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Understanding quantum measurement from the solution of dynamical models

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Abstract

The quantum measurement problem, to wit, understanding why a unique outcome is obtained in each individual experiment, is currently tackled by solving models. After a general introduction to the subject, we review the many dynamical models proposed over the years for elucidating quantum measurements. The approaches range from standard quantum theory, relying for instance on quantum statistical mechanics or on decoherence, to quantum-classical methods, to consistent histories and to modifications of the quantum equations. Next, a flexible and rather realistic quantum model is introduced, describing the measurement of the z-component of a spin through interaction with a magnetic memory simulated by a Curie–Weiss magnet, including $N \gg 1$ spins weakly coupled to a phonon bath. Initially prepared in a metastable paramagnetic state, it may transit to either its up or down ferromagnetic state, triggered by its coupling with the tested spin, so that its magnetization acts as a pointer. A detailed solution of the dynamical equations is worked out, exhibiting several time scales. Conditions on the parameters of the model are found, which ensure that the process satisfies all the features of ideal measurements. Various imperfections of the measurement due to violations of some conditions are discussed, as well as attempts of incompatible measurements. The first steps consist in the solution of the Hamiltonian dynamics for the spin-apparatus density matrix $\hat{D}(t)$. Its off-diagonal blocks in a basis selected by the spin-pointer coupling, rapidly decay owing to the many degrees of freedom of the pointer. Recurrences are ruled out either by some randomness of that coupling, or by the interaction with the bath. On a longer time scale, the trend towards equilibrium of the magnet produces a final state $\hat{D}(t_f)$ that involves correlations between the system and the indications of the pointer, thus ensuring registration. Although $\hat{D}(t_f)$ has the form expected for ideal measurements, it only describes a large set of runs, whereas consideration of individual runs is needed to elucidate the quantum measurement problem. This question is approached within a specified version of the statistical interpretation. There the difficulty lies in a quantum ambiguity: There exist many incompatible decompositions of the density matrix $\hat{D}(t_f)$ into a sum of sub-matrices, so that one cannot infer from its sole determination the states that would describe small subsets of runs. This difficulty is overcome by a dynamical mechanism based on suitable interactions within the apparatus, which produce a special combination of relaxation and decoherence associated with the broken invariance of the pointer. Any subset of runs thus reaches over a brief delay a stable state which satisfies the same hierarchic property as in classical probability theory; the reduction of the state for each individual run follows. Standard quantum statistical mechanics alone appears sufficient to explain the occurrence of a unique answer in each run and the emergence of classicality in a measurement process. Finally, pedagogical exercises are proposed and lessons for future works on models are suggested, while the statistical interpretation is promoted for teaching.

Keywords: quantum measurement problem, statistical interpretation, apparatus, pointer, dynamical models, ideal and imperfect measurements, collapse of the wavefunction, decoherence, truncation, reduction, registration

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Dedicated to our teachers and inspirers Nico G. van Kampen and Albert Messiah

Chi va piano va sano; chi va sano va lontano¹ Italian saying

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 $^{^1\}ensuremath{\mathsf{Who}}$ goes slowly goes safely; who goes safely goes far

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1. General features of quantum measurements

2	For this thing is too heavy for thee,
3	thou art not able to perform it thyself alone
4	Exodus 18.18

In spite of a century of progress and success, quantum mechanics still gives rise to passionate discussions about its 5 interpretation. Understanding quantum measurements is an important issue in this respect, since measurements are a 6 priviledged means to grasp the microscopic physical quantities. Two major steps in this direction were already taken 7 in the early days. In 1926, Born gave the expression of the probabilities² of the various possible outcomes of an ideal 8 quantum measurement [1]. In 1927 Heisenberg conceived the first models of quantum measuruments [2, 3] that were 9 five years later extended and formalized by von Neumann [4]. Since then, many theorists have worked out models of 10 quantum measurements, with the aim of understanding not merely the dynamics of such processes, but in particular 11 solving the so-called measurement problem. This problem is raised by a conceptual contrast between quantum theory, 12 which is irreducibly probabilistic, and our macroscopic experience, in which an individual process results in a well 13 defined outcome. If a measurement is treated as a quantum physical process, in which the tested system interacts 14 with an apparatus, the superposition principle seems to preclude the occurrence of a unique outcome, whereas each 15 single run of a quantum measurement should yield a unique result. The challenge has remained to fully explain how 16 this property emerges, ideally without introducing new ingredients, that is, from the mere laws of quantum mechanics 17 alone. Many authors have tackled this deep problem of measurements with the help of models so as to get insight 18 on the interpretation of quantum mechanics. For historical overviews of the respective steps in the development of 19 the theory and its interpretation, see the books by Jammer [5, 6] and by Mehra and Rechenberg [7]. The tasks we 20 undertake in this paper are first to review these works, then to solve in full detail a specific family of dynamical models 21 and to finally draw conclusions from their solutions. 22

²Born wrote: "Will man dieses Resultat korpuskular umdeuten, so ist nur eine Interpretation möglich: $\Phi_{n,m}(\alpha,\beta,\gamma)$ bestimmt die Wahrscheinlichkeit¹) dafür, daß das aus der z-Richtung kommende Elektron in die durch α, β, γ bestimmte Richtung [...] geworfen wird", with the footnote: "1) Anmerkung bei der Korrektur: Genauere Überlegung zeigt, daß die Wahrscheinlichkeit dem Quadrat der Größe Φ_{nm} proportional ist". In translation from Wheeler and Zurek [8]: "Only one interpretation is possible: $\Phi_{n,m}$ gives the probability¹) for the electron ...", and the footnote: "1) Addition in proof: More careful consideration shows that the probability is proportional to the square of the quantity $\Phi_{n,m}$."

23 1.1. Measurements and interpretation of quantum mechanics

25

26

When you can measure what you are speaking about, and express it in numbers, you know something about it Lord Kelvin

Few textbooks of quantum mechanics dwell upon questions of interpretation or upon quantum measurements, in 27 spite of their importance in the comprehension of the theory. Generations of students have therefore stumbled over 28 the problem of measurement, before leaving it aside when they pursued research work. Most physicists have never 29 returned to it, considering that it is not worth spending time on a problem which "probably cannot be solved" and 30 which has in practice little implication on physical predictions. Such a fatalistic attitude has emerged after the efforts of 31 32 the brightest physicists, including Einstein, Bohr, de Broglie, von Neumann and Wigner, failed to lead to a universally accepted solution or even viewpoint; see for reviews [4, 8, 9, 10, 11, 12, 13, 14]. However, the measurement problem 33 has never been forgotten, owing to its intimate connection with the foundations of quantum mechanics, which it may 34 help to formulate more sharply, and owing to its philosophical implications. 35

In this review we shall focus on the simplest measurements, ideal projective measurements [1], and shall consider non-idealities and unsuccessful processes only occasionally and in section 8. While standard courses deal mainly with this type of measurement, it is interesting to mention that the first experiment based on a nearly ideal measurement was carried out only recently [15]. An optical analog of a von Neumann measurement has been proposed too [16].

Experimentalists meet the theoretical discussions about quantum measurements with a feeling of speaking differ-40 ent languages. While theorists ponder about the initial pure state of the apparatus, the collapse of its wave packet 41 and the question "when and in which basis does this collapse occur" and "how does this collapse agree with the 42 Schrödinger equation", experimentalists deal with different issues, such as choosing an appropriate apparatus for the 43 desired experiment or stabilizing it before the measurement starts. If an experimentalist were asked to describe one 44 cubic nanometer of his apparatus in theoretical terms, he would surely start with a quantum mechanical approach. 45 But this raises the question whether it is possible to describe the whole apparatus, and also its dynamics, i. e., the 46 dynamics and outcome of the measurement, by quantum mechanics itself. It is this question that we shall answer 47 positively in the present work, thus closing the gap between what experimentalists intuitively feel and the formulation 48 of the theory of ideal quantum measurements. To do so, we shall consider models that encompass the points relevant 49 to experimentalists. 50

As said above, for theorists there has remained another unsolved paradox, even deeper than previous ones, the so-51 called quantum measurement problem: How can quantum mechanics, with its superposition principle, be compatible 52 with the fact that each individual run of a quantum measurement yields a well-defined outcome? This uniqueness 53 is at variance with the description of the measurement process by means of a pure state, the evolution of which is 54 governed by the Schrödinger equation. Many workers believe that the quantum measurement problem cannot be answered within quantum mechanics. Some of them then hope that a hypothetical "sub-quantum theory", more basic 56 than standard quantum mechanics, might predict what happens in individual systems [17, 18, 19, 20]. Our purpose 57 is, however, to prove that the probabilistic framework of quantum mechanics is sufficient to explain that the outcome 58 of a single measurement is unique although unpredictable within this probabilistic framework (section 11). We thus 59 wish to show that quantum theory not only predicts the probabilities for the various possible outcomes of a set of 60 measurements – as a minimalist attitude would state – but also accounts for the uniqueness of the result of each run. 61 A measurement is the only means through which information may be gained about a physical system S [4, 8, 9, 10, 62 11, 12, 13, 14, 21, 22]. Both in classical and in quantum physics, it is a dynamical process which couples this system 63

Figure 11, 12, 13, 14, 21, 22]. Both in classical and in quantum physics, it is a dynamical process which couples this system S to another system, the apparatus A. Some correlations are thereby generated between the initial (and possibly final) state of S and the final state of A. Observation of A, in particular the value indicated by its pointer, then allows us to gain by inference some quantitative information about S. A measurement thus involves, in one way or another, the observers³. It also has statistical features, through unavoidable uncertainties and, more deeply, through the irreducibly probabilistic nature of our description of quantum systems.

