\,\tau_J$. The region of $m_{\rm c} = 0.268$, where the drift velocity is small, is a bottleneck: around this point, reached at the time $t = \frac{1}{2}\tau_{\rm reg}$, the average magnetization $\mu(t)$ lingers according to (7.53) where $\delta m_{\rm c} = 0.11$. It then increases rapidly so as to reach at the time $\tau_{\rm reg}$ a value close to $m_{\rm F} \simeq 1$, and finally reaches $m_{\rm F}$ exponentially on the time scale τ_J.

where m' is the initial point of the trajectory that reaches m at the time t. For a distribution (7.10) peaked at all times, we recover from (7.32) the motion (7.30) of the maximum of $P_{\uparrow\uparrow}(m,t)$ and the first term of the variance (7.31). If only the drift term were present, the width of the peak would vary as $1/v(\mu)$: Indeed, in a range of m where the drift velocity increases with m, the front of the peak progresses more rapidly than its tail so that the width increases, and conversely.

The second term of (7.31) arises from the term in w. Since $w(m)$ is positive, it describes a diffusion which widens the distribution. This effect of w is enhanced when v is small. In particular, by the end of the evolution when $\mu(t)$ tends to a zero m_i of $v(m)$ with $\partial v/\partial m < 0$, the competition between the narrowing through v and the widening through w leads to the equilibrium variance $D = -(\mathrm{d}v/\mathrm{d}\mu)^{-1}w$, irrespective of the initial width. This value is given by $D^{-1} = (1-m_i^2)^{-1} - (q-1)\beta J m_i^{q-2}$, in agreement with (3.56) and with (7.27).

We have noted that the drift velocity $v(m)$ has at each point the same sign as $-\mathrm{d}F/\mathrm{d}m$, where F is the free energy (3.54), and that the zeroes m_i of $v(m)$, which are the fixed points of the drift motion, coincide with the extrema of F. At such an extremum, given by (3.55), we have

$$-\frac{\mathrm{d}v}{\mathrm{d}m} = \frac{\gamma}{N\hbar}\frac{2\phi(m_i)}{\sinh 2\phi(m_i)}\frac{\mathrm{d}^2F}{\mathrm{d}m^2}. \tag{7.33}$$

The minima of F correspond to *attractive* fixed points, with negative slope of $v(m)$, its maxima to repulsive points, that is, *bifurcations*. In the present case of a narrow distribution, $\mu(t)$ thus increases from $\mu(0) = 0$ to the smallest positive minimum m_i of $F(m)$, which is reached asymptotically for large times. However, the present hypothesis of a single narrow peak is valid only if $P(m,t)$ lies entirely and at all times in a region of m free of bifurcations. We will discuss in subsection 7.3 the situation where P lies astride a bifurcation, either at the initial time or a little later on, if a tail due to diffusion crosses the bifurcation.

7.2.2. Threshold for the system-apparatus coupling; possibilities of failure

If you are not big enough to lose, you are not big enough to win
Walter Reuther

The measurement is successful only if $P(m,t) \equiv P_{\uparrow\uparrow}(m,t)/r_{\uparrow\uparrow}(0)$, which is interpreted as the conditional probability distribution for m if $s_z = +1$, approaches for large times the narrow normalized peak (3.56) located at the positive ferromagnetic solution $m_{\Uparrow}$ of (3.55), close to m_{F} for $g \ll T$ [45]. This goal can be achieved only if *(i)* the center $\mu(t)$ of the peak approaches $m_{\Uparrow}$; *(ii)* its width remains small at all times so that the above derivation is valid.

(*i*) The first condition is relevant only for a first-order transition ($q = 4$), since $m_{\Uparrow}$ is the only attractive fixed point in the region $m > 0$ for a second-order transition ($q = 2$). For quartic interactions, the first minimum of $F(m)$ that occurs for increasing m is not necessarily $m_{\Uparrow}$ (Fig. 7.2). Indeed, we have seen (end of § 3.3.4 and Fig. 3.4) that for a field lower than

$$h_{\mathrm{c}} = T\operatorname{arctanh} m_{\mathrm{c}} - Jm_{\mathrm{c}}^3 \approx \frac{2}{3}Tm_{\mathrm{c}}, \qquad m_{\mathrm{c}}^2 = \frac{1}{2} - \frac{1}{2}\sqrt{1-\frac{4T}{3J}} \approx \frac{T}{3J} + \frac{T^2}{9J^2}, \tag{7.34}$$

$F(m)$ has not only a ferromagnetic minimum at $m_{\Uparrow}$, but also a local paramagnetic minimum m_{P} at a smaller value of m. Hence, if the spin-apparatus coupling g is smaller than h_{c}, $\mu(t)$ reaches for large times the locally stable point m_{P} in the sector $\uparrow\uparrow$. It reaches $-m_{\mathrm{P}}$ in the sector $\downarrow\downarrow$, so that the apparatus seems to distinguish the values $s_z = \pm 1$ of S. However, it the coupling is switched off at the end of the process, the magnetization m of M returns to 0 in both cases. The result of the measurement thus cannot be registered robustly for $g < h_{\mathrm{c}}$.

The center $\mu(t)$ of the peak may escape the region of the origin only if $g > h_{\mathrm{c}}$ (Fig. 7.2). Relying on the smallness of $T/3J$ (equal to 0.121 at the transition temperature), we can simplify the expression of h_{c} as in (7.34), so that this threshold for g is ($q = 4$):

$$g > h_{\mathrm{c}} \simeq \frac{2T}{3}\sqrt{\frac{T}{3J}}. \tag{7.35}$$

[45] We recall that m_{F} and $-m_{\mathrm{F}}$ were defined as the limits for a vanishing coupling ($g \to 0$) of $m_{\Uparrow}$ and $m_{\Downarrow}$, respectively

Under this condition, the peak $\mu(t)$ of $P_{\uparrow\uparrow}(m,t)$ reaches for large times $m_{\Uparrow}$, close to the magnetization $m_{\rm F}$ of the ferromagnetic state. If the coupling g is removed sufficiently after $\mu(t)$ has passed the maximum of $F(m)$, the peak is expected to end up at $m_{\rm F}$. Likewise, the peak of $P_{\downarrow\downarrow}(m,t)$ reaches $-m_{\rm F}$ at the end of the same process. The apparatus is non-ergodic and the memory of its triggering by S may be kept forever under the necessary (but not sufficient) condition (7.35).

(*ii*) The second requirement involves the width of the distribution $P_{\uparrow\uparrow}(m,t)$ and the location $-m_{\rm B} < 0$ of the repulsive fixed point, at which $F(m)$ is maximum. Consider first the pure drift flow (7.32) without diffusion, for which $-m_{\rm B}$ is a bifurcation. The part $m > -m_{\rm B}$ of $P_{\downarrow\downarrow}(m,0)$ is properly shifted upwards so as to reach eventually the vicinity of the positive ferromagnetic value $+m_{\rm F}$; however its tail $m < -m_{\rm B}$ is pushed towards the negative magnetization $-m_{\rm F}$. If the relative weight of this tail is not negligible, *false measurements*, for which the value $-m_{\rm F}$ is registered by A although s_z equals +1, can occur with a sizeable probability. Such a failure is excluded for $q = 4$, because $m_{\rm B}$ is then much larger than the width $1/\sqrt{N}$ of $P_{\uparrow\uparrow}(m,0)$; for instance, in the case $q = 4$ we have $m_{\rm B} = 0.544$ for the parameters $T = 0.2J$ and $g = 0.045J$ (which satisfy (7.35)). However, in the case $q = 2$ and $g \ll J - T$, the point $-m_{\rm B}$ with

$$m_{\rm B} \simeq \frac{g}{J-T}, \tag{7.36}$$

lies close to the origin (Fig. 7.1), and a risk exists that the initial Gaussian distribution in $\exp(-Nm^2/2\delta_0^2)$ extends below $-m_{\rm B}$ if g is too small. The probability of getting a wrong result is significant if the condition $\delta_0 \ll m_{\rm B}\sqrt{N}$ is not fulfilled. We return to this point in § 7.3.3.

Moreover, in this case $q = 2$, the lower bound thus guessed for the coupling,

$$g = (J-T)m_{\rm B} \gg \frac{(J-T)\delta_0}{\sqrt{N}}, \tag{7.37}$$

is not sufficient to ensure a faithful registration. The diffusive process, which tends to increase $D(t)$ and thus to thicken the dangerous tail $m < -m_{\rm B}$ of the probability distribution $P_{\uparrow\uparrow}(m,t)$, raises the probability of a false registration towards $-m_{\rm F}$ instead of $+m_{\rm F}$. In order to trust the Ansatz (7.10) and the ensuing solution (7.30), (7.31) for $P_{\uparrow\uparrow}(m,t)$, we need $D(t)$ to remain at all times sufficiently small so that $P_{\uparrow\uparrow}(m,t)$ is negligible for $m < -m_{\rm B}$. This is expressed, when taking $\mu(t)$ as a variable instead of t, as

$$\frac{D(\mu)}{N(m_{\rm B}+\mu)^2} \ll 1 \tag{7.38}$$

for any μ between 0 and $m_{\rm F}$: The width $\sqrt{D/N}$ of the peak of $P_{\uparrow\uparrow}(m,t)$ should not increase much faster than its position μ. For sufficiently small g, we have $m_{\rm B} \ll m_{\rm F}$, and we only need to impose (7.38) for times such that $\mu(t)$ lies in an interval $0 < \mu(t) < \mu_{\max}$ such that $m_{\rm B} \ll \mu_{\max} \ll T/J$. In this range we can evaluate $D(\mu)$ from (7.31) by simplifying $\tanh\phi(\mu)$ into $\phi(\mu)$, which yields

$$\frac{D(\mu)}{(m_{\rm B}+\mu)^2} = \frac{\delta_0^2}{m_{\rm B}^2} + \frac{T}{J-T}\left[\frac{1}{m_{\rm B}^2} - \frac{1}{(m_{\rm B}+\mu)^2}\right]. \tag{7.39}$$

This ratio increases in time from $\delta_0^2/m_{\rm B}^2$ to $\delta_1^2/m_{\rm B}^2$, where

$$\delta_1^2 = \delta_0^2 + \frac{T}{J-T} = \frac{T_0}{T_0-J} + \frac{T}{J-T}, \tag{7.40}$$

so that the left-hand side of (7.38) remains at all times smaller than $\delta_1^2/Nm_{\rm B}^2$. The lower bound on g required to exclude false registrations is therefore ($q = 2$)

$$g \gg \frac{(J-T)\delta_1}{\sqrt{N}}, \tag{7.41}$$

a condition more stringent than (7.37) if $J - T \ll J$. Altogether, for $q = 2$ the system-apparatus coupling may for large N be small, for instance as $N^{1/3}$, provided it satisfies (7.41).

For $q = 4$, and more generally for a first-order transition ($3J_4 > J_2$), the lower bound found as (7.35) remains finite for large N: A free energy barrier of order N has to be overpassed. Moreover, the diffusion hinders the trend

of m to increase and may push part of the distribution $P_{\uparrow\uparrow}(m,t)$ leftwards, especially its left tail, while its peak moves rightwards. The widening of $P_{\uparrow\uparrow}(m,t)$ when the barrier is being reached should not be too large, and this effect raises further the threshold for g. We shall show that the condition (7.35) should thus be strengthened into (7.57).

Another difference between first- and second order transitions lies in the possible values of the temperature. For $q = 2$, if T lies near the critical temperature J, the minima m_i of $F(m)$ are very sensitive to g and the ferromagnetic value $m_{\rm F}$ in the absence of a field is small as $\sqrt{3(J-T)/J}$. Using M as the pointer of a measurement apparatus requires the temperature to lie sufficiently below J. For $q = 4$, registration is still possible if T lies near the transition temperature, and even above, although in this case the ferromagnetic states are not the most stable ones for $h = 0$. However, the coupling g should then be sufficiently strong.

7.2.3. The registration process for a second-order transition

Assuming g to satisfy (7.41) and $m_{\rm F}$ to be significantly large, we resume the dynamics of $P_{\uparrow\uparrow}(m,t)$ for $q = 2$ so as to exhibit its characteristic times. After a short delay of order $\hbar/T$, most of the process takes place in the Markovian regime, and the Gaussian Ansatz (7.10) is justified. We can distinguish three stages in the evolution of $P_{\uparrow\uparrow}(m,t)$, which are exhibited on the example of Figs. 7.3 and 7.5.

(*i*) During the first stage, as long as $\mu(t) \ll m_{\rm F}$, we can replace $\phi(m)\coth\phi(m)$ by 1 in v and w, so that the drift velocity v behaves (Fig. 7.1) as

$$v(m) \approx \frac{\gamma T}{\hbar}\left[\frac{g+Jm}{T} - m\right] = \frac{\gamma(J-T)(m_{\rm B}+m)}{\hbar}, \tag{7.42}$$

and the diffusion coefficient as $w \approx \gamma T/\hbar$. Integration of (7.30) then yields the motion

$$\mu(t) \sim m_{\rm B}(e^{t/\tau_{\rm reg}} - 1) = \frac{g}{J-T}(e^{t/\tau_{\rm reg}} - 1) \tag{7.43}$$

for the center of the peak, with the characteristic time

$$\tau_{\rm reg} = \frac{\hbar}{\gamma(J-T)}. \tag{7.44}$$

After beginning to move as $\mu \sim \gamma g t/\hbar$, the distribution shifts away from the origin faster and faster. Once μ has reached values of the order of several times $m_{\rm B}$, $(J-T)\,\mu$ becomes larger than g, so that $v(\mu)$ does not depend much on g. It little matters for the subsequent evolution whether the coupling g is present or not. Thus, after $t/\tau_{\rm reg}$ reaches 2 or 3, *the spin-apparatus coupling may be switched off* and the increase of μ goes on nearly unchanged. In fact, the distribution moves towards $m_{\rm F}$ rather than $m_{\Uparrow}$, but $m_{\rm F} - m_{\Uparrow}$ is small, less than g/J. We shall call $\tau_{\rm reg}$ the *first registration time*. After it, M will necessarily reach the ferromagnetic state $+m_{\rm F}$, independent of S, although the evolution is not achieved yet.

We have seen that during this first stage the width (7.39) is governed both by the drift which yields the factor $(m_{\rm B}+\mu)^2$, increasing as $e^{2t/\tau_{\rm reg}}$, and by the diffusion which raises δ_0 up to δ_1.

(*ii*) During the second stage $\mu(t)$ *rises rapidly* from $m_{\rm B}$ to $m_{\rm F}$, since the drift velocity $v(\mu)$ is no longer small. The distribution has become wide, and its width is now governed mainly by the drift term. Matching $D(\mu)$ with (7.39) for μ larger than $m_{\rm B}$ yields the width

$$\sqrt{\frac{D(\mu)}{N}} \sim \frac{\tau_{\rm reg}\delta_1}{m_{\rm B}\sqrt{N}}v(\mu) = \frac{\hbar\delta_1}{\gamma g\sqrt{N}}v(\mu), \tag{7.45}$$

which varies proportionally to $v\,(\mu)$.The drift velocity $v(m)$ first increases and then decreases as function of m (Fig. 7.1), down to 0 for $m = m_{\Uparrow} \simeq m_{\rm F}$. Accordingly, the width $D(t)$ increases as function of time, then decreases (Fig. 7.5). The time dependence (7.30) of $\mu(t)$ and hence of $D(t)$ is evaluated explicitly in the Appendix E.1, where μ is related to t through Eq. (E.2), that is,

$$\frac{t}{\tau_{\rm reg}} = \ln\frac{m_{\rm B}+\mu}{m_{\rm B}} + a\ln\frac{m_{\rm F}^2}{m_{\rm F}^2-\mu^2}, \tag{7.46}$$

where the coefficient a, given by

$$a = \frac{T(J-T)}{J[T - J(1-m_{\rm F}^2)]}, \tag{7.47}$$

lies between $\frac{1}{2}$ and 1.

We define the *second registration time* $\tau'_{\rm reg}$ as the delay taken by the average magnetization $\mu(t)$ to go from the paramagnetic value $\mu = 0$ to the value $m_{\rm F} - \frac{1}{2}m_{\rm B}$ close to $m_{\rm F}$. From the equation (7.46) that relates μ to t, we find this second registration time, the duration of the second stage, much longer than the first, as

$$\tau'_{\rm reg} = \tau_{\rm reg}(1+a)\ln\left(\frac{m_{\rm F}}{m_{\rm B}} + \frac{1}{2}\right), \tag{7.48}$$

(*iii*) The third stage of the registration, the *establishment of thermal equilibrium*, has been studied in § 7.1.4 and § 7.1.5. While $\mu(t)$ tends exponentially to $m_{\Uparrow}$ (or to $m_{\rm F}$ if the coupling g has been switched off), we saw that the equilibrium width of $P_{\uparrow\uparrow}(m,t)$ is reached as a result of competition between the drift, which according to (7.45) narrows the distribution, and the diffusion which becomes again relevant and tends to widen it. It is shown in the Appendix E.1 that the final relaxation takes place, for times $t - \tau'_{\rm reg} \sim \tau_{\rm reg}$, according to

$$\mu(t) = m_{\rm F}\left[1 - \frac{1}{2}\left(\frac{m_{\rm F}}{m_{\rm B}}\right)^{1/a} \exp\left(-\frac{t}{a\tau_{\rm reg}}\right)\right]. \tag{7.49}$$

At low temperatures, $T \ll J$, we have $m_{\rm F} \sim 1$, $m_{\rm B} \sim g/J$, $a \sim 1$. If T lies close to the transition temperature, $J - T \ll J$, we have $m_{\rm F}^2 \sim 3(J-T)/J$, $m_{\rm B} = g/(J-T)$ and $a \sim \frac{1}{2}$.

The above scenario for the registration process is illustrated by Fig. 7.5 which represents a numerical solution of the equation for $P(m,t) = P_{\uparrow\uparrow}(m,t)/r_{\uparrow\uparrow}(0)$. The curves exhibit the motion from 0 to $m_{\rm F}$ of the center $\mu(t)$ of the peak (also shown by Fig. 7.3), its large initial widening, the intermediate regime where the width $\sqrt{D(t)/N}$ is proportional to $\mu(t)$, and the final adjustment of μ and D to their equilibrium values in the ferromagnetic state. Except near the initial and final state, the width is not small although we have taken a fairly large value $N = 1000$, but one can see that the Gaussian approximation used for $P_{\uparrow\uparrow}(m,t)$ is sufficient and that the resulting formulae given above for $\mu(t)$ and $D(t)$ fit the curves. While a mean-field theory neglecting the fluctuations is satisfactory at equilibrium, the dynamics entails large fluctuations of m at intermediate times.

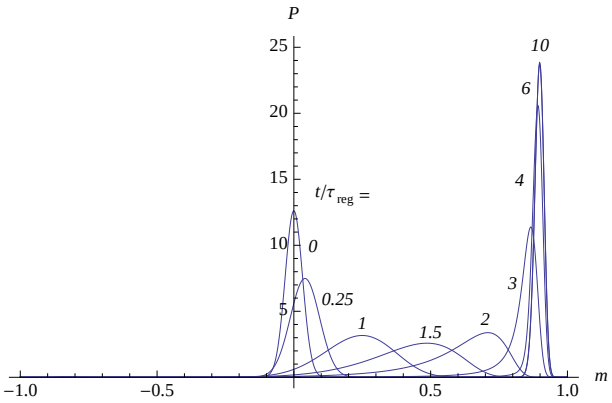

Figure 7.5: The registration process for a quadratic interaction ($q = 2$). The probability density $P(m,t) = P_{\uparrow\uparrow}(m,t)/r_{\uparrow\uparrow}(0)$ for the magnetization m of M is represented at different times. The parameters were chosen as $N = 1000$, $T = 0.65J$ and $g = 0.05J$ as in Figs. 7.1 and 7.3. The time scale is here the registration time $\tau_{\rm reg} = \hbar/\gamma(J-T) = 2.86\,\tau_J$. After a few times $\tau_{\rm reg}$ the evolution is no longer sensitive to the system-apparatus coupling g. In the initial fully disordered paramagnetic state ($T_0 = \infty$), $P(m,0)$ is a Gaussian centered at $m = 0$ with width $1/\sqrt{N}$. In the course of time, the peak of P considerably widens, then narrows and reaches eventually the equilibrium ferromagnetic distribution with positive magnetization $m_{\Uparrow} = 0.90$, which is given by (3.56). The repulsive fixed point lies at $-m_{\rm B}$ with $m_{\rm B} = 0.14$ and no weight is found below this. The second registration time, at which $\mu(t)$ reaches 0.80, is $\tau'_{\rm reg} = 3\tau_{\rm reg}$. It is seen that beyond this, the peak at $m_{\Uparrow}$ quickly builds up.

7.2.4. The registration process for a first-order transition

The process is different when the interaction is quartic ($q = 4$), a case that we chose to exemplify the first-order transitions which occur when $3J_4 > J_2$. The spin-apparatus coupling g must then at least be larger than the threshold (7.35) to ensure that $v(m)$ remains positive up to $m_{\Uparrow}$, which now lies near $m_{\rm F} \simeq 1$ (Figs. 3.3 and 7.2). At the beginning of the evolution, we find from $v(m) \approx (\gamma/\hbar)(g - Tm)$, using $g \ll T$, the motion

$$\mu(t) \approx \frac{g}{T}(1 - e^{-t/\tau_1}), \qquad \tau_1 = \frac{\hbar}{\gamma T}. \tag{7.50}$$

Like for $q = 2$, the peak shifts first as $\mu \sim \gamma g t/\hbar$, but here its motion slows down as t increases, instead of escaping more and more rapidly off the paramagnetic region, as exhibited on the example of Figs. 7.4 and 7.6. Extrapolation of (7.50) towards times larger than τ_1 is not possible, since μ would then not go beyond g/T, and could not reach $m_{\rm F}$. In fact, $v(m)$ does not vanish at $m = g/T$ as implied by the above approximation but only decreases down to a positive minimum near $m_{\rm c} \simeq 3g/2T$ according to (7.34). The vicinity of $m_{\rm c}$ is thus a *bottleneck* for the motion from $\mu = 0$ to $\mu = 1$ of the peak of $P_{\uparrow\uparrow}(m, t)$: This motion is the slowest around $m_{\rm c}$. The determination of the evolution of $P_{\uparrow\uparrow}(m, t)$, embedded in $\mu(t)$ and $D(t)$, and the evaluation of the registration time thus require a control of the shape of $v(m)$, not only near its zeroes, but also near its minimum (Fig. 7.2).

Let us recall the parameters which characterize $v(m)$. For $g = 0$, it has 5 zeroes. Three of them correspond to the attractive fixed points $\pm m_{\rm F} \simeq \pm 1$ and 0 associated with the ferromagnetic and paramagnetic states. The other two are repulsive, producing a bifurcation in the flow of $P(m, t)$; they are located at $m \simeq \pm\sqrt{T/J}$, that is, at $m \simeq \pm m_{\rm c}\sqrt{3}$ according to (7.34). When g increases and becomes larger than $h_{\rm c}$, there remain the two ferromagnetic points, while the repulsive point $-m_{\rm c}\sqrt{3}$ is shifted towards $-m_{\rm B} \simeq -2m_{\rm c}$. The paramagnetic point and the repulsive point $m_{\rm c}\sqrt{3}$ converge towards each other, giving rise to the minimum of $v(m)$ near $m = m_{\rm c}$. The value of $v(m)$ at this minimum is expressed by

$$\frac{\hbar}{\gamma T} v(m_{\rm c}) \simeq \frac{\delta m_{\rm c}^2}{m_{\rm c}}, \qquad \delta m_{\rm c} \simeq \sqrt{\frac{(g - h_{\rm c})m_{\rm c}}{T}}, \qquad h_{\rm c} \simeq \frac{2}{3} T m_{\rm c}, \qquad m_{\rm c} = \sqrt{\frac{T}{3J}}. \tag{7.51}$$

We construct in Appendix E.2, for $\delta m_{\rm c} \ll m_{\rm c}$ and $m_{\rm c}$ small, a parametrization of $v(m)$ which reproduces all these features, so as to derive an algebraic approximation (E.12) which expresses the time dependence of $\mu(t)$ over all times. After the initial evolution (7.50) of $\mu(t)$ for $t \ll \tau_1 = \hbar/\gamma T$, the motion of the peak $P_{\uparrow\uparrow}(m, t)$ is characterized by a much larger time scale. We define the *registration time* as

$$\tau_{\rm reg} = \frac{\pi\hbar}{\gamma T}\sqrt{\frac{m_{\rm c} T}{g - h_{\rm c}}}. \tag{7.52}$$

The bottleneck stage takes place around $\frac{1}{2}\tau_{\rm reg}$. Between the times $t = \frac{1}{4}\tau_{\rm reg}$ and $t = \frac{3}{4}\tau_{\rm reg}$, the average magnetization $\mu(t)$ lingers in the narrow range $m_{\rm c} \pm \delta m_{\rm c}$, according to (Fig. 7.4)

$$\mu(t) = m_{\rm c} - \delta m_{\rm c} \mathrm{cotan} \frac{\pi t}{\tau_{\rm reg}}. \tag{7.53}$$

It is shown in Appendix E.2 that, under the considered conditions on the parameters, $\mu(t)$ rises thereafter rapidly according to (E.15), and that the full time taken by the peak $\mu(t)$ of $P_{\uparrow\uparrow}(m, t)$ to go from 0 to the close vicinity of 1 is $\tau_{\rm reg}$ (Eq. (7.52)). It is also shown in Appendix E.2 that the final relaxation takes place on the short time scale $\hbar/\gamma J$.

We have focused on the location of the peak of $P_{\uparrow\uparrow}(m, t)$. The consideration of its width $D(t)$ is essential to determine when S and A may be decoupled. During the bottleneck stage, the sole drift effect would produce a narrowing of $D(t)$ around $t = \frac{1}{2}\tau_{\rm reg}$ expressed by the first term of (7.31), but the smallness of $v(m)$ enhances the second term, so that the diffusion acts during a long time and produces a large widening of $D(t)$. By using the

parabolic approximation for $v(m)$, which is represented by the first term of (E.11), and by replacing $w(m)$ by $\gamma T/\hbar$, we obtain, with $\mu(t)$ expressed by (7.53),

$$D(\mu) \sim 2m_c\left[(\mu - m_c)^2 + \delta m_c^2\right]\int_0^\mu \frac{d\mu'}{[(\mu' - m_c)^2 + \delta m_c^2]^3}. \tag{7.54}$$

After the bottleneck has been passed, the diffusion may again be neglected. From (7.31) and (7.54), we find for all values of $\mu(t)$ such that $\mu - m_c \gg \delta m_c$

$$D(\mu) \sim \frac{3\pi\hbar^2 m_c^3}{4\gamma^2 T^2 \delta m_c^5} v^2(\mu) = \frac{3\pi\sqrt{Tm_c}}{4(g-h_c)^{5/2}}(J\mu^3 + g)^2[1 - \mu\coth\beta(J\mu^3 + g)]^2, \tag{7.55}$$

where we used (7.6), (7.51) and (4.24). Without any diffusion, the coefficient of $v^2(\mu)$ would have been $1/v^2(0) = 9\hbar^2/4\gamma^2T^2m_c^2$; both factors $v(\mu)$ are multiplied by the large factor $\sqrt{\pi/3}(m_c/\delta m_c)^{5/2}$ due to diffusion.

The distribution $P_{\uparrow\uparrow}(m,t)$ thus extends, at times larger than $\frac{3}{4}\tau_{reg}$, over the region $\mu(t) \pm \sqrt{D(t)/N}$. The *first registration time* has been defined in § 7.2.3 as the time after which S *and* A *can be decoupled* without affecting the process. When g is switched off ($g \mapsto 0$), a repulsive fixed point appears at the zero $m = m_c\sqrt{3}$ of $v(m)$. In order to ensure a proper registration we need this decoupling to take place after the whole distribution $P_{\uparrow\uparrow}(m,t)$ has passed this bifurcation, that is, at a time t_{off} such that

$$\mu(t_{off}) - \sqrt{D(t_{off})/N} > m_c\sqrt{3}. \tag{7.56}$$

The time dependence (E.15) of μ shows that the lower bound of t_{off} is equal to τ_{reg} within a correction of order $\tau_1 \ll \tau_{reg}$. Moreover, we need the distribution to be sufficiently narrow so that (7.56) is satisfied after g is switched off. Taking for instance $\mu(t_{off}) = 2m_c$, which according to (E.15) is reached at the time $t_{off} = \tau_{reg}(1 - 0.25\,\delta m_c/m_c)$, we thus find, by inserting (7.55) with $\mu = 2m_c$ and $g \simeq h_c$ into (7.56), by using (7.51) and evaluating the last bracket of (7.55) for $m_c = 0.268$, a *further lower bound* for the coupling g in our first order case $q = 4$:

$$\frac{g - h_c}{h_c} \gg 8\left(\frac{J}{NT}\right)^{2/5}. \tag{7.57}$$

The first registration time, which governs the possibility of decoupling, and the second one, which is the delay after which the pointer variable approaches the equilibrium value, are therefore nearly the same, namely τ_{reg}, contrary to the case $q = 2$ of a second order transition (§ 7.2.3).

The registration process for $q = 4$ is illustrated by Figs. 7.4 and 7.6, obtained through numerical integration. The time dependence of $\mu(t)$ as well as the widening of the distribution are influenced by the existence of the minimum for the drift velocity. Although in this example g lies above the threshold h_c, N is not sufficiently large to fulfil the condition (7.57). The widening is so large that a significant part of the weight $P(m,t)$ remains for a long time below the bifurcation $m_c\sqrt{3}$ which appears when g is switched off. The bound (7.57) was evaluated by requiring that such a switching off takes place after the average magnetization μ passes $2m_c = 0.54$. Here however, for $N = 1000$, $T = 0.2J$ and $g = 0.045J$, the bound is very stringent, since we cannot switch off g before μ has reached (at the time $1.09\tau_{reg}$ found from (E.15)) the value $1 - 13\cdot10^{-5}$, close to the equilibrium value $m_F = 1 - 9\cdot10^{-5}$.

Altogether, for $q = 2$ as well as for $q = 4$, we can check that the approximate algebraic treatment of §§ 7.2.3 and 7.2.4 fits the numerical solution of Eq. (7.1) exemplified by the figures 7.3 to 7.6. In both cases, the registration times (7.44) and (7.48) for $q = 2$ or (7.52) for $q = 4$, which characterize the evolution of the diagonal blocks of the density matrix of the total system $\hat{\mathcal{D}}$, are much longer than the reduction time (5.6) over which the off-diagonal blocks decay. Two reasons conspire to ensure this large ratio: the weakness of the coupling γ between magnet and bath, which makes τ_{reg} large; and the large value of N, which makes τ_{red} small.

7.3. Giant fluctuations of the magnetization

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Mohandas Gandhi

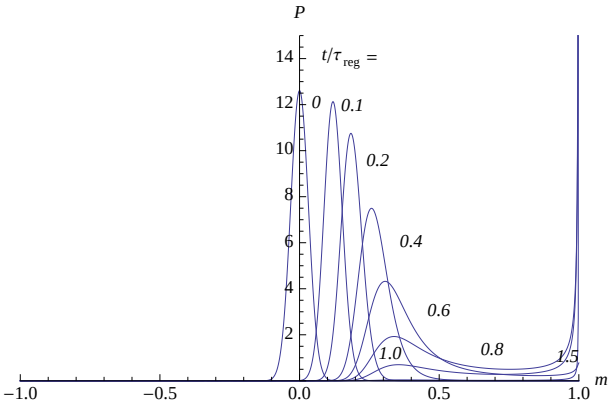

Figure 7.6: The registration process for quartic interactions ($q = 4$). The probability density $P(m,t) = P_{\uparrow\uparrow}(m,t)/r_{\uparrow\uparrow}(0)$ as function of m is represented at different times up to $t = 1.5\,\tau_{\rm reg}$. The parameters are chosen as $N = 1000$, $T = 0.2J$ and $g = 0.045J$ as in Fig 7.4. The time scale is here the registration time $\tau_{\rm reg} = 38\tau_J = 38\hbar/\gamma J$, which is large due to the existence of a bottleneck around $m_{\rm c} = 0.268$. The coupling g exceeds the critical value $h_{\rm c} = 0.0357J$ needed for proper registration, but since $(g - h_{\rm c})/h_{\rm c}$ is small, the drift velocity has a low positive minimum at 0.270 near $m_{\rm c}$ (Fig. 7.2). Around this minimum, reached at the time $\frac{1}{2}\tau_{\rm reg}$, the peak shifts slowly and widens much. Then, the motion fastens and the peak narrows rapidly, coming close to ferromagnetism around the time $\tau_{\rm reg}$, after which equilibrium is exponentially reached.

We have studied in subsection 7.2 the evolution of the probability distribution $P(m,t) = P_{\uparrow\uparrow}(m,t)/r_{\uparrow\uparrow}(0)$ of the magnetization of M in case this distribution presents a single peak (7.10) at all times. This occurs when $P(m,t)$ always remains entirely located, except for negligible tails, on a single side of the bifurcation $-m_{\rm B}$ of the drift flow $v(m)$. We will now consider the case of an *active bifurcation* [213, 214, 215, 216, 217]: The initial distribution is split during the evolution into two parts evolving towards $+m_{\rm F}$ and $-m_{\rm F}$. This situation is relevant to our measurement process for $q = 2$ in regard to two questions: (*i*) How fast should one perform the cooling of the bath before the initial time, and the switching on of the system-apparatus interaction around the initial time? (*ii*) What is the percentage of errors of registration if the coupling g is so small that it violates the condition (7.41)?

7.3.1. Dynamics of the invariance breaking

In order to answer the above two questions, we first determine the Green's function for the equation of motion (7.1) which governs $P(m,t)$ for $q = 2$ in the Markovian regime. This will allow us to deal with an arbitrary initial condition. The *Green's function* $G(m,m',t-t')$ is characterized by the equation

$$\frac{\partial}{\partial t}G(m,m',t-t') + \frac{\partial}{\partial m}[v(m)G(m,m',t-t')] - \frac{1}{N}\frac{\partial^2}{\partial m^2}[w(m)G(m,m',t-t')] = \delta(m-m')\delta(t-t'), \tag{7.58}$$

with $G(m,m',t-t') = 0$ for $t < t'$. We have replaced the initial time 0 by a running time t' in order to take advantage of the convolution property of G. The functions $v(m)$ and $w(m)$ defined by (7.6) and (7.7) involve a field h which stands either for an applied external field if A = M + B evolves alone, or for $\pm g$ if we consider $P_{\uparrow\uparrow}$ or $P_{\downarrow\downarrow}$ if A is coupled to S. We wish to face the situation in which $P(m,t)$ lies, at least after some time, astride the bifurcation point $-m_{\rm B} = -h/(J-T)$. Such a situation has extensively been studied [213, 214, 215, 216, 217], and we adapt the existing methods to the present problem which is similar to Suzuki's model.

We first note that the initial distribution $P(m,t' = 0)$ is concentrated near the origin, a property thus satisfied by the variable m' in $G(m,m',t)$. In this region, it is legitimate to simplify $v(m')$ and $w(m')$ into

$$v(m') \approx \frac{\gamma}{\hbar}[h + (J-T)m'], \qquad w(m') \approx \frac{\gamma T}{\hbar}, \tag{7.59}$$

where we also used $h \ll T$. In order to implement this simplification which holds only for $m' \ll 1$, we replace the forward equation (7.58) in terms of t which characterizes $G(m,m',t-t')$ by the equivalent *backward equation*, for $\partial G(m,m',t-t')/\partial t'$, in terms of the initial time t' which runs down from t to 0. This equation is written and solved in Appendix F. The distribution $P(m,t)$ is then given by

$$P(m,t) = \int \mathrm{d}m' G(m,m',t) P(m',0). \tag{7.60}$$

We derive below several approximations for $P(m,t)$, which are valid in limiting cases. These various results are encompassed by the general expression (F.13)–(F.15) for $P(m,t)$, obtained through the less elementary approach of Appendix F.

As in § 7.2.3, the evolution takes place in three stages [213, 214, 215, 216, 217]: (i) *widening* of the initial distribution, which here takes place over the bifurcation $-m_{\mathrm{B}}$; (ii) *drift* on both sides of $-m_{\mathrm{B}}$ towards $+m_{\mathrm{F}}$ and $-m_{\mathrm{F}}$; (iii) narrowing around $+m_{\mathrm{F}}$ and $-m_{\mathrm{F}}$ of the two final peaks, which evolve *separately towards equilibrium*. We shall not need to consider here the last stage, the approach to quasi-equilibrium, that we studied in § 7.1.5.

The probability distribution $P(m,t)$ is thus expressed in terms of the initial distribution $P(m,0)$ by (7.60), at all times, except during the final equilibration. If $P(m,0)$ is a narrow Gaussian peak centered at $m = \mu_0$ with a width $\delta_0/\sqrt{N}$, we find can use the expression (F.10) of G, which yields

$$P(m,t) = \frac{v(\mu')}{v(m)} \sqrt{\frac{N}{2\pi}} \frac{1}{\delta_1(t)} \exp\left[-\frac{N}{2} \frac{(\mu'-\mu_0)^2}{\delta_1^2(t)} \right]. \tag{7.61}$$

The function $\mu'(m,t)$ is defined for arbitrary values of m by

$$t = \int_{\mu'(m,t)}^{m} \frac{\mathrm{d}m''}{v(m'')}, \tag{7.62}$$

while the variance that enters (7.61) is determined by

$$\delta_1^2(t) \equiv \delta_0^2 + \frac{T}{J-T}(1 - e^{-2t/\tau_{\mathrm{reg}}}) \equiv \delta_1^2 - \frac{T}{J-T} e^{-2t/\tau_{\mathrm{reg}}}, \qquad \delta_1^2 \equiv \frac{T_0}{T_0 - J} + \frac{T}{J-T}. \tag{7.63}$$

With time, it increases from δ_0^2/N to δ_1^2/N.

7.3.2. Spontaneous relaxation of the paramagnetic state

The registration process that we studied in § 7.2.3 is the same as the relaxation, for $q = 2$ and $T < J$, of the initial paramagnetic state (3.48) towards the positive ferromagnetic state $+m_{\mathrm{F}}$ in the presence of a sufficiently large positive external field h. We now consider the situation in which A evolves *in the absence of a field*. The process will describe the *dynamics of the spontaneous symmetry breaking*, which leads from the unstable symmetric paramagnetic distribution $P_{\mathrm{M}}(m,0)$ to the ferromagnetic distribution (3.56) for $+m_{\mathrm{F}}$ and $-m_{\mathrm{F}}$, occurring with equal probabilities. We present below an approximate analytic solution, and illustrate it by Fig. 7.7 which relies on a numerical solution.

Apart from the final stage, the result is given by (7.61) with $\mu_0 = 0$, $\delta_0^2 = T_0/(T_0 - J)$, and $v(m) = (\gamma/\hbar) Jm[1 - m \coth(Jm/T)]$. During the first stage, we have $v(m) \sim m/\tau_{\mathrm{reg}}$ and hence $\mu' \sim m e^{-t/\tau_{\mathrm{reg}}}$, so that (7.61) reduces to

$$P_{\mathrm{M}}(m,t) = \sqrt{\frac{N}{2\pi}} \frac{e^{-t/\tau_{\mathrm{reg}}}}{\delta_1(t)} \exp\left[-\frac{N m^2 e^{-2t/\tau_{\mathrm{reg}}}}{2\delta_1^2(t)} \right]. \tag{7.64}$$

On the time scale $\tau_{\mathrm{reg}} = \hbar/\gamma(J-T)$, this distribution widens exponentially, with the variance

$$\frac{1}{N}\left[\delta_1^2 e^{2t/\tau_{\mathrm{reg}}} - \frac{T}{J-T} \right], \qquad \delta_1^2 \equiv \frac{T_0}{T_0 - J} + \frac{T}{J-T}. \tag{7.65}$$

As in § 7.2.3, the widening is first induced by the diffusion term, which is then relayed by the gradient of the drift velocity $v(m)$. However, the effect is much stronger here because the distribution remains centered around $m = 0$.

In fact, at times of order $\tau_{\mathrm{reg}} \ln \sqrt{N}$, the width of the peak of $P_{\mathrm{M}}(m,t)$ is no longer of order $1/\sqrt{N}$, but it is *finite for large N*. If we define the lifetime τ_{para} of the paramagnetic state as the delay during which this width is less that α, say $\alpha = 1/10$, it equals (for $\alpha\sqrt{N} \gg 1$)

$$\tau_{\mathrm{para}} = \tau_{\mathrm{reg}} \ln \alpha \sqrt{N} = \frac{\hbar}{\gamma(J-T)} \ln \alpha \sqrt{N}. \tag{7.66}$$

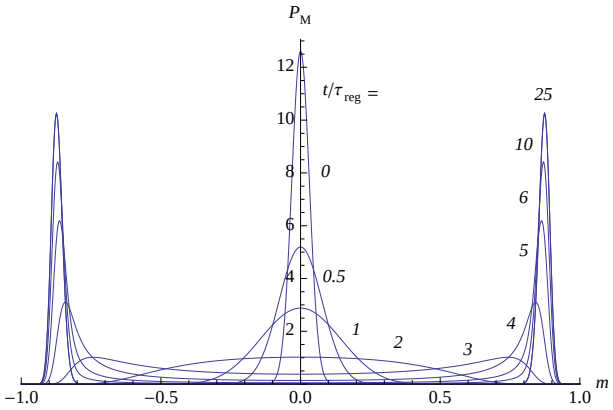

Figure 7.7: Relaxation of an unstable paramagnetic state ($q = 2$) in the absence of a field ($g = 0$). The probability distribution $P_M(m,t)$ is represented at several times. As in Figs. 7.3 and 7.5 the parameters are $N = 1000$ and $T = 0.65J$. First the Gaussian paramagnetic peak around $m = 0$ with width $1/\sqrt{N}$ widens considerably. Around $t = \tau_{flat} = 2.2\tau_{reg}$, the distribution extends over most of the interval $-m_F, +m_F$ ($m_F = 0.872$) and is nearly flat. Then, two peaks progressively build up, moving towards $-m_F$ and $+m_F$. Finally each peak tends to the Gaussian ferromagnetic equilibrium shape, the curves at $t = 10\tau_{reg}$ and $25\tau_{reg}$ basically coincide.

The second stage of the evolution is then reached (Fig. 7.7). An analytic expression of $P_M(m,t)$ can then be found by using the Mittag-Leffler approximation (E.1) for $v(m)$ (with $h = 0$). The relation between μ', m and t becomes

$$\frac{t}{\tau_{reg}} = \ln\frac{m}{\mu'} + a\ln\frac{m_F^2 - \mu'^2}{m_F^2 - m^2}, \tag{7.67}$$

where the coefficient a, defined by (7.47), lies between $\frac{1}{2}$ for $J - T \ll J$ and 1 for $T \ll J$. In this stage, when $t \gg \tau_{reg}$, the distribution

$$P_M(m,t) = \frac{\mu'(m_F^2 - \mu'^2)}{m(m_F^2 - m^2)}\,\frac{m_F^2 + (2a-1)m^2}{m_F^2 + (2a-1)\mu'^2}\sqrt{\frac{N}{2\pi\delta_1^2}}\exp\left(-\frac{N\mu'^2}{2\delta_1^2}\right), \tag{7.68}$$

depends on time only through μ'. It flattens while widening. In particular, around the time

$$\tau_{flat} = \tau_{reg}\ln\left(\frac{m_F}{\delta_1}\sqrt{\frac{N}{6a}}\right), \tag{7.69}$$

it behaves for small m as

$$P_M(m,t) \approx \frac{1}{m_F}\sqrt{\frac{3}{\pi}}e^{-(t-\tau_{flat})/\tau_{reg}}\left\{1 + \frac{3am^2}{m_F^2}\left[1 - e^{-2(t-\tau_{flat})/\tau_{reg}}\right] + O\left(\frac{m^4}{m_F^4}\right)\right\}. \tag{7.70}$$

When t reaches τ_{flat}, the distribution $P_M(m,\tau_{flat})$ has widened so much that it has become *nearly flat*: The probabilities of the possible values (3.22) of m are nearly the same on a range which extends over most of the interval $-m_F, +m_F$. This property agrees with the value of $\frac{1}{2}NP_M(0,\tau_{flat}) = 0.98/m_F$; the coefficient of the term in $(m/m_F)^4$, equal to $-a(8a - \frac{5}{2})$, yields a correction $-(0.93\,m/m_F)^4$ for small m_F, $-(1.53\,m/m_F)^4$ for large m_F.

When t increases beyond τ_{flat}, the distribution begins to deplete near $m = 0$ and two originally not pronounced maxima appear there (Fig. 7.7), which move apart as

$$m = \pm m_F\sqrt{\frac{6(t - \tau_{flat})}{(16a-5)\tau_{reg}}}. \tag{7.71}$$

They then become sharper and sharper as they move towards $\pm m_F$. When they get well separated, $P_M(m,t)$ is concentrated in two symmetric regions, below m_F and above $-m_F$, and it reaches a *scaling regime* [213, 214, 215, 216, 217]

in which (for $m > 0$)

$$\mu'(m,t) \sim m_{\rm F} e^{-t/\tau_{\rm reg}} \left[\frac{m_{\rm F}}{2(m_{\rm F}-m)}\right]^2 \tag{7.72}$$

is small, of order $1/\sqrt{N}$. If we define, with a ($\frac{1}{2} < a < 1$) given by Eq. (7.47),

$$\xi(m,t) \equiv \sqrt{\frac{N}{2}}\frac{\mu'(m,t)}{\delta_1} = \sqrt{3a}\left[\frac{m_{\rm F}}{2(m_{\rm F}-m)}\right]^a e^{-(t-\tau_{\rm flat})/\tau_{\rm reg}}, \tag{7.73}$$

$P_{\rm M}(m,t)$ takes in the region $m > 0$, $\xi > 0$, the form

$$P_{\rm M}(m,t) \approx \frac{1}{\sqrt{\pi}}\frac{\partial \xi}{\partial m} e^{-\xi^2}. \tag{7.74}$$

Its maximum lies at the point $m_{\rm max}$ given by

$$\xi(m,t) = \sqrt{\frac{a+1}{2a}}, \qquad \frac{m_{\rm F}-m_{\rm max}}{m_{\rm F}} = \frac{1}{2}\left(\frac{6a^2}{a+1}\right)^{1/(2a)} e^{-(t-\tau_{\rm flat})/a\tau_{\rm reg}}, \tag{7.75}$$

which approaches $m_{\rm F}$ exponentially, and its shape is strongly asymmetric. In particular, its tail above $m_{\rm max}$ is short, whereas its tail below $m_{\rm max}$ extends far as $1/(m_{\rm max}-m)^{a+1}$; only moments $\langle (m_{\rm F}-m)^\lambda \rangle$ with $\lambda < a$ exist.

After a delay of order $a\tau_{\rm reg} \ln \sqrt{N}$, the width of the peaks of $P_{\rm M}(m,t)$ and their distance to $\pm m_{\rm F}$ reach an order of magnitude $1/\sqrt{N}$. The diffusion term becomes active, and each peak tends to the Gaussian shape (3.56) as in § 7.1.5. This crossover could be expressed explicitly by writing the Green's function for m and m' near $m_{\rm F}$ (as we did near 0 in § 7.3.1) and by taking (7.74) as initial condition. All the above features fit the numerical solution shown by Fig. 7.7.

In our measurement problem, $q = 2$, the above evolution begins to take place at the time $-\tau_{\rm init}$ at which the apparatus is initialized (§ 3.3.3). Before $t = -\tau_{\rm init}$, paramagnetic equilibrium has been reached at the temperature $T_0 > J$, and the initial distribution of m is given by (3.48), (3.49) (3.51). The sudden cooling of the bath down to the temperature $T < J$ lets the evolution (7.64) start at the time $-\tau_{\rm init}$. We wish that, at the time $t = 0$ when the coupling g is switched on and the measurement begins, the distribution $P_{\rm M}(m,0)$ is still narrow, close to (3.48). We thus need $\delta_1(\tau_{\rm init})$ to be of the order of δ_0, that is,

$$\frac{2\tau_{\rm init}}{\tau_{\rm reg}} < \delta_0^2 \frac{J-T}{T} = \frac{T_0}{T_0-J}\frac{J-T}{T}. \tag{7.76}$$

The bath should be cooled down and the system-apparatus interaction $\hat{H}_{\rm SA}$ should be switched on *over a delay* $\tau_{\rm init}$ *not larger than the registration time* $\tau_{\rm reg} = \hbar/\gamma(J-T)$.

The situation is more favourable in case the initial depolarized state of the spins of M is generated by a radiofrequency field rather than through equilibration with the phonon bath at a high but finite temperature T_0. In this case, a sudden cooling of the bath at the time $-\tau_{\rm init}$ is not needed. The bath can beforehand be cooled at the required temperature T lower than $T_c = J$. At the time $-\tau_{\rm init}$, the spins are suddenly set by the field into their most disordered state, a process which hardly affects the bath since $\gamma \ll 1$. The above discussion then holds as if T_0 were infinite.