Throughout decades many thoughts were therefore devoted to quantum measurements in relation to the interpretation of quantum theory. Both Einstein [23] and de Broglie [24] spent much time on such questions after their first

³We shall make the case that observation itself does not influence the outcome of the quantum measurement

discovery; the issue of quantum measurements was formulated by Heisenberg [2, 3] and put in a mathematically pre-71 cise form by von Neumann [4]; the foundations of quantum mechanics were reconsidered in this light by people like 72 Bohm [18, 19] or Everett [25, 26] in the fifties; hidden variables were discussed by Bell in the sixties [27]; the use of a 73 statistical interpretation to analyze quantum measurements was then advocated by Park [28], Blokhintsev [10, 11] and 74 Ballentine [9] (subtleties of the statistical interpretation are underlined by Home and Whitaker [29]); the most relevant 75 papers were collected by Wheeler and Zurek in 1983 [8]. Earlier reviews on this problem were given by London and 76 Bauer [30] and Wigner [13]. We can presently witness a renewed interest for measurement theory; among many recent 77 contributions we may mention the book of de Muynck [31] and the review articles by Schlosshauer [32] and Zurek 78 [33]. Extensive references are given in the pedagogical article [34] and book [35] by Laloë which review paradoxes 79 and interpretations of quantum mechanics. Indeed, these questions have escaped the realm of speculation owing to 80 progresses in experimental physics which allow to tackle the foundations of quantum mechanics from different an-81 gles. Not only Bell's inequalities [27, 34, 36] but also the Greenberger–Horne–Zeilinger (GHZ) logical paradox [37] 82 have been tested experimentally [38]. Moreover, rather than considering cases where quantum interference terms (the 83 infamous "Schrödinger cat problem" [8, 13, 39]) vanish owing to decoherence processes [40], experimentalists have 84 become able to control these very interferences [41], which are essential to describe the physics of quantum superpo-85 sitions of macroscopic states and to explore the new possibilities offered by quantum information [22, 42]. Examples 86 include left and right going currents in superconducting circuits [15, 43, 44, 45], macroscopic atomic ensembles [41] 87 and entangled mechanical oscillators [46]. 88

89 1.1.1. Quantum versus classical measurements

90 91 When the cat and the mouse agree, the grocer is ruined Iranian proverb

The difficulties arise from two major differences between quantum and classical measurements, stressed in most textbooks [4, 3, 47, 48].

(*i*) In classical physics it was legitimate to imagine that quantities such as the position and the momentum of a
 structureless particle like an electron might in principle be measured with increasingly large precision; this allowed
 us to regard both of them as well-defined physical quantities. (We return in section 10 to the meaning of physical
 quantities and of states within the statistical interpretation of quantum mechanics.) This is no longer true in quantum mechanics, where we cannot forget about the random nature of physical quantities. Statistical fluctuations are
 unavoidable, as exemplified by Heisenberg's inequality [2, 3]: we cannot even think of values that would be taken
 simultaneously by non-commuting quantities whether or not we measure them. In general both the theory and the

Consider a measurement of an observable \hat{s} of the system S of interest⁴, having eigenvectors $|s_i\rangle$ and eigenvalues s_i . It is an experiment in which S interacts with an apparatus A that has the following property [4, 13, 30, 47]. A physical quantity \hat{A} pertaining to the apparatus A may take at the end of the process one value among a set A_i which are in one-to-one correspondence with s_i . If initially S lies in the state $|s_i\rangle$, the final value A_i will be produced with certainty, and a repeated experiment will always yield the observed result A_i , informing us that S was in $|s_i\rangle$ However, within this scope, S should generally lie initially in a state represented by a wave function which is a linear combination,

$$|\psi\rangle = \sum_{i} \psi_i |s_i\rangle, \qquad (1.1)$$

of the eigenvectors $|s_i\rangle$. Born's rule then states that the probability of observing in a given experiment the result A_i equals $|\psi_i|^2$ [1]. An axiomatic derivation of the Born's rule is given in [50]; see [32, 33] for a modern perspective on the rule. Quantum mechanics does not allow us to predict which will be the outcome A_i of an individual measurement, but provides us with the full statistics of repeated measurements of \hat{s} performed on elements of an ensemble described by the state $|\psi\rangle$. The frequency of occurrence of each A_i in repeated experiments informs us about the moduli $|\psi_i|^2$, but not about the phases of these coefficients. In contrast to a classical state, a quantum state $|\psi\rangle$, even pure, always refers to an ensemble, and cannot be determined by means of a unique measurement performed on a single system [49]. It

⁴The eigenvalues of \hat{s} are assumed here to be non-degenerate. The general case will be considered in § 1.2.3

cannot even be fully determined by repeated measurements of the single observable \hat{s} , since only the values of the amplitudes $|\psi_i|$ can thus be estimated.

(ii) A second qualitative difference between classical and quantum physics lies in the *perturbation of the system* S 117 brought in by a measurement. Classically one may imagine that this perturbation could be made weaker and weaker, 118 so that S is practically left in its initial state while A registers one of its properties. However, a quantum measurement 119 is carried on with an apparatus A much larger than the tested object S; an extreme example is provided by the huge 120 detectors used in particle physics. Such a process may go so far as to destroy S, as for a photon detected in a 121 photomultiplier. It is natural to wonder whether the perturbation of S has a lower bound. Much work has therefore 122 been devoted to the *ideal measurements*, those which preserve at least the statistics of the observable \hat{s} in the final 123 state of S, also referred to as non-demolition experiments or as measurements of the first kind [31]. Such ideal 124 measurements are often described by assuming that the apparatus A starts in a pure state⁵. Then by writing that, if S 125 lies initially in the state $|s_i\rangle$ and A in the state $|0\rangle$, the measurement leaves S unchanged: the compound system S + A 126 evolves from $|s_i\rangle |0\rangle$ to $|s_i\rangle |A_i\rangle$, where $|A_i\rangle$ is an eigenvector of \hat{A} associated with A_i . If however, as was first discussed 127 by von Neumann, the initial state of S has the general form (1.1), S + A may reach any possible final state $|s_i\rangle |A_i\rangle$ 128 depending on the result A_i observed. In this occurrence the system S is left in $|s_i\rangle$ and A in $|A_i\rangle$, and according to 129 Born's rule, this occurs with the probability $|\psi_i|^2$. As explained in § 1.1.4, for this it is necessary (but not sufficient) to 130 require that the final density operator describing S + A for the whole set of runs of the measurement has the diagonal 131 form 132

$$\sum_{i} |s_{i}\rangle |A_{i}\rangle |\psi_{i}|^{2} \langle A_{i}| \langle s_{i}|, \qquad (1.2)$$

Ashes to ashes, dust to dust

Genesis 3:19

rather than the full form (1.3) below. Thus, not only is the state of the apparatus modified in a way controlled by the object, as it should in any classical or quantal measurement that provides us with information on S, but the marginal state of the quantum system is also necessarily modified (it becomes $\sum_{i} |s_i\rangle |\psi_i|^2 \langle s_i|$), even by such an ideal measurement (except in the trivial case where (1.1) contains a single term, as it happens when one repeats the measurement).

137 1.1.2. Truncation versus reduction

The rules of quantum measurements that we have recalled display a well known contradiction between the principles of quantum mechanics. On the one hand, if the measurement process leads the initial pure state $|s_i\rangle|0\rangle$ into $|s_i\rangle|A_i\rangle$, the linearity of the wave functions of the compound system S + A and the unitarity of the evolution of the wave functions of S + A governed by the Schrödinger equation imply that the final density operator of S + A issued from (1.1) should be⁵

$$\sum_{ij} |s_i\rangle |A_i\rangle \psi_i \psi_j^* \langle A_j | \langle s_j |.$$
(1.3)

On the other hand, according to Born's rule [1] and von Neumann's analysis [4], each run of an ideal measurement should lead from the initial pure state $|\psi\rangle |0\rangle$ to one or another of the pure states $|s_i\rangle |A_i\rangle$ with the probability $|\psi_i|^2$; the final density operator accounting for a large statistical ensemble \mathcal{E} of runs should be the mixture (1.2) rather than the superposition (1.3). In the orthodox Copenhagen interpretation, two separate postulates of evolution are introduced, one for the hamiltonian motion governed by the Schrödinger equation, the other for measurements which lead the system from $|\psi\rangle$ to one or the other of the states $|s_i\rangle$, depending on the value A_i observed. This lack of consistency is unsatisfactory and other explanations have been searched for (§ 1.3.1 and section 2).

It should be noted that the loss of the off-diagonal elements takes place in a well-defined basis, the one in which both the tested observable \hat{s} of S and the pointer variable \hat{A} of A are diagonal (such a basis always exists since the joint Hilbert space of S + A is the tensor product of the spaces of S and A). In usual decoherence processes, it is

¹³⁸

¹³⁹

⁵ Here we follow a current line of thinking in the literature called von Neumann-Wigner theory of ideal measurements. In subsection 1.2 we argue that it is not realistic to assume that A may start in a pure state and end up in a pure state

the interaction between the system and some external bath which selects the basis in which off-diagonal elements are truncated [32, 33]. We have therefore to elucidate this *preferred basis paradox*, and to explain why the truncation which replaces (1.3) by (1.2) takes place in the specific basis selected by the measuring apparatus.

The occurrence in (1.3) of the off-diagonal $i \neq j$ terms is by itself an essential feature of an interaction process between two systems in quantum mechanics. There exist numerous experiments in which a pair of systems is left after interaction in a state of the form (1.3), not only at the microscopic scale, but even for macroscopic objects, involving for instance quantum superpositions of superconducting currents. Such experiments allow us to observe purely quantum coherences represented by off-diagonal terms $i \neq j$.

However, such off-diagonal "Schrödinger cat" terms, which contradict both Born's rule [1] and von Neumann's 163 reduction [4], must disappear at the end of a measurement process. Their absence is usually termed as the "reduction" 164 or the "collapse" of the wave packet, or of the state. Unfortunately, depending on the authors, these words may have 165 different meanings; we need to define precisely our vocabulary. Consider first a large set & of runs of a measurement 166 performed on identical systems S initially prepared in the state $|\psi\rangle$, and interacting with A initially in the state $|0\rangle$. 167 The density operator of S + A should evolve from $|\psi\rangle|0\rangle\langle0|\langle\psi||$ to (1.2). We will term as "truncation" the elimination 168 during the process of the off-diagonal blocks $i \neq j$ of the density operator describing the *joint system* S + A for the 169 whole set \mathcal{E} of runs. If instead of the full set \mathcal{E} we focus on a *single run*, the process should end up in one among 170 the states $|s_i\rangle |A_i\rangle$. We will designate as "reduction" the transformation of the initial state of S + A into such a final 171 "reduced state", for a single run of the measurement. 172

One of the paradoxes of the measurement theory lies in the existence of several possible final states issued from the same initial state. Reduction thus seems to imply a *bifurcation* in the dynamics, whereas the Schrödinger equation entails a one-to-one correspondence between the initial and final states of the isolated system S + A.

We stress that both above definitions refer to S + A. Some authors apply the words reduction or collapse to the *sole* 176 tested system S. To avoid confusion, we will call "weak reduction" the transformation of the initial state $|\psi\rangle\langle\psi|$ of S into 177 the pure state $|\psi_i\rangle\langle\psi_i|$ for a single run, and "weak truncation" its transformation into the mixed state $\sum_i |\psi_i\rangle\langle\psi_i|^2\langle\psi_i|$ 178 for a large ensemble \mathcal{E} of runs. In fact, the latter marginal density operator of S can be obtained by tracing out A, 179 not only from the joint truncated state (1.2) of S+A, but also merely from the non-truncated state (1.3), so that the 180 question seems to have been eluded. However, such a viewpoint, in which the apparatus is disregarded, cannot provide 181 an answer to the measurement problem. The very aim of a measurement is to create correlations between S and A 182 and to read the indications of A so as to derive indirectly information about S; but the elimination of the apparatus 183 suppresses both the correlations between S and A and the information gained by reading A. 184

Physically, a set of repeated experiments involving interaction of S and A can be regarded as a measurement only 185 if we observe on A in each run some well defined result A_i, with probability $|\psi_i|^2$. For an ideal measurement we 186 should be able to predict that S is then left in the corresponding state $|s_i\rangle$. Explaining these features requires that 187 the considered dynamical model produces in each run one of the reduced states $|s_i\rangle |A_i\rangle$. The quantum measurement 188 problem thus amounts to the proof, not only of truncation, but also of reduction. This will be achieved in section 11 for 189 a model of quantum statistical mechanics. As stressed by Bohr and Wigner, the reduction, interpreted as expressing 190 the "uniqueness of physical reality", is at variance with the superposition principle which produces the final state 191 (1.3). The challenge is to solve this contradiction, answering Wigner's wish: "The simplest way that one may try to 192 reduce the two kinds of changes of the state vector to a single kind is to describe the whole process of measurement 193 as an event in time, governed by the quantum mechanical equations of motion". Our purpose is to show that this is 194 feasible, contrary to Wigner's own negative conclusion [13]. 195

196 1.1.3. Selection of outcomes

197

198

Non-discrimination is a cross-cutting principle United Nations human rights, 1996

¹⁹⁹ When after a run of an ideal measurement, S is left in $|s_i\rangle$, a second measurement performed on the same system ²⁰⁰ leaves this state unchanged and yields the same indication A_i of the pointer. Reduction, even weak, thus implies ²⁰¹ *repeatability*. Conversely, repeatability implies weak truncation, that is, the loss in the marginal density of S of the ²⁰² elements $i \neq j$ during the first one of the successive measurement processes [52].

Apart from having been truncated, the final density operator (1.2) of S + A for the whole set \mathcal{E} of runs displays an essential feature, the complete correlation between the indication A_i of the pointer and the state $|s_i\rangle$ of S. We will term

as "registration" the establishment of these correlations. If they are produced, we can ascertain that, if the pointer 205 takes a well defined value A_i in some run, its observation will imply that \hat{s} takes with certainty the corresponding 206 eigenvalue s_i at the end of this run. Sorting the runs according to the outcome A_i allows us to split the ensemble 207 \mathcal{E} into subensembles \mathcal{E}_i , each one labelled by i and described by the state $|s_i\rangle|A_i\rangle^5$. Selection of the subensemble 208 \mathcal{E}_i by filtering the values A_i therefore allows us to set S into this subensemble \mathcal{E}_i described by the density operator 209 $|s_i\rangle|A_i\rangle\langle A_i|\langle s_i|$. It is then possible to sort the runs according to the indication A_i of the pointer. Selecting thus the 210 subensemble \mathcal{E}_i by filtering A_i allows us to set S into the given state $|s_i\rangle$ with a view to future experiments on S. An 211 ideal measurement followed by filtering can therefore be used as a preparation of the state of S [53]. We will make 212 the argument more precise in § 10.2.2 and § 11.3.2. 213

Note that some authors call "measurement" a single run of the experiment, or a repeated experiment in which the 214 occurrence of some given eigenvalue of \hat{s} is detected, and in which only the corresponding events are selected for the 215 outcoming system S. Here we use the term "measurement" to designate a repeated experiment performed on a large 216 ensemble of identically prepared systems which informs us about all possible values s_i of the observable \hat{s} of S, and 217 the term "ideal measurement" if the process perturbs S as little as allowed by quantum mechanics, in the sense that 218 it does not affect the statistics of the observables that commute with \hat{s} . We do not regard the sorting as part of the 219 measurement, but as a subsequent operation, and prefer to reserve the word "preparation through measurement" to 220 such processes including a selection. 221

1.2. The need for quantum statistical mechanics 222

223	<i>Om een paardendief te vangen heb je een paardendief nodig⁶</i>
224	Un coupable en cache un autre ⁷
225	Dutch and French proverbs

We wish for consistency to use quantum mechanics for treating the dynamics of the interaction process between the apparatus and the tested system. However, the apparatus must be a macroscopic object in order to allow the outcome to be read off from the final position of its pointer. The natural framework to reconcile these requirements is non-equilibrium quantum statistical mechanics, and not quantum mechanics of pure states as presented above. It will appear that not only the registration process can be addressed in this way, but also the truncation and the reduction.