If a weak field h_0 is accidentally present during the preparation by thermalization of the initial paramagnetic state, it should not produce a bias in the measurement. This field shifts the initial expectation value $\langle m \rangle$ of m from 0 to $\mu_0 = h_0/(T_0 - J)$, which enters (7.61). At the time 0, $\langle m \rangle$ has become $\mu_0 \exp(\tau_{\rm init}/\tau_{\rm reg})$, so that the residual field h_0 is ineffective provided $\mu_0 < \delta_0$, that is for

$$h_0 < \sqrt{\frac{T_0(T_0-J)}{N}}. \tag{7.77}$$

The success of the measurement process thus requires the conditions (7.76) and (7.77) on the parameters $\tau_{\rm init}$, T_0, h_0 that characterize the preparation of the initial state of the apparatus.

For a quartic interaction ($q = 4$), the initial paramagnetic state is metastable rather than unstable. Its spontaneous decay in the absence of a field requires m to cross the potential barrier of the free energy which ensures metastability,

as shown by Fig. 3.3. At temperatures T below the transition point but not too low, the dynamics is governed by an activation process, with a characteristic duration of order $(\hbar/\gamma J)\exp(\Delta F/T)$, where ΔF is the height of the barrier, for instance $\Delta F = 0.054NT$ for $T = 0.2J$. The lifetime of the paramagnetic state is thus exponentially larger than the registration time for large N, so that there is no hurry in performing the measurement after preparation of the initial state.

7.3.3. Probability of wrong registrations for second order phase transitions of the magnet

We have seen (§ 7.2.3) how the magnet M, under the conjugate effect of B and S, reaches quasi certainly the final magnetization $+m_{\rm F}$ in the sector $\uparrow\uparrow$ where $s_z = +1$, provided g is not too small. We expect that if the condition (7.41) on g is violated, the apparatus will indicate, with some probability $\mathcal{P}_-$, the wrong magnetization $-m_{\rm F}$, although $s_z = +1$. The evolution of $P_{\uparrow\uparrow}(m,t)$ in such a situation is illustrated by Fig. 7.8. A similar failure may occur if the average magnetization μ_0 in the initial state is not 0 but takes a negative value due to a *biased preparation*.

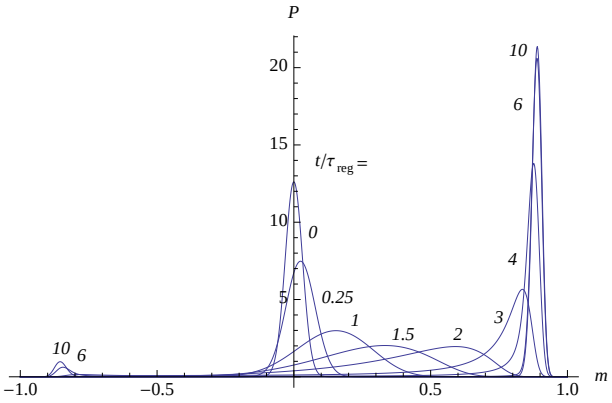

Figure 7.8: Wrong registration for quadratic interactions ($q = 2$). The probability distribution $P(m,t)$ is represented at different times for the same parameters $N = 1000$ and $T = 0.65$ as in Fig. 7.5, but the coupling $g = 0.03J$ is now sufficiently weak so that the apparatus registers the magnetization $-m_{\rm F}$ with a significant probability $\mathcal{P}_-$, although the system has a spin $s_z = +1$. Like in Fig. 7.7, the probability distribution flattens before the two ferromagnetic peaks emerge (with weights $\mathcal{P}_+$ and $\mathcal{P}_-$).

The *probability $\mathcal{P}_-$ of a wrong registration* $-m_{\rm F}$ for $s_z = +1$ arises from values $m < m_{\rm B} < 0$ and reads

$$\mathcal{P}_- = \int_{-1}^{m_{\rm B}} {\rm d}m\, \frac{P_{\uparrow\uparrow}(m,t)}{r_{\uparrow\uparrow}(0)} \equiv \int_{-1}^{m_{\rm B}} {\rm d}m\, P(m,t)\,, \tag{7.78}$$

where the time t is in principle such that $P_{\uparrow\uparrow}(m,t)$ has reached its equilibrium shape, with two peaks around $+m_{\rm F}$ and $-m_{\rm F}$. In fact, we do not need the final equilibrium to have been reached since (7.78) remains constant after $P_{\uparrow\uparrow}$ has split into two separate parts. And even the latter condition is not necessary: After the time $\tau_{\rm reg}$ the diffusion term becomes inactive and the evolution of $P_{\uparrow\uparrow}(m,t)$ is governed by the pure drift Green's function (F.7); then there is no longer any transfer of weight across the bifurcation $-m_{\rm B} = -g/(J-T)$. We can therefore evaluate (7.78) at the rather early stage when the distribution has not yet spread out beyond the small m region where (7.59) holds, provided we take $t \gg \tau_{\rm reg}$.

We thus use the expression (7.61) of $P_{\uparrow\uparrow}(m,t)$ valid during the first stage of the process, which reads

$$P(m,t) = e^{-t/\tau_{\rm reg}} \sqrt{\frac{N}{2\pi}}\frac{1}{\delta_1(t)} \exp\left\{-\frac{N}{2\delta_1^2(t)}\left[(m+m_{\rm B})e^{-t/\tau_{\rm reg}} - m_{\rm B} - \mu_0\right]^2\right\}. \tag{7.79}$$

By taking $(m+m_{\rm B})e^{-t/\tau_{\rm reg}}$ as variable we check that the integral (7.78) depends on time only through the exponential in (7.63), so that it remains constant as soon as $t \gg \tau_{\rm reg}$, when the second stage of the evolution is reached. We eventually find:

$$\mathcal{P}_- = \frac{1}{2}{\rm erfc}\,\lambda, \qquad \lambda \equiv \sqrt{\frac{N}{2}}\frac{1}{\delta_1}(m_{\rm B}+\mu_0), \tag{7.80}$$

where the error function, defined by

$$\operatorname{erfc}\lambda = \frac{2}{\sqrt{\pi}}\int_{\lambda}^{\infty} \mathrm{d}\xi e^{-\xi^2}, \tag{7.81}$$

behaves for $\lambda \gg 1$ as

$$\operatorname{erfc}\lambda \sim \frac{1}{\sqrt{\pi}\lambda} e^{-\lambda^2}. \tag{7.82}$$

The diffusion which takes place during the first stage of the evolution has changed in (7.80) the initial width δ_0 into δ_1, given by (7.40).

For $\mu_0 = 0$, the probability of error becomes sizeable when $\sqrt{N}g/J$ is not sufficiently large. For example, for $T = 0.65J$ and $g = 0.03J$, we find numerically $\mathcal{P}_- = 21\%$, 13%, 5.4%, 1.15% and 0.065% for $N = 250$, 500, 1000, 2000 and 4000, respectively. These data are reasonably fitted by the approximation $\mathcal{P}_-(N) = 1.2\,N^{-1/4}\exp(-0.0014N)$ for (7.80). The result for $N = 1000$ is illustrated by the weight of the peak near $-m_{\mathrm{F}}$ in Fig. 7.8. False registrations were also present with the data of Fig. 7.5 ($N = 1000$, $T = 0.65J$, $g = 0.05J$), with a probability $\mathcal{P}_- = 0.36\%$, but the effect is too small to be visible on the scale of the figure.

The occurrence of a negative μ_0 increases $\mathcal{P}_-$, an effect which, with the above data, becomes sizeable for $|\mu_0| \sim 0.05$. For $P_{\downarrow\downarrow}$ the percentage of errors is given by (7.80) with μ_0 changed into $-\mu_0$ in λ.

We write for completeness in Appendix F the evolution of the shape of $P(m,t)$. This is not crucial for the measurement problem (for which $P_{\uparrow\uparrow}(m,t) = r_{\uparrow\uparrow}(0)\,P(m,t)$), but it is relevant for the dynamics of the phase transition, depending on the initial conditions and on the presence of a parasite field. Here again, Suzuki's regime [213, 214, 215, 216, 217], where the distribution is no longer peaked, is reached for $t \gg \tau_{\mathrm{reg}}$. Now $P(m,t)$ is asymmetric, but it still has a quasi linear behavior in a wide range around $m = 0$ when $\tau \simeq \tau_{\mathrm{flat}}$ (see Eqs. (7.69), (7.70)).

7.3.4. Possible failure of registration for first order transitions

If you dont have a horse, ride a cow
Portuguese proverb

The situation is quite different for first-order transitions ($q = 4$) as regards the possibility of wrong registrations. Note first that $F(m)$ has a high maximum for negative m between 0 and $m_{\Downarrow} < 0$ (Figs. 3.3 and 3.4), which constitutes a practically impassable barrier that diffusion is not sufficient to overcome. Accordingly, the zero of $v(m)$ at $m = -m_{\mathrm{B}} \simeq -2m_{\mathrm{c}}$ is a repulsive fixed point (Fig.7.2 and § 7.2.4), which prevents the distribution from developing a tail below it. We shall therefore never find any registration with negative ferromagnetic magnetization in the sector $s_z = +1$.

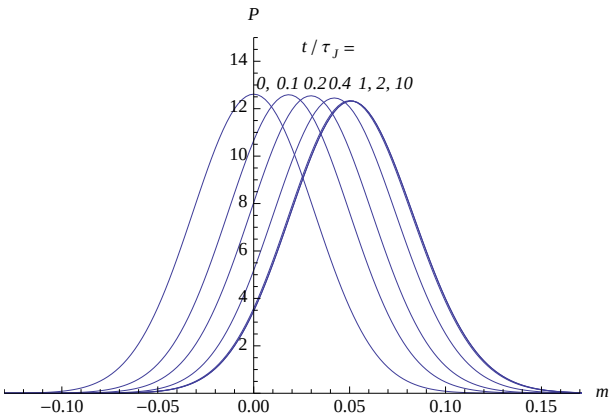

Figure 7.9: Failure of measurement for quartic interactions ($q = 4$). The probability distribution $P(m,t)$ is represented at times up to $10\,\tau_J$, where $\tau_J = \hbar/\gamma J$. The parameters are $N = 1000$ and $T = 0.2$ as in Figs. 7.4 and 7.6, but here $g = 0.01J$ lies below the threshold h_{c}. The peak evolves towards metastable paramagnetic equilibrium in the presence of the field g, but g is too small to allow crossing the barrier and reaching the more stable ferromagnetic equilibrium around $m_{\mathrm{F}} \simeq 1$. Switching off the coupling g brings back the distribution to its original place around 0, so that no proper registration is achieved.

Nevertheless, we have seen (§ 7.2.2) that registration is possible only if the coupling g exceeds h_c. For $g < h_c$, the peak of $P(m)$ initially at $m = 0$ moves upwards in the sector $\uparrow\uparrow$ associated with $s_z = +1$ (Fig. 7.9), and ends up by stabilizing at the first attractive point encountered, at $m = m_P$ (Fig. 7.2). Symmetrically, the distribution of $P_{\downarrow\downarrow}(m)$ ends up at $-m_P$. However this difference between the two values of s_z cannot be regarded as a registration since switching off the coupling g between S and A brings back both distributions $P_{\uparrow\uparrow}$ and $P_{\downarrow\downarrow}$ to the initial Gaussian shape around $m = 0$. The apparatus A then always relaxes back finally to the locally stable paramagnetic state.

Finally, if the coupling g, although larger than the threshold h_c, is close to it, the registration takes place correctly provided this coupling remains active until the distribution $P_{\uparrow\uparrow}(m)$ has completely passed the bifurcation $m_c\sqrt{3}$ occurring for $g = 0$ (Eq. (7.56)). The lower bound t_{off} of the time when g can thus be safely switched off is close to τ_{reg} (which is also close to the time needed to reach ferromagnetic equilibrium).

In case S and A are decoupled too early, so that the condition (7.56) is violated, the tail of $P_{\uparrow\uparrow}(m,t)$ lying below the bifurcation $m = m_c\sqrt{3}$ is pushed back towards the paramagnetic region $m \approx 0$. If the decoupling $g \to 0$ is made suddenly at the time t_{off}, the probability $\mathcal{P}_0$ of such events can be evaluated as in § 7.3.3 in terms of the error function by integration of $P_{\uparrow\uparrow}(m, t_{off})$ from $m = -1$ up to m_c. It represents the *probability of aborted measurement processes*, for which the apparatus returns to its neutral paramagnetic state without giving any indication, while S is left in the state $|\uparrow\rangle$. In a set of repeated measurements, a proportion $\mathcal{P}_0$ of runs are not registered at all, the other ones being registered correctly.

7.3.5. Erasure of the pointer indication

As shown in §§ 7.2.3 and 7.2.4, the registration is achieved at a time t_f sufficiently larger than the delay τ_{reg} after which S and M have been decoupled. The state $\hat{\mathcal{D}}(t_f)$ of S + A is then given by the expected expression (1.7). Within the considered approximations, the distributions $P_{\uparrow\uparrow}(m,t)$ and $P_{\downarrow\downarrow}(m,t)$ no longer evolve for $t > t_f$, and remain fully concentrated near m_F and $-m_F$, respectively, so that the results can be read out or processed at any observation time $t_{obs} > t_f$. However, the breaking of invariance, on which we rely to assert that the two ferromagnetic states of the pointer are stationary, is rigorous only in the large N limit. Strictly speaking, for finite N, the states $\hat{R}_{M\Uparrow}$ and $\hat{R}_{M\Downarrow}$ reached by M at this stage in each sector are not in equilibrium (though they may have a long lifetime). Indeed, in the Markovian regime, we have shown in § 7.1.4 that the evolution of M under the influence of the thermal bath cannot stop until $P_M(m,t)$ becomes proportional to $G(m)\exp[-\beta E(m)]$, with $E(m) = -JNq^{-1}m^q$. Otherwise, the time-derivative (7.21) of the free energy $F(m)$ of the state $\hat{R}_M(t)$ cannot vanish. The limit reached by $\hat{R}_{\uparrow\uparrow}(t)/r_{\uparrow\uparrow}(0)$ (and of $\hat{R}_{\downarrow\downarrow}(t)/r_{\downarrow\downarrow}(0)$) is then $\frac{1}{2}(\hat{R}_{M\Uparrow} + \hat{R}_{M\Downarrow})$. Hence, when the latter true equilibrium state for finite N is attained, the indication of the pointer is completely random. We have lost all information about the initial state of S, and the spin S has been completely depolarized whatever its initial state: the result of the measurement has been washed out. We denote as τ_{eras} the characteristic time which governs this erasure of the indication of the pointer.

It is therefore essential to read or process the registered data before such a loss of memory begins to occur[46]. The observation must take place at a time t_{obs} much shorter than the erasure time:

$$\tau_{reg} < t_f < t_{obs} \ll \tau_{eras}. \tag{7.83}$$

The dynamics of the erasure, a process leading M from $\hat{R}_{M\Uparrow}$ or $\hat{R}_{M\Downarrow}$ to the state $\frac{1}{2}(\hat{R}_{M\Uparrow} + \hat{R}_{M\Downarrow})$ of complete equilibrium, is governed by the Eq. (4.16) for $P_M(m,t)$ (with $\tilde{K}_t(\omega)$ replaced by $\tilde{K}(\omega)$ and $g = 0$), which retains the quantum character of the apparatus. We will rely on this equation in subsection 8.1 where studying the Curie–Weiss model in the extreme case of $N = 2$. For the larger values of N and the temperatures considered here, we can use its continuous semi-classical limit (7.1), to be solved for an initial condition expressed by (3.54) with $m_i = m_F$ or $-m_F$. Here we have to deal with the progressive, very slow leakage of the distribution $P_M(m,t)$ from one of the ferromagnetic states to the other through the free energy barrier that separates them. This mechanism, disregarded in §§ 7.2.3, 7.2.4, 7.3.3 and 7.3.4, is controlled by the weak tail of the distribution $P_M(m,t)$ which extends into the regions of m where $F(m)$ is largest. The drift term of (7.1) alone would repel the distribution $P_{\uparrow\uparrow}(m,t)$ and keep it concentrated near m_F. An essential role is now played by the diffusion term, which tends to flatten this distribution over the whole range of

[46] Photographs on film or paper fade out after some time

m, and thus allows the leak towards $-m_{\rm F}$. Rather than solving this equation, it will be sufficient for our purpose to rely on a semi-phenomenological argument: Under the considered conditions, the full equilibration is an activation process governed by the height of the free energy barrier. Denoting as ΔF the difference between the maximum of $F(m)$ and its minimum, $F_{\rm ferro}$, we thus estimate the time scale of erasure as:

$$\tau_{\rm eras} \sim \frac{\hbar}{\gamma J} \exp \frac{\Delta F}{T}, \tag{7.84}$$

which is large as an exponential of N. In order to use the process as a measurement, we need this time to be much larger than the registration time so that we are able to satisfy (7.83), which yields

$$\frac{J}{J-T} \ll \exp \frac{\Delta F}{T}, \quad (q=2); \qquad \frac{J}{T}\sqrt{\frac{m_{\rm c} T}{g-h_{\rm c}}} \ll \exp \frac{\Delta F}{T}, \quad (q=4). \tag{7.85}$$

From (3.54) (taken for $h = 0$), we find the numerical value of $\Delta F/T$ for the examples of figs. 7.5 and 7.6, namely $0.130N$ for $q = 2$, $T = 0.065J$, and $0.607N$ for $q = 4$, $T = 0.2J$ (see fig. 3.3). The condition (7.85) sets again a lower bound on N to allow successful measurements, $N \gg 25$ for the example with quadratic interactions, $N \gg 7$ for the example with quartic interactions. Such a condition is violated for a non-macroscopic apparatus, in particular in the model with $N = 2$ treated below in subsection 8.1 which will require special care to ensure registration.

7.3.6. "Buridan's ass"effect: hesitation

In the case of a second-order transition ($q = 2$), the subsections 7.2 and 7.3, illustrated by Figs. 7.5, 7.7 and 7.8, show off the occurrence, for the evolution of the probability distribution $P(m, t)$, of two contrasted regimes, depending whether the bifurcation $-m_{\rm B}$ is active or not. The mathematical problem is the same as for many problems of statistical mechanics involving dynamics of instabilities, such as directed Brownian motion near an unstable fixed point, and it has been extensively studied [213, 214, 215, 216, 217]. The most remarkable feature is the behavior exemplified by Fig. 7.7: For a long duration, the random magnetization m hesitates so much between the two stable values $+m_{\rm F}$ and $-m_{\rm F}$ that a wide range of values of m in the interval $-m_{\rm F}$, $+m_{\rm F}$ have nearly equal probabilities. We have proposed to term this anomalous situation *Buridan's ass effect* [192], referring to the celebrated argument attributed to Buridan, a dialectician of the first half of the XIVth century: An ass placed just half way between two identical bales of hay would theoretically stay there indefinitely and starve to death, because the absence of causal reason to choose one bale or the other would let it hesitate for ever, at least according to Buridan[47].

In fact, major qualitative differences distinguish the situation in which the final state $+m_{\rm F}$ is reached with probability $\mathcal{P}_+ = 1$ (subsection 7.2) from the situation in which significant probabilities $\mathcal{P}_+$ and $\mathcal{P}_-$ to reach either $+m_{\rm F}$ or $-m_{\rm F}$ exist (subsection 7.3). In the first case, the peak of $P(m, t)$ moves simply from 0 to $+m_{\rm F}$; the *fluctuation of m remains of order* $1/\sqrt{N}$ at all times, even when it is largest, at the time when the average drift velocity $v(\mu)$ is maximum (Eq. (7.45)). In the second case, the exponential rise of the fluctuations of m leads, during a long period, to a broad and flat distribution $P(m, t)$, with a *shape independent of N*.

In both cases, we encountered (for $q = 2$) the same *time scale* $\tau_{\rm reg} = \hbar/\gamma(J - T)$, which characterizes the first stage of the motion described by either (7.43), (7.39) or (7.79). However, in the first case, $\mathcal{P}_+ \simeq 1$, the duration (7.48) of the whole process is just the product of $\tau_{\rm reg}$ by a factor independent of N, of order $2 \ln[m_{\rm F}(J - T)/g]$, as also shown by (E.6), whereas in the second case, $\mathcal{P}_+ < 1$, the dynamics becomes infinitely slow in the large N limit. The characteristic time $\tau_{\rm flat}$ at which the distribution is flat, given by (7.69), is of order $\tau_{\rm reg} \ln[\sqrt{N} m_{\rm F}(J - T)/J]$. Suzuki's scaling regime [213, 214, 215, 216, 217] is attained over times of order $\tau_{\rm flat}$. Then $P(m, t)$ does not depend on N for $N \to \infty$, but the duration of the relaxation process is *large as* $\ln \sqrt{N}$. It is this long delay which allows the initial distribution, narrow as $1/\sqrt{N}$, to broaden enormously instead of being shifted towards one side.

Buridan's argument has been regarded as a forerunner of the idea of probability. The infinite time during which the ass remains at $m = 0$ is recovered here for $N \to \infty$. An infinite duration of the process is also found in the absence

[47] The effect was never observed, though, at the farm where the last author of this work grew up

of diffusion in the limit of a narrow initial distribution ($\delta_0 \to 0$). The flatness of $P(m,t)$ at times of order $\tau_{\rm flat}$ means that at such times *we cannot predict* at all where the ass will be on the interval $-m_{\rm F}$, $+m_{\rm F}$, an idea that Buridan could not emit before the elaboration of the concept of probability. The counterpart of the field h, for Buridan's ass, would be a strong wind which pushes it; the counterpart of μ_0 would be a different distance from the two bales of hay; in both of these cases, the behavior of the ass becomes predictable within small fluctuations.

Since the slowing factor which distinguishes the time scales in the two regimes is *logarithmic*, very large values of N are required to exhibit a large ratio for the relaxation times. In Figs. 7.5, 7.7 and 7.8 we have taken $N = 1000$ so as to make the fluctuations in $1/\sqrt{N}$ visible. As a consequence, the duration of the registration is hardly larger in Fig. 7.7 than in Fig. 7.5.

Except during the final equilibration, the magnet keeps during its evolution some memory of its initial state through δ_1 (Eq. (7.40)). If the bifurcation is inactive (§ 7.2.3), this quantity occurs through the variance (7.45) of the distribution. If it is active (§ 7.3.2), it occurs through the time scale $\tau_{\rm flat}$, but not through the shape of $P(m,t)$.

Our model of the ferromagnet is well-known for being exactly solvable at equilibrium in the large N limit by means of a static mean-field approach. In the single peak regime, the dynamics expressed by (7.30) is also the same as the outcome of a time-dependent mean-field approach. However, in the regime leading to two peaks at $+m_{\rm F}$ and $-m_{\rm F}$, *no mean-field approximation can describe the dynamics* even for large N, due to the giant fluctuations. The intuitive idea that the variable m, because it is macroscopic, should display fluctuations small as $1/\sqrt{N}$ is then wrong, except near the initial time or for each peak of $P(m,t)$ near the final equilibrium.

The giant fluctuations of m which occur in Buridan's ass regime may be regarded as a dynamic counterpart of the fluctuations that occur at equilibrium at the critical point $T = J$ [196, 197]. In both cases, the order parameter, although macroscopic, presents large fluctuations in the large N limit, so that its treatment requires statistical mechanics. Although no temperature can be associated with M during the relaxation process, the transition from $T_0 > J$ to $T < J$ involves intermediate states which behave as in the critical region. The well-known critical fluctuations and critical slowing down manifest themselves here by the large uncertainty on m displayed during a long delay by $P(m,t)$.

Suzuki's slowing down and flattening [213, 214, 215, 216, 217] take place not only in the symmetric case (§ 7.3.2), but also in the asymmetric case (§ 7.3.3), provided $\mathcal{P}_-$ is sizeable. Thus the occurrence of Buridan's ass effect is governed by the non vanishing of the probabilities $\mathcal{P}_+$ and $\mathcal{P}_-$ of $+m_{\rm F}$ and $-m_{\rm F}$ *in the final state*. Everything takes place as if the behavior were governed by final causes: The process is deterministic if the target is unique; it displays large uncertainties and is slow if hesitation may lead to one target or to the other. These features reflect in a probabilistic language, first, the slowness of the pure drift motion near the bifurcation which implies a long random delay to set m into motion, and, second, the importance of the diffusion term there.

8. Imperfect measurements and failures

Niet al wat blinkt is goud [48]
Tout ce qui brille n'est pas or [48]
Dutch and french proverbs

In sections 5 to 7, we have solved our model under conditions on the various parameters which ensure that the measurement is ideal. We will resume these conditions in section 9.4. We explore beforehand some situations in which they may be violated, so as to set forth how each violation prevents the dynamical process from being usable as a quantum measurement. We have already seen that, in case the spin-apparatus interaction presents no randomness, the *magnet-bath interaction* should not be too small; otherwise, recurrence would occur in the off-diagonal blocks of the density operator, violating von Neumann's reduction (§ 5.1). We have also shown how a spin-apparatus coupling that is too weak may prevent the registration to take place for $q = 4$ (§ 7.2.2 and § 7.2.4), or may lead to wrong results for $q = 2$ (§ 7.3.3). We study below what happens if the number of *degrees of freedom of the pointer* is small, by letting $N = 2$ (subsection 8.1); see [112, 114] for model studies along this line. We then examine the importance of the *commutation* [2, 8, 195, 218, 219, 220] of the measured observable with the Hamiltonian of the system (subsection 8.2). Finally we exhibit a process which might allow imperfect *simultaneous measurements of non-commuting observables* (subsection 8.3) [221, 222, 223, 224, 225, 226].

[48] All that glitters is not gold

The solution of these extensions of the Curie–Weiss model involves many technicalities that we could not skip. The reader interested only in the results will find them in subsection 9.5.

8.1. Microscopic pointer

Ce que je sais le mieux, c'est mon commencement[49]
Jean Racine, Les Plaideurs

In the above sections, we have relied on the large number N of degrees of freedom of the magnet M. As the statistical fluctuations of the magnetization m are then weak, the magnet can behave as a macroscopic pointer with classical features. Moreover the reduction time $\tau_{\rm red}$ is the shortest among all the characteristic times (section 5) because it behaves as $1/\sqrt{N}$. The large value of N was also used (section 7) to describe the registration process by means of a partial differential equation. It is natural to wonder whether a small value of N can preserve the characteristic properties of a quantum measurement. Actually the irreversibility of any measurement process (subsection 6.2) requires the apparatus to be large. In subsection 6.1, we showed that the irreversibility of the reduction can be ensured by a large value of N and a randomness in the couplings g_n, $n = 1, ...N$ (subsection 6.2) ; but this irreversibility, as well as that of the registration (section 7), can also be caused by the large size of the bath. For small N, the irreversibility of both the reduction and the registration should be ensured by B. We now study the extreme situation in which $N = 2$.

8.1.1. Need for a low temperature

For $N = 2$ the magnetization $\hat{m}$ has the eigenvalue $m = 0$ with multiplicity 2, regarded as "paramagnetic", and two non-degenerate eigenvalues $m = +1$ and $m = -1$ regarded as "ferromagnetic". Since $\hat{m}^4 = \hat{m}^2$, we may set $J_4 = 0$ and denote $J_2 = J$. The corresponding eigenenergies of $\hat{H}_M$ are 0 and $-J$, and those of the Hamiltonian $\hat{H}_i$ of Eq. (4.6) are $-2gs_im - Jm^2$.

The equations of motion of § 4.4.2 involve only the two frequencies $\omega_\pm$, defined by

$$\hbar\omega_\pm \equiv J \pm 2g, \tag{8.1}$$

and they have the detailed form (Notice that $P \equiv \frac{1}{2}NP^{\rm dis} = P^{\rm dis}$ for $N = 2$)

$$\frac{{\rm d}P_{\uparrow\uparrow}(0,t)}{{\rm d}t} = \frac{2\gamma}{\hbar^2}\left\{2P_{\uparrow\uparrow}(1,t)\tilde{K}_t(\omega_+) + 2P_{\uparrow\uparrow}(-1,t)\tilde{K}_t(\omega_-) - P_{\uparrow\uparrow}(0,t)\left[\tilde{K}_t(-\omega_+) + \tilde{K}_t(-\omega_-)\right]\right\}, \tag{8.2}$$

$$\frac{{\rm d}P_{\uparrow\uparrow}(\pm1,t)}{{\rm d}t} = \frac{2\gamma}{\hbar^2}\left[P_{\uparrow\uparrow}(0,t)\tilde{K}_t(-\omega_\pm) - 2P_{\uparrow\uparrow}(\pm1,t)\tilde{K}_t(\omega_\pm)\right], \tag{8.3}$$

$$\frac{{\rm d}P_{\uparrow\downarrow}(0,t)}{{\rm d}t} = \frac{2\gamma}{\hbar^2}\left\{2P_{\uparrow\downarrow}(1,t)\left[\tilde{K}_{t>}(\omega_+) + \tilde{K}_{t<}(\omega_-)\right] + 2P_{\uparrow\downarrow}(-1,t)\left[\tilde{K}_{t>}(\omega_-) + \tilde{K}_{t<}(\omega_+)\right] - P_{\uparrow\downarrow}(0,t)\left[\tilde{K}_t(-\omega_+) + \tilde{K}_t(-\omega_-)\right]\right\}, \tag{8.4}$$

$$\frac{{\rm d}P_{\uparrow\downarrow}(\pm1,t)}{{\rm d}t} \mp \frac{4ig}{\hbar}P_{\uparrow\downarrow}(\pm1,t) = \frac{2\gamma}{\hbar^2}\left\{P_{\uparrow\downarrow}(0,t)\left[\tilde{K}_{t>}(-\omega_\pm) + \tilde{K}_{t<}(-\omega_\mp)\right] - 2P_{\uparrow\downarrow}(\pm1,t)\left[\tilde{K}_{t>}(\omega_\pm) + \tilde{K}_{t<}(\omega_\mp)\right]\right\}. \tag{8.5}$$

As initial state for M we take the "paramagnetic" one, $P_{\rm M}(0) = 1$, $P_{\rm M}(\pm1) = 0$, prepared by letting $T_0 \gg J$ or with a radiofrequency field as in § 3.3.3. (We recall that $P_{\rm M} = P_{\uparrow\uparrow} + P_{\downarrow\downarrow}$.) The initial conditions are thus $P_{ij}(m,0) = r_{ij}(0)\delta_{m,0}$.

In order to identify the process with an ideal measurement, we need to find at sufficiently large times (*i*) the von Neumann reduction, expressed by $P_{\uparrow\downarrow}(m,t) \to 0$; (*ii*) the Born rule, expressed by the system-pointer correlations $P_{\uparrow\uparrow}(m,t) \to r_{\uparrow\uparrow}(0)\delta_{m,1}$ and $P_{\downarrow\downarrow}(m,t) \to r_{\downarrow\downarrow}(0)\delta_{m,-1}$. This requires, for the magnet in contact with the bath, a long lifetime for the "ferromagnetic" states $m = +1$ and $m = -1$. However, the breaking of invariance, which for large N allows the ferromagnetic state where m is concentrated near $+m_{\rm F}$ to be stable, cannot occur here: Nothing hinders here the coupling with the bath to induce transitions from $m = +1$ to $m = -1$ through $m = 0$, so that for large times $P(+1,t)$ and $P(-1,t)$, where $P(m,t) \equiv P_{\uparrow\uparrow}(m,t)/r_{\uparrow\uparrow}(0)$, tend to a common value close to $\frac{1}{2}$ for $T \ll J$.

[49] What I know the best I shall begin with

This is made obvious by the expression of (7.21) of the H-theorem. The dissipation in the Markovian regime [145, 199, 200] reads here

$$\frac{\mathrm{d}F(t)}{\mathrm{d}t} = -\frac{\gamma}{2\beta}\frac{\omega_+ e^{-|\omega_+|/\Gamma}}{e^{\beta\hbar\omega_+}-1}\Big[P(0,t)e^{\beta\hbar\omega_+}-2P(1,t)\Big]\ln\frac{P(0,t)e^{\beta\hbar\omega_+}}{2P(1,t)} + [\omega_+ \mapsto \omega_-, \quad P(1,t)\mapsto P(-1,t)], \quad (8.6)$$

and the free energy decreases until the equilibrium $2P_{\mathrm{M}}(\pm1) = P_{\mathrm{M}}(0)\exp\beta\hbar\omega_\pm$ is reached. The only possibility to preserve a long lifetime for the state $m = +1$ is to have a low transition rate from $m = +1$ to $m = 0$, that is, according to (8.2), a small $\tilde{K}_t(\omega_+)$. This quantity is dominated in the Markovian regime by a factor $\exp(-\beta\hbar\omega_+)$. Hence, unless $T \ll J$, the apparatus cannot keep the result of the measurement registered during a significant time, after the interaction with S has been switched off. If this condition is satisfied, we may expect to reach for some lapse of time a state where $P(1,t) = P_{\uparrow\uparrow}(1,t)/r_{\uparrow\uparrow}(0)$ remains close to 1 while $P(0,t)$ is small as $P(-1,t)$.

Moreover, a faithful registration requires that the coupling g with S is sufficiently large so that the final state, in the evolution of $P_{\uparrow\uparrow}(m,t)$, has a very small probability to yield $m = -1$. Since in the Markovian regime the transition probabilities in (8.2) and (8.3) depend on g through $\omega_\pm = J \pm 2g$ in $\tilde{K}(\omega_\pm)$ and $\tilde{K}(-\omega_\pm)$ [145, 199, 200], and since this dependence arises mainly from $\exp\beta\hbar\omega_\pm$, we must have $\exp 4\beta g \gg 1$. The coupling g should moreover not modify much the spectrum, so that we are led to impose the conditions

$$T \ll 4g \ll J. \quad (8.7)$$

8.1.2. Spontaneous relaxation

As we did in § 7.3.2 for large N, we focus here on the evolution of the probabilities $P(m,t) \equiv P_{\uparrow\uparrow}(m,t)/r_{\uparrow\uparrow}(0)$ for the apparatus alone. It is governed by equations (8.2) and (8.3) in which $\omega_+ = \omega_- = J/\hbar$. For a weak coupling γ we expect that the Markovian regime, where $\tilde{K}_t(\omega) = \tilde{K}(\omega)$ will be reached before the probabilities have deviated much from their initial value. The equations of motion then reduce to

$$\tau\frac{\mathrm{d}P(0,t)}{\mathrm{d}t} = e^{-J/T}\,[P(1,t)+P(-1,t)] - P(0,t), \qquad \tau\frac{\mathrm{d}P(\pm1,t)}{\mathrm{d}t} = \frac{1}{2}P(0,t) - e^{-J/T}P(\pm1,t), \quad (8.8)$$

where we made use of

$$\tilde{K}\left(\frac{J}{\hbar}\right) = e^{-J/T}\,\tilde{K}\left(-\frac{J}{\hbar}\right) = \frac{\hbar J}{4}\frac{e^{-J/\hbar\Gamma}}{e^{J/T}-1}, \quad (8.9)$$

as well as $J/T \gg 1$ and $J/\hbar\Gamma \ll 1$, and where we defined a characteristic time related to the spin-spin coupling as

$$\tau \equiv \tau_J = \frac{\hbar}{\gamma J}. \quad (8.10)$$

The Markovian approximation is justified provided this characteristic time scale τ is longer than the time t after which $\tilde{K}_t(\omega) = \tilde{K}(\omega)$, that is, for

$$\gamma \ll \frac{T}{J}. \quad (8.11)$$

The general solution of (8.8), (8.8), obtained by diagonalization, is expressed by

$$\begin{aligned} &P(0,t)+P(1,t)+P(-1,t) = 1,\\ &P(0,t) - e^{-J/T}\,[P(1,t)+P(-1,t)] \propto \exp\left[-\frac{t}{\tau}(1+e^{-J/T})\right] \approx \exp\left(-\frac{t}{\tau}\right),\\ &P(1,t) - P(-1,t) \propto \exp\left(-\frac{t}{\tau}e^{-J/T}\right). \end{aligned} \quad (8.12)$$

Let us first consider the *relaxation of the initial paramagnetic state*, for which $P(0,0) = 1$. We find from the above equations

$$P(0,t) = \frac{e^{-t/\tau}+e^{-J/T}}{1+e^{-J/T}}, \qquad P(1,t) = P(-1,t) = \frac{1-e^{-t/\tau}}{2(1+e^{-J/T})}. \quad (8.13)$$

The lifetime of this initial unstable state is therefore $\tau = \hbar/\gamma J$. In a measurement, the interaction g between S and A must thus be switched on rapidly after the preparation (§ 3.3.3), in a delay $\tau_{\rm init} \ll \tau$ so that $P(0)$ is still close to 1 when the measurement process begins.

We now evaluate the delay $\tau_{\rm obs}$ *during which the pointer keeps its value and can be observed*, after the measurement is achieved and after the coupling with S is switched off. If in the sector $\uparrow\uparrow$ the value $m = 1$ is reached at some time t_1 with a near certainty, the probabilities evolve later on, according to the above equations, as

$$P(0, t_1 + t) = \frac{(1 - e^{-t/\tau})e^{-J/T}}{1 + e^{-J/T}}, \qquad P(\pm 1, t_1 + t) = \frac{1}{2}\left[\frac{1 + e^{-t/\tau}e^{-J/T}}{1 + e^{-J/T}} \pm \exp\left(-\frac{t}{\tau}e^{-J/T}\right)\right]. \tag{8.14}$$

As expected, the information is lost for $t \to \infty$, or, more precisely, for $t \gg \tau \exp(J/T)$, since $P(1, t)$ and $P(-1, t)$ then tend to $\frac{1}{2}$. However, during the time lapse $\tau \ll t \ll \tau \exp(J/T)$, $P(1, t)$ retains a value $1 - \frac{1}{2}\exp(-J/T)$ close to 1, so that the probability of a false registration is then weak. Although microscopic, the pointer is a rather robust and reliable device provided $T \ll J$, on the time scale $t \ll \tau_{\rm obs}$ where the *observation time* is

$$\tau_{\rm obs} = \tau e^{J/T} = \frac{\hbar}{\gamma J}e^{J/T}. \tag{8.15}$$

8.1.3. Registration

We now study the time-dependence of the registration process, and determine the probability to reach a false result, that is, to find $m = -1$ in the sector $\uparrow\uparrow$. In the Markovian regime and under the conditions (8.7), the equations of motion (8.2), (8.3) for the probabilities $P(m, t) = P_{\uparrow\uparrow}(m, t)/r_{\uparrow\uparrow}(0)$ read

$$\tau\frac{{\rm d}P(0,t)}{{\rm d}t} = e^{-(J+2g)/T}P(1,t) + e^{-(J-2g)/T}P(-1,t) - P(0,t), \tag{8.16}$$

$$\tau\frac{{\rm d}P(\pm 1,t)}{{\rm d}t} = \frac{1}{2}P(0,t) - e^{-(J\pm 2g)/T}P(\pm 1,t). \tag{8.17}$$

We have disregarded in each term contributions of relative order $\exp(-J/T)$ and $2g/J$. The general solution of Eqs. (8.16), (8.17) is obtained by diagonalizing their 3×3 matrix. Its three eigenvalues $-z$ are the solutions of

$$z^3 - z^2\left(1 + 2e^{-J/T}\cosh\frac{2g}{T}\right) + ze^{-J/T}\left(\cosh\frac{2g}{T} + e^{-J/T}\right) = 0, \tag{8.18}$$

that is, apart from $z = 0$,

$$z = \frac{1}{2} + e^{-J/T}\cosh\frac{2g}{T} \pm \frac{1}{2}\sqrt{1 + 4e^{-2J/T}\sinh^2\frac{2g}{T}}, \tag{8.19}$$

which under the conditions (8.7) reduce to $z \simeq 1$ and $z \simeq \exp(-J/T)\cosh 2g/T \simeq \frac{1}{2}\exp[-(J - 2g)/T]$. The corresponding characteristic times τ/z are therefore $\tau = \hbar/\gamma J$ and

$$\tau_{\rm reg} = 2\tau e^{(J-2g)/T} = \frac{2\hbar}{\gamma J}e^{(J-2g)/T}. \tag{8.20}$$

The solutions of (8.16),(8.17) are then given by

$$P(0,t) + P(1,t) + P(-1,t) = 1, \tag{8.21}$$

$$P(0,t) - e^{-(J+2g)/T}P(1,t) - e^{-(J-2g)/T}P(-1,t) \propto e^{-t/\tau}, \tag{8.22}$$

$$P(1,t) - P(-1,t) - \tanh\frac{2g}{T} \propto e^{-t/\tau_{\rm reg}}. \tag{8.23}$$

The decay time τ associated with the combination (8.22) is much shorter than the time $\tau_{\rm reg}$ which occurs in (8.23).

With the initial condition $P(0,0) = 1$ we obtain, dropping contributions small as $\exp(-J/T)$,

$$P(0,t) = e^{-t/\tau}, \tag{8.24}$$

$$P(1,t) = \frac{1}{2}\left[\left(1 - e^{-t/\tau}\right) + \tanh\frac{2g}{T}\left(1 - e^{-t/\tau_{\rm reg}}\right)\right], \tag{8.25}$$

$$P(-1,t) = \frac{1}{2}\left[\left(1 - e^{-t/\tau}\right) - \tanh\frac{2g}{T}\left(1 - e^{-t/\tau_{\rm reg}}\right)\right]. \tag{8.26}$$

The evolution takes place in two stages, first on the time scale $\tau = \hbar/\gamma J$, then on the much larger time scale $\tau_{\rm reg} = 2\tau\exp[(J-2g)/T]$.

During the first stage, M relaxes from the paramagnetic initial state $m = 0$ *to both "ferromagnetic" states* $m = +1$ and $m = -1$, with equal probabilities, as in the spontaneous process where $g = 0$. At the end of this stage, at times $\tau \ll t \ll \tau_{\rm reg}$ we reach a nearly stationary situation in which $P(0,t)$ is small as $2\exp(-J/T)$, while $P(1,t)$ and $P(-1,t)$ are close to $\frac{1}{2}$. Unexpectedly, in spite of the presence of the coupling g which is large compared to T, the magnet M remains for a long time in a state close to the equilibrium state which would be associated to $g = 0$, *without any invariance breaking*. This behavior arises from the large value of the transition probabilities from $m = 0$ to $m = \pm 1$, which are proportional to $\tilde{K}(-\omega_\pm)$. For $J \pm 2g \gg T$, the latter quantity reduces to $\hbar(J \pm 2g)/4$, which is not sensitive to g for $2g \ll J$.

In contrast to the situation for large N, the magnet thus begins to *lose memory* of its initial state. For $N \gg 1$, it was the coupling g which triggered the evolution of M, inducing the motion of the peak of $P_{\uparrow\uparrow}(m,t)$, initially at $m = 0$, towards larger and larger values of m. Only an initial state involving values $m < -m_B$ led to false results at the end of the process. Here, rather surprisingly, the two possible results $m = +1$ and $m = -1$ come out nearly symmetrically after the first stage of the process, for $\tau \ll t \ll \tau_{\rm reg}$. In fact we do not even need the initial state to be "paramagnetic". On this time scale, any initial state for which $P(1,0) = P(-1,0)$ leads to $P(1,t) = P(-1,t) \simeq \frac{1}{2}$. (An arbitrary initial condition would lead to $P(\pm 1,t) = P(\pm 1,0) + \frac{1}{2}P(0,0)$.)

Fortunately, when t approaches $\tau_{\rm reg}$ the effect of g is felt. For $t \gg \tau_{\rm reg}$ the probabilities $P(m,t)$ reach the values

$$P(1,t) = \frac{1}{1+e^{-4g/T}}, \qquad P(-1,t) = \frac{e^{-4g/T}}{1+e^{-4g/T}}, \qquad P(0,t) = 2e^{-J/T}, \tag{8.27}$$

which correspond to the thermal equilibrium of M in the field g. Thus, the *probability of a false measurement* is here

$$\mathcal{P}_- = e^{-4g/T}, \tag{8.28}$$

and it can be small. On the other hand, the *registration time* is $\tau_{\rm reg}$, and the registration can be achieved only if the interaction $\hat{H}_{\rm SA}$ remains switched on during a delay larger than $\tau_{\rm reg}$. After this delay, if we switch off the coupling g, the result remains registered for a time which allows observation, since $\tau_{\rm obs}$, determined in § 8.1.2, is much larger than $\tau_{\rm red}$.

Thus, not only the first stage of the registration process is odd, but also the second one. The mechanism at play in section 7 was a *dynamical breaking of invariance* whereas here we have to rely on the *establishment of thermal equilibrium in the presence of* g. The coupling should be kept active for a long time until the values (8.27) are reached, whereas for $N \gg 1$, only the beginning of the evolution of $P_{\uparrow\uparrow}(m,t)$ required the presence of the coupling g; afterwards $P_{\uparrow\uparrow}$ reached the ferromagnetic peak at $m = m_{\rm F}$, and remained there stably.

For $N = 2$ the possibility of registration on the time scale $\tau_{\rm reg}$ relies on the form of the transition probabilities from $m = \pm 1$ to $m = 0$, which are proportional to $\tilde{K}(\omega_\pm)$. Although small as $\exp(-\beta\hbar\omega_\pm)\tilde{K}(-\omega_\pm)$, these transition probabilities contain a factor $\exp(-\beta\hbar\omega_\pm) \propto \exp(\mp 2g/T)$ which, since $2g \gg T$, strongly distinguishes $+1$ from -1, whereas $\tilde{K}(-\omega_+) \simeq \tilde{K}(-\omega_-)$. Hence the transition rate from $m = -1$ to $m = 0$, behaving as $\exp[-(J-2g)/T]$, allows $P(0)$ to slowly increase at the expense of $P(-1)$, then to rapidly decay symmetrically. Since the transition rate from $m = +1$ to $m = 0$, behaving as $\exp[-(J+2g)/T]$, is much weaker, the resulting increase of $P(1)$ remains gained. Altogether $P(1,t)$ rises in two steps, from 0 to $\frac{1}{2}$ on the time scale τ, then from $\frac{1}{2}$ to nearly 1 on the time scale $\tau_{\rm reg}$, as shown by (8.25). In the meanwhile, $P(-1,t)$ rises from 0 to $\frac{1}{2}$, then decreases back to 0, ensuring a correct registration only at the end of the process, while $P(0,t)$ remains nearly 0 between τ and $\tau_{\rm reg}$.

8.1.4. Reduction

It remains to study the evolution of the off-diagonal blocks of the density operator $\hat{D}$, which are characterized by the three functions of time $P_{\uparrow\downarrow}(m,t)$. Their equations of motion (8.4), (8.5) involve oscillations in $P_{\uparrow\downarrow}(\pm1,t)$ with frequency $2g/\pi\hbar$ generated by the coupling g with S and by a relaxation process generated by the bath. Since the oscillations are not necessarily rapid, and since γ is small, the damping effect of the bath is expected to occur over times large compared to $\hbar/T$, so that we can again work in the Markovian regime. Moreover, since $g \ll J$, we are led to replace ω_+ and ω_- in $\tilde{K}_{t>}$ and $\tilde{K}_{t<}$ by $J\hbar$. Hence, we can replace, for instance, $\tilde{K}_{t>}(\omega_+) + \tilde{K}_{t<}(\omega_-)$ by $\tilde{K}(J/\hbar)$.

The equations of motion for the set $P_{\uparrow\downarrow}(m,t)$ are thus simplified into

$$\tau\frac{\mathrm{d}P_{\uparrow\downarrow}(0,t)}{\mathrm{d}t} = \varepsilon\left[P_{\uparrow\downarrow}(1,t) + P_{\uparrow\downarrow}(-1,t)\right] - P_{\uparrow\downarrow}(0,t), \tag{8.29}$$

$$\tau\frac{\mathrm{d}P_{\uparrow\downarrow}(\pm1,t)}{\mathrm{d}t} = \pm i\lambda P_{\uparrow\downarrow}(\pm1,t) + \frac{1}{2}P_{\uparrow\downarrow}(0,t) - \varepsilon P_{\uparrow\downarrow}(\pm1,t), \tag{8.30}$$

where ε and λ are defined by

$$\varepsilon = e^{-J/T}, \qquad \lambda = \frac{4g}{\gamma J}, \tag{8.31}$$

with $\gamma \ll 1$, $g \ll T \ll J$. The reduction process is governed by the interplay between the oscillations in $P(\pm1,t)$, generated by the coupling g between M and S, and the damping due to the bath. The two dimensionless parameters λ and ε characterize these effects.