1.2.1. Irreversibility of measurement processes

 232
 The first time ever I saw your face

233
 I thought the sun rose in your eyes

234
 Written by Ewan MacColl, sung by Roberta Flack

Among the features that we wish to explain, the *truncation* compels us to describe states by means of density operators. The sole use of pure states (quantum states describable by a wave function or a ket), is prohibited by the form of (1.2), which is in general a statistical mixture. Even if we start from a pure state $|\psi\rangle|0\rangle$, we must end up with the truncated mixed state (1.2) through an *irreversible* process. This irreversibility is also exhibited by the fact the same final state (1.2) is reached if one starts from different initial states of the form (1.1) deduced from one another through changes of the phases of the coefficients ψ_i . Such a feature is associated with the disappearance of the specifically quantum correlations between S and A described by the off-diagonal terms of (1.3).

Actually, there is a second cause of irreversibility in any effective measurement process. The apparatus A should 242 register the result A_i in a robust and permanent way, so that it can be read off by any observer. Such a registration, 243 which is often overlooked in the literature on measurements, is needed for practical reasons especially since we 244 wish to explore microscopic objects. Moreover, its very existence allows us to disregard the observers in quantum 245 measurements. Once the measurement has been registered, the result becomes objective and can be read off at any time 246 by any observer. It can also be processed automatically, without being read off. Registration requires an *amplification* 247 within the apparatus of a signal produced by interaction with the microscopic system S. For instance, in a bubble 248 chamber, the apparatus in its initial state involves a liquid, overheated in a metastable phase. In spite of the weakness 249

11

⁶To catch a horse thief, you need a horse thief ⁷One culprit hides another

of the interaction between the particle to be detected and this liquid, amplification and registration of its track can be achieved owing to local transition towards the stable gaseous phase. This stage of the measurement process thus requires an irreversible phenomenon. It is governed by the kinetics of bubble formation under the influence of the particle and implies a dumping of free energy. Similar remarks hold for photographic plates, photomultipliers and other types of detectors.

Since the amplification and the registration of the measurement results require the apparatus A to be a large object 255 so as to behave irreversibly, we must use quantum statistical mechanics to describe A. In particular, the above as-256 sumption that A lay initially in a pure state $|0\rangle$ was unrealistic – nevertheless this assumption is frequent in theoretical 257 works on measurements, see e.g. [25, 32, 33]. Indeed, preparing an object in a pure state requires controlling a 258 complete set of commuting observables, performing their measurement and selecting the outcome (§ 1.1.3). While 259 such operations are feasible for a few variables, they cannot be carried out for a macroscopic apparatus nor even for 260 a mesoscopic apparatus involving, say, 1000 particles. What the experimentalist does in a quantum measurement is 261 quite the opposite [10, 11, 3, 31]: rather than purifying the initial state of A, he lets it stabilize macroscopically by 262 controlling a few collective parameters such as the temperature of the apparatus. The adequate theoretical represen-263 tation of the initial state of A, which is a *mixed state*, is therefore a *density operator* denoted as $\hat{\mathcal{R}}(0)$. Using pure 264 states in thought experiments or models would require averaging so as to reproduce the actual situation (§ 10.2.3 and 265 § 12.1.4). Moreover the initial state of A should be *metastable*, which requires a sudden change of, e.g., temperature. 26 Likewise the final possible stable marginal states of A are not pure. As we know from quantum statistical physics, 267 each of them, characterized by the value of the pointer variable A_i that will be observed, should again be described 268 by means of a density operator $\hat{\mathcal{R}}_i$, and not by means of pure states $|A_i\rangle$ as in (1.3). Indeed, the number of state 269 vectors associated with a sharp value of the *macroscopic* pointer variable A_i is huge for any actual measurement: As 270 always for large systems, we must allow for small fluctuations, negligible in relative value, around the mean value 271 $A_i = \text{tr}_A \hat{\mathcal{R}}_i$. The fact that the possible final states $\hat{\mathcal{R}}_i$ are exclusive is expressed by $\text{tr}_A \hat{\mathcal{R}}_i \hat{\mathcal{R}}_i \simeq 0$ for $i \neq i$, which 272 implies 273

$$\hat{\mathcal{R}}_i \hat{\mathcal{R}}_i \to 0 \quad \text{for } N \to \infty \text{ when } i \neq j.$$
 (1.4)

²⁷⁴ In words, these macroscopic pointer states are practically orthogonal.

275 1.2.2. The paradox of irreversibility

276 277 La vida es sueño⁸ Calderón de la Barca

If we disregard the system S, the irreversible process leading A from the initial state $\hat{\mathcal{R}}(0)$ to one among the final 278 states $\hat{\mathcal{R}}_i$ is reminiscent of relaxation processes in statistical physics, and the measurement problem raises the same 279 type of puzzle as the paradox of irreversibility. In all problems of statistical mechanics, the evolution is governed at 280 the microscopic level by equations that are invariant under time-reversal: Hamilton or Liouville equations in classical 28 physics, Schrödinger, or Liouville-von Neumann equations in quantum physics. Such equations are reversible and 282 conserve the von Neumann entropy, which measures our missing information. Nevertheless we observe at our scale 283 an irreversibility, measured by an increase of macroscopic entropy. The explanation of this paradox, see, e.g., [54, 284 55, 56, 57, 58, 59], relies on the *large number* of microscopic degrees of freedom of thermodynamic systems, on 285 statistical considerations and on plausible assumptions about the *dynamics* and about the *initial state* of the system. 286

Let us illustrate these ideas by recalling the historic example of a classical gas, for which the elucidation of the paradox was initiated by Boltzmann [54, 55, 56]. The microscopic state of a set of *N* structureless particles enclosed in a vessel is represented at each time by a point $\xi(t)$ in the 6*N*-dimensional phase space, the trajectory of which is generated by Hamilton's equations, the energy *E* being conserved. We have to understand why, starting at the time t = 0 from a more or less arbitrary initial state with energy *E*, we always observe that the gas reaches at the final time t_f a state which macroscopically has the equilibrium properties associated with *N* and *E*, to wit, homogeneity and Maxwellian distribution of momenta – whereas a converse transformation is never seen in spite of the reversibility of the dynamics. As we are not interested in a single individual process but in generic features, we can resort to

statistical considerations. We therefore consider an initial macroscopic state S_{init} characterized by given values of the 295 (non uniform) densities of particles, of energy, and of momentum in ordinary space. Microscopically, S_{init} can be 296 realized by any point ξ_{init} lying in some volume Ω_{init} of phase space. On the other hand, consider the volume Ω_E in 297 phase space characterized by the total energy E. A crucial fact is that the immense majority of points ξ with energy 298 E have macroscopically the equilibrium properties (homogeneity and Maxwellian distribution): the volume Ω_{eq} of 299 phase space associated with equilibrium occupies nearly the whole volume $\Omega_{eq}/\Omega_E \simeq 1$. Moreover, the volume Ω_E 300 is enormously larger than Ω_{init} . We understand these properties by noting that the phase space volumes characterized 301 by some macroscopic property are proportional to the exponential of the thermodynamic entropy. In particular, the 302 ratio $\Omega_{eq}/\Omega_{init}$ is the exponential of the increase of entropy from S_{init} to S_{eq} , which is large as N. We note then that 303 Hamiltonian dynamics implies Liouville's theorem. The bunch of trajectories issued from the points $\xi(0)$ in Ω_{init} 304 therefore reach at the time t_f a final volume $\Omega_f = \Omega_{init}$ that occupies only a tiny part of Ω_E , but which otherwise is 305 expected to have nothing special owing to the complexity of the dynamics of collisions. Thus most end points $\xi(t_f)$ 306 of these trajectories are expected to be typical points of Ω_E , that is, to lie in the equilibrium region Ω_{eq} . Conversely, 307 starting from an arbitrary point of Ω_E or of Ω_{eq} , the probability of reaching a point that differs macroscopically 308 from equilibrium is extremely small, since such points occupy a negligible volume in phase space compared to the 309 equilibrium points. The inconceivably large value of Poincaré's recurrence time is also related to this geometry of 310 phase space asociated with the macroscopic size of the system. 31 The above argument has been made rigorous [54, 55, 56] by merging the dynamics and the statistics, that is, 312 by studying the evolution of the density in phase space, the probability distribution which encompasses the bunch of 313 trajectories evoked above. Indeed, it is easier to control theoretically the Liouville equation than to study the individual 314 Hamiltonian trajectories and their statistics for random initial conditions. The initial state of the gas is now described 315 by a non-equilibrium density in the 6N-dimensional phase space. Our full information about this initial state, or the 316 full order contained in it, is conserved by the microscopic evolution owing to the Liouville theorem. However, the 317 successive collisions produce correlations between larger and larger numbers of particles. Thus, while after some 318 time the gas reaches at the macroscopic scale the features of thermodynamic equilibrium, the initial order gets hidden 319 into microscopic variables, namely many-particle correlations, that are inaccessible. Because the number of degrees of 320 freedom is large – and it is actually gigantic for any macroscopic object – this order cannot be retrieved (except in some 321 exceptional controlled dynamical phenomena such as spin echoes [60, 61, 62, 63, 64, 65]). In any real situation, it is 322 therefore impossible to recover, for instance, a non-uniform density from the very complicated correlations created 323 during the relaxation process. For all practical purposes, we can safely keep track, even theoretically, only of the 324 correlations between a number of particles small compared to the total number of particles of the system: the exact 325 final density in phase space cannot then be distinguished from a thermodynamic equilibrium distribution. It is this 326 dropping of information about undetectable and ineffective degrees of freedom, impossible to describe even with the 327

³²⁸ largest computers, which raises the macroscopic entropy [54, 55, 56, 57, 58]. Such approximations can be justified ³²⁹ mathematically through limiting processes where $N \rightarrow \infty$.

Altogether, irreversibility can be derived rigorously for the Boltzmann gas under assumptions of smoothness and approximate factorization of the single particle density. The change of scale modifies qualitatively the properties of the dynamics, for all accessible times and for all accessible physical variables. The *emergence of an irreversible relaxation* from the reversible microscopic dynamics is a statistical phenomenon which becomes nearly deterministic owing to the large number of particles. We shall encounter similar features in quantum measurement processes.

1.2.3. Quantum measurements in the language of statistical physics

336

Now the whole earth was of one language and of one speech⁹

⁹Metaphorically, the discovery of quantum theory and the lack of agreement about its interpretation may be phrased in the next lines of Genesis 11 [66]: 2. And it came to pass, as they journeyed from the east, that they found a plane in the land of Shinar; and they dwelt there. 3. And they said one to another, Go to, let us make brick, and burn them throughly. And they had brick for stone, and slime had they for mortar. 4. And they said, Go to, let us build a city, and a tower whose top *may reach* unto heaven; and let us make us a name, lest we be scattered abroad upon the face of the whole earth. 5. And the Lord came down to see the city and the tower, which the children of men builded. 6. And the Lord said, Behold, the people *is* one, and they have all one language; and this they begin to do: and now nothing will be restrained from them, which they have imagined to do. 7. Go to, let us go down, and there confound their language, that they may not understand one another's speech. 8. So the Lord scattered them abroad from thence upon the face of all the earth: and from thence did the Lord scatter them abroad upon the face of all the earth:

Genesis 11:1

The theoretical description of a measurement process should be inspired by the above ideas. Actually, a measure-338 ment process looks like a relaxation process, but with several complications. On the one hand, the final stable state of 339 A is not unique, and the dynamical process can have *several possible outcomes* for A. In photodetection (the eye, a 340 photomultiplier), one just detects whether an avalanche has or not been created by the arrival of a photon. In a mag-341 netic dot, one detects the direction of the magnetization. The apparatus is therefore comparable to a material which, 342 in statistical physics, has a broken invariance and can relax towards one equilibrium phase or another, starting from 343 a single metastable phase. On the other hand, the evolution of A towards one among the final states $\hat{\mathcal{R}}_i$ characterized 344 by the variable A_i should be triggered by interaction with S, in a way depending on the initial microscopic state of S 345 and, for an ideal measurement, the final microscopic state of S should be *correlated* to the outcome A_i . Thus, contrary 346 to theories of standard relaxation processes in statistical physics, the theory of a measurement process will require a 347 simultaneous control of microscopic and macroscopic variables. In the coupled evolution of A and S which involves 348 truncation and registration, coarse graining will be adequate for A, becoming exact in the limit of a large A, but not 349 for S. Moreover the final state of S + A keeps *memory* of the initial state of S, at least partly. The very essence of a 350 measurement lies in this feature, whereas memory effects are rarely considered in standard relaxation processes. 351

³⁵² Denoting by $\hat{r}(0)$ and $\hat{\mathcal{R}}(0)$ the density operators of the system S and the apparatus A, respectively, before the ³⁵³ measurement, the initial state of S + A is characterized in the language of quantum statistical mechanics by the density ³⁵⁴ operator

$$\hat{\mathcal{D}}(0) = \hat{r}(0) \otimes \hat{\mathcal{R}}(0). \tag{1.5}$$

In the Schrödinger picture, where the wave functions evolve according to the Schrödinger equation while observables are time-independent, the density operator $\hat{D}(t) = \exp(-i\hat{H}t/\hbar)\hat{D}(0)\exp(i\hat{H}t/\hbar)$ of the compound system S + A evolves according to the Liouville-von Neumann equation of motion

$$i\hbar\frac{\mathrm{d}\hat{D}}{\mathrm{d}t} = [\hat{H},\hat{D}] \equiv \hat{H}\hat{D} - \hat{D}\hat{H},\tag{1.6}$$

where \hat{H} is the Hamiltonian of S + A including the interaction between S and A. By solving (1.6) with the initial condition (1.5), we find the expectation value $\langle \hat{A}(t) \rangle$ of any observable \hat{A} of S + A at the time t as tr[$\hat{D}(t)\hat{A}$] (see subsection 10.1 and Appendix G).

We first wish to show that, for an ideal measurement, the final density operator of S + A which represents the outcome af a large set \mathcal{E} of runs at the time t_f has the form

$$\hat{\mathcal{D}}(t_{\rm f}) = \sum_{i} \left(\hat{\Pi}_{i} \hat{r}(0) \, \hat{\Pi}_{i} \right) \otimes \hat{\mathcal{R}}_{i} = \sum_{i} p_{i} \hat{r}_{i} \otimes \hat{\mathcal{R}}_{i}, \tag{1.7}$$

where $\hat{\Pi}_i$ denotes the projection operator (satisfying $\hat{\Pi}_i \hat{\Pi}_j = \delta_{ij} \hat{\Pi}_i$) on the eigenspace s_i of \hat{s} in the Hilbert space of S, with $\hat{s} = \sum_i s_i \hat{\Pi}_i$ and $\sum_i \hat{\Pi}_i = \hat{I}$. (If the eigenvalue s_i is non-degenerate, $\hat{\Pi}_i$ is simply equal to $|s_i\rangle\langle s_i|$.) We have denoted by

$$\hat{r}_{i} = \frac{1}{p_{i}} \hat{\Pi}_{i} \hat{r}(0) \hat{\Pi}_{i}$$
(1.8)

the corresponding normalized projected state (which reduces to $|s_i\rangle\langle s_i|$ if s_i is non-degenerate), and by

$$p_i \equiv \mathrm{tr}_{\mathrm{S}} \hat{r}(0) \hat{\Pi}_i \tag{1.9}$$

the normalizing factor (which reduces to $r_{ii}(0)$ if s_i is non-degenerate). The expression (1.7) generalizes (1.2) to arbitrary density operators; we will use the same vocabulary as in § 1.1.2 to designate its various features. This generalization was first conceived by Lüders [67]. The lack in (1.7) of off-diagonal blocks $i \neq j$ in a basis where \hat{s} is diagonal expresses *truncation*. The correlations between the states \hat{r}_i for S and the states $\hat{\mathcal{R}}_i$ for A, displayed in its diagonal blocks, express *registration*; they are encoded in $\langle \hat{\Pi}_i (\hat{A} - A_i)^2 \rangle = 0$ for each *i*, a consequence of (1.7), which means that in an ideal measurement \hat{s} takes the value s_i when \hat{A} takes the value A_i .