The eigenvalues of the matrix relating $-\tau\mathrm{d}P_{\uparrow\downarrow}(m,t)/\mathrm{d}t$ to $P_{\uparrow\downarrow}(m,t)$ are the solutions of the equation

$$(z-1)[(z-\varepsilon)^2 + \lambda^2] - \varepsilon(z-\varepsilon) = 0. \tag{8.32}$$

The largest eigenvalue behaves for $T \ll J$ as

$$z_0 \approx 1 + \frac{\varepsilon}{1+\lambda^2} + \frac{\varepsilon^2\lambda^2(1-\lambda^2)}{(1+\lambda^2)^3}, \tag{8.33}$$

whereas the other two eigenvalues z_1 and z_2, obtained from

$$z^2 - z\varepsilon\left(\frac{1+2\lambda^2}{1+\lambda^2} + \frac{\varepsilon^2\lambda^2(\lambda^2-1)}{(1+\lambda^2)^3}\right) + \lambda^2\left(1 - \frac{\varepsilon}{1+\lambda^2} + \frac{\varepsilon^2(1+\lambda^4)}{(1+\lambda^2)^3}\right) = 0, \tag{8.34}$$

have a real part small as ε. The solution of (8.29), (8.30), with the initial condition $P_{\uparrow\downarrow}(m,0) = r_{\uparrow\downarrow}(0)\delta_{m,0}$ is given by

$$P_{\uparrow\downarrow}(0,t) = r_{\uparrow\downarrow}(0)\left[e^{-z_0t/\tau} - \frac{(z_1-\varepsilon)^2+\lambda^2}{(z_0-z_1)(z_1-z_2)}(e^{-z_1t/\tau} - e^{-z_0t/\tau}) - \frac{(z_2-\varepsilon)^2+\lambda^2}{(z_0-z_2)(z_2-z_1)}(e^{-z_2t/\tau} - e^{-z_0t/\tau})\right] \tag{8.35}$$

$$P_{\uparrow\downarrow}(\pm1,t) = r_{\uparrow\downarrow}(0)\left[\frac{z_1-\varepsilon\mp i\lambda}{2(z_0-z_1)(z_1-z_2)}\left(e^{-z_1t/\tau} - e^{-z_0t/\tau}\right) + \frac{z_2-\varepsilon\mp i\lambda}{2(z_0-z_2)(z_2-z_1)}\left(e^{-z_2t/\tau} - e^{-z_0t/\tau}\right)\right]. \tag{8.36}$$

According to (8.35), the first term of $P_{\uparrow\downarrow}(0,t)$ is damped for $\varepsilon \ll 1$ over the time scale $\tau = \hbar/\gamma J$, just as $P_{\uparrow\uparrow}(0,t)$ in the registration process. However, here again, the two other quantities $|P_{\uparrow\downarrow}(\pm1,t)|$ increase in the meanwhile and the reduction of the state is far from being achieved after the time τ. In fact, all three components $P_{\uparrow\downarrow}(m,t)$ survive over a much longer delay, which depends on the ratio $2\lambda/\varepsilon$.

In the overdamped situation $2\lambda < \varepsilon$ or $8g < \gamma J\exp(-J/T)$, the eigenvalues

$$z_{1,2} = \frac{1}{2}\varepsilon \pm \frac{1}{2}\sqrt{\varepsilon^2 - 4\lambda^2} \tag{8.37}$$

are real, so that we get, in addition to the relaxation time τ, two much longer off-diagonal relaxation times, $\tau_{1,2} = \tau/z_{1,2}$. The long-time behavior of $P_{\uparrow\downarrow}(m,t)$, governed by z_2, is

$$P_{\uparrow\downarrow}(0,t) \sim r_{\uparrow\downarrow}(0)\frac{\varepsilon(\varepsilon+\sqrt{\varepsilon^2-4\lambda^2})}{2\sqrt{\varepsilon^2-4\lambda^2}}e^{-t/\tau_{\mathrm{red}}}, \qquad P_{\uparrow\downarrow}(\pm1,t) \sim r_{\uparrow\downarrow}(0)\frac{\varepsilon \pm 2i\lambda + \sqrt{\varepsilon^2-4\lambda^2}}{4\sqrt{\varepsilon^2-4\lambda^2}}e^{-t/\tau_{\mathrm{red}}}. \tag{8.38}$$

The reduction time

$$\tau_{\rm red} = \frac{\tau}{2\lambda^2}\left(\varepsilon + \sqrt{\varepsilon^2 - 4\lambda^2}\right) = \frac{\hbar\gamma J}{32g^2}\left(e^{-J/T} + \sqrt{e^{-2J/T} - \frac{64g^2}{\gamma^2 J^2}}\right) \tag{8.39}$$

which characterizes the decay of $\langle\hat{s}_x\rangle$, $\langle\hat{s}_x\rangle$, and of their correlations with $\hat{m}$, is here much longer than the registration time (10), since $\tau_{\rm red}/\tau_{\rm reg}$ is of order $(\varepsilon/2\lambda)^2\exp(2g/T)$, and even larger than $\tau_{\rm obs}$. The quantities $P_{\uparrow\downarrow}(m,t)$ remain for a long time proportional to $r_{\uparrow\downarrow}(0)$, with a coefficient of order 1 for $P_{\uparrow\downarrow}(\pm1,t)$, of order ε for $P_{\uparrow\downarrow}(0,t)$. Reduction is thus here a much slower process than registration: equilibrium is reached much faster for the diagonal elements (8.27) than for the off-diagonal ones which are long to disappear. Let us stress that for the present case of a small apparatus, they disappear due to the bath ("environment-induced decoherence" [28, 29, 35, 147, 148, 149, 150]) rather than, as in our previous discussion of a large apparatus, due to fast dephasing caused by the large size of M.

For $2\lambda > \varepsilon$, we are in an oscillating situation, where the eigenvalues

$$z_{1,2} = \frac{\varepsilon}{2}\frac{1+2\lambda^2}{1+\lambda^2} \pm i\sqrt{\lambda^2 - \frac{\varepsilon^2}{4} - \frac{\varepsilon\lambda^2}{1+\lambda^2}} \tag{8.40}$$

are complex conjugate. (Nothing prevents λ from being large.) The long-time behavior is given by

$$\begin{aligned} P_{\uparrow\downarrow}(0,t) &\sim \frac{\varepsilon r_{\uparrow\downarrow}(0)}{(1+\lambda^2)^2} e^{-t/\tau_{\rm red}}\left[(1-\lambda^2)\cos\frac{2\pi t}{\theta} + \frac{2\lambda^2}{\sqrt{\lambda^2-\varepsilon^2/4}}\sin\frac{2\pi t}{\theta}\right], \\ P_{\uparrow\downarrow}(\pm1,t) &\sim \frac{r_{\uparrow\downarrow}(0)}{2(1\pm i\lambda)} e^{-t/\tau_{\rm red}}\left[\cos\frac{2\pi t}{\theta} \pm \frac{i\lambda}{\sqrt{\lambda^2-\varepsilon^2/4}}\sin\frac{2\pi t}{\theta}\right], \end{aligned} \tag{8.41}$$

with a reduction time

$$\tau_{\rm red} = \frac{2(1+\lambda^2)}{\varepsilon(1+2\lambda^2)}\tau = \frac{2\hbar e^{J/T}(1+\lambda^2)}{\gamma J(1+2\lambda^2)} = \frac{1+\lambda^2}{1+2\lambda^2}\tau_{\rm reg}e^{2g/T}, \tag{8.42}$$

again much larger than the registration time. While being damped, these functions oscillate with a period

$$\theta = \frac{2\pi\tau}{\sqrt{\lambda^2-\varepsilon^2/4}} \tag{8.43}$$

shorter than $\tau_{\rm red}$ if $2\lambda > \varepsilon\sqrt{4\pi^2+1}$. The *reduction time* (8.42) practically does not depend on g (within a factor 2 when 2λ varies from ε to ∞), in contrast to both the reduction time of section 5 and the irreversibility time of section 6. The present reduction time is comparable to the lifetime $\tau_{\rm obs}$ of an initial pure state $m = +1$ when it spontaneously decays towards $m = \pm1$ with equal probabilities (§ 8.1.2). Hence in both cases the reduction takes place over the delay during which the result of the measurement can be observed.

For $\lambda \gg \varepsilon$ and $t \gg \tau$, the off-diagonal contributions (8.41) to $\hat{D}$ are governed by

$$P_{\uparrow\downarrow}(\pm1,t) \sim \frac{r_{\uparrow\downarrow}(0)}{2(1\pm i\lambda)} e^{-t/\tau_{\rm red} \pm i\lambda t/\tau}. \tag{8.44}$$

The effects on M of S and B are well separated: the oscillations are the same as for $\gamma = 0$, while the decay, with characteristic time $\tau/\varepsilon = (\hbar/\gamma J)\exp(J/T)$, is a pure effect of the bath. The amplitude becomes small for $\lambda \gg 1$, that is, $g \gg \gamma J$.

8.1.5. Is this process with bath-induced decoherence a measurement?

When the number N of degrees of freedom of the pointer is small as here, the present model appears as a specific example among the general class of models considered by Spehner and Haake [130, 131]. As shown by these authors, the reduction is then governed by the large number of degrees of freedom *of the bath*, *not of the pointer*; the reduction

is then not faster than the registration. Our detailed study allows us to compare the mechanisms of two types of measurements, for large N and for small N.

We have seen (§ 8.1.3) that for $N = 2$ as for $N \gg 1$ both couplings g and γ between S, M and B establish the diagonal correlations between $\hat{s}_z$ and $\hat{m}$ that characterize Born's rule. This result is embedded in the values reached by $P_{\uparrow\uparrow}$ and $P_{\downarrow\downarrow}$ after the time $\tau_{\rm reg} = (2\hbar/\gamma J)\exp[(J-2g)/T]$, much longer than the lifetime $\tau = \hbar/\gamma J$ of the initial state in the absence of a field or a coupling. Although this property is one important feature of a quantum measurement, its mechanism is here only a relaxation towards thermal equilibrium. The registration is fragile and does not survive beyond a delay $\tau_{\rm obs} = (\hbar/\gamma J)\exp(J/T)$ once the coupling with S is switched off. For larger N, the existence of a spontaneously broken invariance ensured the long lifetime of the ferromagnetic states, and hence the robust registration of the measurement.

The other feature of a quantum measurement, von Neumann's reduction of the state, has also been recovered for $N = 2$, but with an unsatisfactorily long time scale. For large N, the reduction process took place rapidly and was achieved before the registration in the apparatus really began, but here, whatever the parameters ε and λ, the expectation values $\langle\hat{s}_x\rangle$, $\langle\hat{s}_y\rangle$ and the off-diagonal correlations embedded in $P_{\uparrow\downarrow}$ and $P_{\downarrow\uparrow}$ fade out over a reduction time $\tau_{\rm red}$ given by (8.39) or (8.42), which is longer than the registration time and even than the observation time if $2\lambda \ll \varepsilon$. It is difficult to regard such a slow decay as the "collapse" of the state.

By studying the case $N = 2$, we wished to test whether an *environment-induced decoherence* [28, 29, 35, 147, 148, 149, 150] might cause von Neumann's reduction. Here the "environment" is the bath B, which is the source of irreversibility. It imposes thermal equilibrium to S + M, hence suppressing gradually the off-diagonal elements of $\hat{D}$ which vanish at equilibrium, a suppression that we defined as "reduction". However, usually, decoherence time scales are the shortest of all; here, for $N = 2$, contrary to what happened for $N \gg 1$, the reduction time is not shorter than the registration time.

The effect of the bath in the reduction process is therefore quite different for large and for small N. We have found for $N \gg 1$ all properties of ideal measurements. As discussed in § 5.1.2 and § 6.2.4, the rapid initial reduction was then ensured by the large size of the pointer M, whereas bath-induced decoherence played only a minor role, being only one among the two possible mechanisms of suppression of recurrences. For $N = 2$, the reduction itself is caused by the bath, but we cannot really distinguish decoherence from thermal equilibration: Although the dynamics of the diagonal and off-diagonal blocks of $\hat{D}$ are decoupled, there is no neat separation of time scales for the reduction and the registration.

Anyhow, the process that we described cannot be regarded for $N = 2$ as a full measurement. Being microscopic, the pair of spins M is not a "pointer" that can be observed directly. In order to get a stable signal, which provides us with information and which we may use at a macroscopic level, we need to couple M to a genuine macroscopic apparatus. This should be done after the time $\tau_{\rm reg} = (2\hbar/\gamma J)\exp[(J-2g)/T]$ when the correlations $P_{\uparrow\uparrow}(m,t) = r_{\uparrow\uparrow}(0)\delta_{m,1}$ and $P_{\downarrow\downarrow}(m,t) = r_{\downarrow\downarrow}(0)\delta_{m,-1}$ have been created between S and M. Then, S and M should be decoupled, and the measurement of m should be performed in the delay $\tau_{\rm obs} = (\hbar/\gamma J)\exp(J/T)$. In this hypothetical process, the decoupling of S and M will entail reduction, the correlations which survive for the duration $\tau_{\rm red} = 2\tau_{\rm obs}$ in $P_{\uparrow\downarrow}$ and $P_{\downarrow\uparrow}$ being destroyed.

Altogether, it is not legitimate for small N to regard $M + B$ as a "measurement apparatus", since registering robustly the outcomes of the process so as to read them during a long delay requires a *further apparatus* involving a *macroscopic pointer*. The system M, even accompanied with its bath, is not more than a quantum device coupled to S. However, its marginal state is represented by a diagonal density matrix, in the basis which diagonalizes $\hat{m}$, so that the respective probabilities of $m = 0$, $m = +1$ and $m = -1$, from which we will infer $r_{\uparrow\uparrow}(0)$ and $r_{\downarrow\downarrow}(0)$, can be determined by means of an apparatus with classical features.

8.1.6. Can one simultaneously "measure" non-commuting variables?

Although the process described above cannot be regarded as an ideal measurement, we have seen that it allows us to determine the diagonal elements $r_{\uparrow\uparrow}(0)$ and $r_{\downarrow\downarrow}(0)$ of the density matrix of S at the initial time. Surprisingly, the same device may also give us access to the off-diagonal elements, owing to the pathologically slow reduction. Imagine S and M are decoupled at some time $\tau_{\rm dec}$ of order $\tau_{\rm red}$. For $2\lambda \ll \varepsilon$, this time can be shorter than the observation time, so that a rapid measurement of m will inform us statistically on $r_{\uparrow\uparrow}(0)$ and $r_{\downarrow\downarrow}(0)$. However, the transverse components of the spin S have not disappeared on average, and $r_{\uparrow\downarrow}(\tau_{\rm dec})$ is given at the decoupling time and later on by

$$r_{\uparrow\downarrow}(\tau_{\rm dec}) = \sum_m P_{\uparrow\downarrow}(m, \tau_{\rm dec}) = r_{\uparrow\downarrow}(0) \frac{\varepsilon + \sqrt{\varepsilon^2 - 4\lambda^2}}{2\sqrt{\varepsilon^2 - 4\lambda^2}} e^{-\tau_{\rm dec}/\tau_{\rm red}}. \tag{8.45}$$

A measurement of S in the x-direction at a time $t > \tau_{\rm dec}$ will then provide $r_{\uparrow\downarrow}(t) + r_{\downarrow\uparrow}(t) = 2\Re r_{\downarrow\uparrow}(\tau_{\rm dec})$. If the various parameters entering (8.45) are well controlled, we can thus, through repeated measurements, determine indirectly $r_{\uparrow\downarrow}(0) + r_{\downarrow\uparrow}(0)$, as well as $r_{\uparrow\uparrow}(0)$ and $r_{\downarrow\downarrow}(0)$.

Thus a unique experimental setting may be used to determine the statistics of the non-commuting observables $\hat{s}_x$ and $\hat{s}_z$. This possibility is reminiscent of a general result [237]; see also [238, 239]. Suppose we wish to determine all the matrix elements r_{ij} of the unknown $n \times n$ density matrix of a system S at the initial time. Coupling during some delay S with a similar auxiliary system S′, the initial state of which is known, leads to some density matrix for the compound system. The set r_{ij} is thus mapped onto the n^2 diagonal elements of the latter. These diagonal elements may be measured simultaneously by means of a single apparatus, and inversion of the mapping yields the whole set r_{ij}. Here the magnet M plays the role of the auxiliary system S'; we can thus understand the paradoxical possibility of determining the statistics of both $\hat{s}_x$ and $\hat{s}_z$ with a single device.

In this context we note that the simultaneous measurement of non-commuting observables is an important chapter of modern quantum mechanics. Its recent developments are given in [221, 222, 223, 224, 225] (among other references) and reviewed in [226].

With this setup we can also repeat measurements in the z-direction and see how much lapse should be in between to avoid non-idealities.

8.2. Measuring a non-conserved quantity

L'homme est plein d'imperfections, mais ce n'est pas étonnant si l'on songe à l'époque où il a été créé[50]
Alphonse Allais

It has been stressed by Wigner [228] that an observable that does not commute with some conserved quantity of the total system (tested system S plus apparatus A) cannot be measured exactly, and the probability of unsuccessful experiments has been estimated by Araki and Yanase [195, 229]. (Modern developments of this Wigner-Araki-Yanase limitation are given in [218, 219, 220].) However, neither the irreversibility nor the dynamics of the measurement process were considered. We focus here on the extreme case in which Wigner's conserved quantity is the energy itself. We have assumed till now that the measured observable $\hat{s}_z$ commuted with the full Hamiltonian of S + A. This has allowed us to split the dynamical analysis into two separate parts: The diagonal blocks $R_{\uparrow\uparrow}, R_{\downarrow\downarrow}$ of the full density matrix of S + A are not coupled to the off-diagonal blocks $R_{\uparrow\downarrow}, R_{\downarrow\uparrow}$. This gives rise, for $N \gg 1$, to a large ratio between the time scales that characterize the reduction and the registration.

We will discuss, by solving a slightly modified version of our model, under which conditions one can still measure a quantity which is not conserved. We allow therefore transitions between different eigenvalues of $\hat{s}_z$, by introducing a magnetic field that acts on S. The part $\hat{H}_{\rm S}$ of the Hamiltonian, instead of vanishing as in (3.4), is taken as

$$\hat{H}_{\rm S} = -b\hat{s}_y. \tag{8.46}$$

We take as measuring device a large, Ising magnet, with $q = 2$ and $N \gg 1$.

We wish to study how the additional field affects the dynamics of the measurement. We shall therefore work out the equations at lowest order in b, which however need not be finite as $N \to \infty$. In fact, a crucial parameter turns out to be the combination $b/g\sqrt{N}$.

8.2.1. The changes in the dynamics

The formalism of subsection 4.2 remains unchanged, the unperturbed Hamiltonian being now

$$\hat{H}_0 = \hat{H}_{\rm S} + \hat{H}_{\rm SA} + \hat{H}_{\rm M} = -b\hat{s}_y - Ng\hat{m}\hat{s}_z - \frac{1}{2}JN\hat{m}^2. \tag{8.47}$$

[50] Man is full of imperfections, but this is not surprising if one considers when he was created

The additional contribution (8.46) enters the basic equation (4.5) in two different ways.

(*i*) On the left-hand side, the term $-\left[\hat{H}_{\rm S},\hat{D}\right]/i\hbar$ yields a contribution

$$\frac{b}{\hbar}\begin{pmatrix} \hat{R}_{\uparrow\downarrow}+\hat{R}_{\downarrow\uparrow} & \hat{R}_{\downarrow\downarrow}-\hat{R}_{\uparrow\uparrow} \\ \hat{R}_{\downarrow\downarrow}-\hat{R}_{\uparrow\uparrow} & -\hat{R}_{\uparrow\downarrow}-\hat{R}_{\downarrow\uparrow}\end{pmatrix} \tag{8.48}$$

to $\mathrm{d}\hat{D}/\mathrm{d}t$ which couples the diagonal and off-diagonal sectors of (3.18). Accordingly, we must add to the right-hand side of the equation of motion (4.16) for $\mathrm{d}P_{\uparrow\uparrow}/\mathrm{d}t$ the term $\hbar^{-1}b(P_{\uparrow\downarrow}+P_{\downarrow\uparrow})$, and subtract it from the equation for $\mathrm{d}P_{\downarrow\downarrow}/\mathrm{d}t$; we should add to the equations (4.18) for $\mathrm{d}P_{\uparrow\downarrow}/dt$ and $\mathrm{d}P_{\downarrow\uparrow}/\mathrm{d}t$ the term $\hbar^{-1}b(P_{\downarrow\downarrow}-P_{\uparrow\uparrow})$.

(*ii*) The presence of $\hat{H}_{\rm S}$ in $\hat{H}_0$ has another, indirect effect. The operators $\hat{\sigma}_a^{(n)}(u)$ defined by (4.4), which enter the right-hand side of eq. (4.5), no longer commute with $\hat{s}_z$. In fact, while $\hat{\sigma}_z^{(n)}(u)$ still equals $\hat{\sigma}_z^{(n)}$, the operators

$$\hat{\sigma}_+^{(n)}(u)=\left[\hat{\sigma}_-^{(n)}(u)\right]^\dagger=\hat{\sigma}_+^{(n)}e^{-i\hat{H}_0(\hat{m}+\delta m)u/\hbar}e^{i\hat{H}_0(\hat{m})u/\hbar}=e^{-i\hat{H}_0(\hat{m})u/\hbar}e^{i\hat{H}_0(\hat{m}-\delta m)u/\hbar}\hat{\sigma}_+^{(n)} \tag{8.49}$$

now contain contributions in $\hat{s}_x$ and $\hat{s}_y$, which can be found by using the expression (8.47) of $\hat{H}_0$ and the identity $\exp i\,\mathbf{a}\cdot\hat{\mathbf{s}}=\cos a+i\sin a\,\mathbf{a}\cdot\hat{\mathbf{s}}/a$. For $N\gg 1$ and arbitrary b, we should therefore modify the bath terms in $\mathrm{d}P_{ij}/\mathrm{d}t$ by using the expression

$$\begin{aligned}\hat{\sigma}_+^{(n)}(u) &= \hat{\sigma}^{(n)}\exp\left(\frac{2iJ\hat{m}u}{\hbar}\right)\left[\exp\left(\frac{2iNg^2\hat{m}\hat{s}_z u}{\hbar\sqrt{N^2g^2\hat{m}^2+b^2}}\right)\right. \\ &+ \left.\frac{ib}{\sqrt{N^2g^2\hat{m}^2+b^2}}\sin\left(\frac{2iNg^2\hat{m}u}{\hbar\sqrt{N^2g^2\hat{m}^2+b^2}}\right)\left(\hat{s}_y-\frac{b\hat{s}_z}{Ng\hat{m}+\sqrt{N^2g^2\hat{m}^2+b^2}}\right)\right],\end{aligned} \tag{8.50}$$

instead of (B.7); we have dropped in the square bracket contributions that oscillate rapidly as $\exp\left(2iu\sqrt{N^2g^2\hat{m}^2+b^2}/\hbar\right)$ with factors $\hat{s}_x$ and coefficients of order $1/N$.

Except in § 8.2.5 we assume that S and A *remain coupled* at all times. Their joint distribution $\hat{D}(t)$ is then expected to be driven by the bath B to an equilibrium $\hat{D}(t_{\rm f})\propto\exp(-\hat{H}_0/T)$ at large times. The temperature T is imposed by the factor $K(u)$ that enters the equation of motion (4.5), while $\hat{H}_0$ is imposed by the form of $\hat{\sigma}_a^{(n)}(u)$. The additional terms in (8.50) are needed to ensure that S + M reaches the required equilibrium state. As discussed in § 7.1.5, invariance is broken in the final state. Its density operator involves two incoherent contributions, for which the magnetization of M lies either close to $+m_{\rm F}$ or close to $-m_{\rm F}$. In the first one, the marginal state of S is $\hat{r}(t_{\rm f})\propto\exp\left[\left(b\hat{s}_y+Ngm_{\rm F}\hat{s}_z\right)/T\right]$. If $b\ll Ng$, a condition that we will impose from now on, this state cannot be distinguished from the projection on $s_z=+1$. As when $b=0$, the sign of the observed magnetization $\pm m_{\rm F}$ of M is fully correlated with that of the z-component of the spin S in the final state, while $\langle\hat{s}_x(t_{\rm f})\rangle=\langle\hat{s}_y(t_{\rm f})\rangle=0$. The process is *consistent with von Neumann's reduction*, and it can be used as a preparation.

Nevertheless, nothing warrants the weights of the two possible outcomes, $+m_{\rm F}$, $s_z=+1$ and $-m_{\rm F}$, $s_z=-1$, to be equal to the diagonal elements $r_{\uparrow\uparrow}(0)$ and $r_{\downarrow\downarrow}(0)$ of the *initial density matrix: Born's rule may be violated.* A full study of the dynamics is required to evaluate these weights, so as to determine whether the process is still a faithful measurement.

This study will be simplified by noting that the expression (8.50) depends on b only through the ratio $b/Ng\hat{m}$. Once the registration has been established, at times of order $\tau_{\rm reg}$, the relevant eigenvalues of $\hat{m}$, of order $m_{\rm B}$, are finite for large N and the field b does not contribute to $\hat{\sigma}_a^{(n)}(u)$ since $b\ll Ng$. For short times, during the measurement process, the distribution of m is Gaussian, with a width of order $1/\sqrt{N}$, so that b may contribute significantly to $\hat{\sigma}_a^{(n)}(u)$ if b is of order $g\sqrt{N}$. However, we have shown (section 6) that for the off-diagonal blocks the bath terms in (4.29) have the sole effect of inhibiting the recurrences in $P_{\uparrow\downarrow}(m,t)$. Anyhow, such recurrences are not seen when m is treated as a continuous variable. We shall therefore rely on the simplified equations of motion

$$\frac{\partial P_{\uparrow\downarrow}}{\partial t}-\frac{2iNgm}{\hbar}P_{\uparrow\downarrow}=\frac{\partial P_{\downarrow\uparrow}}{\partial t}+\frac{2iNgm}{\hbar}P_{\downarrow\uparrow}=\frac{b}{\hbar}\left(P_{\downarrow\downarrow}-P_{\uparrow\uparrow}\right). \tag{8.51}$$

As regards the diagonal blocks we shall disregard b not only at times of order $\tau_{\rm reg}$, but even earlier. This is legitimate if $b\ll g\sqrt{N}$; if b is of order $g\sqrt{N}$, such an approximation retains the main effects of the bath, driving the distributions $P_{\uparrow\uparrow}(m,t)$ and $P_{\downarrow\downarrow}(m,t)$ apart from $-m_{\rm B}$ and $+m_{\rm B}$, respectively, and widening them. We write therefore:

$$\frac{\partial P_{\uparrow\uparrow}}{\partial t} + \frac{\partial}{\partial m}\left(v_{\uparrow\uparrow}P_{\uparrow\uparrow}\right) - \frac{1}{N}\frac{\partial^2}{\partial m^2}\left(wP_{\uparrow\uparrow}\right) = \frac{b}{\hbar}\left(P_{\uparrow\downarrow} + P_{\downarrow\uparrow}\right), \tag{8.52}$$

$$\frac{\partial P_{\downarrow\downarrow}}{\partial t} + \frac{\partial}{\partial m}\left(v_{\downarrow\downarrow}P_{\downarrow\downarrow}\right) - \frac{1}{N}\frac{\partial^2}{\partial m^2}\left(wP_{\downarrow\downarrow}\right) = -\frac{b}{\hbar}\left(P_{\uparrow\downarrow} + P_{\downarrow\uparrow}\right). \tag{8.53}$$

(Here we should distinguish the drift velocities $v_{\uparrow\uparrow}$ and $v_{\downarrow\downarrow}$, but the diffusion coefficients are equal.) Since the outcome of the registration is governed by the first stage studied in § 7.2.3(*i*), and since the Markovian regime (§ 7.1.2) is reached nearly from the outset, we shall use the simplified forms

$$v_{\uparrow\uparrow} = \frac{\gamma}{\hbar}[g + (J-T)m] = \frac{1}{\tau_{\rm reg}}(m_{\rm B} + m), \qquad w = \frac{\gamma T}{\hbar} \tag{8.54}$$

for the drift velocity and the diffusion coefficient; $v_{\downarrow\downarrow}$ follows from $v_{\uparrow\uparrow}$ by changing g into $-g$.

We have to solve (8.51), (8.52) and (8.53) with initial conditions $P_{ij}(m,0)/r_{ij}(0) = P_{\rm M}(m,0)$ expressed by (3.48). The drift and diffusion induced by the bath terms are slow since $\gamma \ll 1$, and the distribution $P_{\rm M}(m,t) = P_{\uparrow\uparrow}(m,t) + P_{\downarrow\downarrow}(m,t)$ of the magnetization of M can be regarded as constant on the time scales $\tau_{\rm red}$ and $\tau_{\rm Larmor} = \pi\hbar/b$, which is the period of the precession of the spin S when it does not interact with A. Over a short lapse around any time t, the coupled equations $C_x = P_{\uparrow\downarrow} + P_{\downarrow\uparrow}$, $C_y = iP_{\uparrow\downarrow} - iP_{\downarrow\uparrow}$ and $C_z = P_{\uparrow\uparrow} - P_{\downarrow\downarrow}$ simply describe, for each m, a Larmor precession of S [54, 55, 56, 57, 58] submitted to the field b along $\hat{y}$ and to the field Ngm along $\hat{z}$, where m is a classical random variable governed by the probability distribution $P_{\rm M}(m,t)$. The slow evolution of $P_{\rm M}(m,t)$ is coupled to this rapid precession through (8.52) and (8.53).

8.2.2. Ongoing reduction

We first eliminate the off-diagonal contributions by formally solving (8.51) as

$$P_{\uparrow\downarrow}(m,t) = P^*_{\downarrow\uparrow} = r_{\uparrow\downarrow}(0)e^{2iNgmt/\hbar}P_{\rm M}(m,0) - \frac{b}{\hbar}\int_0^t {\rm d}t'\, e^{2iNgm(t-t')/\hbar}\left[P_{\uparrow\uparrow}(m,t') - P_{\downarrow\downarrow}(m,t')\right]. \tag{8.55}$$

The physical quantities (except for correlations involving a large number of spins of M, see § 5.1.3) are obtained by summing over m with a weight smooth on the scale $1/\sqrt{N}$. The first term of (8.55), the same as in section 5 then yields a factor decaying as $\exp[-(t/\tau_{\rm red})^2]$, with $\tau_{\rm red} = \hbar/g\delta_0\sqrt{2N}$, due to destructive interferences.

However, the second term survives much later because the precession induced by the field b along $\hat{y}$ couples $2P_{\uparrow\downarrow} = C_x - iC_y$ to $C_z = P_{\uparrow\uparrow} - P_{\downarrow\downarrow}$ at all times t. Reduction takes place through the oscillatory factor in the integral, which hinders the effect of precession except at times t' just before t. Reduction is an *ongoing process*, which may take place (if b is sufficiently large) for $t \gg \tau_{\rm red}$: The non-conservation of the measured quantity s_z tends to *feed up the off-diagonal components* $\hat{R}_{\uparrow\downarrow}$ and $\hat{R}_{\downarrow\uparrow}$ of the density matrix $\hat{D}$ of S + M. In compensation, $\hat{R}_{\uparrow\uparrow}$ and $\hat{R}_{\downarrow\downarrow}$ may be progressively eroded through the right-hand side of (8.52) and (8.53).

At lowest order in b, we can rewrite explicitly the second term of (8.55) by replacing $P_{\uparrow\uparrow}$ by

$$\begin{aligned} P^{(0)}_{\uparrow\uparrow}(m,t) &= r_{\uparrow\uparrow}(0)\sqrt{\frac{N}{2\pi D(t)}}\exp\left[-\frac{N}{2D(t)}\left(m + m_{\rm B} - m_{\rm B}e^{t/\tau_{\rm reg}}\right)^2\right], \\ D(t) &= \delta_0^2 e^{2t/\tau_{\rm reg}} + \frac{T}{J-T}\left(e^{2t/\tau_{\rm reg}} - 1\right), \qquad \tau_{\rm reg} = \frac{\hbar}{\gamma(J-T)}, \end{aligned} \tag{8.56}$$

that we evaluated for $b = 0$ in section 7. We have simplified the general expression (7.61) by noting that the final outcome will depend only on the first stage of the registration, when t is of order $\tau_{\rm reg}$. For $P^{(0)}_{\downarrow\downarrow}$ we have to change $r_{\uparrow\uparrow}(0)$ into $r_{\downarrow\downarrow}(0)$ and $m_{\rm B} = g/(J-T)$ into $-m_{\rm B}$.

8.2.3. Leakage

The expectation values of $\hat{s}_x$ or $\hat{s}_y$ and their correlations with the pointer variable $\hat{m}$ are now found as in § 5.1.3 through summation over m of $P_{\uparrow\downarrow}(m,t)e^{\lambda m}$. At times t long compared to $\tau_{\rm red}$ and short compared to $\tau_{\rm reg}$, we find the characteristic function

$$\Psi_{\uparrow\downarrow}(\lambda,t) \equiv \langle \hat{s}_- e^{\lambda\hat{m}}(t)\rangle = \int \mathrm{d}m\, P_{\uparrow\downarrow}(m,t)e^{\lambda m} \simeq -\frac{b}{\hbar}\int \mathrm{d}m \int_0^t \mathrm{d}t' e^{2iNgm(t-t')/\hbar+\lambda m}\left[P^{(0)}_{\uparrow\uparrow}(m,t') - P^{(0)}_{\downarrow\downarrow}(m,t')\right] \quad (8.57)$$

$$= -\frac{b}{\hbar}\int_0^t \mathrm{d}t' r_{\uparrow\uparrow}(0)\exp\left[-\left(\frac{t-t'}{\tau_{\rm red}} - \frac{i\lambda\delta_0}{\sqrt{2N}}\right)^2 + \frac{2it'}{\tau_{\rm leak}}\left(\frac{t-t'}{\tau_{\rm red}} - \frac{i\lambda\delta_0}{\sqrt{2N}}\right)\right] - \{r_{\uparrow\uparrow}\mapsto r_{\downarrow\downarrow},\ \tau_{\rm leak}\mapsto -\tau_{\rm leak}\}\,.$$

We have recombined the parameters so as to express the exponent in terms of two characteristic times, the reduction time $\tau_{\rm red} = \hbar/g\delta_0\sqrt{2N}$ introduced in (5.6) and the *leakage time*

$$\tau_{\rm leak} = \sqrt{\frac{2}{N}}\frac{\hbar\delta_0}{\gamma g} = \sqrt{\frac{2}{N}}\frac{\tau_{\rm red}\delta_0}{m_{\rm B}} = \frac{2\tau_{\rm red}\delta_0^2}{\gamma}. \quad (8.58)$$

Integration over t' can be performed in the limit $\tau_{\rm leak} \gg \tau_{\rm red}$, by noting that the dominant contribution arises from the region $t - t' \ll t$, which yields in terms of the error function (7.81)

$$\Psi_{\uparrow\downarrow}(\lambda,t) = -\frac{b}{2g\delta_0}\sqrt{\frac{\pi}{2N}}e^{-(t/\tau_{\rm leak})^2}\left[r_{\uparrow\uparrow}(0)\mathrm{erfc}\left(-\frac{it}{\tau_{\rm leak}} - \frac{i\lambda\delta_0}{\sqrt{2N}}\right) - r_{\downarrow\downarrow}(0)\mathrm{erfc}\left(\frac{it}{\tau_{\rm leak}} - \frac{i\lambda\delta_0}{\sqrt{2N}}\right)\right]. \quad (8.59)$$

The leakage time characterizes *the dynamics of the transfer of polarization* from the z-direction towards the x- and y-directions. It is much shorter than the registration time, since $N \gg 1$ and $\gamma \ll 1$. It also characterizes the delay over which the distribution $P^{(0)}_{\uparrow\uparrow}(m,t)$ keeps a significant value at the origin: The peak of $P^{(0)}_{\uparrow\uparrow}$ with width $\delta_0/\sqrt{N}$, moves as $m_{\rm B}(e^{t/\tau_{\rm reg}} - 1) \sim m_{\rm B}t/\tau_{\rm reg}$, and at the time $t = \tau_{\rm leak}$ we have $P^{(0)}_{\uparrow\uparrow}(0,\tau_{\rm leak})/P^{(0)}_{\uparrow\uparrow}(0,0) = 1/e$.

Using the properties of the error function we can derive from Eq. (8.59), which is valid at times $t \gg \tau_{\rm red}$ such that the memory of $2r_{\uparrow\downarrow}(0) = \langle\hat{s}_x(0)\rangle - i\langle\hat{s}_y(0)\rangle$ is lost, by expanding the first equality of (8.57) in powers of λ, the results

$$\langle\hat{s}_x(t)\rangle = -\frac{b}{g\delta_0}\sqrt{\frac{\pi}{2N}}\langle\hat{s}_z(0)\rangle\exp\left[-\left(\frac{t}{\tau_{\rm leak}}\right)^2\right], \quad (8.60)$$

$$\langle\hat{s}_y(t)\rangle \approx \frac{b}{g\delta_0}\sqrt{\frac{2}{N}}\frac{t}{\tau_{\rm leak}}\left[1 - \frac{2}{3}\left(\frac{t}{\tau_{\rm leak}}\right)^2\right], \qquad t \ll \tau_{\rm leak}, \quad (8.61)$$

$$\langle\hat{s}_y(t)\rangle \sim \frac{b}{g\delta_0}\frac{1}{\sqrt{2N}}\frac{\tau_{\rm leak}}{t}, \qquad t \gg \tau_{\rm leak}. \quad (8.62)$$

where we also used that $r_{\uparrow\uparrow}(0) - r_{\downarrow\downarrow}(0) = \langle\hat{s}_z(0)\rangle$ and $r_{\uparrow\uparrow}(0) + r_{\downarrow\downarrow}(0) = 1$. For t of order $\tau_{\rm leak}$ these results are of order $b\Delta m/g\delta_0^2$, with $\Delta m = \delta_0/\sqrt{N}$ (see Eq. (3.49)). Because $1 - \mathrm{erfc}(z) = \mathrm{erf}(z)$ is imaginary for imaginary values of z, the correlations $\langle\hat{s}_x\hat{m}^k(t)\rangle$, $k \geq 1$, vanish in this approximation, while $\langle\hat{s}_y\hat{m}^k(t)\rangle$ involves an extra factor Δm^k, for instance:

$$\langle\hat{s}_y\hat{m}(t)\rangle = \frac{b}{gN}\langle\hat{s}_z(0)\rangle = \frac{b\,\Delta m^2}{g\delta_0^2}\langle\hat{s}_z(0)\rangle, \qquad \langle\hat{s}_y\hat{m}^2(t)\rangle = \frac{b\delta_0\sqrt{2}}{gN^{3/2}}\frac{t}{\tau_{\rm leak}} = \frac{b\sqrt{2}\,\Delta m^3}{g\delta_0^2}\frac{t}{\tau_{\rm leak}}. \quad (8.63)$$

To understand these behaviors, we remember that the spin S is submitted to the field b in the y-direction and to the random field Ngm in the z-direction, where m has a fluctuation $\delta_0/\sqrt{N}$ and an expectation value which varies as $\pm m_{\rm B}t/\tau_{\rm reg} = \pm\sqrt{2/N}\delta_0 t/\tau_{\rm leak}$ if the spin S is polarized in the $\pm z$-direction. The stationary value of $\langle\hat{s}_y\hat{m}(t)\rangle$ agrees with the value of the random field applied to S. The precession around $\hat{y}$ explains the factor $-b\langle\hat{s}_z(0)\rangle$ in $\langle\hat{s}_x(t)\rangle$. The rotation around z hinders $\langle\hat{s}_x(t)\rangle$ through randomness of m, its effects are characterized by the parameter Ngm, of order $g\delta_0\sqrt{N}$. This explains the occurrence of this parameter in the denominator. Moreover, this same rotation around $\hat{z}$ feeds up $\langle\hat{s}_y(t)\rangle$ from $\langle\hat{s}_x(t)\rangle$, and it takes place in a direction depending on the sign of m; as soon as registration begins, this sign of m is on average positive for $s_z = +1$, negative for $s_z = -1$.Thus the two rotations around $\hat{y}$ and $\hat{z}$ yield a polarization along $\hat{x}$ with a sign opposite to that along $\hat{z}$, whereas the polarization along $\hat{y}$ is positive whatever that

along $\hat{z}$. When $t \gg \tau_{\rm leak}$, the random values of m are all positive (for $P_{\uparrow\uparrow}$) or all negative (for $P_{\downarrow\downarrow}$), with a modulus larger than $1/\sqrt{N}$. Hence S precesses around an axis close to $+\hat{z}$ or $-\hat{z}$, even if b is of order $g\sqrt{N}$, so that the leakage from C_z towards C_x and C_y is inhibited for such times. Altogether, the duration of the effect is $\tau_{\rm leak}$, and its size is characterized by the dimensionless parameter $b/g\sqrt{N}$.

8.2.4. Possibility of an ideal measurement

We wish to find an upper bound on the field b such that the process can be used as a measurement. Obviously, if the Larmor period $\tau_{\rm Larmor} = \pi\hbar/b$ is longer than the registration time $\tau_{\rm reg} = \hbar/\gamma(J-T)$, we can completely disregard the field. However, we shall see that this condition, $b \ll \pi\gamma(J-T)$, is too stringent and that even large violations of the conservation law of the measured quantity $\hat{s}_z$ do not prevent an ideal measurement.

We therefore turn to the registration, still assuming that S and A remain coupled till the end of the process. At lowest order in b, the right-hand side of (8.52) and (8.53) is expressed by (8.55) with (8.56). The Green's functions $G_\uparrow$ and $G_\downarrow$ of the left-hand sides are given by (F.10) with $h = +g$ and $h = -g$, respectively. We thus find $P_{\uparrow\uparrow}(m,t)$ through convolution of $G_\uparrow(m,m',t-t')$ with the initial condition $\delta(t')P^{(0)}_{\uparrow\uparrow}(m,t')$, with

$$\frac{b}{\hbar}C_x^{(0)}(m',t') = \frac{b}{\hbar}\left[P^{(0)}_{\uparrow\downarrow}(m',t') + P^{(0)}_{\downarrow\uparrow}(m',t')\right], \tag{8.64}$$

and with

$$\frac{b}{\hbar}C_x^{(1)}(m',t') = -\frac{2b^2}{\hbar^2}\int_0^{t'} \mathrm{d}t''\cos\left[2Ngm'(t'-t'')/\hbar\right]\left[P^{(0)}_{\uparrow\uparrow}(m',t') - P^{(0)}_{\downarrow\downarrow}(m',t')\right]. \tag{8.65}$$

For $P_{\downarrow\downarrow}$ we change $G_\uparrow$ into $G_\downarrow$ and C_x into $-C_x$. The zeroth-order contribution, evaluated in section 7, corresponds to an ideal measurement. The first-order correction in b, $P^{(1)}_{\uparrow\uparrow}$ issued from $C^{(0)}_x$, depends on the transverse initial conditions $r_{\uparrow\downarrow}(0)$, while the second-order correction, $P^{(2)}_{\uparrow\uparrow}$ issued from $C^{(1)}_x$, depends, as the main term, on $r_{\uparrow\uparrow}(0) = 1 - r_{\downarrow\downarrow}(0)$.

Performing the Gaussian integrals on m', we find:

$$\begin{aligned}
P^{(1)}_{\uparrow\uparrow}(m,t) &= \frac{2b}{\hbar}\Re\int \mathrm{d}m'\,\mathrm{d}t' G_\uparrow(m,m',t-t')P^{(0)}_{\uparrow\downarrow}(m',t') \\
&= \frac{2b}{\hbar}\Re\int_0^t \mathrm{d}t' r_{\uparrow\downarrow}(0)\sqrt{\frac{N}{2\pi}}\frac{e^{-(t-t')/\tau_{\rm reg}}}{\delta_1(t-t')}\exp\left\{-\frac{N}{2\delta_1^2(t-t')}\left[\mu'^2 + 4g^2\delta_0^2\delta_2^2\frac{t'^2}{\hbar^2} - 4ig\delta_0^2\mu'\frac{t'}{\hbar}\right]\right\}, \qquad (8.66)\\
P^{(2)}_{\uparrow\uparrow}(m,t) &= -\frac{2b^2}{\hbar^2}\Re\int_0^t \mathrm{d}t'\int_0^{t'}\mathrm{d}t'' r_{\uparrow\uparrow}(0)\sqrt{\frac{N}{2\pi}}\frac{e^{-(t-t'+t'')/\tau_{\rm reg}}}{\delta_1(t-t'+t'')} \\
&\times\exp\left\{-\frac{Ne^{-2t''/\tau_{\rm reg}}}{2\delta_1^2(t-t'+t'')}\left[(\mu'-\mu'')^2 + 4g^2e^{2t''/\tau_{\rm reg}}\delta_1^2(t'')\delta_2^2\frac{(t-t'')^2}{\hbar^2} - 4ig\left(e^{2t''/\tau_{\rm reg}}\delta_1^2(t'')\mu' + \delta_2^2\mu''\right)\frac{t'-t''}{\hbar}\right]\right\}, \\
&- \{r_{\uparrow\uparrow}\mapsto r_{\downarrow\downarrow};\quad \mu''\mapsto -\mu''\}, \qquad (8.67)
\end{aligned}$$

where $\delta_1(t)$ was defined by (7.63), where $\delta_2^2 \equiv \delta_1^2(t-t') - \delta_0^2 = T[1-e^{-2(t-t')/\tau_{\rm reg}}]/(J-T)$, where $\mu' \equiv -m_{\rm B} + (m+m_{\rm B})\exp[-(t-t')/\tau_{\rm reg}]$ and where $\mu'' \equiv m_{\rm B}[\exp(t''/\tau_{\rm reg})-1]$. These expressions hold for times t of order $\tau_{\rm reg}$. For later times, the part of $P_{\uparrow\uparrow}(m,t)$ for which m is above (below) the bifurcation $-m_{\rm B}$ (with $m_{\rm B} = g/(J-T)$) develops a peak around $+m_{\rm F}$ ($-m_{\rm F}$). For $P_{\downarrow\downarrow}$, we have to change the sign in $P^{(1)}_{\uparrow\uparrow}$ and $P^{(2)}_{\uparrow\uparrow}$ and to replace $m_{\rm B}$ by $-m_{\rm B}$ in μ'; the bifurcation in $+m_{\rm B}$. The probability of finding $s_z = +1$ and $m \simeq m_{\rm F}$ at the end of the measurement is thus $\int_{-m_{\rm B}}^1 \mathrm{d}m\,P_{\uparrow\uparrow}(m,t)$, while $\int_{-1}^{-m_{\rm B}}\mathrm{d}mP_{\uparrow\uparrow}(m,t)$ corresponds to $s_z = 1$ and $m \simeq -m_{\rm F}$. Since $\int_{-m_{\rm B}}^1 \mathrm{d}m\,P^{(0)}_{\uparrow\uparrow}(m,t) = r_{\uparrow\uparrow}(0)$ and $\int_{-1}^{-m_{\rm B}}\mathrm{d}m\,P^{(0)}_{\uparrow\uparrow}(m,t) = 0$, the contributions $P^{(0)}_{\uparrow\uparrow}$ to $P_{\uparrow\uparrow}$ and $P^{(0)}_{\downarrow\downarrow}$ to $P_{\downarrow\downarrow}$ correspond to an ideal measurement. The corrections of order b and b^2 to $P_{\uparrow\uparrow}$ and $P_{\downarrow\downarrow}$ give thus rise to violations of Born's rule, governed at first order in b by the off-diagonal elements $r_{\uparrow\downarrow}(0)$, $r_{\downarrow\uparrow}(0)$ of the initial density matrix of S, and at second order by the diagonal elements $r_{\uparrow\uparrow}(0)$, $r_{\downarrow\downarrow}(0)$. For instance, $\int_{-m_{\rm B}}^1 \mathrm{d}m\,P^{(2)}_{\uparrow\uparrow}(m,t)$ and $\int_{-1}^{-m_{\rm B}}\mathrm{d}m\,P^{(2)}_{\uparrow\uparrow}(m,t)$ are the contributions of these initial diagonal elements to the wrong counts $+m_{\rm F}$ and $-m_{\rm F}$, respectively, associated with $s_z = +1$ in the final state of S.