We further wish to show that *reduction* takes place, i.e., that the set \mathcal{E} of runs can unambiguously be split into 373 subsets \mathcal{E}_i including a proportion p_i of runs, in such a way that for each subset the final state of S + A is $\hat{\mathcal{D}}_i = \hat{r}_i \otimes \hat{\mathcal{R}}_i$. 374 This property obviously requires that (1.7) is satisfied, since by putting back together the subensembles \mathcal{E}_i we recover 375 for \mathcal{E} the state $\sum_i p_i \hat{\mathcal{D}}_i$ of S + A. Nevertheless, due to a quantum queerness (§ 10.2.3), we cannot conversely infer from 376 the latter state the existence of physical subensembles \mathcal{E}_i described by the reduced states \mathcal{D}_i . In fact, the very *selection* 377 of some specific outcome labelled by the index *i* requires the reading of the indication A_i of the pointer (§ 1.1.3), but 378 it is not granted from (1.7) that each run provides such a well-defined indication. This problem will be solved for the 379 Curie-Weiss model in section 11. 380

Tracing out the apparatus from (1.7) provides the marginal state for the tested system S after measurement, which is represented for the whole set of runs by the density operator

$$\hat{r}(t_{\rm f}) \equiv \operatorname{tr}_{\rm A}\hat{\mathcal{D}}(t_{\rm f}) = \sum_{i} p_{i}\hat{r}_{i} = \sum_{i} \hat{\Pi}_{i}\hat{r}(0)\hat{\Pi}_{i} = \sum_{i} p_{i}|s_{i}\rangle\langle s_{i}| = \sum_{i} r_{ii}(0)|s_{i}\rangle\langle s_{i}|.$$
(1.10)

The last two expressions in (1.10) hold when the eigenvalues s_i of \hat{s} are non-degenerate. Symmetrically, the final marginal state of the apparatus

$$\hat{\mathcal{R}}(t_{\rm f}) = \operatorname{tr}_{\rm S} \hat{\mathcal{D}}(t_{\rm f}) = \sum_{i} p_i \hat{\mathcal{R}}_i$$
(1.11)

is consistent with the occurrence with a probability p_i of its indication A_i . The expression (1.10) is the result of *weak truncation*, while the selection of the runs characterized by the outcome A_i produces for S the *weak reduction* into the state \hat{r}_i . The latter process constitutes a *preparation* of S. As already noted in § 1.1.2, the fact that simply tracing out A may lead to a (weakly) truncated or a reduced state for S solves in no way the physics of the measurement process, a well known weakness of some models [10, 11, 31, 68, 69].

³⁹⁰ *1.2.4. Entropy changes in a measurement*

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Discussions about entropy have produced quite a bit of heat Anonymous

When von Neumann set up in 1932 the formalism of quantum statistical mechanics [4], he introduced density 393 operators \hat{D} as quantum analogues of probability distributions, and he associated with any of them a number, its 394 entropy $S[\hat{\mathcal{D}}] = -\text{tr}\,\hat{\mathcal{D}}\ln\hat{\mathcal{D}}$. In case $\hat{\mathcal{D}}$ describes a system in thermodynamic equilibrium, $S[\hat{\mathcal{D}}]$ is identified with 395 the entropy of thermodynamics¹⁰. Inspired by these ideas, Shannon founded in 1948 the theory of communication, 396 which relies on a quantitative estimate of the amount of information carried by a message [70]. Among the various 397 possible messages that are expected to be emitted, each one *i* has some probability p_i to occur; by receiving the specific 398 message *i* we gain an amount $-\ln p_i$ of information. Shannon's entropy $S[p] = -\sum_i p_i \ln p_i$ characterizes the average 399 amount of information which is missing when the message has not yet been acknowledged. Returning to quantum 400 mechanics, a new interpretation of von Neumann's entropy is thus obtained [71, 72, 73]. When a system (or rather 401 a statistical ensemble of systems prepared under similar conditions, in which the considered system is embedded) is 402 described by some density operator $\hat{\mathcal{D}}$, the associated von Neumann entropy can be regarded as an extension of the 403 Shannon entropy: it characterizes a lack of information due to the probabilistic description of the system. It has thus a 404 partly subjective nature, since it measures our uncertainty. One can also identify it with disorder [58, 72, 73, 74, 75]. 405 As measurement processes are means for gathering information, quantitative estimates of the amounts of information 406 involved are provided by the changes of the von Neumann entropies of the systems S, A and S + A. We gather below 407 the various results found in the literature and their interpretation. 408

The equation of motion of S + A is deterministic and reversible, and some manipulations justified by the large size of A are necessary, as in any relaxation problem, to understand how the state of S + A may end as (1.7). Strictly speaking, the Liouville-von Neumann evolution (1.6) conserves the von Neumann entropy $-\text{tr} \hat{D} \ln \hat{D}$ associated with

 $^{^{10}}$ With this definition, S is dimensionless. In thermodynamic units, S is obtained by multiplying its present expression by Boltzmann's constant $1.38 \cdot 10^{-23}$ JK⁻¹. Likewise, if we wish to express Shannon's entropy in bits, its expression should be divided by ln 2

the whole set of degrees of freedom of S + A; in principle no information is lost. However, in statistical physics, 412 irreversibility means that information (identified with order) is transferred towards inaccessible degrees of freedom, 413 in the form of many-particle correlations, without possibility of return in a reasonable delay. A measure of this loss 414 of information is provided by the "relevant entropy" [58, 72, 73, 74, 75], which is the von Neumann entropy of the 415 state that results from the elimination of the information about such inaccessible correlations. Here the truncated state 416 $\hat{D}(t_f)$ should have the latter status: As regards all accessible degrees of freedom, $\hat{D}(t_f)$ should be equivalent to the state 417 issued from $\hat{\mathcal{D}}(0)$ through the equation of motion (1.6), but we got rid in $\hat{\mathcal{D}}(t_f)$ of the irrelevant correlations involving 418 a very large number of elements of the macroscopic apparatus A; such correlations are irremediably lost. 419

We can therefore *measure the irreversibility* of the measurement process leading from $\hat{D}(0)$ to $\hat{D}(t_f)$ by the following entropy balance. The von Neumann entropy of the initial state (1.5) is split into contributions from S and A, respectively, as

$$S\left[\hat{\mathcal{D}}(0)\right] = -\text{tr}\,\hat{\mathcal{D}}(0)\,\ln\hat{\mathcal{D}}(0) = S_{S}\left[\hat{r}(0)\right] + S_{A}\left[\hat{\mathcal{R}}(0)\right],\tag{1.12}$$

whereas that of the final state (1.7) is

$$S\left[\hat{\mathcal{D}}(t_{\rm f})\right] = S_{\rm S}\left[\hat{r}\left(t_{\rm f}\right)\right] + \sum_{i} p_{i} S_{A}\left[\hat{\mathcal{R}}_{i}\right],\tag{1.13}$$

where $\hat{r}(t_f)$ is the marginal state (1.10) of S at the final time¹¹. This equality entails separate contributions from S and A. The increase of entropy from (1.12) to (1.13) clearly arises from the two above-mentioned reasons, truncation and registration. On the one hand, when the density operator $\hat{r}(0)$ involves off-diagonal blocks $\hat{\Pi}_i \hat{r}(0) \hat{\Pi}_j$ ($i \neq j$), their truncation raises the entropy. On the other hand, a robust registration requires that the possible final states $\hat{\mathcal{R}}_i$ of A are more stable than the initial state $\hat{\mathcal{R}}(0)$, so that their entropy is larger. The latter effect dominates because the apparatus is large, typically S_A will be macroscopic and S_S microscopic.

An apparatus is a device which allows us to *gain some information* on the state of S by reading the outcomes A_i . The *price we have to pay* for being thus able to determine the probabilities (1.9) is a complete *loss of information* about the off-diagonal elements $\hat{\Pi}_i \hat{r}(0) \hat{\Pi}_j$ ($i \neq j$) of the initial state of S¹², and a rise in the thermodynamic entropy of the apparatus. More generally, in other types of quantum measurements, some information about a system may be gained only at the expense of erasing other information about this system [76] (see subsection 2.5).