In order to estimate these deviations due to non-conservation of $\hat{s}_z$, we evaluate, as we did for the transverse quantities (8.60-8.62), the expectation values $\langle \hat{s}_z(t)\rangle$, $\langle \hat{m}(t)\rangle$, $\langle \hat{s}_z\hat{m}(t)\rangle$ issued from (8.66) and (8.67). For times $t \gg \tau_{\rm red}$ and t not much longer than $\tau_{\rm reg}$, we find

$$\begin{aligned}\langle \hat{s}_z(t)\rangle &= \int {\rm d}m\,[P_{\uparrow\uparrow}(m,t) - P_{\downarrow\downarrow}(m,t)] = r_{\uparrow\uparrow}(0) - r_{\downarrow\downarrow}(0) + \frac{4b}{\hbar}\Re\int_0^t {\rm d}t' r_{\uparrow\downarrow}(0)\exp\left[-\left(\frac{t'}{\tau_{\rm red}}\right)^2\right]\\ &\quad -\frac{4b^2}{\hbar^2}\Re\int_0^t {\rm d}t'\int_0^{t'} {\rm d}t''[r_{\uparrow\uparrow}(0) - r_{\downarrow\downarrow}(0)]\exp\left[-\left(\frac{t'-t''}{\tau_{\rm red}}\right)^2 + 2i\frac{t''}{\tau_{\rm leak}}\left(\frac{t'-t''}{\tau_{\rm red}}\right)\right]\\ &= \langle \hat{s}_z(0)\rangle + \frac{b}{g\delta_0}\sqrt{\frac{\pi}{2N}}\langle \hat{s}_x(0)\rangle - \frac{b^2}{2N\gamma g^2}\langle \hat{s}_z(0)\rangle\left[1 - {\rm erfc}\left(\frac{t}{\tau_{\rm red}}\right)\right]; \end{aligned} \tag{8.68}$$

we noted that only short times t', t'' and $t' - t''$ contribute. A similar calculation provides

$$\langle \hat{m}(t)\rangle = \int {\rm d}m\, m\,[P_{\uparrow\uparrow}(m,t) + P_{\downarrow\downarrow}(m,t)] = \langle \hat{s}_z(t)\rangle m_{\rm B}\left(e^{t/\tau_{\rm reg}} - 1\right). \tag{8.69}$$

For $t \gg \tau_{\rm leak}$, $\langle \hat{s}_z(t)\rangle$ tends to a constant which differs from the value $\langle \hat{s}_z(0)\rangle$ expected for an ideal measurement. The ratio $\langle \hat{m}(t)\rangle/\langle \hat{s}_z(t)\rangle$ is, however, the same as in section 7 where $b = 0$. Finally the correlation is obtained as

$$\begin{aligned}\langle \hat{s}_z\hat{m}(t)\rangle &= \int {\rm d}m\, m\,[P_{\uparrow\uparrow}(m,t) - P_{\downarrow\downarrow}(m,t)] = m_{\rm B}\left(e^{t/\tau_{\rm reg}} - 1\right) + \frac{4b}{\hbar}\Re\int_0^t {\rm d}t' r_{\uparrow\downarrow}(0) 2ig\delta_0^2\frac{t'}{\hbar}e^{t/\tau_{\rm reg}}\exp\left[-\left(\frac{t'}{\tau_{\rm red}}\right)^2\right]\\ &\quad -\frac{4b^2}{\hbar^2}\Re\int_0^t {\rm d}t'\int_0^{t'} {\rm d}t''\left(\mu'' + 2ig\delta_0^2\frac{t'-t''}{\hbar}e^{t/\tau_{\rm reg}}\right)\exp\left[-\left(\frac{t'-t''}{\tau_{\rm red}}\right)^2 + 2i\frac{t''}{\tau_{\rm leak}}\left(\frac{t'-t''}{\tau_{\rm red}}\right)\right]\\ &= m_{\rm B}\left(e^{t/\tau_{\rm reg}} - 1\right) + \frac{b}{Ng}\langle \hat{s}_y(0)\rangle e^{t/\tau_{\rm red}}; \end{aligned} \tag{8.70}$$

the terms in b^2 cancel out. As in (8.61), (8.62) the correlation $\langle \hat{s}_z\hat{m}(t)\rangle$ is weaker by a factor $\sqrt{N}$ than the expectation value $\langle \hat{s}_z(t)\rangle$.

Altogether, the field b enters all the results (8.50), (8.60-8.62) and (8.68-8.70) through the combination $b/g\sqrt{N}$. However, the dominant deviation from Born's rule, arising from the last term of (8.68), also involves the coupling γ of M with B. The process can therefore be regarded as an ideal measurement provided

$$b \ll g\sqrt{N\gamma}. \tag{8.71}$$

Contrary to the probability of an unsuccessful measurement found in [195], which depended solely on the size of the apparatus, the present condition involves b, which characterizes the magnitude of the violation, as well as the couplings, g between S and M, and γ between M and B, which characterize the dynamics of the process. A large number N of degrees of freedom of the pointer and/or a large coupling g inhibit the transitions between $s_z = +1$ and $s_z = -1$ induced by $\hat{H}_{\rm S}$, making the leakage time short and rendering the field b ineffective. If g is small, approaching the lower bound (7.41), the constraint (8.71) becomes stringent, since $\gamma \ll 1$. Too weak a coupling γ with the bath makes the registration so slow that b has time to spoil the measurement during the leakage delay.

8.2.5. Switching on and off the system-apparatus interaction

Haastige spoed is zelden goed[51]
Dutch proverb

The condition (8.71), which ensures that the process behaves as an ideal measurement although $\hat{s}_z$ is not conserved, has been established by assuming that S and A interact from the time $t = 0$ to the time $t = t_{\rm f}$ at which the pointer has

[51] Being quick is hardly ever good

reached $\pm m_{\rm F}$. However, in a realistic ideal measurement, S and A should be decoupled both before $t = 0$ and after some time larger than $\tau_{\rm reg}$. At such times, the observable $\hat{s}_z$ to be tested suffers oscillations with period $\tau_{\rm Larmor} = \pi b/\hbar$, which may be rapid. Two problems then arise.

(*i*) The repeated process informs us through reading of M about the diagonal elements of the density matrix $\hat{r}$ of S, not at any time, but *at the time when the coupling g is switched on*, that we took as the origin of times $t = 0$. Before this time, the diagonal elements $r_{\uparrow\uparrow}(t)$ and $r_{\downarrow\downarrow}(t)$ oscillate freely with the period $\tau_{\rm Larmor}$. If we wish the outcomes of M to be meaningful, we need to control, within a latitude small compared to $\tau_{\rm Larmor}$, the time at which the interaction is turned on. Moreover, this coupling must occur suddenly: The time during which g rises from 0 to its actual value should be short, much shorter than the leakage time.

(*ii*) Suppose that the coupling g is switched off at some time $\tau_{\rm dec}$ larger than $\tau_{\rm reg}$, the condition (8.71) being satisfied. At this decoupling time $P_{\uparrow\uparrow}(m, \tau_{\rm dec})$ presents a peak for $m > 0$, with weight $\int {\rm d}m\, m\, P_{\uparrow\uparrow}(m, \tau_{\rm dec}) = r_{\uparrow\uparrow}(0)$, $P_{\downarrow\downarrow}(m, \tau_{\rm dec})$ a peak for $m < 0$ with weight $r_{\downarrow\downarrow}(0)$, while $P_{\uparrow\downarrow}(m, \tau_{\rm dec})$ vanishes. Afterwards the system and the apparatus evolve independently. The Larmor precession of S [54, 55, 56, 57, 58] manifests itself through oscillations of $\int {\rm d}m\, [P_{\uparrow\uparrow}(m,t) - P_{\downarrow\downarrow}(m,t)]$ and of $\int {\rm d}m\, [P_{\uparrow\downarrow}(m,t) + P_{\downarrow\uparrow}(m,t)]$, while M relaxes under the influence of the bath B. The two peaks of the probability distribution $P_{\rm M}(m,t) = P_{\uparrow\uparrow}(m,t) + P_{\downarrow\downarrow}(m,t)$ move apart, towards $+m_{\rm F}$ and $-m_{\rm F}$, respectively. At the final time $t_{\rm f}$, once the apparatus has reached equilibrium with broken invariance, we can observe on the pointer the outcomes $+m_{\rm F}$ with probability $r_{\uparrow\uparrow}(0)$, or $-m_{\rm F}$ with probability $r_{\downarrow\downarrow}(0)$. Thus the counting rate agrees with Born's rule. However the process is not an ideal measurement in von Neumann's sense: The outcome of A is correlated not with the state of S at the final reading time, but only with its state $\hat{r}(\tau_{\rm dec})$ at the decoupling time, a state which has been kept unchanged since the reduction owing to the interaction of S with M. Selecting the events with $+m_{\rm F}$ at the time $t_{\rm f}$ cannot be used as a preparation of S in the state $|\uparrow\rangle$, since $\hat{r}(t)$ has evolved since the decoupling.

8.3. Attempt to simultaneously measure non-commutative variables

Je moet niet teveel hooi op je vork nemen[52]
Qui trop embrasse mal étreint [53]
Dutch and French proverbs

Books of quantum mechanics tell that a *precise* simultaneous measurement of non-commuting variables is impossible [5, 6, 27, 43, 73]. It is, however, physically sensible to imagine a setting with which we would try to perform such a measurement approximately [221, 222, 223, 224, 225, 226, 227]. It is interesting to analyze the corresponding dynamical process so as to understand how it differs from a standard measurement.

Consider first successive measurements. In a first stage the component $\hat{s}_z$ of the spin S is tested by coupling S to A between the time $t = 0$ and some time $\tau_{\rm dec}$ at which $\hat{H}_{\rm SA}$ is switched off. If $\tau_{\rm dec}$ is larger than the registration time $\tau_{\rm reg}$, the apparatus A produces $m = m_{\rm F}$ with probability $r_{\uparrow\uparrow}(0)$ and $m = -m_{\rm F}$ with probability $r_{\downarrow\downarrow}(0)$. An interaction $\hat{H}_{\rm SA'}$ is then switched on between S and a second apparatus A′, analogous to A but coupled to the component $\hat{s}_v$ of $\hat{\mathbf{s}}$ in some v-direction. It is the new diagonal marginal state $\hat{r}(\tau_{\rm dec})$, equal to the diagonal part of $\hat{r}(0)$, which is then tested by A′. In this measurement of $\hat{s}_v$ the probability of reading $m' = +m_{\rm F}$ on A′ and finding $\hat{\mathbf{s}}$ in the v-direction is $r_{\uparrow\uparrow}(0)\cos^2\frac{1}{2}\theta + r_{\downarrow\downarrow}(0)\sin^2\frac{1}{2}\theta$, where θ and ϕ are the Euler angles of $\mathbf{v}$. The measurement of $\hat{s}_v$ alone would have provided the additional contribution $\Re r_{\uparrow\downarrow}(0)\sin\theta e^{i\phi}$. We therefore recover dynamically all the standard predictions of quantum mechanics.

Things will be different if the two apparatuses are switched on faster or at the same time.

8.3.1. A model with two apparatuses

Life is really simple, but we insist on making it complicated
Confucius

Let us imagine we attempt to measure simultaneously the non-commuting components $\hat{s}_z$ and $\hat{s}_x$ of the spin $\hat{\mathbf{s}}$. To this aim we extend our model by assuming that, starting from the time $t = 0$, S is coupled with two apparatuses A and A′ of the same type as above, A′ being suited to the measurement of $\hat{s}_x$. We denote by γ', g', N', J', T', ...,

[52] You should not put too much hay on your fork
[53] He who embraces too much fails to catch

the parameters of the second apparatus. The overall Hamiltonian $\hat{H} = \hat{H}_{\rm SA} + \hat{H}_{\rm SA'} + \hat{H}_{\rm A} + \hat{H}_{\rm A'}$ thus involves, in addition to the contributions defined in subsection 3.2, the Hamiltonian $\hat{H}_{\rm A'}$ of the second apparatus A′, analogous to $\hat{H}_{\rm A} = \hat{H}_{\rm M} + \hat{H}_{\rm B} + \hat{H}_{\rm MB}$, with magnetization $m' = (1/N') \sum_{n=1}^{N'} \hat{\sigma}_x'^{(n)}$, and the coupling term

$$\hat{H}_{\rm SA'} = -N' g' \hat{s}_x \hat{m}' \tag{8.72}$$

of A′ and S. The solution of the Liouville–von Neumann equation for S + A+A′ should determine the indications of A and A′ which depend on the initial state $\hat{r}(0)$ of S, as well as the final state of S and its correlations with A and A′.

We readily note that such a dynamical process can not behave as an ideal measurement, since we expect that, whatever the initial state $\hat{r}(0)$ of S, its final state will be perturbed.

The equations of motion are worked out as in section 4. After elimination of the baths B and B′ at lowest order in γ and γ', the density operator $\hat{D}$ of S + M + M′ can be parametrized as in § 3.3.1 and § 4.4.1 by four functions $P_{ij}(m, m', t)$, where $i, j = \uparrow, \downarrow$ refer to S, and where the magnetizations m and m' behave as random variables. However, since the functions P_{ij} are now coupled, it is more suitable to express the dynamics in terms of $P_{\rm MM'}(m, m', t) = P_{\uparrow\uparrow} + P_{\downarrow\downarrow}$, which describes the joint probability distribution of m and m', and of the set $C_a(m, m', t)$ defined for $a = x, y$ and z by (3.29), which describe the correlations between $\hat{s}_a$ and the two magnets M and M′. The density operator $\hat{D}(t)$ of S + M + M′ generalizing (3.28) is

$$\hat{D}(t) = \frac{1}{NG(\hat{m})G(\hat{m}')} \left[P_{\rm MM'}(\hat{m}, \hat{m}', t) + \mathbf{C}(\hat{m}, \hat{m}', t) \cdot \hat{\mathbf{s}} \right]. \tag{8.73}$$

(There is no ambiguity in this definition, since $\hat{m}$ and $\hat{m}'$ commute.) The full dynamics are thus governed by coupled equations for the functions $P_{\rm MM'}(m, m', t)$ and $\mathbf{C}(m, m', t)$ which parametrize $\hat{D}(t)$. The initial state $\hat{D}(0)$ is factorized as $\hat{r}(0) \otimes \hat{R}_{\rm M}(0) \otimes \hat{R}_{\rm M'}(0)$, where $\hat{R}_{\rm M}(0)$ and $\hat{R}_{\rm M'}(0)$ describe the metastable paramagnetic states (3.45) of M and M′, so that the initial conditions are

$$P_{\rm MM'}(m, m', 0) = P_{\rm M}(m, 0) P_{\rm M'}(m', 0), \qquad \mathbf{C}(m, m', 0) = P_{\rm MM'}(m, m', 0) \langle \hat{\mathbf{s}}(0) \rangle, \tag{8.74}$$

where $P_{\rm M}(m, 0)$ and $P_{\rm M'}(m', 0)$ have the Gaussian form (3.48) and where $\langle \hat{\mathbf{s}}(0) \rangle$ is the initial polarization of S.

Two types of contributions enter $\partial P_{\rm MM'}/\partial t$ and $\partial \mathbf{C}/\partial t$, the first one active on the short time scale $\tau_{\rm red}$, and the second one on the long time scale $\tau_{\rm reg}$. On the one hand, for given m and m', the coupling $\hat{H}_{\rm SA} + \hat{H}_{\rm SA'}$ of S with the magnets M and M′ behaves as a magnetic field $\mathbf{b}$ applied to S. This effective field is equal to

$$\mathbf{b}(m, m') = \frac{2Ngm}{\hbar} \hat{\mathbf{z}} + \frac{2N' g' m'}{\hbar} \hat{\mathbf{x}} = b \hat{\mathbf{u}}, \qquad b(m, m') \equiv |\mathbf{b}(m, m')| = \frac{2}{\hbar} \sqrt{N^2 g^2 m^2 + N'^2 g'^2 m'^2}, \tag{8.75}$$

where $\hat{\mathbf{z}}$ and $\hat{\mathbf{x}}$ are the unit vectors in the z- and x-direction, respectively. This yields to $\partial \mathbf{C}/\partial t$ the contribution

$$\left[\frac{\partial \mathbf{C}(m, m', t)}{\partial t} \right]_{\rm MM'} = -\mathbf{b}(m, m') \times \mathbf{C}(m, m', t). \tag{8.76}$$

Both the large Larmor frequency b and the precession axis, characterized by the unit vector $\hat{\mathbf{u}} = \mathbf{b}/b$ in the $x - z$ plane, depend on m and m' (whereas the precession axis was fixed along $\hat{\mathbf{z}}$ for a single apparatus). The distribution $P_{\rm MM'}(m, m', t)$ is insensitive to the part $\hat{H}_{\rm SA} + \hat{H}_{\rm SA'}$ of the Hamiltonian, and therefore evolves slowly, only under the effect of the baths.

On the other hand, $\partial P_{\rm MM'}/\partial t$ and $\partial \mathbf{C}/\partial t$ involve contributions from the baths B and B′, which can be derived from the right-hand sides of (4.30) and (4.29). They couple all four functions $P_{\rm MM'}$ and $\mathbf{C}$, they are characterized by the long time scale $\tau_{\rm reg}$, and they depend on all parameters of the model. In contrast with what happened for a single apparatus, the effects of the precession (8.76) and of the baths can no longer be separated in spite of the large ratio between their time scales. Indeed, the precession tends to eliminate rapidly the components of $\mathbf{C}(m, m', t)$ that are perpendicular to $\mathbf{b}$, but the baths tend to continuously activate the creation of such components. As it occurred already for a single apparatus, see section 6.2, the *reduction* therefore takes place not only at the beginning of the process but also *during the whole registration*, under the simultaneous contradictory effects of the couplings of M and M′ with S and with the baths, a phenomenon reminiscent of the Boltzmann Stosszahlansatz. While for the single apparatus the ongoing reduction takes place along the fixed $\hat{s}_z$ axis, thus only enforcing it, here the reduction along the

moving axis enforces fast decoherence in the instantaneous state. Such an interplay, together with the coupling of four functions $P_{\mathrm{MM}'}$, $\mathbf{C}$ of three variables m, m', t, make the equations of motion difficult to solve, whether analytically or numerically. A qualitative analysis will, however, suffice to provide us with some interesting conclusions.

8.3.2. Structure of the outcome

Note first that the positivity of the density operator (8.73), maintained by the dynamics, is expressed by the condition

$$P_{\mathrm{MM}'}(m, m', t) \geq |\mathbf{C}(m, m', t)|, \tag{8.77}$$

which holds at any time.

The outcome of the process is characterized by the limit, for t larger than the registration time τ_{reg}, of the distributions $P_{\mathrm{MM}'}$ and $\mathbf{C}$. In this last stage of the evolution, the interaction of M with the bath B is expected to drive it towards either one of the two equilibrium states at temperature T, for which the normalized distribution $P_{\mathrm{M}\Uparrow}(m)$ (or $P_{\mathrm{M}\Downarrow}(m)$) expressed by (3.56) is concentrated near $m = +m_{\mathrm{F}}$ (or $m = -m_{\mathrm{F}}$). Likewise, M′ is stabilized into either one of the ferromagnetic states $P'_{\mathrm{M}'\Uparrow}(m')$ (or $P_{\mathrm{M}'\Downarrow}(m')$) with $m' \simeq +m'_{\mathrm{F}}$ (or $m' = -m'_{\mathrm{F}}$). Hence, $P_{\mathrm{MM}'}(m, m', t)$, which describes the statistics of the indications of the pointers, ends up as a sum of four narrow peaks which settle at $m = \varepsilon m_{\mathrm{F}}$, $m' = \varepsilon' m'_{\mathrm{F}}$, with $\varepsilon = \pm 1$, $\varepsilon' = \pm 1$, to wit,

$$P_{\mathrm{MM}'}(m, m', t) \mapsto \sum_{\varepsilon=\pm 1} \sum_{\varepsilon'=\pm 1} \mathcal{P}_{\varepsilon\varepsilon'} P_{\mathrm{M}\varepsilon}(m) P_{\mathrm{M}'\varepsilon'}(m'). \tag{8.78}$$

The weights $\mathcal{P}_{\varepsilon\varepsilon'}$ of these peaks characterize the proportions of counts detected on M and M′ in repeated experiments; they are the only observed quantities.

Over a short time at the beginning of the process, the rapid precession (8.76) together with smoothing over m and m' eliminates the component C_y of $\mathbf{C}$, so that the subsequent evolution keeps no memory of $C_y(m, m', 0)$. Thus, among the initial data (8.74) pertaining to S, only $\langle \hat{s}_x(0) \rangle$ and $\langle \hat{s}_z(0) \rangle$ are relevant to the determination of the final state: the frequencies $\mathcal{P}_{\varepsilon\varepsilon'}$ of the outcomes depend only on $\langle \hat{s}_x(0) \rangle$ and $\langle \hat{s}_z(0) \rangle$ (and on the parameters of the apparatuses).

If $\langle \hat{s}_x(0) \rangle = \langle \hat{s}_z(0) \rangle = 0$ we have $\mathcal{P}_{\varepsilon\varepsilon'} = \frac{1}{4}$ due to the symmetry $m \leftrightarrow -m$, $m' \leftrightarrow -m'$. Likewise, if $\langle \hat{s}_x(0) \rangle = 0$, the symmetry $m' \leftrightarrow -m'$ implies that $\mathcal{P}_{++} = \mathcal{P}_{+-}$ and $\mathcal{P}_{-+} = \mathcal{P}_{--}$. Since the equations of motion are linear, $\mathcal{P}_{++} - \mathcal{P}_{-+}$ is in this situation proportional to $\langle \hat{s}_z(0) \rangle$; we define the proportionality coefficient λ by $\mathcal{P}_{\varepsilon+} = \frac{1}{4}(1 + \varepsilon\lambda \langle \hat{s}_z(0) \rangle)$. In the situation $\langle \hat{s}_z(0) \rangle = 0$ we have similarly $\mathcal{P}_{+\varepsilon'} = \mathcal{P}_{-\varepsilon'} = \frac{1}{4}(1 + \varepsilon'\lambda' \langle \hat{s}_x(0) \rangle)$. Relying on the linearity of the equations of motion, we find altogether for an arbitrary initial state of S the general form for the probabilities $\mathcal{P}_{\varepsilon\varepsilon'}$:

$$\mathcal{P}_{\varepsilon\varepsilon'} = \frac{1}{4} \left(1 + \varepsilon\lambda \langle \hat{s}_z(0) \rangle + \varepsilon'\lambda' \langle \hat{s}_x(0) \rangle\right), \tag{8.79}$$

where $\langle \hat{s}_z(0) \rangle = r_{\uparrow\uparrow}(0) - r_{\downarrow\downarrow}(0)$, $\langle \hat{s}_x(0) \rangle = r_{\uparrow\downarrow}(0) + r_{\downarrow\uparrow}(0)$. We term λ and λ' the *efficiency factors*.

In the long time limit, the functions $\mathbf{C}(m.m', t)$ also tend to sums of four peaks located at $m = \pm m_{\mathrm{F}}$, $m' = \pm m'_{\mathrm{F}}$, as implied by (8.77). With each peak is associated a direction $\mathbf{u}_{\varepsilon\varepsilon'}$, given by (8.75) where $m = \varepsilon m_{\mathrm{F}}$, $m' = \varepsilon' m'_{\mathrm{F}}$, around which the precession (8.76) takes place. The reduction process eliminates over a short delay compared to the considered time scale the component of $\mathbf{C}$ perpendicular to $\mathbf{u}_{\varepsilon\varepsilon'}$, for each peak. Thus, if in their final state the apparatuses M and M′ indicate $\varepsilon m_{\mathrm{F}}$, $\varepsilon' m'_{\mathrm{F}}$, the spin S is lead into a state partly polarized in the direction $\mathbf{u}_{\varepsilon\varepsilon'}$ of the effective field $\mathbf{b}$ generated by the two ferromagnets.

8.3.3. A fully informative statistical process

The above process cannot be regarded as an ideal measurement. On the one hand, the above-mentioned correlations between the final state of S and the indications of the apparatus are weak; they are limited by the inequality (8.77). In an ideal measurement the correlation must be *complete*: if for a given run we read $+m_{\mathrm{F}}$ on the apparatus M measuring $\hat{s}_z$, the spin S has been led by the ideal process into the pure state $|\uparrow\rangle$. Here we cannot make such assertions about an individual system, and we cannot use the process as a preparation.

On the other hand, in an ideal measurement, the outcome of the process is *unique* for both S and M in case S is initially in an eigenstate of the tested quantity. Suppose the spin S is initially oriented up in the z-direction, that is, $\hat{r}(0) = |\uparrow\rangle\langle\uparrow|$. The response of the apparatuses M and M′ is given by (8.79) as

$$\mathcal{P}_{++} = \mathcal{P}_{+-} = \frac{1}{4}(1+\lambda), \qquad \mathcal{P}_{-+} = \mathcal{P}_{--} = \frac{1}{4}(1-\lambda), \tag{8.80}$$

so that there exists a probability $\frac{1}{2}(1-\lambda)$ to *read the wrong result* $-m_{\mathrm{F}}$ on M. Indeed, without even solving the equations of motion to express the efficiency factors λ and λ' in terms of the various parameters of the model, we can assert that λ is smaller than 1: Because all $\mathcal{P}_{\varepsilon\varepsilon'}$ must be non-negative for any initial state of S, λ and λ' should satisfy

$$\lambda^2 + \lambda'^2 \leq 1, \tag{8.81}$$

and because not only $\hat{s}_z$ but also $\hat{s}_x$ are tested, λ' should be non zero so that the probability of failure $\frac{1}{2}(1-\lambda)$ is finite.

It is therefore clear why the attempt to perform a simultaneous *ideal* measurement of $\hat{s}_x$ and $\hat{s}_z$ fails. Both Born's rule and von Neumann's reduction are violated. Nevertheless, consider a set of repeated experiments in which we read simultaneously the indications of the two apparatuses M and M′. If the runs are sufficiently numerous, we can determine the probabilities $\mathcal{P}_{\varepsilon\varepsilon'}$ from the frequencies of occurrence of the four possible outcomes $\pm m_{\mathrm{F}}$, $\pm m_{\mathrm{F}'}$. Under reasonable conditions on the parameters of the model, we expect λ and λ' to take significant values, so that inversion of eq. (8.79) yields

$$\begin{aligned} \langle \hat{s}_z(0) \rangle &= r_{\uparrow\uparrow}(0) - r_{\downarrow\downarrow}(0) = \frac{1}{\lambda}(\mathcal{P}_{++} + \mathcal{P}_{+-} - \mathcal{P}_{-+} - \mathcal{P}_{--}), \\ \langle \hat{s}_x(0) \rangle &= r_{\uparrow\downarrow}(0) + r_{\downarrow\uparrow}(0) = \frac{1}{\lambda'}(\mathcal{P}_{++} - \mathcal{P}_{+-} + \mathcal{P}_{-+} - \mathcal{P}_{--}). \end{aligned} \tag{8.82}$$

Thus, a sequence of repeated experiments reveals the initial expectation values of *both* $\hat{s}_z$ *and* $\hat{s}_x$ although these observables do not commute.

Paradoxically, as regards the determination of an unknown initial density matrix, the present process is *more informative than an ideal measurement* with a single apparatus [226]. Repeated measurements of $\hat{s}_z$ yield $r_{\uparrow\uparrow}(0)$ (and $r_{\downarrow\downarrow}(0)$) through counting of the outcomes $\pm m_{\mathrm{F}}$ of M. Here we moreover find through repeated experiments the real part of $r_{\uparrow\downarrow}(0)$. However, more numerous runs are needed to reach a given precision if λ and λ' are small. (If the parameters of the model are such that λ and λ' nearly vanish, the relaxation of M and M′ is not controlled by S, all $\mathcal{P}_{\varepsilon\varepsilon'}$ lie close to $\frac{1}{4}$, and the observation of the outcomes is not informative since they are fully random.)

More generally, for a repeated process using three apparatuses M, M′ and M″ coupled to $\hat{s}_z$, $\hat{s}_x$ and $\hat{s}_y$, respectively, the statistics of readings allows us to determine simultaneously all matrix elements of the initial density operator $\hat{r}(0)$. The considered *single apparatus* thus provides full statistical information about the state $\hat{r}(0)$ of S. Our knowledge is gained indirectly, through an expression of the type (8.82) which involves both *statistics* and *calibration* so as to determine the parameters λ, λ' and λ''. A process of the present type, although it violates the standard rules of the ideal measurement, can be regarded as a *complete statistical measurement* of the initial state of S. The knowledge of the efficiency factors allows us to determine simultaneously the statistics of the observables currently regarded as incompatible. The price to pay is the loss of precision due to the fact that the efficiency factors are less than 1, which requires a large number of runs.

The dynamics thus establish a one-to-one correspondence between the initial density matrix of S, which embeds the whole quantum probablistic information on S, and the classical probabilities of the various indications that may be registered by the apparatuses at the final time. The possibility of such a mapping was considered in [60]. The size of the domain in which the counting rates may lie is limited; for instance, if S is initially polarized along z in (8.79), no $\mathcal{P}_{\varepsilon\varepsilon'}$ can lie beyond the interval $[\frac{1}{4}(1-\lambda), \frac{1}{4}(1+\lambda)]$. The limited size of the domain for the probabilities of the apparatus indications is needed to reconcile the classical nature of these probabilities with the peculiarities of the quantum probabilities of S that arise from non commutation. It also sets limitations on the precision of the measurement.

8.3.4. Testing Bell's inequality

Bell's inequality for an EPR pair of spins is expressed in the CHSH form as [231]

$$|\langle \hat{s}_a^{(1)} \hat{s}_{a'}^{(2)} \rangle + \langle \hat{s}_b^{(1)} \hat{s}_{a'}^{(2)} \rangle + \langle \hat{s}_a^{(1)} \hat{s}_{b'}^{(2)} \rangle - \langle \hat{s}_b^{(1)} \hat{s}_{b'}^{(2)} \rangle| \leq 2, \tag{8.83}$$

which holds for classical random variables $s = \pm 1$. If $\hat{s}_a^{(1)}$ and $\hat{s}_b^{(1)}$ are the components of a quantum spin $\hat{\mathbf{s}}^{(1)}$ in the two fixed directions a and b, $\hat{s}_{a'}^{(2)}$ and $\hat{s}_{b'}^{(2)}$ the components the other spin $\hat{\mathbf{s}}^{(2)}$ in directions a' and b', the left-hand side of (8.83) can rise up to $2\sqrt{2}$[54].

Standard measurement devices allow us to test simultaneously a pair of commuting observables, for instance $\hat{s}_a^{(1)}$ and $\hat{s}_{a'}^{(2)}$. At least theoretically, the counting rates in repeated runs *directly* provide their correlation, namely $\langle \hat{s}_a^{(1)} \hat{s}_{a'}^{(2)} \rangle$. However, since $\hat{s}_a^{(1)}$ and $\hat{s}_b^{(1)}$, as well as $\hat{s}_{a'}^{(2)}$ and $\hat{s}_{b'}^{(2)}$ do not commute, we need four different settings to determine the four terms of (8.83). Checking the violation of Bells inequalities thus requires combining the outcomes of *four incompatible experimental contexts* [230, 233, 234], in each of which the spin pair is being tested through repeated runs. This necessity may be regarded as a "contextuality loophole" [235, 236]. Either hidden variables exist, and they cannot be governed by ordinary probabilities and ordinary logics, since there is no global distribution function that would yield as marginals the partial results tested in the four different contexts. Or we must admit that quantum mechanics forbids us to put together the results of these different measurements. The latter alternative is favoured by the solution of models, in which the values of physical quantities do not pre-exist but are produced during a measurement process owing to the interaction between the system and the apparatus. Since these values reflect the reality of the system only within its context, it appears inconsistent to put them together [230, 233, 234, 235, 236].

In the present situation it is tempting to imagine using a combination of apparatuses of the previous type so as to *simultaneously* test through repeated runs all four non-commuting observables $\hat{s}_a^{(1)}$, $\hat{s}_b^{(1)}$, $\hat{s}_{a'}^{(2)}$, and $\hat{s}_{b'}^{(2)}$. Such a unique experimental setting would bypass the contextuality loophole. However, as shown in § 8.3.3, the counting rates of the two apparatuses associated with the components $\hat{s}_a^{(1)}$ and $\hat{s}_b^{(1)}$ of the first spin are not directly related to the statistics of these components, but only reflect them through an efficiency factor λ at most equal to $1/\sqrt{2}$. For the pair of spins, one can *deduce* a correlation such as $\langle \hat{s}_a^{(1)} \hat{s}_{a'}^{(2)} \rangle$ from the statistical indications of the corresponding apparatuses, but this *quantum* correlation is at least equal to twice the associated *observed* correlation (since $1/\lambda^2 > 2$).

Thus, with this experimental setting which circumvents the contextuallity loophole, the correlations directly exhibited by the counting rates *satisfy Bell's inequality*; this is natural since they have a *classical* nature. However, from these very observations, we can use standard quantum mechanics to analyse the results. We thus infer indirectly from the observations, by using a mapping of the type (8.82), the tested quantum correlations (8.83) between spins components. Within a *single set of repeated experiments* where the various data are simultaneously registered, we thus acknowledge the *violation* of Bell's inequality. Here this violation no longer appears as a consequence of merging incompatible sets of measurements, but as a consequence of an analysis of the ordinary correlations produced in the apparatus.

9. Analysis of the results

And the rain from heaven was restrained
Genesis 8.2

In sections 3 we have introduced the Curie–Weiss model for the quantum measurement of a spin $\frac{1}{2}$ and in sections 4–8 we have discussed its solution. For the readers who have not desired to go through all the details, and for those who did, we resume here the main points as a separate reading guide, and add pedagogical hints for making students familiar with the matter and techniques.

9.1. Requirements for models of quantum measurements

J'ai perdu mon Eurydice[55]
Che farò senza Euridice?[56]
Christoph Willibald Gluck, Orphée et Eurydice; Orfeo ed Euridice

A model accounting for the various properties of quantum measurements should ideally satisfy the following features ("F"):

[54] For the establishment of Bell-type equalities for SQUIDs, see Jaeger et al. [232]
[55] I lost my Euridice
[56] What shall I do without Euridice?

F1: simulate as much as possible real experiments;

F2: ensure unbiased, robust and permanent registration by the pointer of A, which should therefore be macroscopic;

F3: involve an apparatus initially in a metastable state and evolving towards one or another stable state under the influence of S, so as to amplify this signal; the transition of A, instead of occurring spontaneously, is triggered by S;

F4: include a bath where the free energy released due to the irreversibility of the process may be dumped;

F5: be solvable so as to provide a complete scenario of the joint evolution of S + A and to exhibit the characteristic times;

F6: conserve the tested observable;

F7: lead to a final state devoid of "Schrödinger cats";

F8: satisfy Born's rule for the registered results;

F9: produce, for ideal measurements or preparations, the required diagonal correlations between the tested system S and the indication of the pointer, as coded in the von Neumann expression (1.7) or (9.1) for the final state of S + A;

F10: be sufficiently flexible to allow discussing processes that are not perfect measurements.

These features need not be fulfilled with mathematical rigor. A physical scope is sufficient, where violations may occur over unreachable time scales or with a negligible probability.

9.2. Features of the Curie–Weiss model

The Curie-Weiss model is satisfactory in this respect. Its choice (section 3) has relied on a compromise between two conflicting requirements. On the one hand, the apparatus A simulates a *real object*, a magnetic dot which behaves as a magnetic memory. On the other hand, the Hamiltonian of S + A is sufficiently simple so as to afford an explicit and detailed *dynamical solution*. The registration device is schematized as a set M of N Ising spins (the magnet). The size of the dot is supposed to be much smaller than the range of the interactions, both among the N spins and between them and the tested spin S. We further simplify by taking into account only interactions between the z-components of the spins of M and S. Finally, as in a real magnetic dot, phonons (with a quasi-ohmic behavior [122, 123, 145, 199, 200]) behave as a thermal bath B which ensures equilibrium in the final state (Fig. 3.1). In spite of the schematic nature of the model, its solution turns out to exhibit a rich structure and to display the various features listed in subsection 9.1.

In particular, the choice for A = M + B of a system which can undergo a phase transition implies many properties desirable for a measuring apparatus. The weakness of the interaction γ between each spin of the magnet M and the phonon bath B, maintained at a temperature T lower than T_c, ensures a long lifetime for the initial *metastable* paramagnetic state. By itself, the system M+B would ultimately relax spontaneously towards a stable state, but here its transition is triggered by S. The symmetry breaking in the dynamics of the measurement produces either one of the two possible final *stable* ferromagnetic states, in *one-to-one correspondence* with the eigenvalues of the tested observable $\hat{s}_z$ of the system S, so that the sign of the final magnetization can behave as a pointer. It is this breaking of symmetry which underlies *reduction*, owing to the *irreversibility* of the transition from the paramagnetic to either one of the ferromagnetic macroscopic states. Moreover, the built-in symmetry between the two possible outcomes of A prevents the appearance of *bias*.

An essential property of a measurement, often overlooked, is the ability of the apparatus A to *register* the indication of the pointer. Here this is ensured by the large value of the number N of spins of M, which entails a neat separation between the two ferromagnetic states of M and their extremely long lifetime. This stability warrants a *permanent* and *robust* registration. In both the paramagnetic state and the ferromagnetic states, the pointer variable m presents statistical fluctuations negligible as $1/\sqrt{N}$; breaking of invariance makes quantum influences ineffective. The nature of the order parameter, a macroscopic magnetization, also makes the result *accessible to reading, processing or printing*. These properties cannot be implemented in models for which the pointer is a microscopic object.

The coupling between the tested spin S and the apparatus A has been chosen in such a way that the observable $\hat{s}_z$ is *conserved*, $[\hat{s}_z, \hat{H}] = 0$, so as to remain unperturbed during its measurement. This coupling *triggers* the beginning of the registration process, which thereby ends up in a situation which *informs us* about the the physical state of S at a

certain moment, so that the process be a measurement. This requires a sufficiently large value of the coupling constant g which characterizes the interaction of S and M.

Once the probability distribution of the magnetization m has left the vicinity of $m = 0$ to move towards either $+m_{\mathrm{F}}$ or $-m_{\mathrm{F}}$, the motion of this pointer is *driven by the bath* through the coupling γ between M and B. Somewhat later the interaction g between S and A becomes ineffective and can be switched off. It is the interplay between the metastability of the initial state of A, the initial triggering of M by S, and the ensuing action of B on M which ensures an *amplification* of the initial perturbation. This amplification is necessary since the indication of the pointer M, which is macroscopic, should reflect an effect caused by the tested system S, which is microscopic — the very essence of a measurement.

Such a number of adequate properties makes this model attractive, but technical developments were needed to elaborate a rigorous proof that the final state of S + A has the form (1.7), viz.

$$\hat{\mathcal{D}}(t_{\mathrm{f}}) = \sum_i \left(\hat{\Pi}_i \hat{r}(0)\, \hat{\Pi}_i\right) \otimes \hat{\mathcal{R}}_i = \sum_i p_i \hat{r}_i \otimes \hat{\mathcal{R}}_i, \tag{9.1}$$

which encompasses most among the required specific features of ideal quantum measurements, in particular the absence of off diagonal terms. These developments have allowed us to *discuss the conditions* under which the process can be regarded as a measurement, and also to explore what happens if one or another condition is violated.

9.3. Scenario of the Curie–Weiss ideal measurement: the characteristic time scales

The above study (sections 4–7) of the dynamical process undergone by S + A has revealed several successive steps involving different time scales.

9.3.1. Preparation

What is cooked home is eaten home
Czech proverb

Before S and A are coupled, A should be prepared in a metastable state. Indeed, in the old days of photography the unexposed film was metastable and could not be prevented from evolving in the dark on a time scale of months. In our magnetic case, for quartic interactions within M, the *lifetime of the paramagnetic initial state* is extremely large, exponentially large in N. For quadratic interactions with coupling constant J, it was evaluated in section § 7.3.2 (eq. (7.66)) as

$$\tau_{\mathrm{para}} = \frac{\hbar}{\gamma(J-T)} \ln \alpha \sqrt{N}, \tag{9.2}$$

where α is typically of order $1/10$, and it is larger than all other characteristic times for $\alpha \sqrt{N} \gg 1$. We can thus engage the measurement process by switching on the interaction between S and M during the delay τ_{para} after preparation of A, before the paramagnetic state is spontaneously spoiled.

9.3.2. Reduction

Let us recall our decomposition of the density matrix $\hat{\mathcal{D}}$ of the total system S + A into blocks with definite value $s_z = \uparrow, \downarrow$ of the tested spin component $\hat{s}_z$:

$$\hat{\mathcal{D}} = \begin{pmatrix} \hat{\mathcal{R}}_{\uparrow\uparrow} & \hat{\mathcal{R}}_{\uparrow\downarrow} \\ \hat{\mathcal{R}}_{\downarrow\uparrow} & \hat{\mathcal{R}}_{\downarrow\downarrow} \end{pmatrix}. \tag{9.3}$$

The first stage of the measurement process is the reduction, defined as the disappearance of the off-diagonal blocks $\hat{\mathcal{R}}_{\uparrow\downarrow}$ and $\hat{\mathcal{R}}_{\downarrow\uparrow}$ of the full density matrix (section 5). It takes place during the *reduction time*

$$\tau_{\mathrm{red}} = \frac{\hbar}{\sqrt{2N}\delta_0 g}, \tag{9.4}$$

which is governed by the coupling constant g between S and M and the size N of the pointer (the fluctuation of M in the paramagnetic state is $\delta_0/\sqrt{N}$). This characteristic time is the shortest of all; its briefness reflects an effect

produced by a macroscopic object, the pointer M, on a microscopic one, the tested system S. During the delay $\tau_{\rm red}$, the off-diagonal components $a = x, y$ of the spin S decay on average as $\langle \hat{s}_a(t)\rangle = \langle \hat{s}_a(0)\rangle \exp[-(t/\tau_{\rm red})^2]$.

Over the time scale $\tau_{\rm red}$, only the off-diagonal blocks $\hat{\mathcal{R}}_{\uparrow\downarrow} = \hat{\mathcal{R}}_{\downarrow\uparrow}^\dagger$ of the overall density matrix $\hat{\mathcal{D}}$ of S + A are affected by the evolution. *Correlations* between S and M, involving larger and larger numbers $k = 1, 2, \cdots$ of spins of M, such as $\langle \hat{s}_a \hat{m}^k(t)\rangle_{\rm c} \propto t^k \exp[-(t/\tau_{\rm red})^2]$ are successively created in a *cascade*: They develop later and later, each one reaches a maximum for $t = \tau_{\rm red}\sqrt{k/2}$ and then tends to zero (§ 5.1.3 and Fig. 5.1). The information originally carried by the off-diagonal elements of the initial density matrix of S are thus transferred towards correlations which couple the system S with more and more spins of M and eventually decline (§ 5.1.4). When t increases far beyond $\tau_{\rm red}$, all the matrix elements of $\hat{\mathcal{R}}_{\uparrow\downarrow}$ that contribute to correlations of rank $k \ll N$ tend to zero. Correlations of higher rank k, for large but finite N, are the residue of reversibility of the microscopic evolution generated by $\hat{H}_{\rm SA}$ (§ 5.3.2).

If the total Hamiltonian of S + A did reduce to the coupling $\hat{H}_{\rm SA} = -Ng\hat{s}_z\hat{m}$ which produces the above behavior, the reduction would be provisional, since S + A would periodically return to its initial state with the *recurrence time*

$$\tau_{\rm recur} = \frac{\pi\hbar}{2g}, \tag{9.5}$$

much larger than $\tau_{\rm red}$ (§ 5.3.1). As in spin-echo experiments, the extremely small but extremely numerous correlations created by the interaction between S and the many spins of M would conspire to progressively reconstruct the off-diagonal blocks of the initial uncorrelated state of S + A: The reversibility and simplicity of the dynamics would ruin the initial reduction.

Two possible mechanisms can prevent such recurrences to occur. In subsection 6.1 we slightly modify the model, taking into account the (realistic) possibility of a spread δg in the coupling constants g_n between S and each spin of the magnet M. The Hamiltonian (6.1) with the conditions (6.2) then produces the same initial reduction as with constant g, over the same characteristic time $\tau_{\rm red}$, but recurrences are now ruled out owing to the dispersion of the g_n, which produces an extra damping as $\exp[-(t/\tau_{\rm irrev}^{\rm M})^2]$. The *irreversibility time induced by the fluctuations* δg in the spin-magnet coupling,

$$\tau_{\rm irrev}^{\rm M} = \frac{\hbar}{\sqrt{2N}\delta g}, \tag{9.6}$$

is intermediate between $\tau_{\rm red}$ and $\tau_{\rm recur}$ provided δg is sufficiently large, viz. $g/\sqrt{N} \ll \delta g \ll g$. As usual for a reversible linear evolution, a recurrence phenomenon still occurs here, but the recurrence time is inaccessibly large as shown in § 6.1.2 (see eq. (6.20)). The numerous but weak correlations between S and M, issued from the off-diagonal blocks of the initial density matrix of S, are therefore completely ineffective over any reasonable time lapse.

An alternative mechanism can also rule out any recurrence, even if the couplings between S and the spins are all equal (subsection 6.2). In this case, the required irreversibility is induced by the bath, which produces an extra decay, as $\exp[-NB(t)]$, of the off-diagonal blocks (the shape of $B(t)$ is shown in Fig 6.1). The initial reduction of section 5, for $t \ll 1/\Gamma$, is not affected by the interaction with the bath if $NB(\tau_{\rm red}) \ll 1$, that is, if

$$\frac{\gamma\hbar^2\Gamma^2}{8\pi N\delta_0^4 g^2} \ll 1, \tag{9.7}$$

where Γ is the Debye cutoff on the phonon frequencies. At times t such that $t \gg \hbar/2\pi T$, $B(t)$ is quasi linear and the bath produces an exponential decay, as $\exp(-t/\tau_{\rm irrev}^{\rm B})$, where the *bath-induced irreversibility time* is defined as

$$\tau_{\rm irrev}^{\rm B} = \frac{2\hbar\tanh g/T}{N\gamma g} \simeq \frac{2\hbar}{N\gamma T}. \tag{9.8}$$

This expression is a typical decoherence time, inversely proportional to the temperature T of B, to the bath-magnet coupling γ and to the number N of degrees of freedom of the system S + M. The p-th recurrence is then damped by a factor $\exp(-p\tau_{\rm recur}/\tau_{\rm irrev}^{\rm B})$, so that the phonon bath eliminates all recurrences if $\tau_{\rm irrev}^{\rm B} \ll \tau_{\rm recur}$.

At this stage, *the reduction is achieved* in the sense that the off-diagonal blocks $\hat{\mathcal{R}}_{\uparrow\downarrow}(t)$ and $\hat{\mathcal{R}}_{\downarrow\uparrow}(t)$ of the density operator (9.3) of S + A have practically disappeared in a definitive way. The off-diagonal correlations created during the reduction process have been irremediably destroyed at the end of this process, whereas the diagonal correlations needed to register in A the tested properties of S are not yet created. See also § 11.2.3 below.

9.3.3. Registration

Just after the irreversibility time, the diagonal blocks $\hat{\mathcal{R}}_{\uparrow\uparrow}(t)$ and $\hat{\mathcal{R}}_{\downarrow\downarrow}(t)$ as well as the marginal density operator $\hat{\mathcal{R}}(t) = \mathrm{tr}_S \hat{\mathcal{D}}(t) = \hat{\mathcal{R}}_{\uparrow\uparrow}(t) + \hat{\mathcal{R}}_{\downarrow\downarrow}(t)$ of A *remain nearly unaffected*. The process cannot yet be regarded as a measurement: The pointer gives no indication, m is still small, and no correlation exists between A and the initial state of S.

The registration then starts and proceeds on time scales much larger than the above ones. It is a slower process because it leads to a change of a macroscopic object, the apparatus, triggered by the microscopic S. After a brief transient regime (§ 7.1.1) the process becomes Markovian (§ 7.1.2). The evolution of each of the two diagonal blocks $\hat{\mathcal{R}}_{\uparrow\uparrow}(t)$ or $\hat{\mathcal{R}}_{\downarrow\downarrow}(t)$ can be expressed in terms of that of the corresponding probability distribution $P_{\uparrow\uparrow}(m,t)$ or $P_{\downarrow\downarrow}(m,t)$ for the magnetization of M, which obeys an equation of the Fokker-Planck type [202]. This equation, presenting classical features (§ 7.1.3), is governed for $P_{\uparrow\uparrow}(m,t)$ by a *drift velocity* $v(m)$ given by (7.6) and illustrated by Figs. 7.1 and 7.2, and by a *diffusion* coefficient given by (7.7). The irreversibility of the process is exhibited by an *H-theorem* (§ 7.1.4) which implies the decrease of the free energy of M. Thus, the total entropy of M + B increases, and some energy is dumped from M to B, while the transition leads from the paramagnetic to either one of the ferromagnetic states. The existence of two possible final states is associated with breaking of ergodicity, discussed for finite but large N in § 7.1.5 and subsection 7.3.

For purely quadratic interactions within M (the coupling (3.7) behaving as $J\hat{m}^2$), the registration proceeds in three stages (§ 7.2.3), illustrated by Figs. 7.3 and 7.5. Firstly the distribution $P_{\uparrow\uparrow}(m,t)$, initially a paramagnetic symmetric peak around $m = 0$, is shifted faster and faster towards the positive direction of m and it widens, under the conjugate effects of both S and B. For suitably chosen parameters, after a delay given by Eq. (7.44),

$$\tau_{\mathrm{reg}} = \frac{\hbar}{\gamma(J-T)}, \tag{9.9}$$

that we term the *first registration time*, $P_{\uparrow\uparrow}(m,t)$ is entirely located in the positive region of m, its tail in the region $m < 0$ has then become negligible. Symmetrically, $P_{\downarrow\downarrow}(m,t)$ lies entirely in the $m < 0$ region for $t > \tau_{\mathrm{reg}}$. Thereafter the coupling between M and S becomes ineffective and may be *switched off*, so that the registration is virtually, but not yet fully, achieved at this time τ_{reg}.

The last two stages describe a standard relaxation process for which the tested system S is no longer relevant. The stochastic motion of m is first governed mainly by the contribution of B to the drift of the magnetization m. The distribution $P_{\uparrow\uparrow}(m,t)$ moves rapidly towards $+m_{\mathrm{F}}$, first widening, then narrowing. We term as *second registration time* τ'_{reg} the delay needed for the average magnetization to go from 0 to the vicinity of m_{F}. It is expressed by Eq. (7.48), together with (7.47) and (7.36). During the third stage of the registration, both the drift and the diffusion generated by B establish thermal equilibrium of the pointer in an exponential process, and stabilize the distribution $P_{\uparrow\uparrow}(m,t)$ around $+m_{\mathrm{F}}$. Thus, $\hat{\mathcal{R}}_{\uparrow\uparrow}(t)$ ends up as $r_{\uparrow\uparrow}(0)\hat{\mathcal{R}}_{\Uparrow}$, where $\hat{\mathcal{R}}_{\Uparrow}$ denotes the ferromagnetic equilibrium state with positive magnetization, and, likewise, $\hat{\mathcal{R}}_{\downarrow\downarrow}(t)$ ends up as $r_{\downarrow\downarrow}(0)\hat{\mathcal{R}}_{\Downarrow}$.

For purely quartic interactions within M (coupling as $J\hat{m}^4$), or for $3J_4 > J_2$, the transition is of first order. We can again distinguish in the registration the above three stages (§ 7.2.4), illustrated by Figs 7.4 and 7.6. Here the first stage is slowed down by the need to pass through the bottleneck $m \simeq m_{\mathrm{c}}$ given by (7.34). The widening of the distribution $P_{\uparrow\uparrow}(m,t)$ is much larger than for quadratic interactions, because diffusion is effective during the large duration of the bottleneck stage. Both the first and the second registration times defined above are nearly equal here, and given by (7.51), that is,

$$\tau_{\mathrm{reg}} = \frac{\pi\hbar}{\gamma T}\sqrt{\frac{m_{\mathrm{c}}T}{g-h_{\mathrm{c}}}}, \qquad m_{\mathrm{c}} \simeq \sqrt{\frac{T}{3J}}, \qquad h_{\mathrm{c}} \simeq \frac{2}{3}Tm_{\mathrm{c}}. \tag{9.10}$$

The last stage is again an exponential relaxation towards the ferromagnetic state $+m_{\mathrm{F}}$ for $P_{\uparrow\uparrow}(m,t)$.

The ratio $\tau_{\mathrm{reg}}/\tau_{\mathrm{red}}$ between the registration and reduction times, proportional to $\sqrt{N}/\gamma$, is large for two reasons, the weakness of γ and the large value of N. As usual in statistical mechanics, the coexistence of very different time scales is associated here with exact and approximate conservation laws, expressed by $[\hat{s}_z, \hat{H}] = 0$ and $[\hat{m}, \hat{H}] = [\hat{m}, \hat{H}_{\mathrm{MB}}] \propto \gamma \ll 1$.

If N is finite, the registration is not permanent. However, the characteristic time of erasure τ_{eras} is much larger than the registration time τ_{reg} by a factor behaving as an exponential of N (§ 7.3.5).

The time scales involved in this Curie–Weiss measurement process present some analogy with the relaxation times in nuclear magnetic resonance [152, 153]. The reduction, i. e., the disappearance of the transverse components $\langle \hat{s}_x \rangle$ and $\langle \hat{s}_y \rangle$ and of their correlations with A, can be compared to the transverse relaxation in NMR. The reduction time $\tau_{\rm red}$, as well as is the relaxation time $\mathcal{T}_2^*$ associated in NMR with a dispersion in the precession frequencies of the spins of a sample due to a non-uniformity of the field along z, are durations of dephasing processes in which complex exponentials interfere destructively. By themselves, these phenomena give rise to recurrences in the measurement (in our model) or to spin echoes (in NMR). The bath-induced irreversibility time $\tau_{\rm irrev}^{\rm B}$ is comparable to the relaxation time $\mathcal{T}_2$: both characterize decoherence effects, namely the damping of recurrences in the measurement, and the complete transverse relaxation which damps the echoes in NMR. Finally the registration time characterizes the equilibration of the diagonal blocks of the density matrix $\hat{\mathcal{D}}$, in the same way as the relaxation time $\mathcal{T}_1$ characterizes the equilibration of the longitudinal polarization of the spins submitted to the field along z.

9.4. Conditions for ideality of the measurement

What you do not wish for yourself, do not do to others
Confucius

Strictly speaking, for finite values of the parameters of the model, the process that we have studied is not an ideal measurement in a mathematical sense. However, in a physical sense, the situation is comparable to the solution of the irreversibility paradox, which is found by disregarding correlations between inaccessibly large numbers of particles and by focusing on time scales short compared to the inaccessible Poincaré recurrence time. Here, we can likewise identify physically the process to an ideal measurement, within negligible deviations, provided the parameters of the model satisfy some conditions.

The definition of the apparatus includes a *macroscopic pointer*, so that

$$N \gg 1. \tag{9.11}$$

The temperature T of the bath B should lie below the transition temperature of the magnet M, which equals J for quadratic interactions ($q = 2$) and $0.363\,J$ for quartic interactions ($q = 4$).

Our solution was found by retaining only the *lowest order* in the coupling between B and M. Neglecting the higher order terms is justified provided

$$\gamma \ll \frac{T}{J}. \tag{9.12}$$

This condition ensures that the autocorrelation time of the bath, $\hbar/T$, is short compared to the registration time (9.9) or (9.10). We have also assumed a large value for the *Debye cutoff*, a natural physical constraint expressed by

$$\hbar\Gamma \gg J. \tag{9.13}$$

The *irreversibility of the reduction*, if it is ensured by a dispersion δg of the couplings between tested spin and apparatus spins, requires a neat separation of the time scales $\tau_{\rm red} \ll \tau_{\rm irrev}^{\rm M} \ll \tau_{\rm recur}$, that is

$$\delta_0 \gg \frac{\delta g}{g} \gg \frac{1}{\pi}\sqrt{\frac{2}{N}}. \tag{9.14}$$

The coefficient δ_0, the width of the initial paramagnetic distribution of $m\sqrt{N}$, is somewhat larger than 1 for $q = 2$ (quadratic Ising interactions, Eq. (3.51) and equal to 1 for $q = 4$ (quartic interactions) or when using a strong RF field to initialize the magnet, so that the condition (9.14) is readily satisfied.

If the irreversibility of the reduction is ensured by the bath, we should have $NB(\tau_{\rm recur}) = \tau_{\rm recur}/\tau_{\rm irrev}^{\rm B} \gg 1$, that is

$$\gamma \gg \frac{4}{\pi N}\tanh\frac{g}{T}. \tag{9.15}$$

This condition provides a lower bound on the bath-magnet coupling. An upper bound is also provided by (9.7) if we wish the initial reduction to be controlled by M only. Both bounds are easily satisfied for $N \gg 1$.

The *coupling* g between S and M has been assumed to be rather weak,

$$g < T. \tag{9.16}$$

However, this coupling should be sufficiently strong to initiate the registration, and to ensure that the final indication of the pointer after decoupling will be $+m_{\mathrm{F}}$ if S lies initially in the state $|\uparrow\rangle$, $-m_{\mathrm{F}}$ if it lies initially in the state $|\downarrow\rangle$. For $q = 2$, this condition is not very stringent. We have seen in § 7.2.2 that it is expressed by (7.41), namely

$$g \gg \frac{(J-T)\delta_1}{\sqrt{N}}, \qquad \delta_1^2 = \delta_0^2 + \frac{T}{J-T} = \frac{T_0}{T_0 - J} + \frac{T}{J-T}. \tag{9.17}$$

For purely quartic interactions $-\frac{1}{4}J\hat{m}^4$ (or for $3J_4 > J_2$) the paramagnetic state is locally stable in the absence of interaction with S. The coupling g should therefore be larger than some threshold, finite for large N,

$$g > h_{\mathrm{c}} \simeq \sqrt{\frac{4T^3}{27J}}, \tag{9.18}$$

so as to trigger the phase transition from $m = 0$ to $m = \pm m_{\mathrm{F}}$ during the delay (9.10). Moreover, if we wish the decoupling between S and A to take place before the magnet has reached ferromagnetic equilibrium, g must lie sufficiently above h_{c} (see Eq. (7.57)).

If all the above conditions are satisfied, the final state reached by S + A is physically indistinguishable from the surmise (9.1), which encompasses the required properties of ideal measurements: reduction, Born's rule expressed by unbiased registration, full correlation between the indication of the apparatus and the final state of the tested system.

9.5. Processes differing from ideal measurements

In de beperking toont zich de meester[57]
Le mieux est l'ennemi du bien[58]
Dutch and French sayings

Violations of some among the conditions of subsection 9.4 or modifications of the model allow us to get a better insight on quantum measurements, by evaluating deviations from ideality and exploring processes which fail to be measurements, but are still respectable evolutions of coupled quantum mechanical systems.

In subsection 5.2, we modify the initial state of the apparatus, assuming that it is not prepared in an equilibrium paramagnetic state. This discussion leads us to understand reduction as a consequence of the *disordered nature of the initial state* of M, whether or not this state is pure (§ 5.2.2). For some peculiar initial states, the reduction mechanism can even fail (§ 5.2.3).

Imperfect preparation may also produce another kind of failure. In § 7.3.3 we consider a *bias in the initial state* due to the presence during the preparation stage of a parasite magnetic field which produces a paramagnetic state with non-zero average magnetization. Wrong registrations, for which M reaches for instance a negative magnetization $-m_{\mathrm{F}}$ in the final state although it is coupled to a tested spin in the state $s_z = +1$, may then occur with a probability expressed by (7.79).

Section 6 shows that *recurrences* are not washed out if the conditions Eq. (9.14) or (9.15) are not fulfilled. The probability for the p-th recurrence to occur is $\exp[-(p\tau_{\mathrm{recur}}/\tau^{\mathrm{M}}_{\mathrm{irrev}})^2]$ in the first case, $\exp(-p\tau_{\mathrm{recur}}/\tau^{\mathrm{B}}_{\mathrm{irrev}})$ in the second case. The process is not an ideal measurement if recurrences are still present when the outcome is read.

The violation of the condition (9.17) for $q = 2$ or (9.18) for $q = 4$ prevents the registration from taking place properly. For $q = 2$, if the coupling g is too weak to satisfy (9.17), the apparatus does relax towards either one of the ferromagnetic states $\pm m_{\mathrm{F}}$, but it may provide a *false indication*. The probability for getting wrongly $-m_{\mathrm{F}}$ for an initial state $|\uparrow\rangle$ of S, evaluated in § 7.3.3, is given by (7.79). For $q = 4$, the registration is aborted if (9.18) is violated: the magnet M does not leave the paramagnetic region, and its magnetization returns to 0 when the coupling is switched off.

[57] Conciseness exposes the master
[58] Best is the enemy of good

The *large number N* of elements of the pointer M is essential to ensure a faithful and long-lasting registration. It also warrants a brief reduction time, and an efficient suppression of recurrences by the bath. We study in subsection 8.1 the extreme situation with $N = 2$, for which $\hat{m}$ has only two "paramagnetic" eigenstates with $m = 0$ and two "ferromagnetic" eigenstates with $m = \pm 1$. Although correlations can be established at the time (8.20) between the initial state of S and the magnet M in agreement with Born's rule, there is no true registration. The indication of M reached at that time is lost after a delay $\tau_{\rm obs}$ expressed by (8.15); moreover, a macroscopic extra apparatus is needed to observe M itself during this delay. On the other hand, the reduction process, governed here by the bath, is more akin to equilibration than to decoherence; it has an anomalously long characteristic time, longer than the registration time. These non-idealities of the model with $N = 2$ are discussed in § 8.1.5. However, such a device might be used (§ 8.1.6) to implement the idea of determining all four elements of the density matrix of S by means of repeated experiments using a single apparatus [237, 238, 239].

In subsection 8.2 we tackle the situation in which the measured observable $\hat{s}_z$ is *not conserved* during the evolution. An ideal measurement is still feasible under the condition (8.71), but it fails if S and A are not decoupled after some delay (§ 8.2.5).

The model can also be extended (subsection 8.3) by simultaneously coupling S with *two apparatuses* A and A′ which, taken separately, would measure $\hat{s}_z$ and $\hat{s}_x$, respectively. The simultaneous measurement of such non-commuting observables is of course impossible. However, here again, repeated runs can provide full information of both $\hat{s}_z$ and $\hat{s}_x$ in the initial state $\hat{r}(0)$ (§ 8.3.3). More generally, all the elements of the density matrix $\hat{r}(0)$ characterizing an ensemble of identically prepared spins S can be determined by repeated experiments involving a compound apparatus A+A′+A″, where A, A′ and A″ are simultaneously coupled to the observables $\hat{s}_x$, $\hat{s}_y$ and $\hat{s}_z$, respectively. Indirect tests of Bell's inequalities may rely on this idea (§ 8.3.4).

9.6. Pedagogical hints

The path is made by walking
African proverb

Models of quantum measurements give rise to many exercises of tutorial interest, which help the students to better grasp quantum (statistical) mechanics. We have encountered above several questions which may inspire teachers. The exercises that they suggest require the use of density operators. As quantum mechanics is often taught only in the language of pure states, we present in appendix G an introduction for students on this topic.

For instance, the *treatment of a thermal bath* at lowest order in its coupling with the rest of the system (subsection 4.2 and Appendix A), although standard, deserves to be worked out by advanced students.

For a general class of models of measurement involving a pointer with many degrees of freedom, the *reduction mechanism* exhibited in § 5.1.2 shows how dephasing can eliminate the off-diagonal blocks of the density matrix of S + A over a short time through interferences.

The evaluation of the *recurrence time* for the pointer coupled with the tested system, or more generally for an arbitrary quantum system (or for a linear dynamical system) having a random spectrum (§ 6.1.2 and Appendix C) is also of general interest.

We now give two further examples of exercises for students which highlight the central steps of the quantum measurement.

9.6.1. End of "Schrödinger cats"

Focusing on the Curie-Weiss model, we present here a simpler derivation of the processes which first lead to reduction and which prevent recurrences from occurring. We showed in section 6 and Appendix D that the interactions J_2 and J_4 between the spins $\hat{\sigma}^{(n)}$ of M play little role here, so that we neglect them. We further assume that M lies initially in the most disordered state (3.46), that we write out, using the notation (3.1), as

$$\hat{R}_{\rm M}(0) = \frac{1}{2^N} \hat{\sigma}_0^{(1)} \otimes \hat{\sigma}_0^{(2)} \otimes \cdots \otimes \hat{\sigma}_0^{(N)}. \tag{9.19}$$

This occurs for $q = 4$ and in the general case of $J_2 > 0$ provided the temperature of preparation T_0 in (3.51) is much higher than J_2, so that $\delta_0 = 1$. Then, since the Hamiltonian $\hat{H}_{\rm SA} + \hat{H}_{\rm B} + \hat{H}_{\rm MB}$ is a sum of independent contributions

associated with each spin $\hat{\boldsymbol{\sigma}}^{(n)}$, the spins of M behave independently at all times, and the off-diagonal block $\hat{R}_{\uparrow\downarrow}(t)$ of $\hat{D}(t)$ has the form

$$\hat{R}_{\uparrow\downarrow}(t) = r_{\uparrow\downarrow}(0)\hat{\rho}^{(1)}(t) \otimes \hat{\rho}^{(2)}(t) \otimes \cdots \otimes \hat{\rho}^{(N)}(t), \tag{9.20}$$

where $\hat{\rho}^{(n)}(t)$ is a 2×2 matrix in the Hilbert space of the spin $\hat{\boldsymbol{\sigma}}^{(n)}$. This matrix will depend on $\hat{\sigma}_z^{(n)}$ but not on $\hat{\sigma}_x^{(n)}$ and $\hat{\sigma}_y^{(n)}$, and it will neither be hermitean nor normalized.

The task starts with keeping the effect of the bath as in subsection 6.2, but leave open the possibility for the coupling g_n to be random as in subsection 6.1, whence the coupling between S and A reads $\hat{H}_{\rm SA} = -\hat{s}_z \sum_{n=1}^{N} g_n \hat{\sigma}_z^{(n)}$ instead of (3.5). (As simpler preliminary exercises, one may keep the $g_n = g$ as constant, and/or disregard the bath.) Each factor $\hat{\rho}^{(n)}(t)$, initially equal to $\frac{1}{2}\hat{\sigma}_0^{(n)}$, evolves according to the same equation as (4.8) for $\hat{R}_{\uparrow\downarrow}(t)$, rewritten with $N = 1$. (To convince oneself of the product structure (9.20), it is instructive to work out the cases $N = 1$ and $N = 2$ in Eq. (4.8) or (4.18).) Admit, as was proven in subsection 6.2 and appendix D, that the effect of the bath is relevant only at times $t \gg \hbar/2\pi T$, and that in this range $\hat{\rho}^{(n)}$ evolves according to

$$\frac{\mathrm{d}\hat{\rho}^{(n)}(t)}{\mathrm{d}t} - \frac{2ig_n}{\hbar}\hat{\rho}^{(n)}\hat{\sigma}_z^{(n)} = -\frac{2\gamma}{\hbar^2}\left[\tilde{K}_-\left(\frac{2g_n}{\hbar}\right) + \tilde{K}_+\left(-\frac{2g_n}{\hbar}\right)\right]\left[\hat{\rho}^{(n)} - \frac{1}{2}\hat{\sigma}_0^{(n)}\mathrm{tr}\hat{\rho}^{(n)}\right]. \tag{9.21}$$

(Advanced students may derive this equation by noting that for $N = 1$, $\hat{\rho}^{(n)}$ can be identified to $P_{\uparrow\downarrow}(\hat{m} = \hat{\sigma}_z)$; starting then from Eq. (4.17) for $N = 1$, keeping in mind that $P_{\uparrow\downarrow}(\pm 3) = 0$ and verifying that, in the non-vanishing terms, Eq. (4.13) implies that $\Omega_i^{\pm} = \mp 2g_n s_i/\hbar$, they should show that the factors $\tilde{K}_{t>}(\Omega_\uparrow^-) + \tilde{K}_{t<}(\Omega_\downarrow^-)$ and $\tilde{K}_{t>}(\Omega_\uparrow^+) + \tilde{K}_{t<}(\Omega_\downarrow^+)$ of (4.17) reduce for $t \gg \hbar/2\pi T$ and for $J_2 = 0$ to the symmetric part of $\tilde{K}(2g_n/\hbar)$ according to (4.18) and (D.21).)

Next parameterize $\hat{\rho}^{(n)}$ as

$$\hat{\rho}^{(n)}(t) = \frac{1}{2}\exp\left[-B_n(t) + i\Theta_n(t)\hat{\sigma}_z^{(n)}\right], \tag{9.22}$$

and derive from (9.21) the equations of motion

$$\begin{aligned}
\frac{\mathrm{d}\Theta_n}{\mathrm{d}t} &= \frac{2g_n}{\hbar} - \frac{\gamma}{\hbar^2}\left[\tilde{K}\left(\frac{2g_n}{\hbar}\right) + \tilde{K}\left(-\frac{2g_n}{\hbar}\right)\right]\sin 2\Theta_n, \\
\frac{\mathrm{d}B_n}{\mathrm{d}t} &= \frac{2\gamma}{\hbar^2}\left[\tilde{K}\left(\frac{2g_n}{\hbar}\right) + \tilde{K}\left(-\frac{2g_n}{\hbar}\right)\right]\sin^2\Theta_n,
\end{aligned} \tag{9.23}$$

with initial conditions $\Theta_n(0) = 0$, $B_n(0) = 0$. Keeping only the dominant contributions for $\gamma \ll 1$, use the expression (3.37) for $\tilde{K}$, find the solution

$$\Theta_n(t) \simeq \frac{2g_n t}{\hbar}, \qquad B_n(t) \simeq \frac{\gamma g_n}{2\hbar}\coth\frac{g_n}{T}\left(t - \frac{\hbar}{4g_n}\sin\frac{4g_n t}{\hbar}\right), \tag{9.24}$$

and compare B_n with (6.28) for B.

Eqs. (9.22), (9.24) provide the evolution of the density matrix of the spin n from the paramagnetic initial state $\hat{\rho}^{(n)}(0) = \frac{1}{2}\mathrm{diag}(1,1)$ to

$$\hat{\rho}^{(n)}(t) = \frac{1}{2}\mathrm{diag}\left(e^{2ig_n t/\hbar}, e^{-2ig_n t/\hbar}\right)\exp\left[-\frac{\gamma g_n}{2\hbar}\coth\frac{g_n}{T}\left(t - \frac{\hbar}{4g_n}\sin\frac{4g_n t}{\hbar}\right)\right]. \tag{9.25}$$

By inserting (9.25) into (9.20) and tracing out the pointer variables, one finds the transverse polarization of S as

$$\frac{1}{2}\langle\hat{s}_x(t) - i\hat{s}_y(t)\rangle \equiv \mathrm{tr}_{\rm S,A}\hat{\mathcal{D}}(t)\frac{1}{2}(\hat{s}_x - i\hat{s}_y) = r_{\uparrow\downarrow}(t) \equiv r_{\uparrow\downarrow}(0)\,\mathrm{Evol}(t), \tag{9.26}$$

where the temporal evolution is coded in the function

$$\mathrm{Evol}(t) \equiv \left(\prod_{n=1}^{N}\cos\frac{2g_n t}{\hbar}\right)\exp\left[-\sum_{n=1}^{N}\frac{\gamma g_n}{2\hbar}\coth\frac{g_n}{T}\left(t - \frac{\hbar}{4g_n}\sin\frac{4g_n t}{\hbar}\right)\right]. \tag{9.27}$$

To see what this describes, the student can first take $g_n = g$, $\gamma = 0$ and plot the factor $|\mathrm{Evol}(t)|$ from $t = 0$ to $5\tau_{\mathrm{recur}}$, where $\tau_{\mathrm{recur}} = \pi\hbar/2g$ is the time after which $|r_{\uparrow\downarrow}(t)|$ has recurred to its initial value $|r_{\uparrow\downarrow}(0)|$. By increasing N, e.g., $N = 1, 2, 10, 100$, he/she can convince him/herself that the decay near $t = 0$ becomes close to a Gaussian decay, over the characteristic time τ_{red} of Eq. (9.4). The student may demonstrate this analytically by setting $\cos 2g_n t/\hbar \approx \exp(-2g_n^2 t^2/\hbar^2)$ for small t. This time characterizes decoherence, that is, disappearance of the off-diagonal blocks of the density matrix; we called it "reduction time" rather than "decoherence time" to distinguish it from usual decoherence, which is induced by a thermal environment and coded in the second factor of Evol(t).

The exercise continues with the aim to show that $|\mathrm{Evol}| \ll 1$ at $t = \tau_{\mathrm{recur}}$ in order that the model describes a faithful quantum measurement. To this aim, keeping $\gamma = 0$, the student can in the first factor of Evol decompose $g_n = g + \delta g_n$, where δg_n is a small Gaussian random variable with $\langle \delta g_n \rangle = 0$ and $\langle \delta g_n^2 \rangle \equiv \delta g^2 \ll g^2$, and average over the δg_n. The Gaussian decay (6.10) will thereby be recovered, which already prevents recurrences. The student may also take e.g. $N = 10$ or 100, and plot the function to show this decay and to estimate the size of Evol at later times.

Next by taking $\gamma > 0$ the effect of the bath in (9.27) can be analyzed. For values γ such that $\gamma N \gg 1$ the bath will lead to a suppression. Several further tasks can be given now: Take all g_n equal and plot the function Evol(t); take a small spread in them and compare the results; make the small-g_n approximation $g_n \coth g_n/T \approx T$, and compare again.

At least one of the two effects (spread in the couplings or suppression by the bath) should be strong enough to prevent recurrences, that is, to make $|r_{\uparrow\downarrow}(t)| \ll |r_{\uparrow\downarrow}(0)|$ at any time $t \gg \tau_{\mathrm{red}}$, including the recurrence times. The student can recover the conditions (9.14) or (9.15) under which the two mechanisms achieve to do so. The above study will show him/her that, in the dynamical process for which each spin $\hat{\boldsymbol{\sigma}}^{(n)}$ of M independently rotates and is damped by the bath, the reduction, which destroys the expectation values $\langle \hat{s}_a \rangle$ and all correlations $\langle \hat{s}_a \hat{m}^k(t) \rangle$ ($a = x$ or y, $k \geq 1$), arises from the precession of the tested spin $\hat{s}$ around the z-axis; this is caused by the conjugate effect of the many spins $\hat{\sigma}^{(n)}$ of M, while the suppression of recurrences is either due to dephasing if the g_n are non-identical, or due to damping by the bath.

A less heavy exercise is to derive (5.27) from (5.26); hereto the student first calculates $\langle m \rangle$ and then $\langle m^2 \rangle$. Many other exercises may be inspired by sections 5 and 6, including the establishment and disappearance of the off-diagonal spin-magnet correlations (§ 5.1.3); the numerical or analytical derivation of the damping function $B(t)$ (Appendix D); its short-time behavior obtained either as for (D.9) or from the first two terms of the short-time expansion of $K(t)$; the analytical study of the autocorrelation functions $K(t)$, $K_{>t}$ and $K_{<t}$ of the bath for different time scales using the complex plane technique of Appendix D.

9.6.2. Simplified description of the registration process

We have seen in § 7.1.3 that the registration process looks, for the diagonal block $R_{\uparrow\uparrow}(t)$, as a classical relaxation of the magnet M towards the stable state with magnetization $+m_{\mathrm{F}}$ under the effect of the coupling g which behaves in this sector as a positive field. This idea can be used to describe the registration by means of the classical Fokker-Planck equation (7.1) which governs the evolution of the probability distribution $P(m,t) = P_{\uparrow\uparrow}(m,t)/r_{\uparrow\uparrow}(0)$.

By assuming explicit expressions for the drift and the diffusion coefficient which enter this equation of motion, one can recover some of the results of section 7 in a form adapted to teaching.

In particular, if we keep aside the shape and the width of the probability distribution, which has a narrow peak for large N (§ 7.2.1), the center $\mu(t)$ of this peak moves according to the mean-field equation

$$\frac{\mathrm{d}\mu(t)}{\mathrm{d}t} = v[\mu(t)], \tag{9.28}$$

where $v(m)$ is the local drift velocity of the flow of m. This equation can be solved once $v(m)$ is given, and its general properties do not depend on the precise form of $v(m)$. The first choice is phenomenological: we take $v(m)$ proportional to $-\mathrm{d}F/\mathrm{d}m$, where F is the free energy (3.54), resulting in

$$v(m) = \frac{C(m)}{\hbar}\left(Jm^{q-1} + g - \frac{T}{2}\ln\frac{1+m}{1-m}\right), \tag{9.29}$$

with a dimensionless, positive function $C(m)$ which may depend smoothly on m in various ways (§ 7.1.3), or even be approximated as a constant. An alternative phenomenological choice consists in deriving from detailed balance, as in § 7.1.2, the expression (7.14) for $v(m)$, that is, within a multiplicative factor $\theta(m)$,

$$v(m) = \frac{1}{\theta(m)} \left(\tanh \frac{g + Jm^{q-1}}{T} - m \right). \tag{9.30}$$

possibly approximating θ as a constant. A more precise way is to derive $v(m)$ from the autocorrelation function of the bath (Eq. (7.6)) as

$$v(m) = \frac{\gamma}{\hbar}(g + Jm^{q-1}) \left(1 - m \coth \frac{g + Jm^{q-1}}{T} \right). \tag{9.31}$$

An introductory exercise is to show that the $C(m)$ (or the $\theta(m)$) obtained from equating (9.29) (or (9.30)) to (9.31) is a smooth positive function, finite at the stable or unstable fixed points of Eq. (9.28), given by the condition $v(m) = 0$, which can in all three cases be written as $m = \tanh[(g + Jm^{q-1})/T]$.

If the coupling g is large enough, the resulting dynamics will correctly describe the transition of the magnetization from the initial paramagnetic value $m = 0$ to the final ferromagnetic value $m = m_{\mathrm{F}}$. Comparison between quadratic interactions ($q = 2$) and quartic interactions ($q = 4$) is instructive. The student can determine in the latter case the minimum value of the coupling g below which the registration cannot take place, and convince him/herself that it does not depend on the form of $C(m)$. Approaching this threshold from above, one observes the slowing down of the process around the crossing of the bottleneck. This feature is made obvious by comparing the Figs 7.3 and 7.4 which illustrate the two situations $q = 2$ and $q = 4$, respectively, and which were evaluated by using the form (9.30)) of $v(m)$.

The above exercise overlooks the broadening and subsequent narrowing of the profile at intermediate times, which is relevant for finite values of N. More advanced students may be proposed to numerically solve the time evolution of $P(m, t)$, i. e., the whole registration process, at finite N, taking in the rate equations Eq. (4.16) e.g. $N = 10$, 100 and 1000. For the times of interest, $t \gg \hbar/\Gamma$, one is allowed to employ the simplified form of the rates from (4.33) and (4.14), and to set $\Gamma = \infty$. The relevant rate coefficients are listed at the end of Appendix B.

10. Implications of the measurement models on the interpretation of quantum mechanics

A man should first direct himself in the way he should go.
Only then should he instruct others
Buddha

The elucidation of the paradox of irreversibility and of the Maxwell demon has provided a deeper understanding of the Second Law of thermodynamics and an interpretation of entropy as missing information at the microscopic scale [51, 52, 62, 65, 70, 240, 241, 242]. Likewise the whole literature devoted to the quantum measurement problem has as a background the interpretation of quantum mechanics. Solving the Curie-Weiss model has reconciled all the required properties of measurements with the principles of quantum mechanics; this required the use of quantum statistical mechanics. This adequacy of the latter theory leads us to prefer, among the various interpretations of quantum mechanics [27, 30, 32, 243], a statistical interpretation. Before discussing this preference, we review the main features of this statistical interpretation, in a form slightly different from the one advocated by Ballentine [4] and supported by other authors. For a related historic perspective, see Plotnitsky [246].

10.1. Statistical interpretation

10.1.1. Statistical ensembles of systems

The statistical interpretation highlights the fact that quantum mechanics provides us only with probabilities [4, 5, 6, 25, 27, 46, 52]. Although a probabilistic theory may produce some predictions with certainty, most quantities that we deal with at the microscopic scale are subject to statistical fluctuations: expectation values, correlations at a given time, or autocorrelations at different times when we observe for instance the successive transitions of a trapped ion. Thus, explicitly or implicitly, our descriptions refer to statistical ensembles of systems and to repeated experiments [4, 27].

Even when we describe a single object we should imagine that it belongs to a thought ensemble $\mathcal{E}$ [243], all elements of which are considered to be prepared under similar conditions characterized by the same set of data. Notice the similarity with ensemble theory in classical statistical physics, which also allows probabilistic predictions on single systems [49, 50]. However, there is no quantum system devoid of any statistical fluctuations [4, 27]. Individual events resulting from the same preparation are in general not identical but obey some probability law, even when the preparation is as complete as possible.

Note that we adhere to the so-called subjective interpretation of probabilities, initiated by Bayes and Laplace, and later on advocated by Cox [247], de Finetti [248] and Jaynes [241]. According to this conception, probabilities are defined as the mathematical measure of likelihood of events. They are not inherent to the considered object alone, but are tools for making reasonable predictions about this object through consistent inference. They do not pertain to a system *in itself*, but characterize *our information* on it or on the ensemble to which it belongs. In fact, information has turned out to be a central concept in statistical physics [51, 52, 62, 65, 70, 240, 241, 242]. This idea is exemplified by spin-echo experiments [54, 55, 56, 57, 58, 152, 153]. After the initial relaxation, an observer not aware of the history of the system cannot describe its spins better than by means of a completely random probability distribution. However, the experimentalist, who is able to manipulate the sample so as to let the original magnetization revive, includes in his probablistic description the hidden correlations that keep track of the ordering of the initial state. Likewise, we can assign different probabilities to the content of a coded message that we have intercepted, depending on our knowledge about the coding [62]. Since quantum theory is irreducibly probabilistic, it has thus a partly subjective nature — or rather "inter-subjective" since under similar conditions all observers, using the same knowledge, will describe a quantum system in the same way and will make the same probabilistic predictions about it. The recent developments about the use of quantum systems as information processors [37] enforce this information-based interpretation [74, 249] (see end of § 10.2.2).

10.1.2. Observables and evolution

Hello, Dolly!
It's so nice to have you back where you belong
Written by Jerry Herman, sung by Louis Armstrong

In classical physics, the physical quantities are represented by c-numbers, that is, scalar commuting variables, possibly random in stochastic dynamics or classical statistical mechanics. In quantum physics, scalar variables are inadequate for representing physical quantities.

The mathematical representation of a quantum system involves a Hilbert space $\mathcal{H}$, and the *physical quantities* such as the position, the momentum or the spin of a particle are represented by Hermitian *operators* $\hat{O}$ acting in this space, along the lines of Heisenberg's matrix theory [5, 6, 27, 30, 32, 43, 73, 243]. These operators, called the "observables", play in quantum mechanics the same rôle as *random variables* in classical statistical mechanics, except for the essential fact that they constitute a *non-commutative algebra*, the structure of which fully characterizes the system [105]. Ordinary reasoning and macroscopic experience do not help us to develop intuition about such non-commuting physical quantities, and this is the main incentive for proposals of alternative interpretations of quantum mechanics [12, 14, 160, 162, 163, 250].

In some circumstances, when the observables of interest constitute a commutative subset, the peculiar aspects of quantum mechanics that raise difficulties of interpretation do not appear [105]. For instance, the classical probability theory is sufficient for working out the statistical mechanics of non-interacting Fermi or Bose gases at thermal equilibrium. This simplification occurs because we deal there only with commuting observables, the occupation number operators $\hat{n}_k$ for the single particle states $|k\rangle$, which can be treated as random c-numbers taking the discrete values $n_k = 0$ or 1 for fermions, $n_k = 0, 1, 2, \cdots$ for bosons. However, even in this simple case, it is the underlying non-commutative algebra of the creation and annihilation operators $\hat{a}_k^\dagger$ and $\hat{a}_k$ which explains why the eigenvalues of $\hat{n}_k = \hat{a}_k^\dagger \hat{a}_k$ are those integers. A similar situation occurs for macroscopic systems, for which classical behaviors emerge from the hidden microscopic fundamental quantum theory. The variables controlled in practice then commute, at least approximately, so that classical concepts are sufficient. Macroscopic properties such as electronic conduction versus insulation, magnetism, heat capacities, superfluidity, or the very existence of crystals have a quantum origin but obey equations of the "classical" type, in the sense that they involve only commutative variables. Non commutation, the

essence of quantum mechanics, may manifest itself only exceptionally in systems that are not microscopic, see [287] and references therein.

What one calls "quantum" and "classical" depends, though, on one's definition of these terms. We have identified above a "truly quantum" behavior with non-commutativity, a deep but restrictive definition, and other viewpoints are currently expressed, such as dependence on $\hbar$. Moreover, the quantal or classical nature of a given concept may depend on the specific situation. The center of mass of a small metallic grain can be described by its "classical" value, while the shape of its heat capacity requires a quantum description, such as the Debye model. Still the concept of specific heat, its measurement, its thermodynamic aspects, are all "pre-quantal". On the other hand, in atomic clocks one needs to control the quantum fluctuations of the position of the center of mass, which is therefore not so classical. An extreme case of quantal center of mass is a mechanical resonator in its ground state or excited by one phonon [251].

Although we have worked above in the Schrödinger picture, where the observables remain constant while the states (pure or mixed) evolve according to the Liouville-von Neumann equation, it is conceptually enlightening to account for the evolution of an isolated system in the Heisenberg picture, as a change in time of its observables. In the space of observables, this transformation should be a linear mapping and should leave invariant the algebraic relations between the whole set of observables [5, 6, 27, 30, 32, 43, 73]. This implies that the transformation is unitary, its infinitesimal generator being the Hamiltonian $\hat{H}$. The motion of observables describes a perfect transfer of information along time. The Heisenberg picture thus exhibits a *deterministic structure* within quantum mechanics [105]. It also allows to define correlations of observables taken at different times and pertaining to the same system [243]. Such autocorrelations characterize statistically the behavior of each individual system of the considered ensemble when it is followed in its evolution.

10.1.3. States

L' Etat c'est moi[59]
Louis XIV

In this scope, the definition of a quantum state is conceptually the same as in statistical mechanics [252, 65]: A state of the considered system (or more precisely a state of the real or virtual statistical ensemble $\mathcal{E}$ of systems to which it belongs) is characterized by specifying the *collection of expectation values* $\langle\hat{O}\rangle$ of all possible observables $\hat{O}$ of this system, that is, by the *correspondence* $\hat{O} \mapsto \langle\hat{O}\rangle$. This correspondence has the following properties [46, 52]: it is linear, it associates a real number to hermitean operators, a non-negative number to the square of an observable, and the number 1 to the unit operator. Such properties have a natural interpretation, entailing in particular that variances cannot be negative [46, 52]. The probabilistic nature of this correspondence is exemplified by the fact that it includes statistical fluctuations $\langle\hat{O}^2\rangle - \langle\hat{O}\rangle^2$. The probability of finding for $\hat{O}$ some eigenvalue O_i is also expressed as the expectation value $\langle\hat{\Pi}_i\rangle$ of the projection operator $\hat{\Pi}_i$ on the corresponding eigenspace of $\hat{O}$.

For infinite systems or fields, this definition of a state as a mapping of the algebra of observables onto commuting c-numbers has given rise to mathematical developments in the theory of C^*-algebras [105]. For finite systems the above properties are implemented in an elementary way. The mapping is represented by a *density operator* $\hat{\mathcal{D}}$, which is hermitean, non-negative and normalized, and which generates all the expectation values through [46, 52]

$$\hat{O} \mapsto \langle\hat{O}\rangle = \mathrm{tr}\hat{\mathcal{D}}\hat{O}. \tag{10.1}$$

In fact, according to Gleason's theorem [44], the linearity of this correspondence for any pair of commuting observables is sufficient to ensure the existence of $\hat{\mathcal{D}}$. (We use the notation $\hat{\mathcal{D}}$ for the generic system considered here; no confusion should arise with the state of S + A in the above sections.)

A density operator which characterizes a state plays the rôle of a probability distribution for the non-commuting physical quantities $\hat{A}$ since it gathers through (10.1) our whole information about an ensemble of quantum systems [241, 46, 52]. As in probability theory, the amount of *missing information* associated with the state $\hat{\mathcal{D}}$ is measured by its von Neumann entropy [46, 52, 241].

[59] The State, that's me

$$S(\hat{\mathcal{D}}) = -\mathrm{tr}\hat{\mathcal{D}}\ln\hat{\mathcal{D}}. \tag{10.2}$$

For time-dependent predictions on an isolated system, Eq. (10.1) still holds in the Heisenberg picture, with a fixed $\hat{\mathcal{D}}$ and observables evolving unitarily. Correlation and autocorrelation functions are then obtained as $\mathrm{tr}\,\hat{\mathcal{D}}\hat{O}_1(t_1)\hat{O}_2(t_2)$ (both, depending on the choice of observables and on the system) [105].

In this interpretation, what we call the state *"of a system"*, whether it is pure or not, is not a property of *the considered system in itself*, but it characterizes the statistical properties of the real or virtual *ensemble* to which this system belongs [46, 52, 241]. The word "*state*" itself is also misleading, since we mean by it the summary of *our knowledge* about the ensemble, from which we wish to make probabilistic predictions The conventional expression "the state of the system" is therefore doubly improper in quantum physics, especially within the statistical interpretation [46, 52], and we should not be misled by this wording — although we cannot help to use it when teaching.

Density operators *differ* from distributions of the probability theory taught in mathematical courses, because the quantum physical quantities have a *non-commutative* nature [5, 6, 27, 30, 32, 43, 46, 52, 73, 243]. This algebraic feature, compelled by experiments in microphysics, lies at the origin of the odd properties which make quantum mechanics counterintuitive. It implies *quantization*. It entails *Heisenberg's inequality* and hence Bohr's complementarity: since the product of the variances of two non-commuting observables has a lower bound, it is only in a fuzzy way that we can think simultaneously of quantities such as the position and the momentum (or the wavelength) of a particle, contrary to what would happen in classical statistical mechanics. Thus the non-commutation of observables implies the existence of intrinsic fluctuations, and the quantum theory is irreducibly probabilistic [5, 6, 27, 30, 32, 43, 73, 243].

One should note, however, that the non-commutation of two observables does not necessarily imply that they present quantum fluctuations. For instance, if two operators do not commute, there may exist states (their common eigenstates) in which both have well-defined values. As an example, in states with orbital momentum zero, the components $\hat{L}_x$ and $\hat{L}_y$ vanish without any statistical fluctuation. (This does not contradict Heisenberg's inequality $\Delta\hat{L}_x^2\Delta\hat{L}_y^2 \geq (\hbar^2/4)\langle\hat{L}_z\rangle$, because $\langle\hat{L}_z\rangle$ also vanishes.) Conversely, two commuting observables may fluctuate in some states, even pure ones.

In the statistical interpretation, we should refrain from imagining that the observables might take well-defined but undetectable values in a given state, and that the uncertainties about them might be a mere result of incomplete knowledge. The very concept of physical quantities has to be dramatically changed. We should accept the idea that quantum probabilities, as represented by a density operator, do not simply reflect as usual our ignorance about supposedly preexisting values of physical quantities (such as the position and the momentum of a particle), but arise because *our very conception of physical quantites as scalar numbers*, inherited from macroscopic experience, *is not in adequacy with microscopic reality* [5, 6, 27, 30, 32, 43, 73, 243]. Macroscopic physical quantities take scalar values that we can observe, in particular for a pointer, but the scalar values that we are led to attribute to microscopic (non-commuting) observables are the outcome of inferences which are indirectly afforded by our measurement processes.

10.1.4. Quantum logics and contextuality

Non-commutativity gives rise to odd phenomena that force us to overturn some of our ways of thinking. According to the above viewpoint, the violation of Bell's inequalities [24, 25, 27, 30, 231] should be attributed to the non-commutative nature of the distribution $\hat{\mathcal{D}}$ rather than to non-locality; quantum mechanics does not involve ordinary probabilities nor ordinary correlations. The violation of the classical inequality, observed experimentally [244, 245], arises when one puts together outcomes of measurements performed in different experimental contexts, and this may itself be a probem [230, 233, 234, 235, 236]. The discussion of 8.3.4 shows how quantum and ordinary correlations may be reconciled in the context of a thought experiment where one attempts to measure simultaneously, with a unique setting, all spin components.

Other quantum phenomena, involving properties satisfied exactly rather than statistically, may be regarded as failures of ordinary logics. This is exemplified by the GHZ paradox [30, 32, 253], that we now recall.

The GHZ setup is as follows: Consider six observables $\hat{B}_i$ and $\hat{C}_i$ ($i = 1, 2, 3$) such that $\hat{B}_i^2 = \hat{C}_i^2 \equiv \hat{I}$, $\hat{C}_1\hat{C}_2\hat{C}_3 \equiv \hat{I}$, and with commutators $[\hat{B}_i, \hat{B}_j] = [\hat{C}_i, \hat{C}_j] = 0$, $[\hat{B}_i, \hat{C}_i] = 0$ and $\hat{B}_i\hat{C}_j = -\hat{C}_j\hat{B}_i$ for $i \neq j$. (A physical realization is provided with 3 spins, employing their $\hat{\sigma}_x$ and $\hat{\sigma}_z$ operators by taking $\hat{B}_1 = \hat{\sigma}_x^{(1)}$, $\hat{C}_1 = \hat{\sigma}_z^{(2)}\hat{\sigma}_z^{(3)}$, and likewise[60]. In the

[60] More precisely, $\hat{B}_1 = \hat{\sigma}_x^{(1)}\hat{\sigma}_0^{(2)}\hat{\sigma}_0^{(3)}$, $\hat{C}_1 = \hat{\sigma}_0^{(1)}\hat{\sigma}_z^{(2)}\hat{\sigma}_z^{(3)}$, etc., so that $\hat{I} = \hat{\sigma}_0^{(1)}\hat{\sigma}_0^{(2)}\hat{\sigma}_0^{(3)}$

pure state $|\varphi\rangle$ characterized by $\hat{B}_i\hat{C}_i|\varphi\rangle = |\varphi\rangle$, each one of the three statements "B_i takes the same value as C_i", where $B_i = \pm 1$ and $C_i = \pm 1$ are the values taken by the observables $\hat{B}_i$ ând $\hat{C}_i$, is *separately* true, and can be experimentally checked. However, these three statements cannot be true *together*, since the identity $\hat{C}_1\hat{C}_2\hat{C}_3 \equiv \hat{I}$ seems to entail that $B_1B_2B_3 = +1$ in the considered state, whereas the algebra implies $B_1B_2B_3 = -1$, which is confirmed experimentally [33]. Indeed we are not allowed to think simultaneously about the values of B_1 and C_2, for instance, since these observables do not commute. It is not only impossible to measure them simultaneously but it is even "forbidden" (i. e., devoid of any physical meaning) to imagine in a given system the simultaneous existence of numerical values for them[61]. Thus the value taken by an observable is not a property of the system alone, but can be defined only with reference to an apparatus that might measure it. The statements of quantum mechanics are meaningful and can be logically combined only if one can imagine an *unique experimental context* in which the quantities involved might be simultaneously measured.

10.1.5. Preparations of states

Que sera, sera[62]
Jay Livingston and Ray Evans; sung by Doris Day in *The man who knew too much*

In order to analyze theoretically quantum phenomena, we need to associate with the considered situation the state that describes adequately the system (or rather the set of systems of the considered ensemble). In particular, to study a dynamical process in the Schrödinger picture, we must specify the initial state. Such an assignment can be performed in various ways, depending on the type of preparation of the system.

Textbooks often stress complete preparations, in which a complete set of commuting observables is controlled. The state $\hat{\mathcal{D}}$ is then the projection on the common eigenvector of these observables determined by their given eigenvalues. (This unambiguous determination of $\hat{\mathcal{D}}$ should not hide its probabilistic nature.) The control of a single observable may in fact be sufficient to allow a complete preparation of a pure state, in case one is able to select a non-degenerate eigenvalue that characterizes this state. Atoms or molecules are currently prepared thereby in their non-degenerate ground state.

As indicated in § 1.1.4, the ideal measurement of an observable $\hat{s}$ (like the spin component $\hat{s}_z$ in the Curie–Weiss model considered in the bulk of the present work) of a system S, followed by the selection of the outcome A_i of the pointer constitutes a preparation through measurement. If the density operator of S before the process is $\hat{r}(0)$, this selection produces the filtered state $\Pi_i\hat{r}(0)\Pi_i$, where Π_i denotes the projection operator onto the eigenspace associated with the eigenvalue s_i of $\hat{s}$ (see § 12.1.3).

Macroscopic preparations are much more incomplete [49, 50]. Usually they provide on the quantum system of interest a number of data much too small to characterize a single density operator. As in ordinary probability theory one must rely on some criterion to select among the allowed $\hat{\mathcal{D}}$'s the least biased one [241]. For instance, if the only known data are the expectation values of some observables, Laplace's "principle of insufficient reason" yields the least biased density operator, among all those compatible with the available data, as the one that maximizes the entropy (10.2) [254, 255, 256]. In particular, the energy of a small object can be controlled by macroscopic means, exchange of heat or of work; depending on the type of control, the maximum entropy criterion leads us to assign a different distribution to this object [52, 65, 240]. For instance, if one controls only the expectation value of its energy, which is free to fluctuate owing to exchanges with a large bath, the least biased state is the canonical one. Alternatively, for a non-extensive system such that the logarithm of its level density is not concave, another type of thermal equilibrium (locally more stable) can be established [257] through a different preparation involving the confinement of the energy in a narrow range. Within this range, the maximum entropy criterion leads us to attribute the same probability to all allowed levels and to adopt a microcanonical distribution.

The fact that macroscopic states cannot be characterized completely entails that in measurement models the apparatus should be supposed to have initially been prepared in a mixed state.

[61] In popular accounts the attempts to do so are explained to fail, after which the situation is often called "mind boggling" [9]. But the only mind boggling point is that one should *not* make these attempts

[62] What will be, will be

10.1.6. Mixed states and pure states

Something is rotten in the state of Denmark
Shakespeare, Hamlet

Most textbooks introduce the principles of quantum mechanics by relying on pure states $|\psi\rangle$, which evolve according to the Schrödinger equation and from which the expectation value of any observable $\hat{O}$ can be evaluated as $\langle\psi|\hat{O}|\psi\rangle$ [2, 73]. Mixed states, represented by density operators, are then constructed from pure states [2, 73]. This form of the principles entail the above-mentioned laws, namely, the Liouville–von Neumann equation of motion and the properties of the mapping (10.1) (linearity, reality, positivity and normalization).