The quantitative estimation of the gains and losses of information in the measurement process is provided by an 435 entropic analysis, reviewed in [22, 72, 77]. Applications of entropy for quantifying the uncertainties in quantum 436 measurements are also discussed in [78]. We recall here the properties of the entropy of the marginal state of S and 437 their interpretation in terms of information. We have just noted that $S_S[\hat{r}(t_f)] - S_S[\hat{r}(0)]$, which is non-negative, 438 measures the increase of entropy of S due to weak truncation. This means that, in case we know $\hat{r}(0)$, the interaction 439 with A (without reading the pointer) lets us loose the amount of information $S_{S}[\hat{r}(t_{f})] - S_{S}[\hat{r}(0)]$ about all observables 440 that do not commute with \hat{s} [72, 77]. In fact, this loss is the largest possible among the set of states that preserve 441 the whole information about the observables commuting with \hat{s} . Any state of S that provides, for all observables 442 commuting with \hat{s} , the same expectation values as $\hat{r}(t_f)$ is less disordered than $\hat{r}(t_f)$, and has an entropy lower than 443 $S_{\rm S}[\hat{r}(t_{\rm f})]$. In other words, among all the processes that leave the statistics of the observables commuting with \hat{s} 444 unchanged, the ideal measurement of \hat{s} is the one which *destroys the largest amount of information* (about the other 445 observables of S). 446

Reading the pointer value A_i , which occurs with probability p_i , allows us to ascertain (for the considered ideal measurement) that S is in the state \hat{r}_i after the measurement. By acknowledging the outcomes of a large sequence of runs of the measurement, we *gain* therefore on average *some amount of information* given on the one hand by the Shannon entropy $-\sum_i p_i \ln p_i$, and equal on the other hand to the difference between the entropies of the final state and of its separate components,

¹¹The latter expression is found by using the orthogonality $\hat{\mathcal{R}}_i \hat{\mathcal{R}}_j = 0$ for $i \neq j$, so that $-\hat{\mathcal{D}}(t_f) \ln \hat{\mathcal{D}}(t_f)$ is equal to the sum of its separate blocks, $\sum_i p_i \hat{r}_i \otimes \hat{\mathcal{R}}_i (-\ln p_i - \ln \hat{r}_i)$, and hence the entropy of $\hat{\mathcal{D}}(t_f)$ is a sum of contributions arising from each *i*. The trace over A of the first two terms leads to $\sum_i p_i \hat{r}_i (-\ln p_i - \ln \hat{r}_i)$, the trace over S of which may be identified with the entropy $S_S[\hat{r}(t_f)]$ of (1.10); the trace of the last term leads to the last sum in (1.13)

¹²In the language of section 1.1: Loss of information about the phases of the ψ_i

$$S_{S}[\hat{r}(t_{f})] - \sum_{i} p_{i} S_{S}[\hat{r}_{i}] = -\sum_{i} p_{i} \ln p_{i} \ge 0.$$
(1.14)

The equality expresses additivity of information, or of uncertainty, at the end of the process, when we have not yet read 452 the outcomes A_i : Our uncertainty $S_{\rm S}[\hat{r}(t_{\rm f})]$, when we know directly that $\hat{r}(t_{\rm f})$, the density operator of the final state, 453 encompasses all possible marginal final states \hat{r}_i , each with its probability p_i , is given by the left-hand side of (1.14). 454 It is the same as if we proceed in two steps. As we have not yet read A_i , we have a total uncertainty $S_S[\hat{r}(t_f)]$ because 455 we miss the corresponding amount of Shannon information $-\sum_i p_i \ln p_i$ about the outcomes; and we miss also, with 456 the probability p_i for each possible occurrence of A_i , some information on S equal to $S[\hat{r}_i]$, the entropy of the state 457 \hat{r}_i . As it stands, the equality (1.14) also expresses the equivalence between negentropy and information [74, 79, 80]: 458 sorting the ensemble of systems S according to the outcome *i* lowers the entropy by a quantity equal on average to 459 the left-hand side of (1.14), while *reading* the indication A_i of the pointer provides, in Shannon's sense, an additional 460 amount of information $-\ln p_i$, on average equal to the right-hand side. 461

⁴⁶² Two inequalities are satisfied in the whole process, including the sorting of results:

$$-\sum_{i} p_{i} \ln p_{i} \ge S_{S} [\hat{r}(0)] - \sum_{i} p_{i} S_{S} [\hat{r}_{i}] \ge 0.$$
(1.15)

The first inequality expresses that the *additivity* of the information gained on the *final* state $\hat{r}(t_f)$ of S by acknowledging the probabilities p_i , as expressed by (1.14), is *spoiled in quantum mechanics* when one considers the whole process, due to the quantum perturbation of the initial state of S which eliminates its off diagonal sectors. The second inequality, derived in [81], expresses that measurements yield a *positive balance of information* about S in spite of the losses resulting from the perturbation of S. Indeed, this inequality means that, on average over many runs of the measurement process, and after sorting of the outcomes, the entropy of S has decreased, i. e., more information on S is available at the time t_f than at the initial time. The equality holds only if all possible final states \hat{r}_i of S have the same entropy.

470 Note finally that, if we wish to perform repeated quantum measurements in a closed cycle, we must reset the
 471 apparatus in its original metastable state. As for a thermal machine, this requires lowering the entropy and costs some
 472 supply of energy.

473 1.3. Towards a solution of the measurement problem?

В гостя́х хорошо́, а до́ма лу́чше. ¹³ Russian proverb

The quantum measurement problem arises from the acknowledgement that individual measurements provide welldefined outcomes. Standard quantum mechanics yields only probabilistic results and thus seems unable to explain such a behavior. We have advocated above the use of quantum statistical physics, which seems even less adapted to draw conclusions about individual systems. Most of the present work will be devoted to show how a statistical approach may nevertheless solve the measurement problem as will be discussed in section 11. We begin with a brief survey of the more current approaches.

482 1.3.1. Various approaches

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Ջորին յոթը գետում լողալ գիտի, բայց ջուր տեսնելիս բոլորը մոռանում է։¹⁴ Armenian proverb

In the early days of quantum mechanics, the apparatus was supposed to behave classically, escaping the realm of quantum theory [82, 83, 84]. A similar idea survives in theoretical or experimental works exploring the possible

¹³Visiting is good, but home is better

¹⁴The mule can swim over seven rivers, but as soon as it sees the water it forgets everything

existence of a border between small or large, or between simple and complex objects, which would separate the domains of validity of quantum and classical physics (Heisenberg's cut [3]).

Another current viewpoint has attributed the reduction¹⁵ in a measurement to the "act of observing the result". 489 Again, the observer himself, who is exterior to the system, is not described in the framework of quantum mechanics. In 490 Rovelli's relational interpretation [85] a quantum mechanical description of some object is regarded as a codification of 491 its properties which is "observer-dependent", that is, relative to a particular apparatus. Then, while a first "observer" A 492 who gathers information about S regards reduction as real, a second observer testing S + A can consider that reduction 493 has not taken place. In the many-worlds interpretation, reduction is even denied, and regarded as a delusion due to the 494 limitations of the human mind [25, 26]. From another angle, people who wish to apply quantum theory to the whole 495 universe, even have a non-trivial task in defining what is observation. A more rational attitude is taken within the consistent histories approach, in which one is careful with defining when and where the events happen, but in which 497 one holds that the measurements simply reveal the pre-existing values of events (this approach is discussed below in 498 section 2.9). For interpretations of quantum mechanics, see Bohm's textbook [86] and for interpretations based on 499 entanglement and information, see Peres [22] and Jaeger [87]. 500

The reduction may be regarded as a bifurcation in the evolution of the considered system, which may end up 501 in different possible states $|s_i\rangle$ although it has been prepared in the single initial state $|\psi\rangle$. In the de Broglie–Bohm 502 interpretation involving both waves and classical-like trajectories, the wave function $|\psi(t)\rangle$ appears both as arising 503 from the density of trajectories and as guiding their dynamics. The randomness of quantum mechanics then arises 504 merely from a randomness in the initial points of the set of trajectories. During a measurement process, the single 505 initial bundle of trajectories, associated with $|\psi\rangle$, is split into separate bundles, each of which is associated with a wave 506 function $|s_i\rangle$. While this interpretation accounts for the bifurcation and for the uniqueness of the outcome of each run 507 of a measurement process, it is not widely accepted [18, 19, 24, 35, 88]. 508

A more recent line of thought, going "beyond the quantum" [20] relies on modification of the Schrödinger mechanics by additional non-linear and stochastic terms; see Refs. [17, 89, 90] for review. Such generalizations are based in the belief, emphasized in the standard Copenhagen interpretation of quantum mechanics, that the Schrödinger equation is unable to describe the joint evolution of a system S and an apparatus A, so that a special separate postulate is needed to account for the rules of quantum measurements, in particular reduction. Indeed, a hamiltonian evolution seems to preclude the emergence of a single result in each single realization of a measurement [4, 13, 30].

We will focus below on the most conservative approach where S + A is treated as an isolated quantum object 515 governed by a Hamiltonian, and yet where reduction can be understood. The measurement is not considered on formal 516 and general grounds as in many conceptual works aimed mainly at the interpretation of quantum mechanics, but it 517 is fully analyzed as a dynamical process. Unfortunately the theory of specific experimental measurement processes 518 based on hamiltonian dynamics is made difficult by the complexity of a real measuring apparatus. One can gain full 519 insight only by solving models that mimic actual measurements. The formal issue is first to show how S + A, which 520 starts from the state (1.5) and evolves along (1.6), may reach a final state of the truncated and correlated form (1.7), 521 then to explain how dynamics may provide for each run of the experiment one among the reduced states $\hat{\mathcal{D}}_i$. 522

The realization of such a program should meet the major challenge raised long ago by Bell [91]: "So long as the wave packet reduction is an essential component, and so long as we do not know exactly when and how it takes over from the Schrödinger equation, we do not have an exact and unambiguous formulation of our most fundamental physical theory". Indeed, a full understanding of quantum mechanics requires knowledge of the time scales involved in measurements.