In fact, within the statistical interpretation, there is little conceptual difference between pure states and mixed states, since in both cases the density operator behaves as a non-Abelian probability distribution that realizes the correspondence (10.1) [4, 5, 6, 27, 43, 46, 52, 65]. As a mathematical specificity, pure states are those for which all eigenvalues but one of the density operator $\hat{\mathcal{D}}$ vanish, or equivalently those for which the von Neumann entropy $S(\hat{\mathcal{D}})$ vanishes. They appear thus as extremal among the set of Hermitean positive normalized operators, in the form $|\psi\rangle\langle\psi|$.

Historically, von Neumann [2], relying on the existing concept of pure states, constructed density operators by analogy with densities in phase space of classical statistical mechanics. Each individual system of an ensemble $\mathcal{E}$ (that will be described by the density operator $\hat{\mathcal{D}}$) is assumed to have been extracted from an ensemble $\mathcal{E}_k$ described by the pure state $|\phi_k\rangle$; the same ket $|\phi_k\rangle$ occurs n_k times in the whole ensemble $\mathcal{E}$. The expectation value of an observable $\hat{O}$ in either one of the $n = \sum_k n_k$ systems of the ensemble $\mathcal{E}$ is then given by $\langle\hat{O}\rangle = \sum_k q_k\langle\phi_k|\hat{O}|\phi_k\rangle = \mathrm{tr}\,\hat{\mathcal{D}}\hat{O}$ as in (10.1), where $q_k = n_k/n$, if we define $\hat{\mathcal{D}}$ as

$$\hat{\mathcal{D}} = \sum_k |\phi_k\rangle q_k \langle\phi_k|. \tag{10.3}$$

It is important to note that the whole information on the systems belonging to $\mathcal{E}$ is embedded solely in the combination (10.3) of the kets $|\phi_k\rangle$ and of their probabilities q_k, and that the same combination $\hat{\mathcal{D}}$ can be obtained in different ways. For instance, the 2×2 density operator $\hat{\mathcal{D}} = \frac{1}{2}\hat{\sigma}_0$ which represents an unpolarized spin $\frac{1}{2}$ can be interpreted as describing a spin polarized either along $+z$ with probability $\frac{1}{2}$ or along $-z$ with probability $\frac{1}{2}$; but these two possible directions of polarization may also be taken as $+x$ and $-x$, or as $+y$ and $-y$; the same isotropic state $\hat{\mathcal{D}} = \frac{1}{2}\hat{\sigma}_0$ can also be interpreted by assuming that the direction of polarization is fully random [27, 43]. Within the statistical interpretation, we regard the above situations, described differently but involving the same state $\hat{\mathcal{D}}$, as being physically the same, since no experiment on the systems of the ensemble $\mathcal{E}$, and no theoretical argument can allow us to distinguish them [46, 65][63]. We return in § 10.2.2 to the meaning of mixed states in other interpretations.

Density operators have also been introduced by Landau in a different context [27, 43, 73]. Consider a compound system $S_1 + S_2$. Its observables are the operators that act in the Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, and its states $\hat{\mathcal{D}}$ are characterized by the correspondence (10.1) in the space $\mathcal{H}$. If we are interested only in the subsystem S_1, disregarding the properties of S_2 and the correlations between S_1 and S_2, the relevant observables constitute the subalgebra of operators acting in $\mathcal{H}_1$, and the correspondence (10.1) is implemented in the subspace $\mathcal{H}_1$ by means of the mixed density operator $\hat{\mathcal{D}}_1 = \mathrm{tr}_2\hat{\mathcal{D}}$. Suppose for instance that in an ensemble $\mathcal{E}$ of pairs S_1, S_2 of spins $\frac{1}{2}$ prepared in the singlet pure state, we wish to describe only the spin S_1. Its state in the considered ensemble $\mathcal{E}$ is again the unpolarized state, represented by $\hat{\mathcal{D}}_1 = \frac{1}{2}\hat{\sigma}_0$. Isotropy is here built in, from this definition of the state of the spin S_1.

Let us now generalize von Neumann's construction and consider within the statistical interpretation a collection of ensembles, each one prepared in a pure or mixed state $\hat{\mathcal{D}}_k$ [27, 258]. If we select among each of these ensembles some sub-ensemble $\mathcal{E}_k$ containing n_k systems, and if we constitute a new ensemble $\mathcal{E}$ by putting these systems together at random, this resulting joint ensemble $\mathcal{E}$ will be described by the density operator

$$\hat{\mathcal{D}} = \sum_k q_k \hat{\mathcal{D}}_k, \qquad q_k = \frac{n_k}{\sum_k n_k}. \tag{10.4}$$

After such a merger, we no longer know to which subensemble $\mathcal{E}_k$ each individual system belonged previously. We have lost track of the building blocks $\hat{\mathcal{D}}_k$ and q_k of (10.4), since repeated experiments performed on samples of the

[63] This may be phrased as follows: although the history that led to a certain density operator is a physically valid issue, it cannot be determined within quantum mechanics

final ensemble $\mathcal{E}$ can provide us with no other information than included in $\hat{\mathcal{D}}$. Thus if we conversely consider an arbitrary mixed state, quantum mechanics implies that its density operator can not only be decomposed in the form (10.3) or (10.4), but that moreover such a decomposition can always be performed *in an infinity of different ways*. In contrast, in classical statistical mechanics, a mixed state, represented by a density in phase space, can be regarded as a weighted sum over pure states localized at given points in phase space, but this decomposition is obviously unique.

This indetermination leads us to acknowledge an important difference between pure and mixed quantum states [4, 27, 43, 73, 258]. If a statistical ensemble $\mathcal{E}$ of systems is described by a pure state, any one of its sub-ensembles is also described by *the same pure state*, since in this case (10.4) can include only a single term. If for instance a set of spins have been prepared in the polarized state $|\uparrow\rangle$, the statistical prediction about any subset are embedded in $|\uparrow\rangle$ as for the whole set. In contrast, the existence of many decompositions (10.3) or (10.4) of a mixed state $\hat{\mathcal{D}}$ describing an ensemble $\mathcal{E}$ implies that there exists many ways of splitting this ensemble into sub-ensembles $\mathcal{E}_k$ that would be described by different states $\hat{\mathcal{D}}_k$. In particular, pure states $|\phi_k\rangle$ that would underlie as in (10.3) a mixed $\hat{\mathcal{D}}$ cannot *a posteriori* be identified unambiguously by means of experiments performed on the ensemble of systems. In the statistical interpretation, such *underlying pure states have no physical meaning*. More generally, if the density operator $\hat{\mathcal{D}}$ is given without any other information, an infinity of decompositions (10.4) always exist, and cannot be distinguished by any experimental procedure [27, 43, 258]. The only possibility of giving a meaning to such a decomposition is to have at one's disposal some extra information which allows one to identify in a unique fashion sub-ensembles $\mathcal{E}_k$ within the considered ensemble $\mathcal{E}$ described by $\hat{\mathcal{D}}$ [258]. This would occur, for instance, for a set $\mathcal{E}$ of spins in an unpolarized state $\hat{\mathcal{D}} = \frac{1}{2}\hat{\sigma}_0$, if this set have been constructed by mixing two subsets $\mathcal{E}_\uparrow$ and $\mathcal{E}_\downarrow$ with equal size, described by the pure states $|\uparrow\rangle$ and $|\downarrow\rangle$, respectively, and if we had kept track of the a priori information that allows us to identify $\mathcal{E}_\uparrow$ and $\mathcal{E}_\downarrow$ within $\mathcal{E}$. A similar remark will be crucial in section 12 where we will show that ideal measurement processes are privileged means for collecting such extra information.

Another essential difference between pure and mixed states is especially appealing to intuition [68, 224]. Consider a system in a state represented by a density operator $\hat{\mathcal{D}}$ whose eigenvalues are non-degenerate and differ from zero. Consider then a set of observables that have non-degenerate spectra.Then none of such observables can produce definite results when measured in the state $\hat{\mathcal{D}}$ [224]. In other words, all such observables have non-zero dispersion in $\hat{\mathcal{D}}$. This statement has been suitably generalized when either $\hat{\mathcal{D}}$ or the observables have degeneracies in their spectra; see Appendix C of Ref. [224]. In contrast, for a pure density operator $|\psi\rangle\langle\psi|$ all observables that have $|\psi\rangle$ as eigenvector are dispersionless. Finally pure and mixed states also differ as regards their determination via measurements (e.g., the number of observables to be measured for a complete state determination) [259].

10.2. Interpretation of measurement processes

I have not failed.
I've just found 10,000 ways that won't work
Thomas A. Edison

Most models treat quantum measurements as standard processes of quantum mechanics involving interaction between two systems, the tested one and an apparatus [see section 2]. Technically, the derivations are independent of any specific intepretation, but the ideas on which the solution relies may provide hints in favour of some interpretation or another. As a measurement is a means for gaining information about a physical quantity pertaining to some state of the system, the meaning of "physical quantity" and of "state" should anyhow be made clear.

10.2.1. Measurements in the statistical interpretation

Լավ է մրջնի գլուխ լինես քան առյուծի պոչ։[64]
Armenian proverb

Since a measurement process is irreversible and requires a macroscopic apparatus or a macroscopic environment, the explicit solution of a model requires in general the description of S + A by means of a density operator (this fact is acknowledged by many authors of measurement models; see section 2). This tool appears to constitute the very

[64] Better to be an ant's head than a lion's tail

definition (10.1) of a state in the statistical interpretation of quantum mechanics, which is thus a natural framework for the analysis of measurements.

A *classical* measurement can be regarded as a means to exhibit, through an apparatus A, some *pre-existing* property of an *individual* system S. In the statistical interpretation of a quantum measurement, we are not allowed to think that the outcome reflects directly a value that the measured observable $\hat{s}$ would be supposed to take before the process. Instead, we have to face the fact that we deal with the joint evolution of an *ensemble* of systems S + A, the outcome of which indirectly reveals only *some probabilistic properties* of the initial state of S [5, 6, 27, 43, 46, 52]. In agreement with the properties of quantum states recalled in § 10.1.3, the value s_i of the tested observable $\hat{s}$ inferred from the observation of the indication A_i of the pointer *did not preexist* the process, even though we can assert that it is taken by S after an ideal measurement where A_i has been registered.

The first stage of a measurement is usually the preparation of the apparatus A in a metastable state, which will eventually evolve towards a stable state after interaction with S to ensure registration. The solution of the model describing this process requires the assignment of an initial density operator $\hat{\mathcal{R}}(0)$ to A. Since the preparation of A can in practice be performed only by macroscopic means, the choice of $\hat{\mathcal{R}}(0)$ should according to the statistical interpretation rely on probabilistic arguments, as indicated in § 10.1.4. For the Curie–Weiss model, we have discussed this point in §§ 3.3.2 and 3.3.3; our choice of canonical equilibrium states relied implicitly on the maximum entropy criterion associated with an external control of the energy of the magnetic dot. Thus, a measurement requires a preliminary preparation of the apparatus; but this is not a preparation through measurement (which would let measurements depend on one another). This is a macroscopic preparation, which is needed to solve the measurement problem within the statistical interpretation. This preparation will cost energy.

Solving a model for an ideal measurement should provide a final state of S + A of the form

$$\hat{\mathcal{D}}\,(t_{\rm f}) = \sum_i \left(\hat{\Pi}_i \hat{r}\,(0)\,\hat{\Pi}_i\right) \otimes \hat{\mathcal{R}}_i. \tag{10.5}$$

The entropy of this state is larger than the entropy of the initial state $\hat{\mathcal{D}}(0)$ (§ 1.2.4 and [63]), which shows that the dynamical process has produced a *breaking of unitarity*. As the breaking of invariance which occurs for equilibrium states in statistical physics [49, 50], this phenomenon arises because the apparatus is macroscopic. For finite values of the parameters, the exact equations of motion would produce a final state $\hat{\mathcal{D}}_{\rm exact}(t_{\rm f})$ unitarily related to $\hat{\mathcal{D}}(0)$, but $\hat{\mathcal{D}}_{\rm exact}(t_{\rm f})$ is equivalent to (10.5) in practice, within corrections negligible under the conditions of subsection 9.4.

The statistical interpretation, with its definition (10.1) of states, enlightens this equivalence through a natural restriction. If we disregard the observables associated with correlations between an inaccessibly large number of particles, which are completely ineffective if no recurrences occur, both states $\hat{\mathcal{D}}(t_{\rm f})$ and $\hat{\mathcal{D}}_{\rm exact}(t_{\rm f})$ realize the same correspondence (10.1) for all other observables $\hat{O}$. The entropy $S\,[\hat{D}(t_{\rm f})]$, larger than $S\,[\hat{D}_{\rm exact}(t_{\rm f})] = S\,[\hat{D}(0)]$, enters the framework of the general concept of relevant entropies associated with a reduced description from which irrelevant variables have been eliminated [52]. See also the discussion on the rôle of entropies in quantum measurements in § 1.2.4.

The possibility of restricting the definition (10.1) of states to accessible observables $\hat{O}$ also allows one, either to eliminate the environment in models for which a decoherence is induced by this environment, or to eliminate the bath B in the Curie–Weiss model. In the latter case (subsection 4.1), we have traced out B from the state $\hat{\mathcal{D}}(t)$ of S + A = S + M + B so as to work out simpler equations of motion for the reduced density operator $\hat{D}(t)$ of S + M. This is justified by the practical equivalence between $\hat{\mathcal{D}}(t)$ and $\hat{D}(t) \otimes \hat{R}_{\rm B}(t)$ (Fig. 3.2), if we disregard the inobservable and ineffective correlations that develop between B and S + M during the process. A further approximation has consisted in the replacement of $\hat{R}_{\rm B}(t)$ by the initial equilibrium distribution of the bath; this is justified because only the temperature of B affects the dynamics and because B is supposed to be sufficiently large so as to stay nearly in equilibrium at its original temperature.

The statistical interpretation is closely connected to information theory (§§ 10.1.1 and 10.1.3) [5, 6, 46, 52, 62, 69, 70]. This is another bridge with quantum measurements, followed or not by selection of the outcomes, which can be regarded as means of *information processing*. The amounts of information brought into play are resumed in § 1.2.4 in the form of entropy balance.

Although the statistical interpretation thus appears in adequation with measurement models, it deals with an ensemble of systems S + A. However, the measurement problem (subsection 1.1) refers to individual processes. When

we pretend to analyze a single measurement, it should be understood that our description is always probabilistic, as regards both the theory of the process and the experimental outcome. We cannot tell what happens in each individual measurement, but we account for repeated experiments. Quantum mechanics does not treat single measurements, it treats statistics of measurements.

Moreover, the results of quantum measurements, including Born's rule and the system-pointer correlations, can be interpreted in everyday's language [260]. It is essential to understand how ordinary logics, ordinary probabilities, ordinary correlations, as well as exact statements about individual systems may emerge at our scale from quantum mechanics in measurement processes, even within the statistical interpretation which is foreign to such concepts. We postpone these major questions to section 12.

10.2.2. Empiricism versus ontology

Einstein, stop telling God what to do
Niels Bohr

There is no general agreement about the purpose of science [261]. Is our task only to explain and predict phenomena? Or are we able to uncover the very nature of things? This old debate, more epistemologic than purely scientific, cannot be skipped since it may inflect our research. The question has become more acute with the advent of quantum physics, which deals with a "veiled reality" [261]. Physicists, including the authors of the present article, balance between two extreme attitudes, illustrated by Bohr's pragmatic question [260]: *"What can we say about..."*? facing Einstein's ontological question [18]: *"What is..."?* The latter position leads one to ask questions about *individual systems* and not only about general properties, to regard quantum mechanics as an *incomplete* theory and to look for hidden "elements of reality".

This opposition may be illustrated by current discussions about the status of pure states. As shown in § 10.1.4, the statistical interpretation, which is purely empirical, puts forward the idea of density operators as tools that fully characterize an ensemble of systems, since they allow us to answer statistically any question about physical quantities $\hat{O}$. They behave as probability distributions, and so do pure states which simply appear in this context as special cases. However, many authors with ontologic aspirations afford pure states a more fundamental status, even though they acknowledge their probabilistic character; see [9] for a clear presentation of this viewpoint[65]. They argue that the mixed density operator $\hat{\mathcal{D}}$, when expressed as (10.3), should be interpreted as a collection of "underlying pure states" $|\phi_k\rangle$ which would carry more "physical reality" than $\hat{\mathcal{D}}$ itself [9]. Each individual system of the considered ensemble $\mathcal{E}$ would have its own ket $|\phi_k\rangle$. In this *realist interpretation* [27], where kets are regarded as building blocks of density operators, two types of probabilities are distinguished [9]: "merely quantal" probabilities are interpreted as properties of the individual objects through the pure states $|\phi_k\rangle$, while the weights q_k are interpreted as ordinary probabilities associated with our ignorance of the structure of the statistical ensemble; these weights play the same rôle as the density in phase space of classical statistical mechanics.

Such an interpretation of the decomposition (10.3) would be natural if, as a preliminary, each individual system of the considered ensemble $\mathcal{E}$ had been prepared in some pure state and if the systems had then been mixed at random [258] – for instance, if the unpolarized state $\hat{\mathcal{D}} = \frac{1}{2}\hat{\sigma}_0$ of an ensemble of spins had been prepared by mixing an equal number of spins polarized in the $+z$ and $-z$ directions. A different interpretation would hold for the same unpolarized state $\hat{\mathcal{D}} = \frac{1}{2}\hat{\sigma}_0$ if it resulted from the mixing of spins polarized in the $+x$ and $-x$ directions; although these two situations are not distinguishable a posteriori, their descriptions in terms of underlying pure states $|\phi_k\rangle$ are different but both are sensible. However, such processes of production of the unpolarized state $\hat{\mathcal{D}} = \frac{1}{2}\hat{\sigma}_0$ are artificial; usually a lack of polarization results from a lack of control of the spins during this process. We also find this same state for each spin of a singlet pair. In all such cases, the ambiguity of the decomposition (10.3) raises difficulties, since it is impossible to decide which is the very collection of pure states $|\phi_k\rangle$ in which the various individual systems would lie. These difficulties can sometimes have a serious impact on the physical understanding of the problem; see [258, 262] for examples.

Landau's approach to mixed states raises another question, if we admit that pure states are closer to reality than mixed states [9, 73]. Interactions between two systems S_1 and S_2 create correlations between them, so that each one is

[65] One may object that the sought underlying description of reality should lie *outside* quantum mechanics

in a mixed state. To produce a pure state, we are led to embed any system into the largest one with which it has interacted in the past, and thus to imagine that the sole pure state is that of the whole Universe, a hazardous extrapolation [5, 6]. Not to mention the introduction in quantum mechanics of hypothetic many worlds or multiverses [22, 263]. Such oddities should lead to the conclusion that pure states are no more real than mixed states, and that the distinction between the two types of probabilities on which decompositions (10.3) rely is artificial and meaningless [5, 6].

The statistical interpretation, either in the form put forward by Blokhintsev [5, 6] and Ballentine [4] or in the form presented above, is empirical and minimalistic. It regards quantum mechanics only as a means for deriving predictions from available data. The same epistemologic attitude is shared by phenomenologically-minded people, and is advocated, for instance, by de Muynck [27]. It can be viewed as a common ground for all physicists, as stressed by Laloë [30], whose "correlation interpretation" emphasizes predictions as correlations between successive experiments. A more extreme philosophical position, the rejection of any interpretation, is even defended in [264]. According to such attitudes, quantum theory has the modest task of accounting for the results of experiments or of predicting them. It deals with what we know about reality, and does not claim to unveil an underlying reality per se. Bohr himself shared [260] this conception when he said (see [265, 266] for a list of Bohr's quotations): *There is no quantum world. There is only an abstract quantum physical description. It is wrong to think that the task of physics is to find out how nature is. Physics concerns what we can say about nature.*

Empirical interpretations are therefore sufficient "for all practical purposes" [267], and should satisfy anyone interested only in the technical aspects. However, they present only a blurred image of the microscopic reality, and one may long for a more fundamental description that would uncover hidden faces of Nature [261]. It is legitimate to look for other interpretations than purely operational ones, as long as the resulting equations fit the experimental facts. Such interpretations have been extensively reviewed [12, 14, 27, 32, 160, 162, 163, 250] (see also references in § 1.1.1). We discuss some of them in connection with models of measurements in § 10.2.3 and in section 11. They range from realist interpretations, involving hidden variables of various kinds (such as Bohm's bunches of trajectories or such as stochastic backgrounds) or hidden structures (such as many worlds or consistent histories), to idealist ones, involving observers. As indicated in § 10.1.1, partly subjective interpretations that focus on information [249] are related to the statistical interpretation, since information is akin to probability.

One should distinguish the interpretations that provide exactly the same outcomes as the conventional quantum mechanics, and that can therefore neither be verified nor falsified, from the attempts to go "beyond the quantum", which resort, for instance, to stochastic electrodynamics [268, 269, 270, 271], to quantum Langevin equations [27], or to nonlinear corrections to quantum mechanics such as in the GRW approach [12, 75, 161]. The search for testable specific predictions in such a hypothetical sub-quantum theory seems to be the sole issue to close convincingly the Einstein–Bohr debate [18, 260]. For the time being, Ockam's razor supports Bohr's position.

10.2.3. Models of quantum measurements and interpretations

In the above prospect, quantum measurements throw bridges between the microscopic reality, conveniently grasped by us only through empirical and mathematical means, and the macroscopic reality, easier to apprehend directly. The images that they provide of the microscopic world obey laws more "natural" (i.e., more customary) than the counter-intuitive laws which govern the quantum scale. We will show in section 12 how so different concepts can be compatible.

As shown by the solution of models, standard quantum mechanics within the statistical interpretation provides a satisfactory explanation of all the properties, including odd ones, of quantum measurement processes. Thus, any other interpretation is of course admissible insofar as it yields the same probabilistic predictions. Discussing measurements does not help us to discriminate between interpretations that are technically equivalent, whether empirical or not. Nevertheless, we will see in section 11, to which we postpone the discussion of some of these interpretations, that they may introduce extra elements which are at variance with conclusions drawn from measurement models.

In particular, we can eliminate the variants of the "orthodox" Copenhagen interpretation in which it is postulated that two different types of evolution may exist, depending on the circumstances, a Hamiltonian evolution if the system is isolated, and a sudden change producing von Neumann's reduction and Born's rule if the system S undergoes an ideal measurement [2, 143]. We can rule out the second type of evolution, because models (at least some models discussed here) demonstrate that all features of a measurement process follow from the Liouville–von Neumann equation alone, provided it is applied to the isolated compound system S + A. The apparent violation of the superposition principle is understood as the result of interactions of S with A, of dynamics, of statistical considerations and

of approximations justified under the conditions of subsection 9.4. It is therefore legitimate to abandon the "postulate of reduction", in the same way as the old "quantum jumps" have been replaced by transitions governed by quantum electrodynamics.

The search for "sub-quantum mechanics" is motivated mainly by a wish to describe individual situations at a fundamental level [15, 27]. As the quantum measurement problem refers to individual processes, a new viewpoint on measurements might thus emerge. However, such drastic changes do not seem to be needed in the context of measurements, as they are not implied by the existing models. Justifications should probably be looked for at scales where quantum mechanics would fail, hopefully at length scales larger than the Planck scale so as to allow experimental tests[66].

Of particular interest in the context of measurements are the information-based interpretations [52, 62, 69, 70, 249]. Indeed, an apparatus can be regarded as a device which processes information about the system S, or rather about the ensemble $\mathcal{E}$ to which S belongs. The initial density operator $\hat{r}(0)$, if given, gathers our information about some preliminary prepration of S. During the process, which leads $\mathcal{E}$ to the final state $\hat{r}(t_f) = \sum_i p_i \hat{r}_i$ (Eq. (1.10)), all the off-diagonal information are lost. However, the correlations created between S and A then allow us to gain indirectly information on S by reading the outcome of the pointer. If we select its indication A_i, the corresponding systems constitute a sub-ensemble $\mathcal{E}_i$ of $\mathcal{E}$, the density operator of which being updated as $\hat{r}_i$. We have thus gained information about the systems S of each sub-ensemble $\mathcal{E}_i$. The amonts of information involved in each step are measured by the entropy balance of § 1.2.4.

11. Lessons from measurement models

Cette leçon vaut bien un fromage, sans doute[67]
Jean de la Fontaine, Le renard et le corbeau

The elucidation of the irreversibility paradox has provided an interpretation of the entropy concept [48, 49, 50, 63]. Likewise, most authors who solve models of quantum measurements (section 2) aim at elucidating the measurement problem so as to get insight on the interpretation of quantum mechanics. We gather below several ideas put forward in this search, using as an illustration the detailed solution of the Curie–Weiss model presented above, and we try to draw consequences on the interpretation of quantum physics. These ideas deserve to be taken into account in future works on measurement models.

11.1. About the nature of the solutions

11.1.1. Approximations are needed

Fire could leave ashes behind
Arab proverb

As stressed in § 1.2.1, a measurement is an *irreversible* process, though governed by the *reversible* von Neumann equation of motion for the coupled system S + A. This apparent contradiction cannot be solved with mathematical rigor if the compound system S + A is finite and all its observables are under explicit control. As in the solution of the irreversibility paradox (§ 1.2.2), some approximations, justified on physical grounds, should be introduced [48, 49, 50, 145, 199]. We accept the idea that theoretical analyses of quantum measurements are approximate [272].

For instance, when solving the Curie–Weiss model, we were led to neglect some contributions, which strictly speaking do not vanish for a finite apparatus A = M + B, but which are very small under the conditions of subsection 9.4. For the diagonal blocks $\hat{\mathcal{R}}_{\uparrow\uparrow}$ and $\hat{\mathcal{R}}_{\downarrow\downarrow}$, the situation is the same as for ordinary thermal relaxation processes [145, 199, 200]: the invariance under time reversal is broken through the *elimination of the bath* B, performed by keeping only the lowest order terms in γ and by treating the spectrum of B as continuous (section 4). Correlations within B and between B and M+S are thus disregarded, and an irreversible nearly exact Fokker–Planck equation [202] for the marginal operators $\hat{R}_{\uparrow\uparrow}$ and $\hat{R}_{\downarrow\downarrow}$ thus arises from the exact reversible dynamics. For the off-diagonal blocks $\hat{R}_{\uparrow\downarrow}$

[66] Incidentally, if experiments show that quantum mechanics happens to fail before the Planck scale is reached, new insights might be offered for its integration with gravitation

[67] Surely, this lesson is worth a cheese

and $\hat{R}_{\uparrow\downarrow}$, *correlations* between S and a large number, of order N, of spins of M are also *discarded* (section 5). Such correlations are ineffective, except for recurrences; but these recurrences are damped either by a randomness in the coupling between S and M (subsection 6.1) or by the bath (subsection 6.2). We return to this point in § 11.2.3. We also showed that, strictly speaking, *false or aborted registrations* may occur but that they are very rare (§ 7.3.4 and § 7.3.5).

Mathematically rigorous theorems can be proved in statistical mechanics by going to the thermodynamic limit of infinite systems [105]. In the Curie–Weiss model, the disappearance of $\hat{R}_{\uparrow\downarrow}$ and $\hat{R}_{\downarrow\uparrow}$ would become exact in the limit where $N \to \infty$ first, and then $t \to \infty$. However, in this limit, we lose track of the time scale $\tau_{\rm red}$, which tends to 0. Likewise, the weak coupling condition $\gamma \to 0$, needed to justify the elimination of the bath, implies that $\tau_{\rm reg}$ tends to ∞. Physically sensible time scales are obtained only *without limiting process* and at the price of approximations.

11.1.2. Probabilities are omnipresent

O Fortuna, imperatrix mundi[68]
Carmina Burana

Although the dynamics of S + A is deterministic, randomness occurs in the solution of measurement models for several reasons. On the one hand, quantum physical quantities are blurred due to the *non-commutation* of the observables which represent them, so that quantum mechanics is irreducibly probabilistic (subsection 10.1 and [5, 6, 27, 43, 46, 52]). On the other hand, the *large size* of the apparatus, needed to ensure registration, does not allow us to describe it at the microscopic scale; for instance it lies after registration in a thermal equilibrium (or quasi-equilibrium) state. Thus, both conceptually and technically, we are compelled to analyse a quantum measurement by relying on the formalism of quantum statistical mechanics.

Moreover, as shown in subsection 5.2, some randomness is needed in the *initial state* of the apparatus. Indeed, for some specific initial pure states, the process is pathological; the measurement may fail, in the same way, for instance, as some exceptional initial configurations of a classical Boltzmann gas with uniform density may produce after some time a configuration with non uniform density. For realistic models of quantum measurements, which are of rising interest for q-bit processing in quantum information theory, *experimental noise* and *random errors* should also accounted for [273].

Recognizing thus that a quantum measurement is a process of quantum statistical mechanics has led us to privilege the statistical interpretation of quantum mechanics, in which an assertion is *"certain" if its probability is close to one.* Still, the statistical solution of the quantum measurement problem does not exclude the existence of a hidden variable theory that would describe individual measurements, the statistics of which would be given by the probabilistic theory, that is, the standard quantum mechanics; see [250] for a recent review of hidden variable theories.

11.1.3. Time scales

De tijd zal het leren[69]
Dutch proverb

Understanding a quantum measurement requires mastering the *dynamics of the process* during its entire duration [77]. This is also important for experimental purposes, especially in the control of quantum information. Even when the number of parameters is small, a measurement is a complex process which takes place over several time scales, as exhibited by the solution of the Curie–Weiss model (subsection 9.3). There, the reduction time turns out to be much shorter than the registration time. This feature arises from the large number of degrees of freedom of the pointer M (directly coupled to S) and from the weakness of the interaction between M and B. The large ratio that we find for $\tau_{\rm reg}/\tau_{\rm red}$ allows us to distinguish in the process a rapid disappearance of the off-diagonal blocks of the density matrix of S + A. After that, the registration takes place as if the density matrix of S were diagonal. The registration times are also not the same for quartic or quadratic interactions within M.

However, in the variant of the Curie–Weiss model with $N = 2$ (subsection 8.1), the orders of magnitude of the reduction and registration times are reversed. In such a case, one has to wait the achievement not only of the

[68] Oh Fortune, empress of the world
[69] Time will tell

registration, but also of the reduction, before observing the results of ideal measurements. A large variety of results have been found in other models for which the dynamics was studied (section 2). This should encourage one to explore the dynamics of other, more and more realistic models.

11.1.4. May one think in terms of underlying pure states?

Als de geest uit de fles is,
krijg je hem er niet makkelijk weer in[70]
Dutch proverb

The solutions of the Curie–Weiss model and of many other models have relied on the use of density operators. We have argued (§§ 10.1.6 and 10.2.2) that, at least in the statistical interpretation, the non unicity of the representations (10.3) of mixed states as superpositions of pure states makes the existence of such underlying pure states unlikely. Here again, Ockam's razor works against such a structure, which is more complicated than the framework of quantum statistical mechanics, and which in general would not permit explicit calculations. Moreover, it is experimentally completely unrealistic to assume that the apparatus has been initially prepared in a pure state. Nevertheless, it is not a priori wrong to rely on other interpretations based on the use of pure states regarded as more fundamental than density operators [9], and to afford the latter a mere status of probabilistic technical tools, used to describe both the initial state and the evolution.

We can compare this situation with that of the irreversibility paradox for a gas (§ 1.2.2). In that case, although it is technically simpler to tackle the problem in the formalism of statistical mechanics, one may equivalently explain the emergence of irreversibility by regarding the time-dependent density in phase space as a mathematical object that synthetizes the trajectories and the random initial conditions [49, 50]. The dynamics is then accounted for by Hamilton's equations instead of the Liouville equation, whereas the statistics bears on the initial conditions. (We stressed, however, in § 10.1.6, that although density operators and densities in phase space have a similar status, quantum pure states differ conceptually from points in phase space due to their probabilistic nature; see also [49, 50] in this context.)

Likewise, in the Curie–Weiss measurement model, one may imagine to take as initial state of A a pure state, S being also in a pure state. Then at all subsequent times S + A lies in pure states unitarily related to one another. However it is impossible in any experiment to prepare A = M + B in a pure state. What can be done is to prepare M and B in thermal equilibrium states, at a temperature higher than the Curie temperature for M, lower for B. Even if one wishes to stick to pure states, one has to explain *generic experiments*. This can be done by weighing the possible initial pure states of A = M + B as in $\hat{\mathcal{R}}(0)$, assuming that M is a typical paramagnetic sample and B a typical sample of the phonons at temperature T. This statistical description in terms of weighted pure states governed by the Schrödinger equation is technically the same as the above one based on the density operator $\hat{\mathcal{D}}(t)$, governed by the Liouville–von Neumann equation, so that the results obtained above are recovered in a statistical sense for most relevant pure states. As regards the expectation values of physical quantities (excluding correlations between too many particles), the typical final pure states are equivalent to $\hat{\mathcal{D}}(t_\mathrm{f})$. Very unlikely events (as those resulting from the pathological initial pure states of M considered in § 5.2.3) will never be observed over reasonable times for most of these pure states. One can therefore resort to a density operator approach as a technical tool even if one wishes to interpret the solution in terms of pure states.

11.2. About reduction

De duivel steekt in het detail[71]
Le diable est dans les détails[71]
Dutch and French proverb

[70] When the genie is out of the bottle, it is not easy to get it in again
[71] The devil is in the details

11.2.1. The reduction must take place for the compound system S + A

Het klopt als een bus[72]
Dutch expression

In §§ 1.1.3 and 1.3.3 we have defined the reduction *in a strong sense*, as the disappearance of the off-diagonal blocks of the density matrix $\hat{\mathcal{D}}(t)$ characterizing the whole compound system S + A, written in a basis where the tested observable $\hat{s}$ is diagonal. The disappearance of the off-diagonal blocks of the marginal density matrix $\hat{r}(t)$ of S expresses a loss of information about a subset of observables of S that do not commute with $\hat{s}$. (In the Curie–Weiss model it expresses the vanishing of $\langle \hat{s}_x \rangle$ and $\langle \hat{s}_y \rangle$.) The full reduction, for S + A, expresses moreover the loss of the correlations between this subset and the observables of the apparatus, in particular between this subset and the pointer variables. Thus, if reduction is achieved, the only correlations remaining between S and A at the end of the process will be the correlations of a classical type between $\hat{s}$ and the pointer. As discussed in section 12, this property is one of the conditions needed to ensure that the process is a bona fide measurement.

In many approaches, starting from von Neumann [2, 12, 16] the word "collapse" or "reduction" is taken in a weak acception, *referring to* S *alone*. Such theoretical analyses involve only a proof that, in a basis that diagonalizes the tested observable, the off-diagonal blocks of the marginal density matrix $\hat{r}(t)$ of S fade out, but not necessarily those of the full density matrix $\hat{\mathcal{D}}(t)$ of S + A. In the Curie–Weiss model, this would mean that $\hat{r}_{\uparrow\downarrow}(t)$ and $\hat{r}_{\downarrow\uparrow}(t)$, or the expectation values of the x- and y-components of the spin S, fade out, but that $\hat{R}_{\uparrow\downarrow}(t)$ and $\hat{R}_{\downarrow\uparrow}(t)$, which characterize the correlations between M and these components, do not necessarily disappear.

However, the presence of non negligible off-diagonal blocks $\hat{\mathcal{R}}_{\uparrow\downarrow}$ and $\hat{\mathcal{R}}_{\downarrow\uparrow}$ in the final state $\hat{\mathcal{D}}(t_f)$ of S + A is a nuisance. In the Curie–Weiss model it would mean that the pointer is correlated not only with the measured observable $\hat{s}_z$, but also with $\hat{s}_x$ and $\hat{s}_y$. This presence would prevent the final state of the compound systems S + A from having the form (10.5), and would thus invalidate the important result of section 12, namely the complete solution of the quantum measurement problem. It is therefore essential, when solving a model of ideal quantum measurement, to prove the *strong reduction* property, as we did in sections 5 and 6.

In the general case, the weak type of reduction is the mere result of disregarding the off-diagonal correlations that exist between S and A. This procedure of tracing out the apparatus has often been considered as a means of circumventing the existence of "Schrödinger cats" issued from the superposition principle [26, 27, 31]. However, this tracing procedure as such does not have a direct physical meaning [9, 38, 53, 91, 221]. For example, in the works studying the measurement problem within the decoherence approach (cf. our discussion in section 2.7), one assumes that the measurement basis is determined by the environment, which is then traced out to ensure the decay of the off-diagonal blocks of the density matrix $\hat{r}$ of S. This is self-contradictory, since then the degrees of freedom of the environment cannot be regarded as irrelevant.

While satisfactory for the statistical predictions about the final marginal state of S, which has the required form $\sum_i p_i \hat{r}_i$, the lack of a complete reduction for S + A keeps the quantum measurement problem open. On the one hand, the correlations represented by $\hat{\mathcal{R}}_{\uparrow\downarrow}$ and $\hat{\mathcal{R}}_{\downarrow\uparrow}$ are sufficiently simple so that they can manifest themselves in a subsequent evolution [274]. On the other hand, and worse, tracing out the apparatus also eliminates the correlations created between S and A by the process in the diagonal blocks $\hat{\mathcal{R}}_{\uparrow\uparrow}$ and $\hat{\mathcal{R}}_{\downarrow\downarrow}$. These correlations are the very essence of a measurement since they allow us, by looking at the change of A in a specific run of the experiment, to derive information on the final state reached by S (section 12). Without them we would not be able to make any statement on S. This is why the elimination of the apparatus in a model is generally considered as a severe weakness of such a model [12], that even led to the commandment "*Thou shalt not trace*" [29].

So indeed, theory and practice are fundamentally related. The elimination of the apparatus in the theory of measurements is no less serious than its elimination in the experiment!

11.2.2. The reduction is a material phenomenon, not a mental one

Weh! Ich ertrag' dich nicht[73]
Johann Wolfgang von Goethe, Faust, part one

[72] It really fits
[73] Beware, I can't stand you

The reduction of the density operator of S + A appears in measurement models as an irreversible change, occurring with a nearly unit probability during the dynamical process. It has a material effect on this compound system, modifying its properties as can be checked by subsequent measurements. Though described statistically for an ensemble, the joint reduction of S + A thus appears as a real dynamical phenomenon.

The interpretations of quantum mechanics that deny the reality of this collapse are therefore inadequate. Since the collapse can fully be accounted for by the standard (statistical) quantum mechanics, the introduction of *conscious observers* appears as useless to explain the reduction. In interpretations of quantum mechanics which focus on the processing of information in a measurement, the reduction should not be regarded as the same type of updating as when we set the probability law for a random phenomenon up to date by acquiring new knowledge: The measurement implies a *real modification* of S, and the value s_i that we infer in the end by acknowledging the indication A_i of the pointer *did not preexist*, although subsequent measurements can prove that it has effectively been reached after the process.

The real nature of the reduction is also at variance with Everett's *many-worlds* interpretation (also called multiple universe or multiverse interpretation), or, in his terms, the relative state [22, 23], with its creation of branches in measurements, an idea supposed to explain the specificity of individual events and to help us get rid of statistical ensembles. However, no dynamical mechanism has been proposed to explain how such a branching might take place during a measurement process, why it occurs with Born's probability, and which are the time scales involved. And in fact, we will show (section 12) how a well defined result emerges in each individual run of a measurement, making the idea of a branching universe superfluous.

11.2.3. Physical extinction versus mathematical survival of the off-diagonal sectors

An expert is a man who has made all the mistakes which can be made, in a narrow field
Niels Bohr

Many works on quantum measurement theory stumble over the folowing paradox. The evolution of the density matrix $\hat{\mathcal{D}}(t)$ of the isolated system S + A is unitary. Hence, if $\hat{\mathcal{D}}$ is written in a representation where the full Hamiltonian $\hat{H}$ is diagonal, each of its matrix elements is proportional to a complex exponential $\exp(i\omega t)$ (where $\hbar\omega$ is a difference of eigenvalues of $\hat{H}$), so that its modulus remains constant in time. In the ideal case where the tested observable $\hat{s}$ commutes with $\hat{H}$, we can imagine writing $\hat{\mathcal{D}}(t)$ in a common eigenbasis of $\hat{s}$ and $\hat{H}$; the moduli of the matrix elements of its off-diagonal block $\hat{\mathcal{R}}_{\uparrow\downarrow}(t)$ are therefore independent of time. Such a basis was used in sections 5 and 6.1 where the bath played no rôle; in section 6.2, the term $\hat{H}_{\rm MB}$ does not commute with $\hat{s}$, and likewise in most other models the full Hamiltonian is not diagonalizable in practice. In such a general case, the moduli of the matrix elements of $\hat{\mathcal{R}}_{\uparrow\downarrow}(t)$, in a basis where only $\hat{s}$ is diagonal, may vary, but we can ascertain that the *norm* $\mathrm{Tr}\,\hat{\mathcal{R}}_{\uparrow\downarrow}(t)\hat{\mathcal{R}}^{\dagger}_{\uparrow\downarrow}(t)$ *remains invariant*. This mathematically rigorous property is in glaring contradiction with the very phenomenon of reduction.

In which sense are we then allowed to say that the off-diagonal block $\hat{\mathcal{R}}_{\uparrow\downarrow}(t)$ decays? The clue was discussed in § 6.1.2: The physical quantities of interest are weighted sums of matrix elements of $\hat{\mathcal{D}}$, or here of its block $\hat{\mathcal{R}}_{\uparrow\downarrow}$. For instance, the off-diagonal correlations between $\hat{s}_x$ or $\hat{s}_y$ and the pointer variable $\hat{m}$ are embedded in the characteristic function (5.14), which reads

$$\Psi_{\uparrow\downarrow}(\lambda, t) \equiv \langle \hat{s}_- e^{\lambda\hat{m}} \rangle = \mathrm{Tr}_{\rm A}\hat{\mathcal{R}}_{\uparrow\downarrow}(t)e^{\lambda\hat{m}}, \tag{11.1}$$

where the trace is taken over A = B + M. Likewise, the elimination of the bath B, which is sensible since we cannot control B and have no access to its corelations with M and S, produces $\hat{R}_{\uparrow\downarrow} = \mathrm{Tr}_{\rm B}\hat{\mathcal{R}}_{\uparrow\downarrow}$, which contains our whole off-diagonal information, and which is a sum of matrix elements of the full density matrix $\hat{\mathcal{D}}$. We are therefore interested only in weighted sums of complex exponentials, that is, in *almost periodic functions* (in the sense of Harald Bohr[74]). For a large apparatus, these sums involve *a large number of terms*, which will usually have incommensurable frequencies. Depending on the model, their large number reflects the large size of the pointer or that of some environment.

[74]The mathematician and olympic champion Harald Bohr, younger brother of Niels Bohr, founded the field of almost periodic functions. For a recent discussion of his contributions, see the expository talk "The football player and the infinite series" of H. Boas [275].

The situation is the same as for a large set of coupled harmonic oscillators [122, 123, 124, 145, 199, 200], which in practice present damping although some exceptional quantities involving a single mode or a few modes oscillate. In § 6.1.2 we have studied a generic situation where the frequencies of the modes are random. The random almost periodic function $F(t)$ defined by (6.14) then exhibits a decay over a time scale that rises rapidly with N; Poincaré *recurrences* are not excluded, but occur only after *enormous times* — not so enormous as for chaotic evolutions but still large as $\exp(\exp N)$.

The above contradiction is therefore apparent. The off-diagonal blocks cannot vanish in a mathematical sense since their norm is constant. However, all quantities of physical interest in the measurement process combine many complex exponentials which interfere destructively, so that everything takes place as if $\hat{\mathcal{R}}_{\uparrow\downarrow}$ did vanish at the end of the process. The exact final state of S + A and its reduced final state are thus equivalent with respect to all physically reachable quantities in the sense of Jauch [79]. Admittedly, one may imagine some artificial quantities involving few exponentials; or one may imagine processes with huge durations. But such irrealistic circumstances will never be encountered by experimentalists.

Note that the matrix elements of the marginal state $\hat{r}(t)$ of S, obtained by tracing out the apparatus, are again obtained but summing a very large number of matrix elements of $\hat{\mathcal{D}}(t)$. We can thus understand that the decay of the off-diagonal elements of $\hat{r}$ is easier to prove than the reduction of the full state of S + A.

11.2.4. The preferred basis paradox

Lieverkoekjes worden niet gebakken[75]
Dutch saying

It is a challenge to explain why the reduction does not take place in an arbitrary basis but in the specific basis in which the tested observable of S is diagonal. This leads to the question of determining which mechanism selects the basis in which the off-diagonal blocks of S + A disappear; intuitively, it is the very apparatus that the experimentalist has chosen and, in the Hamiltonian, the form of the interaction between S and A. One has, however, to understand precisely for each specific apparatus how the dynamics achieve this property. For the Curie–Weiss model, the tested observable is directly coupled through (3.5) with the pointer observable $\hat{m}$, and the preferred basis problem is readily solved because the initial reduction is a mere result of the form of this coupling and of the large number of degrees of freedom of the pointer M. The finite expectation values $\langle \hat{s}_x \rangle$ and $\langle \hat{s}_y \rangle$ in the initial state of S are thereby transformed into correlations with many spins of the pointer, which eventually vanish (sections 5 and 6). Pointer-induced reduction thus takes place, as it should, in the eigenbasis of the tested observable, whereas a decoherence generated by a random environment would have no reason to select this basis [28, 29, 35, 147, 148, 149, 150].

We have also shown (§ 6.2.4) that in this model the suppression of the recurrences by the bath, although a *decoherence* phenomenon, is *piloted by the spin-magnet interaction* which selects the decoherence basis. When it is extended to a microscopic pointer, the Curie–Weiss model itself exhibits this difficulty (§ 8.1.5).

11.2.5. Direct reduction or bath-induced decoherence?

We keep here the word "reduction" in its technical sense, the decay of the off-diagonal blocks $\hat{\mathcal{R}}_{\uparrow\downarrow}$ and $\hat{\mathcal{R}}_{\downarrow\uparrow}$ of the density matrix $\hat{\mathcal{D}}$ of S + A, and we reserve the word "decoherence" to a process generated by a random environment, such as a thermal bath. We have just recalled that, in the Curie-Weiss model with large N, the reduction is ensured mainly by the pointer; the bath B only provides one of the two mechanisms that prevent recurrences from occurring after reduction (subsection 6.2). We have contrasted this direct mechanism with bath-induced decoherence (§ 5.1.2). In particular, our reduction time $\tau_{\rm red}$ does not depend on the temperature as does a decoherence time, and it is so short that the bath B is not yet effective. Later on, the prohibition of recurrences by the bath in this model is a subtle decoherence process, which involves all three objects, the tested spin, the magnet and the bath (§ 6.2.4)

We have shown (§ 5.1.2 and § 6.1.2) that more general models with macroscopic pointers can also give rise to direct reduction. However, in models involving a microscopic pointer (section 2 and subsection 8.1), the reduction mechanism can only be a bath-induced decoherence [28, 29, 35, 147, 148, 149, 150], and the occurrence of a preferred reduction basis is less easy to understand.

[75] "I-prefer-this" cookies are not baked, i.e., you won't get what you want

11.3. About registration

In order to regard a dynamical process as an ideal measurement, we need it to produce reduction. However we also need it to include registration, so as to let us sort the successive events according to their outcome. This point is too often overlooked. Indeed, we will show in section 12 that both reduction and registration are essential features involved in the elucidation of the quantum measurement problem.

11.3.1. The pointer must be macroscopic

Like the reduction, the registration is a material process, which affects the apparatus, creating correlations between it and the tested system. This change of A must be *detectable*: We should be able to *read, print or process* the result registered by the pointer. In the Curie-Weiss model, the apparatus simulates a magnetic memory, and, under the conditions of subsection 9.4, it satisfies these properties required for registrations (section 7). The apparatus is *faithful*, since the probability of a wrong registration, in which the distribution $P_{\uparrow\uparrow}(m, \tau_{\rm reg})$ would be sizeable for negative values of m, is negligible, though it does not vanish in a mathematical sense. The registration is *robust* since both ferromagnetic states represented by density operators yielding magnetizations located around $+m_{\rm F}$ and $-m_{\rm F}$ are stable against weak perturbations, such as the ones needed to read or to process the result.

The registration is also *permanent*. This is an essential feature, not only for experimental purposes but also because the solution of the quantum measurement problem (section 12) will rely on the possibility of processing the indication of the pointer after registration in each individual run. However, this permanence, or rather quasi-permanence, may again be achieved only in a physical sense (§ 11.1.1), just as the broken invariance associated with phase transitions is only displayed at physical times and not at "truly infinite times" for finite materials. Indeed, in the Curie-Weiss model, thermal fluctuations have some probability to induce in the magnetic dot transitions from one ferromagnetic state to the other. More generally, information may spontaneously be erased after some delay in any finite registration device, but this delay can be extremely long, sufficiently long for our purposes. For our magnetic dot, it behaves as an exponential of N owing to invariance breaking, see Eq. (7.84).

The enhancement of the effect of S on A is ensured by the metastability of the initial state of A, and by the irreversibility of the process, which leads to a stable final state.

All these properties require a *macroscopic* pointer (§ 1.2.1), and not only a macroscopic apparatus. In principle, the models involving a large bath but a small pointer are therefore unsatisfactory for the aim of describing ideal measurements. In many models of quantum measurement (section 2), including the Curie-Weiss model for $N = 2$ (subsection 8.1), the number of degrees of freedom of the pointer is not large. We have discussed this situation, in which an ideal measurement can be achieved, but only if the small pointer is coupled at the end of the process to a *further, macroscopic apparatus* ensuring amplification and true registration of the signal.

Altogether the macroscopic pointer behaves in its final state as a *classical object* which may lie in either one or the other of the states characterized within negligible fluctuation by the value A_i of the pointer observable $\hat{A}$. (In the Curie-Weiss model, $A_i \simeq \pm m_{\rm F}$ is semiclassical, while $s_i = \pm 1$ is quantal). This point will be crucial in the solution of the measurement problem (section 12), and calls for explanations. Nothing prevents, for instance, a magnetic dot from lying in an equilibrium state which incoherently superposes the two ferromagnetic states. We have seen in § 7.1.5 and § 7.3.3 how such mixed states can arise for suitably chosen initial conditions and parameters which make the bifurcation active. Likewise, nothing prevents us from imagining, for the pointer M, or the apparatus A, or for the compound system S + A, coherent quantum states for which $\hat{m}$ takes random values having either sign. However, as shown is section 7, such situations do not appear under the conditions recalled in subsection 9.4. Indeed, it is the *dynamical process* governed by quantum statistical mechanics which *rules out all superpositions*, whether incoherent or coherent. In the sector $\uparrow\uparrow$, the bifurcation is inactive so that the distribution $P_{\uparrow\uparrow}(m, t)$ (associated with $R_{\uparrow\uparrow}(t)$) is pushed nearly completely towards positive values of m. In the sector $\downarrow\downarrow$, $\hat{m}$ takes likewise only negative values with probability nearly equal to 1, whereas $R_{\uparrow\downarrow}$ and $R_{\downarrow\uparrow}$ have practically disappeared after the time $\tau_{\rm red}$. Thus we acknowledge two well separated sectors for $s_z = +1$ and -1, in each of which m takes (within fluctuations of order $1/\sqrt{N}$ a well-defined value, $+m_{\rm F}$ and $-m_{\rm F}$, respectively, so that both the quantum and the statistical natures of the pointer do not manifest themselves, at least not at the end of the process.

11.3.2. Do we need observers?

Hij stond erbij en keek er naar[76]
Dutch saying

We have seen that reduction does not involve observers. Likewise, conscious observers are irrelevant for the registration, which is a physical process, governed by a Hamiltonian. Once the registration of the outcome has taken place, the values of $A_i \simeq \pm m_F$ and hence $s_i = \pm 1$ take an *objective* character, since any observer will read the same indication. "Forgetting" to read off the registered result will not modify it in any way. Anyhow, nothing prevents the *automatic processing* of the registered data, in view of further experiments on the tested system.

We thus cannot agree with the idealist statement that "the state is a construct of the observer". Although we interpret the concept of probabilities as a means for making predictions from available data (§ 10.1.1), a state reflects real properties of the physical system acquired through its preparation, within some indeterministic effects due to the non-commutative nature of the observables.

11.3.3. What does "measuring an eigenvalue" mean?

A measurement process is an experiment which creates in the apparatus an image of some property of the tested system. From a merely experimental viewpoint alone, one cannot know the observable of S that is actually tested, but experience as well as theoretical arguments based on the form of the interaction Hamiltonian may help to determine which one. From the observed value A_i of the pointer variable, one can then infer the corresponding eigenvalue s_i of the measured operator (that appears in the interaction Hamiltonian), provided the correlation between A_i and s_i is complete. In the Curie-Weiss model the observed quantity is the magnetization of M; we infer from it the eigenvalue of $\hat{s}_z$. The statement of some textbooks "only eigenvalues of an operator can be measured" refers actually to the pointer values, which are in one-to-one correspondence with the eigenvalues of the tested observable provided the process is an ideal measurement. The eigenvalues of an observable as well as the quantum state of S are abstract mathematical objects associated with a microscopic probabilistic description, whereas the physical measurement that reveals them indirectly relates to the macroscopic pointer variable.

11.3.4. Did the registered results preexist in the system?

Don't cross the stream to find water
Norwegian proverb

After the measurement process has taken place and after the outcome of the apparatus has been read, we can assert that the apparatus lies in the state $\hat{\mathcal{R}}_i$ characterized by the value A_i of the pointer while the system lies in the final projected state $\hat{r}_i$ (Eq. (12.1)). We can also determine the weights p_i from the statistics of the various outcomes A_i. However a quantum measurement involves not only a change in A that reflects a property of S, but also a change in S (§ 1.1.2). In an ideal measurement the latter change is minimal, but we have to know precisely which parts of the initial state $\hat{r}(0)$ are conserved during the process so as to extract information about it from the registered data.

Consider first the whole ensemble of runs of the experiment. Together with the theoretical analysis it provides the set of final states $\hat{r}_i$ and their weights p_i. The corresponding marginal density operator $\sum_i p_i \hat{r}_i$ of S is obtained from $\hat{r}(0)$ by keeping only the diagonal blocks, the off-diagonal ones being replaced by 0. We thus find a partial statistical information about the initial state: all probabilistic properties of the tested observable $\hat{s}$ remain unaffected, as well as those pertaining to observables that commute with $\hat{s}$. (The amont of information retained is minimal, see § 1.2.4.) Some retrodiction is thus possible, but it is merely statistical and partial.

Consider now a single run of the measurement that has provided the result A_i. The fact that S is thereafter in the state $\hat{r}_i$ with certainty does not mean that it was initially in the same state. In fact no information about the initial state $\hat{\mathcal{D}}(0)$ is provided by reading the result A_i (except for the fact that the expectation value in $\hat{\mathcal{D}}(0)$ of the projection on the correspnding eigenspace of $\hat{A}$ does not vanish). *Retrodiction* about S in this individual run *is not allowed* (i. e., it is devoid of a physical meaning), due to the irreversibility of the interaction process. The property "$\hat{s}$ takes the value s_i" did not preexist the process. It is only in case all runs provide the outcome A_i that we can tell that S was originally

[76] He stood there and just watched, i.e., he did not assist

in the state $\hat{r}_i$. One should therefore beware of some realist interpretations in which the value s_i is supposed to preexist the individual measurement: they do not take properly into account the perturbation brought in by the measurement [27].

12. The quantum measurement problem explained within the statistical interpretation

All's well that ends well
Shakespeare

In the above sections, we have discussed models of quantum measurements. Here we summarize, within the statistical interpretation, the impact of their dynamical solution on the understanding of quantum mechanics, especially on the questions of interpretation of measurements.

12.1. Emergence of classical features and of uniqueness in measurements

Luctor et emergo[77]
Fluctuat nec mergitur[78]
Device of the often flooded Dutch province Zeeland and city of Paris

We first give an intuitive description of a general ideal quantum measurement. We then properly formulate the quantum measurement problem, which aims at understanding how statements about individual systems may emerge in spite of the fundamentally probabilistic nature of the theory; after which we explain its solution.

12.1.1. A physical argument

Une idée simple mais fausse s'impose toujours face à une idée juste mais compliquée[79]
Alexis de Tocqueville

As shown by the review of section 2 and by the Curie–Weiss example of section 3, many models of ideal quantum measurements rely on the following ideas. The apparatus A is a macroscopic system which has several possible stable states $\hat{\mathcal{R}}_i$ characterized by the value A_i of the (macroscopic) pointer variable. If A is initially set into a metastable state $\hat{\mathcal{R}}(0)$, it may spontaneously switch towards one or another state $\hat{\mathcal{R}}_i$; This would happen after a reasonably long time. In a measurement, this transition is triggered by the coupling with the tested object S. Then the transition happens fast, and it creates correlations such that, if the apparatus reaches the state $\hat{\mathcal{R}}_i$, the tested observable $\hat{s}$ takes the value s_i. The neat separation between the states $\hat{\mathcal{R}}_i$ and their long lifetime, together with the lack of survival of "Schrödinger cats", indicate that each individual process has a unique outcome, characterized by the indication A_i of the pointer and by the value s_i for the observable $\hat{s}$ of the system S.

12.1.2. Where does the difficulty lie?

Δεν βρέτηεκα τίτλο με του όρο[80]
Aesop

Although suggestive, the above simple argument is in fact too simplistic and defective. It includes an idea that survives in the quantum solution of models, namely the possibility of inferring from the observation of A_i that $\hat{s}$ takes the value s_i. However, the classical kind of reasoning used above is known to produce severe mistakes when applied to the subtle properties of quantum objects. We ought to analyze quantum measurements by means of rigorous quantum theoretical arguments. Indeed, quantum mechanics, whether it deals with pure or mixed states, is *irreducibly probabilistic* [5, 6, 27, 43, 46, 52]. As such, it does not govern individual systems, but only statistical ensembles

[77] I fight and I emerge
[78] She is agitated by the stream, but does not sink
[79] A simple but wrong idea always prevails over a right but complex idea
[80] After all is said and done, more is said than done

of systems (10.1.1). Solving an ideal quantum measurement model can provide us with the evolution of the density operator of the compound system S + A, from the initial state

$$\hat{\mathcal{D}}(0) = \hat{r}(0) \otimes \hat{\mathcal{R}}(0), \tag{12.1}$$

to the final state

$$\hat{\mathcal{D}}(t_{\rm f}) = \sum_i p_i \hat{r}_i \otimes \hat{\mathcal{R}}_i, \qquad p_i \hat{r}_i = \hat{\Pi}_i \hat{r}(0) \hat{\Pi}_i, \qquad p_i = {\rm tr}_S \hat{r}(0) \hat{\Pi}_i. \tag{12.2}$$

where $\hat{\Pi}_i$ is the projection operator on the eigenspace of $\hat{s}$ associated with the eigenvalue s_i. A probabilistic account is thus available for the dynamics of a set $\mathcal{E}$ of runs of similarly prepared experiments, but nothing more. Quantum mechanics is our most fundamental theory, but even a complete solution of its dynamical equations refers only to the statistics of an ensemble $\mathcal{E}$. The description of *individual processes* is *excluded* because the state (12.2), as any state in quantum mechanics, necessarily involves statistical fluctuations. We also need to rely on quantum statistical mechanics because the apparatus is macroscopic. More deeply, the very expression (12.2) is always the outcome of approximations that can be justfied only through statistical arguments. In particular, the initial and final states (12.1) and (12.2) cannot be unitarily related to each other due to the absence of off-diagonal blocks in the final density matrix (12.2); however, as ilustrated by the Curie–Weiss model (sections 5 and 6), the reduced expression of $\hat{\mathcal{D}}(t_{\rm f})$ and the mathematically exact result differ only through contributions to physically inobservable correlations. Moreover, we have seen, by studying the possibility of measurement errors in the Curie–Weiss model (section 7), that the pointer may take the value A_i when $\hat{s}$ has in the initial state of S an eigenvalue s_j with $j \neq i$: but this occurs with a small probability if the apparatus is large.

Relying therefore only on the statistical interpretation of the state $\hat{\mathcal{D}}(t_{\rm f})$, the challenge we face in the ideal quantum measurement problem is to explain why *each individual run provides a well defined outcome*, for both the apparatus and the tested system, an experimental fact in apparent contradiction with the superposition principle. As is well known [126, 127, 128, 129, 130, 131, 132] the sole result (12.2), although necessary, is not sufficient to solve this problem, since it has no direct interpretation in terms of single processes.

The subtlety of the measurement problem is showed off by the analysis of an inverse problem, that of *retrodiction* (§ 11.3.4). In classical physics, if we disregard experimental errors, measurements also establish a one-to-one correspondence between the indication A_i of the pointer and the value s_i of the tested quantity s. A single run of the measurement allows us to infer from the observation of A_i that s took before the process the value s_i for this sample. In contrast, consider a set of measurements of the z-component of a spin $\frac{1}{2}$. The observations of the pointer provide the data $p_\uparrow$ and $p_\downarrow$, from which we infer a purely statistical property of the initial state of the ensemble $\mathcal{E}$, the expectation value of the z-component of the polarization. However, detecting in a single process the outcome $\uparrow$ does not mean that $\hat{s}$ took the value $+1$ at the initial time for the tested sample. We can only ascertain that the considered spin was initially not in the pure state $|\downarrow\rangle$; its initial polarization could otherwise have been completely arbitrary. *Retrodiction is impossible for an individual quantum measurement.*

A qualitative difference thus exists between the microscopic quantum theory, which only deals with ensembles, and the expected explanation for the unicity of the outcome of a single measurement. A similar well known opposition between an underlying microscopic theory and the phenomena that it governs is also encountered in statistical mechanics [49, 50, 65]. The qualitative features of the macroscopic laws (irreversibility, continuity of matter, existence of phase transitions, etc.) seem to contradict those of microphysics. Nevertheless a reductionist attitude leads us to admit that the latter are more fundamental and that they should entail the former. The successes of such an approach exhibit the emergence, owing to the change of scale, of the macroscopic properties from the microscopic ones in spite of their evident differences. In particular, the emergence of irreversibility (§ 1.2.2), for sufficiently large systems and not too large times, is explained by using statistical arguments and analyzing which initial states can be practically produced. Similar ideas, the use of quantum statistical mechanics and the randomness of the initial paramagnetic state of A, already underlied the establishment of the form (12.2), devoid of Schrödinger cats, of the final state in the Curie–Weiss model.

12.1.3. The real task: to define proper sub-ensembles

Horresco referens[81]
Virgil, Aeneid

But the task is not over. Indeed, a major difference exists between classical and quantum statistical mechanics. In the former case, we can interpret a probabiliy in phase space as a description of many individual systems, for each of which all physical quantities take well-defined values. Such an interpretation is not available in the quantum case. Within the minimalistic statistical interpretation, it is impossible, within the full ensemble $\mathcal{E}$ of systems S + A described by the state $\hat{\mathcal{D}}(t_f)$, to distinguish individual systems with well-defined properties. Since our measurement definitely involves a microscopic quantum system, the tested system S, one has to follow the general rules, so one can only imagine splitting $\mathcal{E}$ into sub-ensembles $\mathcal{E}_k$, each one described by a less random state according to (10.4) or even by a pure state according to (10.3) [258]. However, the decomposition exhibited in (12.2) is far from being the only one for $\hat{\mathcal{D}}(t_f)$, and a *large degree of arbitrariness* exists for such decompositions of $\hat{\mathcal{D}}(t_f)$ of the type (10.4) [27, 43, 258]. Each possible decomposition is associated with a different splitting of $\mathcal{E}$ into sub-ensembles $\mathcal{E}_k$, and thus cannot a priori be given a meaning. The only issue to physically identify a partition of $\mathcal{E}$ is to find *additional information, not included in* $\hat{\mathcal{D}}(t_f)$ but still available from physical arguments, in order to afford an unambiguous selection of subsets among the whole sequence of runs of the measurement. Owing to such a partitioning, we shall then be in position to understand how statements can be made about individual processes in spite of the subtleties of quantum mechanics lying in its irreducibly probabilistic nature.

12.1.4. Identification of meaningful sub-ensembles

De hond bijt de kat niet[82]
Les chiens ne font pas des chats[82]
Dutch and French sayings

The clue is obviously the different behaviour of microscopic and macroscopic systems. Whereas quantum mechanics is needed at the microscopic scale, macroscopic variables such as the order parameter in a phase transition or the indication of the pointer of an apparatus can take well-defined values within negligible fluctuations (whether quantum or statistical)[83]. These values can remain stable for long times (§ 11.3.1). Neither probabilities nor statistical ensembles are then needed, a classical interpretation can replace the statistical interpretation and we can make definite observations and definite statements about *individual macroscopic systems*. For this reason, ideal measurements, which establish a bridge between microscopic and macroscopic scales, constitute privileged means for gathering information about *individual microscopic systems*[84]. However, such information cannot be complete due to the quantum nature of microscopic objects.

The crucial properties allowing the realization of this goal are illustrated by the Curie–Weiss model of measurement. (i) The magnetic dot is an adequate device for discriminating two a priori equivalent alternatives: it has two ferromagnetic stable states, which may be set in correspondence with the up or down states of the tested spin. (ii) The dynamics of the reduction (sections 5 and 6) eliminate the off-diagonal blocks $\hat{\mathcal{R}}_{\uparrow\downarrow}(t_f)$ and $\hat{\mathcal{R}}_{\downarrow\uparrow}(t_f)$ of $\hat{\mathcal{D}}(t_f)$, so that no correlation exists at the end of the measurement between the apparatus A and the observables $\hat{s}_x$ and $\hat{s}_y$ of S. (iii) The dynamics of the registration (section 7) select for the diagonal block $\hat{\mathcal{R}}_{\uparrow\uparrow}(t_f)$ (or $\hat{\mathcal{R}}_{\downarrow\downarrow}(t_f)$) a distribution of the magnetization of M which stabilizes around the ferromagnetic value $m \simeq +m_F$ (or $-m_F$); generation of coherent or incoherent superpositions in each sector is extremely improbable. (iv) Within negligible errors (§ 7.3.3), *a perfect correlation* is thereby created between the sign of the magnetization $A_i \simeq \pm m_F$ of the apparatus A and the sign of the eigenvalue $s_z = \pm 1$ of the observable $\hat{s}_z$. (v) As discussed in § 11.3.1, the solid registration in a long-lived state of A

[81] I shiver while I am telling it

[82] Dogs do not beget cats

[83] In the Curie–Weiss model, small fluctuations around the final magnetization $\pm m_F$ of the magnet still pertain to the same spin value ± 1

[84] A more artificial link between microscopic and macroscopic scales was established by Bohr [260] – see also [73, 265, 266] – by postulating the classical behavior of the measuring apparatus. Though we consider that the apparatus must be treated as a quantum object, we have noticed that the pointer variable does have some classical features

is a macroscopic phenomenon which allows one to *read* or to *process* the outcomes *of each run* (at least before the indication of the pointer is spoiled or erased).

We are now in position to interpret a measuring experiment, which involves successive runs performed on an ensemble of similarly prepared systems S. The purpose of the experiment is to determine partially the initial density operator $\hat{r}(0)$. For each run of the ensemble $\mathcal{E}$, an experimental information is available, the value A_i registered by the pointer. This allows one to sort out the individual runs according to this value A_i, thus splitting $\mathcal{E}$ into the corresponding *well identified sub-ensembles* $\mathcal{E}_i$. This splitting associates with each sub-ensemble $\mathcal{E}_i$, built unambiguously, the term labelled by i in the representation (12.2) of $\hat{\mathcal{D}}(t_f)$. Among the various possibilities of writing the mixed state $\hat{\mathcal{D}}(t_f)$ as a sum (10.4), the decomposition (12.2) has thus a physical meaning: Each of its terms is associated with a set of runs $\mathcal{E}_i$ identified through the indication A_i given by the pointer in the specific measurement.

The sub-ensemble $\mathcal{E}_i$ is governed by the density operator

$$\hat{\mathcal{D}}_i(t_f) = \hat{r}_i \otimes \hat{\mathcal{R}}_i. \tag{12.3}$$

Whereas S and A were correlated in the full ensemble $\mathcal{E}$, they are uncorrelated in each sub-ensemble $\mathcal{E}_i$. According to (12.2), the relative size, i.e., the statistical weight, of each sub-ensemble $\mathcal{E}_i$ is p_i. These weights p_i are a special case of the coefficients q_k entering the general decompositions (10.4), but they have been generated by a physical protocol (selection after each run of the measurement) and are now physically meaningful [258].

To summarize this point, creating the physical sub-ensembles identified by observation of the pointer will allow us to compare the results of actual measurements with the theoretical formulas (12.2) and (12.3). Only then shall we be allowed to draw conclusions.

12.1.5. Individual processes, individual outcomes and preparations through measurement

Cogito, ergo sum[85]
René Descartes

Consider now an individual run. One can identify on the macroscopic pointer some associated outcome A_i. This run therefore necessarily belongs to the corresponding sub-ensemble $\mathcal{E}_i$, and, applying the Hamiltonian model of the measurement process, we can conclude that the evolution has lead S + A to the final state (12.3). This individual run thus produces a *well-defined result*, not only for the pointer but also *for the tested observable* $\hat{s}$ of S[86]. This fact is a consequence of the special type of correlations between S and A created by the measurement process[87].

As a consequence, an *ideal measurement process* followed by the *selection* of the indication A_i of the pointer constitutes a preparation of the system S (as was anticipated in § 1.1.4) [27]. Lying formerly (for the ensemble $\mathcal{E}$) in the state $\hat{r}(0)$, this system lies after measurement and selection (for the ensemble $\mathcal{E}_i$) in the projected state $\hat{r}_i$ defined by (12.2). In the Curie–Weiss model, this final filtered state is pure, $|\uparrow\rangle$ or $|\downarrow\rangle$[88]. As discussed below, a subsequent measurement of $\hat{s}$ would yield the same value s_i with unit probability. In this circumstance, quantum mechanics, although irreducibly probabilistic and dealing with ensembles, *can provide certainty about $\hat{s}$ for an individual system* after measurement and selection of the indication of the pointer. While answering, within the statistical interpretation, Bohr's modest query "*What can we say about...*?" [260], we have given a partial answer to Einstein's query "*What is....*?" [18]. The solution of models involving interaction of the microscopic object with a macroscopic apparatus thus explains the emergence of a well-defined answer in a single measurement, a property interpreted as the emergence of a "physical reality" [87].

However, although the outcome of each individual process is unique, it could not have been predicted (§ 11.3.4). In fact, this uniqueness is ensured only at the end of the measurement. The current statement "the measurement is responsible for the appearance of the uniqueness of physical reality" holds for the considered single system and for the tested observable, but *only after* measurement and selection of the result. It is questionable or impossible before, depending on the chosen interpretation of quantum mechanics. In the statistical interpretation physical quantities remain fundamentally random till they are measured.

[85] I think, therefore I exist

[86] But of course there are no well-defined results for observables that do not commute with $\hat{s}$

[87] Consequently, one may conclude that the physical argument of section 12.1.1 is *not even wrong*

[88] That is to say, $\hat{r}_\uparrow = |\Uparrow\rangle\langle\Uparrow|$ or $\hat{r}_\downarrow = |\Downarrow\rangle\langle\Downarrow|$

12.1.6. Repeatability of ideal measurements

It is a bad plowman that quarrels with his ox
Korean proverb

Another property that allows us to acknowledge physical reality within the statistical interpretation is the repeatability of ideal measurements[89]. Suppose two successive ideal measurements are performed on the same system S, first with an apparatus A, then, independently, with a similar apparatus A′. The second process does not affect A, and generates for S and A′ the same effect as the first one, as exhibited by Eq (12.2). Hence, the initial state

$$\hat{\mathcal{D}}(0) = \hat{r}(0) \otimes \hat{\mathcal{R}}(0) \otimes \hat{\mathcal{R}}'(0) \tag{12.4}$$

of S + A + A′ becomes

$$\hat{\mathcal{D}}(t_{\mathrm{f}}) = \sum_i p_i \hat{r}_i \otimes \hat{\mathcal{R}}_i \otimes \hat{\mathcal{R}}'(0) \tag{12.5}$$

at the time t_{f} between the two measurements, and

$$\hat{\mathcal{D}}(t'_{\mathrm{f}}) = \sum_i p_i \hat{r}_i \otimes \hat{\mathcal{R}}_i \otimes \hat{\mathcal{R}}'_i \tag{12.6}$$

at the final time t'_{f} following the second process. For the whole statistical ensemble $\mathcal{E}$, a complete correlation is therefore exhibited *between the two pointers*. In an individual process, the second measurement does not affect S. We can even retrodict, from the observation of the value A'_i for the pointer of the apparatus A′, that S lies in the state $\hat{r}_i$ not only at the final time t'_{f}, but already at the time t_{f}, the end of the first measurement.

12.1.7. Classical probabilities

De oudjes doen het nog goed[90]
Dutch expression

Let us now return to the ensemble $\mathcal{E}$ *of many runs*. As shown by (12.2) and (12.3), the proportion of runs of the sub-ensemble $\mathcal{E}_i$ having produced the result A_i is given by Born's rule $p_i = \mathrm{tr}_{\mathrm{S}} \hat{r}(0) \hat{\Pi}_i$. The *statistics of the indications of the pointer* characterized by the rates p_i of occurrence of A_i therefore allow a partial determination of $\hat{r}(0)$ (its diagonal elements for the Curie–Weiss model). The quantity p_i can also be interpreted as the ordinary probability of finding the outcome A_i, s_i in each run. From the density operator $\hat{r}(0)$ which has no classical interpretation, the measurement process has thus extracted *classical probabilities*. Because of the one-by-one selection procedure, these probabilities are defined within the frequency interpretation.

12.1.8. The ingredients of the solution of the measurement problem

Altogether, as in statistical mechanics [49, 50, 65], *qualitatively new features* emerge in an ideal measurement process, with a *probability nearly equal to one*. The explanation of the appearance, within the quantum theory, of properties seemingly in contradiction with this very theory relies on several ingredients, the *macroscopic size* of the apparatus, *statistical considerations* which allow us to disregard unlikely events, and the *special dynamics* of the process which is responsible for *reduction* and permanent *registration*. As shown in § 12.1.4, the latter property, often overlooked, is essential to recognize individual properties within the probabilistic quantum framework. As the symmetry breaking for phase transitions, a breaking of unitarity takes place, entailing an apparent violation of the superposition principle for S + A[91]. Here also, there cannot exist any breaking in the strict mathematical sense

[89] It can be shown that the sole property of repeatability implies reduction in the weak sense, for the marginal state of S [46]
[90] The oldies are doing well still
[91] When taking the apparatus into consideration, the breaking of unitarity for the tested system alone is trivial, since it is not an isolated system

for a finite apparatus and for finite parameters. Nevertheless, this acknowledgement has no physical relevance: the approximations that underlie the effective breaking of unitarity are justified for the evaluation of physically sensible quantities.

However, the type of emergence that we acknowledge here is more subtle than in statistical mechanics. Instead of bearing on *phenomena*, it bears on *concepts*. In the measurement device we lose track of the non-commutative nature of observables, which constitutes the deep originality of quantum mechanics and gives rise to its peculiar types of correlations and of probabilities; we recover familiar macroscopic concepts: ordinary correlations between system and apparatus, classical probabilities p_i, and uniqueness of outcome for an individual experiment.

12.2. What next?

Այս ապուրը դեռ շատ ջուր կպաշի։[92]
Il va couler encore beaucoup d'eau sous les ponts[93]
Er zal nog heel wat water door de Rijn moeten[94]
Armenian, French and Dutch proverbs

We have only dealt in this article with ideal quantum measurements, in which information about the initial state of the tested system S is displayed by the apparatus at some later time, and in which the final state of the system S is projected. Other realistic setups, e.g. of particle detectors or of avalanche proccesses, deserve to be studied through models. Measurements of a more elaborate type, in which some quantum property of S is continuously followed in time, are now being performed owing to experimental progress [276, 277]. For instance, non-destructive (thus non-ideal) repeated observations of photons allow the study of quantum jumps [277], and quantum-limited measurements, in which a mesoscopic detector accumulates information progressively [278], are of interest to optimize the efficiency of the processing of q-bits. It seems timely, not only for conceptual purposes but to help the development of realistic experiments, to work out further models, in particular for such quantum measurements in which the whole history of the process is used to gather information. In this context we should mention the so-called weak measurements that (in a sense) minimize the back-action of the measurement devise on the measured system, and – although they have certain counterintuitive features – can reveal the analogues of classical concepts in quantum mechanics; e.g., state determination with the minimal disturbance, classical causality [279, 280, 281, 282], and even map out the complete wave function [283] or the average trajectories of single photons in a double-slit experiment [284].

Apart from such foreseeable research works, it seems desirable to make educational progress by taking into account the insights provided by the solution of models of quantum measurement processes. The need of quantum statistical mechanics to explain these processes, stressed all along this paper, and the central role that they play in the understanding of quantum phenomena, invite us to a reformation of teaching at the introductory level. The statistical interpretation, as sketched in subsection 10.1, is in keeping with the analysis of measurements. Why not introduce the concepts and bases of quantum mechanics within its framework? This "minimal" interpretation seems more easily assimilable by students than the traditional approaches. It thus appears desirable to foster the elaboration of new courses and of new textbooks, which should hopefully preserve the forthcoming generations from bewilderment when being first exposed to quantum physics...and even later!

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[92] Preparing this porridge still requires much water
[93] There will stream still quite some water under the bridges
[94] There will have to flow still quite some water through the Rhine

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A. Elimination of the bath

De ballast van je afgooien[95]
Dutch expression

Taking $\hat{H}_0 = \hat{H}_S + \hat{H}_{SA} + \hat{H}_M$ and $\hat{H}_B$ as the unperturbed Hamiltonians of S + M and of B, respectively, and denoting by $\hat{U}_0$ and $\hat{U}_B$ the corresponding evolution operators, we consider the full evolution operator associated with $\hat{H} = \hat{H}_0 + \hat{H}_B + \hat{H}_{MB}$ in the interaction representation. We can expand it as

[95] To throw the ballast from you

$$\hat{U}_0^\dagger(t)\,\hat{U}_{\rm B}^\dagger(t)\,e^{-i\hat{H}t/\hbar} \approx \hat{I} - i\hbar^{-1}\int_0^t {\rm d}t'\,\hat{H}_{\rm MB}(t') + O(\gamma)\,, \tag{A.1}$$

where the coupling in the interaction picture is

$$\hat{H}_{\rm MB}(t) = \sqrt{\gamma}\sum_{n,a}\hat{U}_0^\dagger(t)\,\hat{\sigma}_a^{(n)}\hat{U}_0(t)\,\hat{B}_a^{(n)}(t)\,, \tag{A.2}$$

with $\hat{B}_a^{(n)}(t)$ defined by (3.34).

We wish to take the trace over B of the exact equation of motion eq. (4.1) for $\hat{\mathcal{D}}(t)$, so as to generate an equation of motion for the density operator $\hat{D}(t)$ of S + M. In the right-hand side the term ${\rm tr}_{\rm B}\left[\hat{H}_{\rm B},\hat{\mathcal{D}}\right]$ vanishes and we are left with

$$i\hbar\frac{{\rm d}\hat{D}}{{\rm d}t} = \left[\hat{H}_0,\hat{D}\right] + {\rm tr}_{\rm B}\left[\hat{H}_{\rm MB},\hat{\mathcal{D}}\right]. \tag{A.3}$$

The last term involves the coupling $\hat{H}_{\rm MB}$ both directly and through the correlations between S+M and B which are created in $\mathcal{D}(t)$ from the time 0 to the time t. In order to write (A.3) more explicitly, we first exhibit these correlations. To this aim, we expand $\mathcal{D}(t)$ in powers of $\sqrt{\gamma}$ by means of the expansion (A.1) of its evolution operator. This provides, using $\hat{U}_0(t) = \exp[-i\hat{H}_0 t/\hbar]$,

$$\hat{U}_0^\dagger(t)\,\hat{U}_{\rm B}^\dagger(t)\,\hat{\mathcal{D}}(t)\,\hat{U}_{\rm B}(t)\,\hat{U}_0(t) \approx \hat{\mathcal{D}}(0) - i\hbar^{-1}\left[\int_0^t {\rm d}t'\hat{H}_{\rm MB}(t'),\hat{D}(0)\,\hat{R}_{\rm B}(0)\right] + O(\gamma)\,. \tag{A.4}$$

Insertion of the expansion (A.4) into (A.3) will allow us to work out the trace over B. Through the factor $\hat{R}_{\rm B}(0)$, this trace has the form of an equilibrium expectation value. As usual, the elimination of the bath variables will produce memory effects as obvious from (A.4). We wish these memory effects to bear only on the bath, so as to have a short characteristic time. However the initial state which enters (A.4) involves not only $\hat{R}_{\rm B}(0)$ but also $\hat{D}(0)$, so that a mere insertion of (A.4) into (A.3) would let $\hat{D}(t)$ keep an undesirable memory of $\hat{D}(0)$. We solve this difficulty by re-expressing perturbatively $\hat{D}(0)$ in terms of $\hat{D}(t)$. To this aim we note that the trace of (A.4) over B provides

$$U_0^\dagger(t)\,\hat{D}(t)\,\hat{U}_0(t) = \hat{D}(0) + O(\gamma)\,. \tag{A.5}$$

We have used the facts that the expectation value over $\hat{R}_{\rm B}(0)$ of an odd number of operators $\hat{B}_a^{(n)}$ vanishes, and that each $\hat{B}_a^{(n)}$ is accompanied in $\hat{H}_{\rm MA}$ by a factor $\sqrt{\gamma}$. Hence the right-hand side of (A.5) as well as that of (A.3) are power series in γ rather than in $\sqrt{\gamma}$.

We can now rewrite the right-hand side of (A.4) in terms of $\hat{D}(t)$ instead of $\hat{D}(0)$ by means of (A.5), then insert the resulting expansion of $\hat{\mathcal{D}}(t)$ in powers of $\sqrt{\gamma}$ into (A.3). Noting that the first term in (A.4) does not contribute to the trace over B, we find

$$\frac{{\rm d}\hat{D}}{{\rm d}t} - \frac{1}{i\hbar}\left[\hat{H}_0,\hat{D}\right] = -\frac{1}{\hbar^2}{\rm tr}_{\rm B}\int_0^t {\rm d}t'\left[\hat{H}_{\rm MB},\hat{U}_{\rm B}\hat{U}_0\left[\hat{H}_{\rm MB}(t'),\hat{U}_0^\dagger\hat{D}\hat{U}_0\hat{R}_{\rm B}(0)\right]\hat{U}_0^\dagger\hat{U}_{\rm B}^\dagger\right] + O\left(\gamma^2\right), \tag{A.6}$$

where $\hat{D}$, $\hat{U}_{\rm B}$ and $\hat{U}_0$ stand for $\hat{D}(t)$, $\hat{U}_{\rm B}(t)$ and $\hat{U}_0(t)$. Although the effect of the bath is of order γ, the derivation has required only the first-order term, in $\sqrt{\gamma}$, of the expansion (A.4) of $\mathcal{D}(t)$.

The bath operators $\hat{B}_a^{(n)}$ appear through $\hat{H}_{\rm MB}$ and $\hat{H}_{\rm MB}(t')$, and the evaluation of the trace thus involves only the equilibrium autocorrelation function (3.33). Using the expressions (3.10) and (A.2) for $\hat{H}_{\rm MB}$ and $\hat{H}_{\rm MB}(t')$, denoting the memory time $t-t'$ as u, and introducing the operators $\hat{\sigma}_a^{(n)}(u)$ defined by (4.4), we finally find the differential equation (4.5) for $\hat{D}(t)$.

B. Representation of the density operator of S + M by scalar functions

Je moet je niet beter voordoen dan je bent[96]

[96] Don't pretend to be more than you are

Dutch proverb

We first prove that, if the operators $\hat{R}_{ij}(t)$ in the Hilbert space of M depend only on $\hat{m}$, the right hand side of (4.8) has the same property.

The operators $\hat{\sigma}_+^{(n)} = \frac{1}{2}\left(\hat{\sigma}_x^{(n)} + i\hat{\sigma}_y^{(n)}\right)$ and $\hat{\sigma}_-^{(n)} = \left(\hat{\sigma}_+^{(n)}\right)^\dagger$ raise or lower the value of m by $\delta m = 2/N$, a property expressed by

$$[\hat{\sigma}_+^{(n)}, \hat{\sigma}_z^{(n)}] = -2\hat{\sigma}_+^{(n)}, \qquad \hat{\sigma}_+^{(n)}\hat{m} = (\hat{m} - \delta m)\,\hat{\sigma}_+^{(n)}. \tag{B.1}$$

The last identity can be iterated to yield

$$\hat{\sigma}_+^{(n)}\hat{m}^k = (\hat{m} - \delta m)\,\hat{\sigma}_+^{(n)}\hat{m}^{k-1} = \cdots = (\hat{m} - \delta m)^k\,\hat{\sigma}_+^{(n)}, \tag{B.2}$$

so that for every function that can be expanded in powers of $\hat{m}$, but does not otherwise depend on the $\hat{\sigma}_a^{(k)}$, it holds that

$$\hat{\sigma}_\pm^{(n)} f(\hat{m}) = f(\hat{m} \mp \delta m)\hat{\sigma}_\pm^{(n)}. \tag{B.3}$$

In order to write explicitly the time-dependent operators $\hat{\sigma}_a^{(n)}(u, i)$ defined by (4.7) with the definition (4.6), it is convenient to introduce the notations

$$m_\pm = m \pm \delta m = m \pm \frac{2}{N}, \tag{B.4}$$

$$\Delta_\pm f(m) = f(m_\pm) - f(m). \tag{B.5}$$

The time-dependent operators (4.7) are then given by ($u = t - t'$ is the memory time; $i = \uparrow, \downarrow$)

$$\hat{\sigma}_z^{(n)}(u, i) = \hat{\sigma}_z^{(n)}, \tag{B.6}$$

$$\hat{\sigma}_+^{(n)}(u, i) = \tfrac{1}{2}\left[\hat{\sigma}_x^{(n)}(u, i) + i\hat{\sigma}_y^{(n)}(u, i)\right] = e^{-i\hat{H}_i u/\hbar}\hat{\sigma}_+^{(n)}e^{i\hat{H}_i u/\hbar} = \hat{\sigma}_+^{(n)}e^{-i\hat{\Omega}_i^+ u} = e^{i\hat{\Omega}_i^- u}\,\hat{\sigma}_+^{(n)} = [\hat{\sigma}_-^{(n)}(u, i)]^\dagger, \tag{B.7}$$

where we used (B.3) and where the operators $\hat{\Omega}_\uparrow^+, \hat{\Omega}_\uparrow^-, \hat{\Omega}_\downarrow^+, \hat{\Omega}_\downarrow^-$ are functions of $\hat{m}$ defined by $\hat{\Omega}_i^\pm = \Omega_i^\pm(\hat{m})$ and by

$$\hbar\Omega_i^\pm(m) = \Delta_\pm H_i(m) = H_i(m \pm \delta m) - H_i(m). \tag{B.8}$$

If in the right-hand side of (4.8) the operator $\hat{R}_{ij}$ depends only on $\hat{m}$ at the considered time, the terms with $a = z$ cancel out on account of (B.6). The terms with $a = x$ and $a = y$, when expressed by means of (B.7), generate only products of $\hat{\sigma}_+^{(n)}\hat{\sigma}_-^{(n)}$ or $\hat{\sigma}_-^{(n)}\hat{\sigma}_+^{(n)}$ by functions of $\hat{m}$. This can be seen by using (B.3) to bring $\hat{\sigma}_+^{(n)}$ and $\hat{\sigma}_-^{(n)}$ next to each other through commutation with $\hat{R}_{ij}$. Since $\hat{\sigma}_+^{(n)}\hat{\sigma}_-^{(n)} = 1 - \hat{\sigma}_-^{(n)}\hat{\sigma}_+^{(n)} = \frac{1}{2}\left(1 + \hat{\sigma}_z^{(n)}\right)$, we can then perform the summation over n, which yields products of some functions of $\hat{m}$ by the factor

$$\sum_n \hat{\sigma}_+^{(n)}\hat{\sigma}_-^{(n)} = N - \sum_n \hat{\sigma}_-^{(n)}\hat{\sigma}_+^{(n)} = \frac{N}{2}(1 + \hat{m}), \tag{B.9}$$

itself depending only on $\hat{m}$. Hence, if $\hat{R}_{ij}$ is a function of the operator $\hat{m}$ only, this property also holds for $\mathrm{d}\hat{R}_{ij}(t)/\mathrm{d}t$ given by (4.8). Since, except in section 5.2, it holds at the initial time, it holds at any time.

The equations of motion (4.8) for $\hat{R}_{ij}(t)$ are therefore equivalent to the corresponding equations for $P_{ij}(m, t)$ which we derive below. The matrices $\hat{R}_{ij}(t)$ which characterize the density operator of S + M are parametrized as $\hat{R}_{ij}(t) = R_{ij}(\hat{m}) = P_{ij}^{\mathrm{dis}}(\hat{m}, t)/G(\hat{m})$; in the continuum limit, we introduced $P_{ij}(m, t) = (N/2)P_{ij}^{\mathrm{dis}}(m, t)$. We first note that the autocorrelation function $K(t)$ enters (4.8) through integrals of the form

$$\tilde{K}_{t>}(\omega) = \int_0^t \mathrm{d}u e^{-i\omega u}K(u) = (1/2\pi i)\int_{-\infty}^{+\infty}\mathrm{d}\omega'\{[\exp[\,i(\omega' - \omega)\,t] - 1]/(\omega' - \omega)\}\tilde{K}(\omega'),$$
$$\tilde{\mathrm{K}}_{t<}(\omega) = \int_{-t}^0 \mathrm{d}u e^{-i\omega u}K(u) = \int_0^t \mathrm{d}u e^{i\omega u}K(-u) = \left[\tilde{K}_{t>}(\omega)\right]^*. \tag{B.10}$$

As shown above, only the contributions to (4.8) with $a = x$ or $a = y$ survive owing to (B.6). The first term is transformed, by relying successively on (B.7), (B.10), (B.3) and (B.9), into

$$\int_0^t \mathrm{d}u \sum_n \sum_{a=x,y} K(u)\, \hat{\sigma}_a^{(n)}(u,i)\, \hat{R}_{ij} \hat{\sigma}_a^{(n)} = 2\int_0^t \mathrm{d}u \sum_n K(u) \left[e^{i\hat{\Omega}_i^- u} \hat{\sigma}_+^{(n)} R_{ij}(\hat{m})\, \hat{\sigma}_-^{(n)} + e^{i\hat{\Omega}_i^+ u} \hat{\sigma}_-^{(n)} R_{ij}(\hat{m})\, \hat{\sigma}_+^{(n)} \right] \quad \text{(B.11)}$$

$$= N\tilde{K}_{t>}\left(-\hat{\Omega}_i^-\right) R_{ij}(\hat{m} - \delta m)(1+\hat{m}) + N\tilde{K}_{t>}\left(-\hat{\Omega}_i^+\right) R_{ij}(\hat{m} + \delta m)(1-\hat{m}).$$

From the relation $R_{ij}(m) = P_{ij}^{\mathrm{dis}}(m)/G(m)$ (see Eq. (3.28)), we get

$$(1 \mp m) R_{ij}(m_\pm) = (1 \mp m) \frac{P_{ij}^{\mathrm{dis}}(m_\pm)}{G(m_\pm)} = \frac{1 \pm m_\pm}{G(m)} P_{ij}^{\mathrm{dis}}(m_\pm), \quad \text{(B.12)}$$

so that we can readily rewrite (B.11) in terms of $P_{ij}(\hat{m}) = \frac{1}{2} N P_{ij}^{\mathrm{dis}}(\hat{m})$ instead of $\hat{R}_{ij}$. The same steps allow us to express the other three terms of (4.8) in a similar form. Using also $\Delta_+ \Omega_i^- = \Delta_+ [H_i(m - \delta m) - H_i(m)] = -\Omega_i^+$ and $\Delta_- \Omega_i^+ = -\Omega_i^-$, where Δ_+ and Δ_- were defined by (B.4) and (B.5), we find altogether, after multiplying by $G(m)$,

$$\frac{\mathrm{d}}{\mathrm{d}t} P_{ij}(m,t) - \frac{1}{i\hbar}\left[H_i(m) - H_j(m)\right] P_{ij}(m,t) = \frac{\gamma N}{\hbar^2} \Delta_+ \left\{ (1+m) \left[\tilde{K}_{t>}(\Omega_i^-) + \tilde{K}_{t<}\left(\Omega_j^-\right) \right] P_{ij}(m,t) \right\}$$

$$+ \frac{\gamma N}{\hbar^2} \Delta_- \left\{ (1-m) \left[\tilde{K}_{t>}(\Omega_i^+) + \tilde{K}_{t<}\left(\Omega_j^+\right) \right] P_{ij}(m,t) \right\}, \quad \text{(B.13)}$$

For $i = j$ this equation simplifies into Eq. (4.16), due to both the cancellation in the left-hand side and the appearance of the combination (4.17) in the right-hand side.

Since it is an instructive exercise for students to numerically solve the full quantum dynamics of the registration process at finite N, we write out here the ingredients of the dynamical equation (B.13) for $P_{\uparrow\uparrow}$ and $P_{\downarrow\downarrow}$. As we just indicated above, this equation simplifies for $i = j$ into (4.16). Moreover, in the registration regime, we can replace $\tilde{K}_{t>}(\omega) + \tilde{K}_{t<}(\omega) = \tilde{K}_t(\omega)$ by $\tilde{K}(\omega)$, defined in (3.37). The rates entering Eq. (4.16) or Eq. (B.13) for $i = j$ have therefore the form

$$\frac{\gamma N}{\hbar^2} \tilde{K}(\omega) = \frac{N\hbar\omega}{8J\tau_J} \left[\coth\left(\frac{1}{2}\beta\hbar\omega\right) - 1 \right] \exp\left(-\frac{|\omega|}{\Gamma}\right), \quad \text{(B.14)}$$

where the timescale $\tau_J = \hbar/\gamma J$ can be taken as a unit of time. The variable ω in $\tilde{K}(\omega)$ takes the values $\Omega_i^\pm$, with $i = j = \uparrow$ or $\downarrow$, which are explicitly given by (4.14) in terms of the discrete variable m. It can be verified that, for $\Gamma \gg J/\hbar$, the omission of the Debye cut-off in (B.14) does not significantly affect the dynamics.

C. Evaluation of the recurrence time for a general pointer

For what cannot be cured, patience is best
Irish proverb

We consider here general models for which the tested observable $\hat{s}$ is coupled to a pointer through the Hamiltonian (6.12) where the pointer observable $\hat{m}$ has Q eigenvalues behaving as independent random variables. The probability distribution $p(\omega_q)$ for the corresponding eigenfrequencies $\omega_q \equiv Ng(s_i - s_j)m_q/\hbar$ which enter the function $\Re F(t) = Q^{-1}\sum_q \cos\omega_q t$ is taken as (6.16). For shorthand we denote from now on in the present appendix by $F(t)$ the real part $\Re F$ of the function defined in § 6.1.2 by (6.14).

We wish to evaluate the probability $\mathcal{P}(f,t)$ for $F(t)$ to be larger than some number f at a given time $t \gg \Delta\omega$. This probability is deduced from the characteristic function for $F(t)$ through

$$\mathcal{P}(f,t) = \overline{\theta[F(t) - f]} = \int_{-\infty}^{+\infty} \frac{\mathrm{d}\lambda}{2\pi(i\lambda+0)} e^{-iQ\lambda f}\, \overline{e^{iQ\lambda F(t)}} = \int_{-\infty}^{+\infty} \frac{\mathrm{d}\lambda}{2\pi(i\lambda+0)} \left[e^{-i\lambda f} \int \mathrm{d}\omega\, p(\omega)\, e^{i\lambda\cos\omega t} \right]^Q. \quad \text{(C.1)}$$

Since $t \gg 1/\Delta\omega$, the factor $p(\omega)$ in the integrand varies slowly over the period $2\pi/t$ of the exponential factor $\exp i\lambda\cos\omega t$. This exponential may therefore be replaced by its average on ω over one period, which is the Bessel function $J_0(\lambda)$. The integral over ω then gives unity, and we end up with

$$\mathcal{P}(f,t) = \int_{-\infty}^{+\infty} \frac{\mathrm{d}\lambda}{2\pi(i\lambda+0)} \exp\{Q[\ln J_0(\lambda - i0) - i\lambda f]\}. \tag{C.2}$$

For $Q \gg 1$, the exponent has a saddle point λ_s given as function of f by

$$\lambda_s \equiv -iy, \qquad \frac{I_1(y)}{I_0(y)} = f, \qquad \frac{\mathrm{d}f}{\mathrm{d}y} = 1 - \frac{f}{y} - f^2, \tag{C.3}$$

and we find

$$\mathcal{P}(f,t) = \frac{1}{y}\left(2\pi Q \frac{\mathrm{d}f}{\mathrm{d}y}\right)^{-1/2} \exp\{-Q[yf - \ln I_0(y)]\}. \tag{C.4}$$

We now evaluate the average duration $\overline{\delta t}$ of an excursion of $F(t)$ above the value f. To this aim, we determine the average curvature of $F(t)$ at a peak, reached for values of the set ω_q such that $F(t) > f$. The quantity

$$\overline{\theta[F(t) - f] \frac{\mathrm{d}^2F(t)}{\mathrm{d}t^2}} \tag{C.5}$$

is obtained from (C.1) by introducing in the integrand a factor

$$\frac{-\int \mathrm{d}\omega p(\omega)\omega^2 \cos\omega t\ e^{-i\lambda\cos\omega t}}{\int \mathrm{d}\omega p(\omega) e^{i\lambda\cos\omega t}} = \frac{-i\Delta\omega^2 J_1(\lambda)}{J_0(\lambda)}, \tag{C.6}$$

where we used $t\Delta\omega \gg 1$. The saddle-point method, using (C.3), then provides on average, under the constraint $F(t) > f$,

$$\frac{1}{F(t)}\frac{\mathrm{d}^2F(t)}{\mathrm{d}t^2} = -\Delta\omega^2. \tag{C.7}$$

A similar calculation shows that, around any peak of $F(t)$ emerging above f, the odd derivatives of $F(t)$ vanish on average while the even ones are consistent with the gaussian shape (6.17), rewritten for $f^{-1}F(t')$ in terms of $t' - t < 1/\Delta\omega$. This result shows that the shape of the dominant term of (6.19) is not modified by the constraint $F(t) > f$. Hence, if $F(t)$ reaches a maximum $f + \delta f$ at some time, the duration of its excursion above f is

$$\delta t = \frac{2}{\Delta\omega}\sqrt{\frac{2\delta f}{f}}. \tag{C.8}$$

From (C.4) we find the conditional probability density for $F(t)$ to reach $f + \delta f$ if $F(t) > f$, as $Qye^{-Qy\delta f}$, and hence

$$\overline{\delta t} = \frac{1}{\Delta\omega}\sqrt{\frac{2\pi}{Qyf}}. \tag{C.9}$$

Since the probability $\mathcal{P}(f,t)$ for a recurrence to occur at the time t does not depend on this time, and since the average duration of the excursion is $\overline{\delta t}$, the average delay between recurrences is here

$$\tau_{\mathrm{recur}} = \frac{\overline{\delta t}}{\mathcal{P}(f,t)} = \frac{2\pi}{\Delta\omega}\sqrt{\frac{y}{f}\frac{\mathrm{d}f}{\mathrm{d}y}} e^{Q[yf - \ln I_0(y)]}, \tag{C.10}$$

where y is given by $I_1(y) = f\, I_0(y)$.

For f sufficiently small so that $\ln I_0(f) \simeq f^2$ (for $f = 0.2$ the relative error is 1%), we find from (C.3) that $y \simeq 2f$, and this expression of the recurrence time reduces to (6.20), that is exponentially large in Q.

We notice that in this derivation the shape of the eigenvalue spectrum $p(\omega)$ hardly played any role, we only used that it is smooth on the scale $2\pi/t$, where t is the observation time. So after times $t \gg 2\pi/\Delta\omega$, where the individual levels are no longer resolved, there will be an exponentially long timescale for the pointer to recur.

D. Effect of the bath on the off-diagonal sectors of the density matrix of S + M

A jug carries water until its handle breaks off
Polish proverb

D.1. Full expression of $P_{\uparrow\downarrow}$ for large N

In Eq. (6.22) we have parametrized $P_{\uparrow\downarrow}(m,t)$ in terms of the function $A(m,t)$, which satisfies

$$\frac{\partial A}{\partial t} = \frac{2igm}{\hbar} - \frac{1}{NP_{\uparrow\downarrow}}\frac{\partial P_{\uparrow\downarrow}}{\partial t}, \tag{D.1}$$

with $A(m,0) = 0$. In subsection 4.4, we have derived the equation (4.29) for $P_{\uparrow\downarrow}$, from which $A(m,t)$ can be obtained for large N at the two relevant orders (finite and in $1/N$). As we need $A(m,t)$ only at linear order in γ, we can replace in (4.29) the quantity $X_{\uparrow\downarrow}(m,t)$ by its value for $\gamma = 0$,

$$X \equiv X_{\uparrow\downarrow}(m,t) = \frac{2igt}{\hbar} - \frac{m}{\delta_0^2}, \tag{D.2}$$

which contains no $1/N$ term. We then insert (4.29) in (D.1) to obtain

$$\frac{\partial A(m,t)}{\partial t} = \frac{\gamma}{\hbar^2}\left\{\left(1-e^{2X}\right)(1+m)\tilde{K}_- + \left(1-e^{-2X}\right)(1-m)\tilde{K}_+ - \frac{2}{N}\left[\frac{\partial[(1+m)\tilde{K}_- e^X]}{\partial m}e^X - \frac{\partial[(1-m)\tilde{K}_+ e^{-X}]}{\partial m}e^{-X}\right]\right\}, \tag{D.3}$$

where the combinations $\tilde{K}_\pm(m,t) = \tilde{K}_{t>}\left(\Omega_\uparrow^\pm\right) + \tilde{K}_{t<}\left(\Omega_\downarrow^\pm\right)$ were introduced in (4.19). The functions $\tilde{K}_{t>}(\omega)$ and $\tilde{K}_{t<}(\omega) = \tilde{K}_{t>}^*(\omega)$ were defined by (3.36), (3.37), (4.10) and (4.11), and the frequencies $\Omega_\uparrow^\pm$ and $\Omega_\downarrow^\pm$ by (4.14). The initial condition is $A(m,0) = 0$.

D.2. Expansion for small m

The above result holds for arbitrary values of m and t. However, since in $P_{\uparrow\downarrow}(m,t)$ the values of m remain small as $1/\sqrt{N}$, only the first three terms in the expansion

$$A(m,t) \approx B(t) - i\Theta(t)m + \frac{1}{2}D(t)m^2, \tag{D.4}$$

are relevant. The time-dependence of these three functions, which vanish for $t = 0$, will be elementary so that we will work out only their time derivatives, which are simpler and which result from (D.3).

We note as Ω the frequency defined by

$$\Omega \equiv \frac{2g}{\hbar} \equiv \frac{\pi}{\tau_{\rm recur}}, \tag{D.5}$$

which is related to the period $\tau_{\rm recur}$ of the recurrences that arise from the leading oscillatory term $\exp(2iNgmt/\hbar)$ in (6.22) with m taking the discrete values (3.22). We can then rewrite, up to the order m^2 and up to corrections in $1/N$,

$$\Omega_\uparrow^\pm \approx \mp\Omega \mp \frac{2J_2 m}{\hbar}, \qquad \Omega_\downarrow^\pm \approx \pm\Omega \mp \frac{2J_2 m}{\hbar}, \qquad X = i\Omega t - \frac{m}{\delta_0^2}. \tag{D.6}$$

The expressions (4.10) and (4.11) for $\tilde{K}_{t>}(\omega)$ or $\tilde{K}_{t<}(\omega)$ then provide for their combinations (4.19) the expansion

$$\tilde{K}_\pm(m,t) \approx e^{\pm i\Omega t}\int_{-\infty}^{\infty}\frac{{\rm d}\omega}{\pi}\,\tilde{K}\left(\omega \mp \frac{2J_2 m}{\hbar}\right)\frac{\omega\sin\omega t - \Omega\sin\Omega t \mp i\Omega(\cos\Omega t - \cos\omega t)}{\omega^2-\Omega^2} + O\left(\frac{1}{N}\right). \tag{D.7}$$

The required functions $B(t)$, $\Theta(t)$ and $D(t)$ are obtained by inserting (D.4) and (D.7) into (D.3). While the term of order $1/N$ in $B(t)$ provides a finite factor in $P_{\uparrow\downarrow}(m,t)$, the terms of order $1/N$ in $\Theta(t)$ and $D(t)$ provide negligible contributions. However that may be, it will be sufficient for our purpose to evaluate only the finite contribution to $B(t)$ and the large t approximations for $\Theta(t)$ and $D(t)$.

D.3. The damping term $B(t)$

To find $B(t)$, we simply set $m = 0$ in (D.3) and (D.7). Next we employ the expression (3.37) for $\tilde{K}(\omega)$ and take advantage of the symmetry of the integrand with respect to ω, which allows us to keep only the symmetric part of $\tilde{K}(\omega)$. This yields

$$\frac{\mathrm{d}B}{\mathrm{d}t} = \frac{4\gamma\Omega \sin\Omega t}{\hbar^2} \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{\pi}\, \tilde{K}(\omega)\, \frac{\cos\Omega t - \cos\omega t}{\omega^2 - \Omega^2} = \frac{\gamma\Omega \sin\Omega t}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}\omega\, \omega \coth\frac{\hbar\omega}{2T}\, \exp\left(-\frac{|\omega|}{\Gamma}\right) \frac{\cos\Omega t - \cos\omega t}{\omega^2 - \Omega^2}. \quad \text{(D.8)}$$

where we discarded corrections of order $1/N$. This entails the result for $B(t)$ presented in Eq. (6.25) of the main text. For $t \ll 1/\Gamma$, (D.8) reduces to $\mathrm{d}B/\mathrm{d}t \sim (\gamma\Gamma^2\Omega^2/2\pi)t^3$ and hence

$$B(t) \sim \frac{\gamma\Gamma^2\Omega^2}{8\pi} t^4 = \frac{\gamma\Gamma^2 g^2}{2\pi\hbar^2} t^4. \quad \text{(D.9)}$$

The ω integral in Eq. (6.25) for $B(t)$ can be easily carried out numerically and the result is plotted in Fig 6.1 for typical values of the parameters. It is nevertheless instructive to carry out this integral explicitly. This calculation is hindered by the non-analyticity of our Debye cutoff. However, since the result is not expected to depend significantly on the shape of the cutoff (Γ is the largest frequency of the model), we may replace the exponential cutoff in (3.37) by a quasi Lorentzian cutoff,

$$\exp\left(-\frac{|\omega|}{\Gamma}\right) \mapsto \frac{4\tilde{\Gamma}^4}{4\tilde{\Gamma}^4 + \omega^4}; \qquad \tilde{K}(\omega) \mapsto \frac{\hbar^2\omega}{4(e^{\beta\hbar\omega} - 1)} \frac{4\tilde{\Gamma}^4}{4\tilde{\Gamma}^4 + \omega^4}, \quad \text{(D.10)}$$

where the factors 4 are introduced for later convenience. This expression ensures convergence while being analytic with simple poles. The cutoff (D.10) provides for $B(t)$ the same short time behavior as (D.9) if we make the connection

$$\tilde{\Gamma} = \sqrt{\frac{2}{\pi}}\, \Gamma. \quad \text{(D.11)}$$

In order to integrate the thus modified version of (D.8) over ω, we first split $\cos\omega t$ into $\frac{1}{2}\exp i\omega t + \frac{1}{2}\exp -i\omega t$ and then slightly rotate the integation contour so that ω passes below $+\Omega$ and above $-\Omega$, instead of passing through these poles. For each of the terms we can close the contour either in the upper or lower half-plane, such that it decays for $|\omega| \to \infty$, and pick up the residues at the various poles. The first set of poles, arising from the denominator of (D.8), consist of $\pm\Omega$; since they lie on the real ω-axis, they will produce a non-decaying long time behavior. The second set of poles arise from the coth, as exhibited by the expansion

$$\coth\frac{\hbar\omega}{2T} = \sum_{n=-\infty}^{\infty} \frac{2T}{\hbar(\omega - i\Omega_n)}, \qquad \Omega_n \equiv \frac{2\pi n T}{\hbar}, \quad \text{(D.12)}$$

where the sum is meant as principal part for $n \to \pm\infty$; the frequencies Ω_n are known as Matsubara frequencies. Thirdly, the cutoff (D.10) provides the four poles $\pm\tilde{\Gamma} \pm i\tilde{\Gamma}$. We can also take advantage of the symmetry $\omega \to -\omega$, which associates pairwise complex conjugate residues. Altogether, we find

$$\begin{aligned} \frac{1}{\gamma\Omega}\frac{\mathrm{d}B}{\mathrm{d}t} &= \coth\frac{\hbar\Omega}{2T}\, \frac{\tilde{\Gamma}^4}{4\tilde{\Gamma}^4 + \Omega^4}(1 - \cos 2\Omega t) + \frac{T}{\hbar}\sum_{n=1}^{\infty} \frac{\Omega_n}{\Omega_n^2 + \Omega^2} \frac{4\tilde{\Gamma}^4}{4\tilde{\Gamma}^4 + \Omega_n^4} \left[\sin 2\Omega t - 2\exp(-\Omega_n t)\sin\Omega t\right] \\ &\quad + \frac{1}{2}\Im\left\{\coth\frac{(1+i)\hbar\tilde{\Gamma}}{2T}\, \frac{\tilde{\Gamma}^2}{2\tilde{\Gamma}^2 + i\Omega^2} \left[\sin 2\Omega t - 2\exp[-(1-i)\tilde{\Gamma}t]\, \sin\Omega t\right]\right\} + O\left(\frac{1}{N}\right). \end{aligned} \quad \text{(D.13)}$$

Now B is easily obtained by integrating this from 0 to t,

$$B(t) = \frac{\gamma}{2}\coth\frac{g}{T}\,\frac{\tilde\Gamma^4}{4\tilde\Gamma^4+\Omega^4}(2\Omega t-\sin 2\Omega t)+\sum_{n=1}^{\infty}\frac{4\gamma\tilde\Gamma^4\Omega_n T}{\hbar(4\tilde\Gamma^4+\Omega_n^4)}\left[\frac{\sin^2\Omega t}{\Omega^2+\Omega_n^2}+2\Omega\frac{(\Omega\cos\Omega t+\Omega_n\sin\Omega t)\exp(-\Omega_n t)-\Omega}{(\Omega^2+\Omega_n^2)^2}\right]$$
$$-\frac{\gamma\tilde\Gamma^2}{2}\Re\left\{\coth\frac{(1+i)\hbar\tilde\Gamma}{2T}\left[\frac{\sin^2\Omega t}{\Omega^2-2i\tilde\Gamma^2}+2\Omega\frac{\left(\Omega\cos\Omega t+(1-i)\tilde\Gamma\sin\Omega t\right)\exp[-(1-i)\tilde\Gamma t]-\Omega}{(\Omega^2-2i\tilde\Gamma^2)^2}\right]\right\}, \tag{D.14}$$

where we made the residues at $(\pm1\pm i)\tilde\Gamma$ look as much as possible like the ones at Ω_n.

With these exact results in hand, let us discuss the relative sizes of the various terms. The above complete formula exhibits some contributions that become exponentially small for sufficiently large t. Such contributions are essential to ensure the behavior (D.9) of B for $t\ll1/\tilde\Gamma$, and also its behavior for $t\ll\hbar/2\pi T$, but can be neglected otherwise. Moreover, we have $\hbar\tilde\Gamma\gg T$ and $\tilde\Gamma\gg\Omega$; hence, within exponentially small corrections, the third term of (D.13) reduces, for $t\gg1/\tilde\Gamma$, to $-\Omega^2\sin(2\Omega t)/8\tilde\Gamma^2$ and is therefore negligible compared to the first two terms. In the first term of (D.13), the Debye cutoff is irrelevant, but it is needed in the second term to ensure convergence of the series. Restoring our exponential cutoff, we can write this series as

$$\frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{n}{n^2+a^2}e^{-bn},\qquad a\equiv\frac{\hbar\Omega}{2\pi T}\ll1,\qquad b\equiv\frac{2\pi T}{\hbar\Gamma}\ll1, \tag{D.15}$$

which, within corrections of order a^2, is equal to

$$\frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{1}{n}e^{-bn}=-\frac{1}{2\pi}\ln\left(1-e^{-b}\right)\sim\frac{1}{2\pi}\ln\frac{\hbar\Gamma}{2\pi T}. \tag{D.16}$$

Altogether, returning to our original notations through use of (D.5), we find from the first two terms of (D.13), for $t\gg\hbar/2\pi T$:

$$\frac{\tau_{\rm recur}}{\gamma}\frac{\mathrm{d}B}{\mathrm{d}t}=\frac{\pi}{4}\coth\frac{g}{T}\left(1-\cos\frac{2\pi t}{\tau_{\rm recur}}\right)+\frac{1}{2}\ln\frac{\hbar\Gamma}{2\pi T}\sin\frac{2\pi t}{\tau_{\rm recur}}. \tag{D.17}$$

Likewise, the function $B(t)$ itself behaves in this region as

$$B(t)=\frac{\gamma\pi}{4}\coth\frac{g}{T}\left(\frac{t}{\tau_{\rm recur}}-\frac{1}{2\pi}\sin\frac{2\pi t}{\tau_{\rm recur}}\right)+\frac{\gamma}{4\pi}\ln\frac{\hbar\Gamma}{2\pi T}\left(1-\cos\frac{2\pi t}{\tau_{\rm recur}}\right)-\frac{\gamma\zeta(3)}{\pi^3}\frac{g^2}{T^2}, \tag{D.18}$$

where the last piece arises, in the considered approximation, from the last term of the sum in (D.14).

D.4. Approximations for $\Theta(t)$ and $D(t)$.

We have just seen that the dominant contribution to $B(t)$ in the region $t\gg\hbar/2\pi T$ originates from the poles $\omega=\pm\Omega$ of the integrand of (D.8). Likewise, as we need only an estimate of $\Theta(t)$ and $D(t)$, we will evaluate approximately the integral in (D.7) by picking up only the contributions of these poles. As we did for $B(t)$, we deform and close the integration contour in the upper or in the lower half-plane, but we now disregard the singularities of $\tilde K(\omega\mp2J_2m/\hbar)$. This approximation amounts to make the replacements

$$\frac{\omega\sin\omega t-\Omega\sin\Omega t}{\omega^2-\Omega^2}\mapsto\frac{\pi}{2}\cos(\Omega t)[\delta(\omega-\Omega)+\delta(\omega+\Omega)], \tag{D.19}$$
$$\frac{\Omega(\cos\Omega t-\cos\omega t)}{\omega^2-\Omega^2}\mapsto\frac{\pi}{2}\sin(\Omega t)[\delta(\omega-\Omega)+\delta(\omega+\Omega)], \tag{D.20}$$

which as we have seen are justified for $t \gg \hbar/2\pi T$. As a result, we find the time-independent expressions for $\tilde{K}_\pm$,

$$\tilde{K}_\pm \approx \frac{1}{2}[\tilde{K}(\Omega \mp 2J_2 m) + \tilde{K}(-\Omega \mp 2J_2 m)]. \tag{D.21}$$

We now return to our original notations by use of (D.5) for Ω and (D.6) for X, rewriting the dominant part of (D.3) as

$$\frac{\tau_{\rm recur}}{\gamma}\frac{{\rm d}A}{{\rm d}t} = \frac{\pi}{2\hbar g}\left[(1-e^{2X})(1+m)\tilde{K}_- + (1-e^{-2X})(1-m)\tilde{K}_+\right]. \tag{D.22}$$

In order to generate $\Theta(t)$ and $D(t)$ through the expansion (D.4) of $A(m,t)$ in powers of m, we insert into (D.22) the expansions

$$\left[1-e^{\pm 2X}\right](1\pm m) \approx \left[1-e^{\pm 2i\Omega t}\right] \pm \left[1+e^{\pm 2i\Omega t}\left(\frac{2}{\delta_0^2}-1\right)\right]m + 2e^{\pm 2i\Omega t}\left(\frac{1}{\delta_0^2}-\frac{1}{\delta_0^4}\right)m^2, \tag{D.23}$$

$$\frac{4}{\hbar}\tilde{K}_\pm \approx g\coth\frac{g}{T} \pm J_2 m - \frac{J_2^2}{T^2\sinh^2 g/T}\left(\coth\frac{g}{T}-\frac{T}{g}\right)m^2. \tag{D.24}$$

Gathering, in the resulting expansion of $A(m,t)$, the terms in m, we find (for $g \ll T$)

$$\frac{\tau_{\rm recur}}{\gamma}\frac{{\rm d}\Theta}{{\rm d}t} = -\frac{\pi}{4}\left[\left(\frac{2}{\delta_0^2}-1\right)\coth\frac{g}{T}+\frac{J_2}{g}\right]\sin\frac{2\pi t}{\tau_{\rm recur}} \sim -\frac{\pi}{4}\left[\left(\frac{2}{\delta_0^2}-1\right)\frac{T}{g}+\frac{J_2}{g}\right]\sin\frac{2\pi t}{\tau_{\rm recur}}, \tag{D.25}$$

which is integrated as

$$\Theta(t) \sim -\frac{\gamma}{8g}\left[\left(\frac{2}{\delta_0^2}-1\right)T + J_2\right]\left[1-\cos\frac{2\pi t}{\tau_{\rm recur}}\right]. \tag{D.26}$$

Likewise, the terms in m^2 yield

$$\frac{\tau_{\rm recur}}{\gamma}\frac{{\rm d}D}{{\rm d}t} \sim \frac{\pi}{2}\left[\frac{J_2^2}{T^2\sinh^2 g/T}\left(\coth\frac{g}{T}-\frac{T}{g}\right)-\frac{J_2}{g}\right]\left(1-\cos\frac{2\pi t}{\tau_{\rm recur}}\right)+\frac{\pi}{2}\left[2\coth\frac{g}{T}\left(\frac{1}{\delta_0^2}-\frac{1}{\delta_0^4}\right)-\frac{2J_2}{g\delta_0^2}\right]\cos\frac{2\pi t}{\tau_{\rm recur}}. \tag{D.27}$$

The first bracket simplifies for $g \ll T$ into

$$\frac{J_2^2}{T^2\sinh^2 g/T}\left(\coth\frac{g}{T}-\frac{T}{g}\right)-\frac{J_2}{g} \sim \frac{J_2}{g}\left(\frac{J_2}{3T}-1\right). \tag{D.28}$$

We shall only need the values of $D(t)$ at the recurrence times $p\tau_{\rm recur}$. Integration of the factors $\cos 2\pi t/\tau_{\rm recur}$ generates $\sin 2\pi t/\tau_{\rm recur}$, which vanishes at these times. We have therefore the compact result

$$D(p\tau_{\rm recur}) \simeq p \times D(\tau_{\rm recur}) = p\,\frac{\pi\gamma}{2}\frac{J_2}{g}\left(\frac{J_2}{3T}-1\right). \tag{D.29}$$

E. Time dependence of the registration process

De tijd heelt alle wonden[97]
Dutch proverb

The location $\mu(t)$ of the peak of the distribution $P(m,t)$ increases in time according to (7.30) where $\phi(m)$ is defined by (7.25). We wish in § 7.2.3 and § 7.2.4 to obtain an algebraic approximation for $\mu(t)$ at all times. To this aim, we will represent $1/v(\mu)$ by its Mittag-Leffler expansion

$$\frac{\gamma T}{\hbar v(m)} \equiv \frac{1}{\phi(m)[1 - m \coth \phi(m)]} = \sum_i \frac{m_i}{[(1-m_i^2)(\mathrm{d}\phi/\mathrm{d}m_i) - 1]\phi(m_i)} \frac{1}{m - m_i}, \tag{E.1}$$

which sums over all real or complex values $m = m_i$ where $v(m) = 0$.

E.1. Registration for second-order transition of M

For $q = 2$, it is sufficient for our purpose to keep in the expansion (E.1) only the real poles m_i. This truncation does not affect the vicinity of the (stable or unstable) fixed points where the motion of $\mu(t)$ is slowest, and provides elsewhere a good interpolation provided T/J is not too small. Three values m_i occur here, namely $-m_\mathrm{B}$, $m_\Uparrow \simeq m_\mathrm{F}$ and $m_\Downarrow \simeq -m_\mathrm{F}$, with $m_\mathrm{B} \ll m_\mathrm{F}$, so that we find over the whole range $0 < \mu < m_\mathrm{F}$, through explicit integration of (7.30),

$$\frac{t}{\tau_\mathrm{reg}} = \ln \frac{m_\mathrm{B} + \mu}{m_\mathrm{B}} + a \ln \frac{m_\mathrm{F}^2}{m_\mathrm{F}^2 - \mu^2}, \tag{E.2}$$

where the coefficient a, given by

$$a = \frac{T(J-T)}{J[T - J(1 - m_\mathrm{F}^2)]}, \tag{E.3}$$

decreases with temperature from $a = 1$ at $T = 0$ to $a = \frac{1}{2}$ for $T = J$. For short times, such that $\mu \ll m_\mathrm{F}$, we recover from the first term of (E.2) the evolution (7.43) of $\mu(t)$. When μ approaches m_F, the second term dominates, but as long as $m_\mathrm{F} - m$ is of order m_B the time needed for μ to reach m is of order $\tau_\mathrm{reg} \ln(m_\mathrm{F}/m_\mathrm{B})$. We define the cross-over by writing that the two logarithms of (E.2) are equal, which yields $\mu = m_\mathrm{F} - \frac{1}{2}m_\mathrm{B}$. The time τ'_reg during which $\mu(t)$ goes from 0 to $m_\mathrm{F} - \frac{1}{2}m_\mathrm{B}$, termed the second characteristic registration time, is then given by (7.48), that is,

$$\tau'_\mathrm{reg} = \tau_\mathrm{reg}(1+a) \ln \frac{m_\mathrm{B} + m_\mathrm{F}}{m_\mathrm{B}}. \tag{E.4}$$

When μ approaches m_F in the regime $m_\mathrm{F} - \mu \ll m_\mathrm{B}$, we can invert (E.2) as

$$\mu(t) = m_\mathrm{F} \left[1 - \frac{1}{2} \left(\frac{m_\mathrm{F}}{m_\mathrm{B}} \right)^{1/a} \exp\left(-\frac{t}{a\tau_\mathrm{reg}} \right) \right], \tag{E.5}$$

which exhibits the final exponential relaxation. We can also invert this relation in the limiting cases $T \to J$ and $T \to 0$. If T lies close to the transition temperature, we have $m_\mathrm{F} \sim \sqrt{3(J-T)/J}$ and $a = \frac{1}{2}$. Provided the coupling is weak so that $m_\mathrm{B} = g/(J-T) \ll m_\mathrm{F}$, we find

$$\mu(t) = \frac{m_\mathrm{B} m_\mathrm{F}}{m_\mathrm{B}^2 + m_\mathrm{F}^2 e^{-2t/\tau_\mathrm{reg}}} \left[\sqrt{m_\mathrm{B}^2 + (m_\mathrm{F}^2 - m_\mathrm{B}^2) e^{-2t/\tau_\mathrm{reg}}} - m_\mathrm{B} m_\mathrm{F} e^{-2t/\tau_\mathrm{reg}} \right]. \tag{E.6}$$

This expression encompasses all three regimes of § 7.2.3, namely, $\mu \sim m_\mathrm{B} t/\tau_\mathrm{reg}$ for $t \ll \tau_\mathrm{reg}$, μ running from m_B to m_F for t between τ_reg and τ'_reg, and

$$\mu(t) \approx m_\mathrm{F} \left(1 - \frac{m_\mathrm{F}^2}{2m_\mathrm{B}^2} e^{-2t/\tau_\mathrm{reg}} \right) \tag{E.7}$$

[97] Time heals all wounds

for $t - \tau'_{\rm reg} \gg \tau_{\rm reg}$. In the low temperature regime ($T \ll J$, with $m_{\rm B} \sim g/J$ and $a \sim 1$), we can again invert (E.2) as

$$\mu\,(t) = \frac{1}{2m_{\rm B}}\left[\sqrt{4m_{\rm B}^2\left(m_{\rm F}^2 - m_{\rm B}^2\right) + \left(2m_{\rm B}^2 - m_{\rm F}^2 e^{-t/\tau_{\rm reg}}\right)^2} - m_{\rm F}^2 e^{-t/\tau_{\rm reg}}\right], \tag{E.8}$$

encompassing the same three regimes; for $t - \tau'_{\rm reg} \gg \tau_{\rm reg}$, we now have

$$\mu(t) \approx m_F\left(1 - \frac{m_{\rm F}}{2m_{\rm B}} e^{-t/\tau_{\rm reg}}\right). \tag{E.9}$$

E.2. Registration for first-order transition of M

For $J_4 \neq 0$, such as the $q = 4$ case with $J_2 = 0$ and $J_4 = J$, we need to account for the presence of the minimum of $v(m)$ at $m = m_{\rm c}$. To this aim, we still truncate the Mittag-Leffler expansion (E.1) of $1/v(m)$. However, we now retain not only the real poles but also the two complex poles near $m_{\rm c}$ which govern the minimum of $v(m)$. These poles are located at

$$m_{\rm c} \pm i\delta m_{\rm c}, \qquad \delta m_{\rm c}^2 = \frac{m_{\rm c}(1-m_{\rm c}^2)^2}{1+2m_{\rm c}^2}\,\frac{g - h_{\rm c}}{T} \sim m_{\rm c}\left(\frac{g}{T} - \frac{2m_{\rm c}}{3}\right). \tag{E.10}$$

The real pole associated with the repulsive fixed point lies at $-m_{\rm B} \sim -2m_{\rm c}$, and the ferromagnetic poles lie close to $\pm m_{\rm F} \sim \pm 1$. We have thus, at lowest order in $T/J \simeq 3m_{\rm c}^2$ and in $g/T \sim 2m_{\rm c}/3$, but with T/J sufficiently large so that we can drop the other complex poles,

$$\frac{\gamma T}{\hbar v(m)} = \frac{m_{\rm c} - \frac{1}{2}(m - m_{\rm c})}{(m - m_{\rm c})^2 + \delta m_{\rm c}^2} + \frac{1}{3(m + 2m_{\rm c})} + \frac{2Tm}{J(1-m^2)}. \tag{E.11}$$

Hence the time-dependence of the peak $\mu(t)$ of $P_{\uparrow\uparrow}(m,t)$ is given through integration of (7.30) as

$$\frac{t}{\tau_{\rm reg}} = \frac{1}{\pi}\left(\frac{\pi}{2} + \arctan\frac{\mu - m_{\rm c}}{\delta m_{\rm c}}\right) + \frac{\delta m_{\rm c}}{\pi m_{\rm c}}\left[\frac{1}{4}\ln\frac{m_{\rm c}^2}{(\mu - m_{\rm c})^2 + \delta m_{\rm c}^2} + \frac{1}{3}\ln\frac{\mu + 2m_{\rm c}}{2m_{\rm c}} + \frac{T}{J}\ln\frac{1}{1-\mu^2}\right], \tag{E.12}$$

where we introduced the registration time

$$\tau_{\rm reg} \equiv \frac{\pi\hbar m_{\rm c}}{\gamma T \delta m_{\rm c}} = \frac{\pi\hbar}{\gamma T}\sqrt{\frac{m_{\rm c} T}{g - h_{\rm c}}}, \tag{E.13}$$

with $m_{\rm c} = \sqrt{T/3J} = 3h_{\rm c}/2T$.

The initial evolution (7.50) is recovered from (E.12) for $\mu \ll m_{\rm c}$ and $t \ll \hbar/\gamma T$. It matches the bottleneck stage in which $\mu(t)$ varies slowly around the value $m_{\rm c}$ on the time scale $\tau_{\rm reg}$. Then, the right-hand side of (E.12) is dominated by its first term, so that the magnetization increases from $m_{\rm c} - \delta m_{\rm c}$ to $m_{\rm c} + \delta m_{\rm c}$ between the times $t = \tau_{\rm reg}/4$ and $t = 3\tau_{\rm reg}/4$, according to:

$$\mu(t) = m_{\rm c} - \delta m_{\rm c}\,{\rm cotan}\,\frac{\pi t}{\tau_{\rm reg}}. \tag{E.14}$$

After μ passed the bottleneck, for $\mu - m_{\rm c} \gg \delta m_{\rm c}$, (E.12) provides

$$t = \tau_{\rm reg} + \tau_1\left(-\frac{m_{\rm c}}{\mu - m_{\rm c}} + \frac{1}{2}\ln\frac{m_{\rm c}}{\mu - m_{\rm c}} + \frac{1}{3}\ln\frac{\mu + 2m_{\rm c}}{2m_{\rm c}} + \frac{T}{J}\ln\frac{1}{1-\mu^2}\right), \tag{E.15}$$

which is nearly equal to $\tau_{\rm reg}$ within corrections of order $\tau_1 = \hbar/\gamma T$, as long as μ is not very close to 1. The final exponential relaxation takes place on the still shorter scale $\hbar/\gamma J$.

F. Effects of bifurcations

Of je door de hond of de kat gebeten wordt, het blijft om het even[98]
Dutch proverb

In subsection 7.3 we consider situations in which Suzuki's slowing down is present, namely the preparation of the initial metastable state for $q = 2$ and the possibility of false registrations. We gather here some derivations.

The Green's function $G(m, m', t - t')$ associated to the equation (7.1) for $P_{\rm M}(m, t)$ will be obtained from the backward equation

$$\frac{\partial}{\partial t'}G(m, m', t - t') + v(m')\frac{\partial}{\partial m'}G(m, m', t - t') + \frac{1}{N}[w(m')\frac{\partial^2}{\partial m'^2}G(m, m', t - t')] = -\delta(m - m')\delta(t - t'), \tag{F.1}$$

where t' runs down from $t + 0$ to 0. Introducing the time scale $\tau_{\rm reg}$ defined by (7.44) and using the expression (7.42) for $v(m')$ for small m' together with the related $w(m') \approx \gamma g t/\hbar$, we have to solve the equation

$$\left[\tau_{\rm reg}\frac{\partial}{\partial t'} + (m_{\rm B} + m')\frac{\partial}{\partial m'} + \frac{1}{N}\frac{T}{J - T}\frac{\partial^2}{\partial m'^2}\right]G(m, m', t - t') = 0, \tag{F.2}$$

with the boundary condition $G(m, m', 0) = \delta(m - m')$. Its solution in terms of m' has the Gaussian form

$$G(m, m', t) = A(m, t)\sqrt{\frac{N}{2\pi D(m, t)}}\exp\left\{-\frac{N[m' - \mu'(m, t)]^2}{2D(m, t)}\right\}, \tag{F.3}$$

where the coefficients μ', D and A should be found by insertion into (F.2).

As in § 7.2.3, the evolution of $P_{\rm M}(m, t)$ takes place in three stages: (i) *widening* of the initial distribution, which here takes place over the bifurcation $-m_{\rm B}$; (ii) *drift* on both sides of $-m_{\rm B}$ towards $+m_{\rm F}$ and $-m_{\rm F}$; (iii) narrowing around $+m_{\rm F}$ and $-m_{\rm F}$ of the two final peaks, which evolve *separately towards equilibrium*. We are interested here only in the first two stages. During the first stage, the relevant values of m lie in the region where the approximation (7.59) holds. The functions of m and t: μ', D and A, satisfy according to (F.2) the equations

$$\tau_{\rm reg}\frac{\partial\mu'}{\partial t} = -m_{\rm B} - \mu', \qquad \frac{1}{2}\tau_{\rm reg}\frac{\partial D}{\partial t} = \frac{T}{J - T} - D, \qquad \tau_{\rm reg}\frac{\partial A}{\partial t} = -A, \tag{F.4}$$

and the boundary condition $G(m, m', 0) = \delta(m - m')$ for $t' = t - 0$ yields

$$\mu' = -m_{\rm B} + (m + m_{\rm B})e^{-t/\tau_{\rm reg}}, \qquad D = \frac{T}{J - T}(1 - e^{-t/\tau_{\rm reg}}), \qquad A = e^{-t/\tau_{\rm reg}}. \tag{F.5}$$

As function of m, the probability

$$P_{\rm M}(m, t) = \int {\rm d}m' G(m, m', t)P_{\rm M}(m', 0) \tag{F.6}$$

given by (F.3), (F.5) involves fluctuations which increase exponentially as $\exp(t/\tau_{\rm reg})$.

In the second stage, the time is sufficiently large so that $P_{\rm M}(m, t)$ extends over regions of m where the linear approximation (7.59) for $v(m)$ fails; we must account for the decrease of $|v(m)|$, which vanishes at $m = \pm m_F$. We therefore cannot comply directly with the boundary condition for $G(m, m', t - t')$ at $t' = t$ since it requires m' to be large as m. However, during this second stage $P_{\rm M}(m, t)$ is not peaked, so that diffusion is negligible compared to drift. The corresponding Green's function, with its two times t and t' taken during this stage, is given according to (7.32) by

$$G(m, m', t - t') = \frac{1}{v(m)}\delta\left(t - t' - \int_{m'}^{m}\frac{{\rm d}m''}{v(m'')}\right). \tag{F.7}$$

[98] Whether bitten by a dog or a cat, the result is the same

We can now match the final time of (F.3), (F.5) with the initial time of (F.7), using the convolution law for Green's functions. This yields an approximation for $G(m, m', t)$ valid up to the final equilibration stage. We therefore define the function $\mu'(m, t)$ by the equation

$$t = \int_{\mu'(m,t)}^{m} \frac{\mathrm{d}m''}{v(m'')}, \tag{F.8}$$

of which (F.5) is the approximation for small m and μ'. For $m > -m_{\mathrm{B}}$, we have $m > \mu' > -m_{\mathrm{B}}$ and $v(m'') > 0$; for $m < -m_{\mathrm{B}}$ we have $m < \mu' < -m_{\mathrm{B}}$ and $v(m'') < 0$. We also note that the convolution replaces $A = e^{-t/\tau_{\mathrm{reg}}}$ by

$$A(m, t) = \frac{v[\mu'(m, t)]}{v(m)} = \frac{\partial \mu'(m, t)}{\partial m}. \tag{F.9}$$

Altogether the Green's function (F.3) reads

$$G(m, m', t) = \frac{v(\mu')}{v(m)} \sqrt{\frac{N(J-T)}{2\pi T(1-e^{-2t/\tau_{\mathrm{reg}}})}} \exp\left[-\frac{N(J-T)(m'-\mu')^2}{2T(1-e^{-2t/\tau_{\mathrm{reg}}})}\right], \tag{F.10}$$

where $\mu' = \mu'(m, t)$ is found through (F.8). The resulting distribution function $P_{\mathrm{M}}(m, t)$, obtained from (F.6), (F.10) and $P_{\mathrm{M}}(m, 0) \propto \exp[-N(m-\mu_0)^2/2\delta_0^2]$, is expressed by (F.10) or, in the main text, by (7.61) with (7.63) for $\delta_1(t)$. Notice that here we allowed for a finite value μ_0 of the average magnetization in the initial state.

We have studied in § 7.3.2 the evolution of $P_{\mathrm{M}}(m, t)$ for $g = 0$ and for an unbiased initial state. For $m_{\mathrm{B}} = g/(J-T) \neq 0$ and a non-vanishing expectation value of μ_0 of m in the initial state, the dynamics of $P_{\mathrm{M}}(m, t)$ is explicitly found from (F.10) by noting that $m_{\mathrm{B}} \ll m_{\mathrm{F}}$; the expression (E.1) for $v(m)$ thus reduces to

$$\frac{1}{\tau_{\mathrm{reg}} v(m)} = \frac{1}{m + m_{\mathrm{B}}} + \frac{2am}{m_{\mathrm{F}}^2 - m^2}, \tag{F.11}$$

with $\tau_{\mathrm{reg}} = \hbar/\gamma(J-T)$ and a defined by (E.3). Hence, the relation (F.8) between μ', m and t reads

$$\frac{t}{\tau_{\mathrm{reg}}} = \ln \frac{m + m_{\mathrm{B}}}{\mu' + m_{\mathrm{B}}} + a \ln \frac{m_{\mathrm{F}}^2 - \mu'^2}{m_{\mathrm{F}}^2 - m^2}. \tag{F.12}$$

For large N, the quantities μ', m_0 and m_{B} are small as $1/\sqrt{N}$, except at the very large times when $P_{\mathrm{M}}(m, t)$ is concentrated near $+m_{\mathrm{F}}$ and $-m_{\mathrm{F}}$. We can thus write (7.60) as

$$P_{\mathrm{M}}(m, t) = \frac{1}{\sqrt{\pi}} \frac{\partial \xi}{\partial m} e^{-(\xi - \xi_0)^2}, \tag{F.13}$$

where we introduced the functions

$$\xi(m, t) = \sqrt{3a} \frac{m + m_{\mathrm{B}}}{m_{\mathrm{F}}} \left(\frac{m_{\mathrm{F}}^2}{m_{\mathrm{F}}^2 - m^2}\right)^a \frac{\delta_1}{\delta_1(t)} e^{-(t - \tau_{\mathrm{flat}})/\tau_{\mathrm{reg}}}, \tag{F.14}$$

$$\xi_0(t) \equiv \sqrt{\frac{N}{2}} \frac{m_{\mathrm{B}} + \mu_0}{\delta_1(t)}. \tag{F.15}$$

The characteristic time τ_{flat} is the same as (7.69), it is large as $\frac{1}{2} \ln N$. The function $\delta_1(t)$ and the parameter δ_1 are defined in (7.63)

The expression (F.13) encompasses (7.64), (7.70), (7.74) and (7.79), which were established in the special case where the distribution is symmetric ($m_{\mathrm{B}} = \mu_0 = 0$) and/or when m is small as $1/\sqrt{N}$. For $t \gg \tau_{\mathrm{reg}}$ we reach Suzuki's scaling regime characterized by the scaling parameter (F.14), in which $\delta_1(t)$ reduces to the constant δ_1 and in which m_{B} can be disregarded. The asymmetry of $P_{\mathrm{M}}(m, t)$ then arises only from the constant ξ_0. Even in the presence of this assymetry, the time $t = \tau_{\mathrm{flat}}$ still corresponds to a flat $P_{\mathrm{M}}(m, t)$, in the sense that the curvature of $P_{\mathrm{M}}(m, \tau_{\mathrm{flat}})$ at $m = 0$ vanishes.

G. Density operators for beginners

In the beginning was the Word
Genesis 1.1

In elementary courses of quantum mechanics, a state is usually represented by a vector $|\psi\rangle$ in Hilbert space (or a ket, or a wave function). Such a definition is too restrictive. On the one hand, as was stressed by Landau, if the considered system is not isolated and presents quantum correlations with another system, its properties cannot be described by means of a state vector. On the other hand, as was stressed by von Neumann, an incomplete preparation does not allow us to assign a unique state vector to the system; various state vectors are possible, with some probabilities, and the formalism of quantum statistical mechanics is needed. Both of these circumstances occur in a measurement process: The tested system is correlated to the apparatus, and the apparatus is macroscopic. The opinion, too often put forward, that the (mixed) post-measurement state cannot be derived from the Schrödinger equation, originates from the will to work in the restricted context of pure states. This is why we should consider, to understand quantum measurement processes, the realistic case of a mixed initial state for the apparatus, and subsequently study the time-dependent mixed state for the tested system and the apparatus.

In the more general formulation of quantum mechanics that is needed, a state is represented by a *density operator* $\hat{\mathcal{D}}$ or, in a basis $|i\rangle$ of the Hilbert space, by a *density matrix* $\langle i|\hat{\mathcal{D}}|j\rangle$. The *expectation value* in this state of an observable $\hat{O}$ (itself represented on the basis $|i\rangle$ by the matrix $\langle i|\hat{O}|j\rangle$) is equal to

$$\langle\hat{O}\rangle = \mathrm{tr}\,\hat{\mathcal{D}}\hat{O} = \sum_{ij}\langle i|\hat{\mathcal{D}}|j\rangle\langle j|\hat{O}|i\rangle. \tag{G.1}$$

This concept encompasses as a special case that of state vector, as the expectation value of $\hat{O}$ in the state $|\psi\rangle$,

$$\langle\hat{O}\rangle = \langle\psi|\hat{O}|\psi\rangle = \sum_{ij}\langle\psi|j\rangle\langle j|\hat{O}|i\rangle\langle i|\psi\rangle, \tag{G.2}$$

is implemented by associating with $|\psi\rangle$ the density operator $\hat{\mathcal{D}} = |\psi\rangle\langle\psi|$ or the density matrix $\langle i|\hat{\mathcal{D}}|j\rangle = \langle i|\psi\rangle\langle\psi|j\rangle$, referred to as a "pure state" in this context.

Density operators have several characteristic properties. (i) They are *Hermitean*, $\hat{\mathcal{D}} = \hat{\mathcal{D}}^\dagger$, (i. e., $\langle j|\hat{\mathcal{D}}|i\rangle = \langle i|\hat{\mathcal{D}}|j\rangle^*$), implying that the expectation value (G1) of a Hermitean observable is real. (ii) They are *normalized*, $\mathrm{tr}\,\hat{\mathcal{D}} = 1$, meaning that the expectation value of the unit operator is 1. (iii) They are *non-negative*, $\langle\phi|\hat{\mathcal{D}}|\phi\rangle \geq 0\ \forall\,|\phi\rangle$, meaning that the variance $\langle\hat{O}^2\rangle - \langle\hat{O}\rangle^2$ of any Hermitean observable $\hat{O}$ is non-negative. A density operator can be diagonalized; its eigenvalues are real, non negative, and sum up to 1. For a pure state $\hat{\mathcal{D}} = |\psi\rangle\langle\psi|$, all eigenvalues vanish but one, equal to 1.

In the Schrödinger picture, the evolution of the time-dependent density operator $\hat{\mathcal{D}}(t)$ is governed by the Hamiltonian H of the system if it is isolated. The *Liouville–von Neumann equation of motion*,

$$i\hbar\frac{\mathrm{d}\hat{\mathcal{D}}(t)}{\mathrm{d}t} = [\hat{H}, \hat{\mathcal{D}}(t)], \tag{G.3}$$

generalizes the Schrödinger equation $i\hbar\mathrm{d}|\psi\rangle/\mathrm{d}t = \hat{H}|\psi\rangle$, or, in the position basis, $i\hbar\mathrm{d}\psi(x)/\mathrm{d}t = \hat{H}\psi(x)$, which governs the motion of pure states. The evolution of $\hat{\mathcal{D}}(t)$ is unitary; it conserves its eigenvalues.

In quantum statistical mechanics, the *von Neumann entropy*

$$S(\hat{\mathcal{D}}) = -\mathrm{tr}\hat{\mathcal{D}}\ln\hat{\mathcal{D}} \tag{G.4}$$

is associated with $\hat{\mathcal{D}}$. It characterizes the amount of information about the system that is missing when it is described by $\hat{\mathcal{D}}$, the origin of values of S being chosen as $S = 0$ for pure states. If $S(\hat{\mathcal{D}}) \neq 0$, $\hat{\mathcal{D}}$ can be decomposed in an infinite number of ways into a sum of projections onto pure states (§ 10.1.5).

The concept of density operator allows us to define the *state of a subsystem*, which is not feasible in the context of state vectors or pure states. Consider a compound system $S_1 + S_2$, described in the Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$ by a density operator $\hat{\mathcal{D}}$. This state is represented, in the basis $|i_1, i_2\rangle$ of $\mathcal{H}_1 \otimes \mathcal{H}_2$, by the density matrix $\langle i_1, i_2|\hat{\mathcal{D}}|j_1, j_2\rangle$. Suppose we wish to describe the subsystem S_1 alone, that is, to evaluate the expectation values of the observables O_1 pertaining only to the Hilbert space $\mathcal{H}_1$ and thus represented by matrices $\langle i_1|O_1|j_1\rangle$ in $\mathcal{H}_1$, or $\langle i_1|O_1|j_1\rangle\delta_{i_2,j_2}$ in $\mathcal{H}_1 \otimes \mathcal{H}_2$. These expectation values are given by

$$\langle \hat{O}_1 \rangle = tr_1 \hat{\mathcal{D}}_1 \hat{O}_1 = \sum_{i_1, j_1} \langle i_1|\hat{\mathcal{D}}_1|j_1\rangle\langle j_1|\hat{O}_1|i_1\rangle, \tag{G.5}$$

where the matrix $\langle i_1|\hat{\mathcal{D}}_1|j_1\rangle$ in the Hilbert space $\mathcal{H}_1$ is defined by

$$\langle i_1|\hat{\mathcal{D}}_1|j_1\rangle = \sum_{i_2} \langle i_1, i_2|\hat{\mathcal{D}}|j_1, i_2\rangle. \tag{G.6}$$

The partial trace $\hat{\mathcal{D}}_1 = \mathrm{tr}_2\hat{\mathcal{D}}$ on the space $\mathcal{H}_2$ is therefore, according to (G1), the density operator of the subsystem S_1. If the subsystems S_1 and S_2 interact, the evolution of $\hat{\mathcal{D}}_1$ should in principle be determined by solving (G3) for the the density operator $\hat{\mathcal{D}}$ of the compound system, then by taking the partial trace at the final time. The elimination of the bath (subsection 4.1) followed this procedure. The evolution of a subsystem is in general not unitary, because it is not an isolated system.

The formalism of density operators is more flexible than that of pure states: It affords the possibility not only of changing the basis in the Hilbert space, but also of performing linear transformations in the vector space of observables, which mix the left and right indices of observables $\langle i|\hat{O}|j\rangle$ and of density matrices $\langle i|\hat{\mathcal{D}}|j\rangle$. The resulting *Liouville representations of quantum mechanics* [64, 285, 286] are useful in many circumstances. They include for instance the *Wigner representation*, suited to study the semi-classical limit, and the *polarization representation* for a spin, currently used by experimentalists, in which any operator is represented by its coordinates on the basis (3.1) of the space of operators; in the present work, the parametrization of the state $\hat{D}$ of S + M by $P_{\mathrm{M}}^{\mathrm{dis}}(m)$ and $C_a^{\mathrm{dis}}(m)$ enters this framework (Eqs. (3.18), (3.26), (3.28), (3.29)).