Knowing how the truncation, then the reduction proceed in time, how long they take, is a prerequisite for clearing up the meaning of this phenomenon. On the other hand, the registration is part of the measurement; it is important to exhibit the time scale on which it takes place, to determine whether it interferes with the reduction or not, and to know when and how the correlations between S and A are established. These are the tasks we undertake in the body of this work on a specific but flexible model. We resume in sections 9 and 11 how the solution of this model answers such questions.

¹⁵We use the word "reduction" in the meaning specified in § 1.1.2, although the same word is often used in the literature to designate what we call "truncation" so as to distinguish the two concepts

534 1.3.2. Outline

We review in section 2 the works that tackled the program sketched above, and discuss to which extent they satisfy the various features that we stressed in the introduction. For instance, do they explain reduction by relying on a full dynamical solution of the Liouville–von Neumann equation for the considered model, or do they only invoke environment-induced decoherence? Do they solve the preferred basis paradox? Do they account for a robust registration? Do they produce the time scales involved in the process?

In section 3 we present the Curie–Weiss model, which encompasses many properties of the previous models and
 on which we will focus afterwards. It is sufficiently simple to be completely solvable, sufficiently elaborate to account
 for all characteristics of ideal quantum measurements, and sufficiently realistic to resemble actual experiments: The
 apparatus simulates a magnetic dot, a standard registering device.

The detailed dynamical solution of this model is worked out in sections 4 to 7, some calculations being given in appendices. After analysing the equations of motion of S + A (section 4), we exhibit several time scales. The truncation rapidly takes place (section 5). It is then made irremediable owing to two alternative mechanisms (section 6). Amplification and registration require much longer delays since they involve a macroscopic change of the apparatus and energy exchange with the bath (section 7).

Solving several variants of the Curie–Weiss model allows us to explore various dynamical processes which can be interpreted either as imperfect measurements or as failures (section 8). In particular, we study what happens when the pointer has few degrees of freedom or when one tries to simultaneously measure non-commuting observables. The calculations are less simple than for the original model, but are included in the text for completeness.

The results of sections 4 to 8 are resumed and analysed in section 9, which also presents some simplified derivations suited for tutorial purposes. However, truncation and registration, explained in sections 5 to 7 for the Curie– Weiss model, are only prerequisites for elucidating the quantum measurement problem, which moreover requires an explanation of reduction.

⁵⁵⁷ Before we tackle this remaining task, we need to make more precise the conceptual framework on which we rely, ⁵⁵⁸ since reduction is tightly related with the interpretation of quantum mechanics. The statistical interpretation, in a form ⁵⁵⁹ presented in section 10, appears as the most natural and consistent one in this respect.

We are then in position to work out the occurrence of reduction within the framework of the statistical interpretation. This is achieved in section 11 for a modified Curie–Weiss model, in which very weak but still sufficiently elaborate interactions within the apparatus are implemented. The uniqueness of the result of a single measurement, as well as the occurrence of classical probabilities, are thus seen to emerge only from the dynamics of the measurement process.

Lessons for future work are drawn in section 12, and some open problems are suggested in section 13.

The reader interested only in the outcomes may skip the technical sections 4 to 8, and focus upon section 9, which gathers all the outcomes of the dynamics in the Curie–Weiss model. This section can be regarded as a *self-contained reading guide*, and as a basis on which the conclusions of sections 10 to 12 will be founded.

569 1.3.3. Terminology employed for the basic concepts

Authors do not always assign the same meaning to some current words. In order to avoid misunderstandings, we gather here the definitions that we are using throughout.

- *Observable*: an operator that represents a physical quantity of a system
- *Statistical ensemble*: a real or virtual set of systems prepared under identical conditions.

Quantum state: a mathematical object from which all the probabilistic properties of a statistical ensemble of systems can be obtained. (Strictly speaking, the state of an individual system refers to a thought ensemble in which it is embedded, since this state has a probabilistic nature.) States are generally represented by a density operator (or, in a given basis, a density matrix) which encompasses the expectation values of all the observables. Pure states are characterized by an absence of statistical fluctuations for some complete set of commuting observables.

• Measurement: a dynamical process which involves an apparatus A coupled to a tested system S and which

- provides information about one observable \hat{s} of S. The time-dependent state of the compound system S + A 581 describes a statistical ensemble of runs, not individual runs. With this definition, the reading of the outcomes 582 and the selection of the results are not encompassed in the "measurement", nor in the "truncation" and the 583 "registration". 584 • Individual run of a measurement: a single interaction process between tested system and apparatus (prepared 585 in a metastable state), followed by the reading of the outcome. 586 • *Ideal measurement*: a measurement which does not perturb the observables of S that commute with \hat{s} . 587 • Truncation; disappearance of Schrödinger cat states: the disappearance, at the end of the measurement process, 588 of the off-diagonal blocks of the density matrix of S + A describing the whole set of runs, in a basis where \hat{s} is 589 diagonal16,17 590 • *Reduction*: for an individual run of the measurement, assignment of a state to S+A at the end of the process^{16,17}. 591 • Decoherence: in general, a decay of the off-diagonal blocks of a density matrix under the effect of a random 592 environment, such as a thermal bath. 593
- *Pointer*; *pointer variable*: a part of the apparatus which undergoes a change that can be read off or registered. In general the pointer should be macroscopic and the pointer variable should be collective.
- *Registration*: the creation during a measurement of correlations between S and the macroscopic pointer of A.
- Selection: the sorting of the runs of an ideal measurement, after truncation and registration, according to the indication of the pointer. The original ensemble that underwent the process is thus split into subensembles characterized by a well-defined value of \hat{s} . Measurement followed by selection may constitute a *preparation*.
- *Hierarchic structure of subensembles*: a property required to solve the quantum measurement problem. Namely, the final state associated with any subset of runs of the measurement should have the same form as for the whole set (§ 11.2.1).
- 603 2. The approach based on models

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604		Point n'est besoin d'espérer pour entreprendre,
605		ni de réussir pour persévérer ¹⁸
606		Charles le Téméraire and William of Orange

We have briefly surveyed in § 1.3.1 many theoretical ideas intended to elucidate the problem of quantum measurements. We feel that it is more appropriate to think along the lines of an experimentalist who performs measurements in his laboratory. For this reason, it is instructive to formulate and solve models with this scope. We review in this section various models in which S + A is treated as a compound system which evolves during the measurement process according to the standard rules of quantum mechanics. The existing models are roughly divided into related classes. Several models serve to elucidate open problems. Besides specific models, we shall discuss several more general approaches to quantum measurements (e.g., the decoherence and consistent histories approaches).

¹⁶ We will refrain from using popular terms such as "collapse of the wave function" or "reduction of the wave packet"

¹⁷ We use the terms "weak truncation" and "weak reduction" for the same operations as truncation and reduction, but perfomed on the marginal density matrix of the tested system S, and not on the density matrix of the compound system S+A

¹⁸It is not necessary to hope for undertaking, neither to succeed for persevering

614 2.1. Heisenberg-von Neumann setup

615

616

Before you milk a cow, tie it up South African proverb

A general set-up of quantum measurement was proposed and analysed by Heisenberg [2, 3]. His ideas were formalized by von Neumann who proposed the very first mathematically rigorous model of quantum measurement [4]. An early review on this subject is by London and Bauer [30], in the sixties it was carefully reviewed by Wigner [13]; see [92] for a modern review.

Von Neumann formulated the measurement process as a coupling between two quantum systems with a specific 621 interaction Hamiltonian that involves the (tensor) product between the measured observable of the tested system and 622 the pointer variable, an observable of the apparatus. This interaction conserves the measured observable and ensures a 623 correlation between the tested quantity and the pointer obervable. In one way or another the von Neumann interaction 624 Hamiltonian is applied in all subsequent models of ideal quantum measurements. However, von Neumannn's model 625 does not account for the differences between the microscopic [system] and macroscopic [apparatus] scales. As a main 626 consequence, it does not have a mechanism to ensure the specific classical correlations (in the final state of the system 627 + apparatus) necessary for the proper interpretation of a quantum measurement. Another drawback of this approach 628 is its requirement for the initial state of the measuring system (the apparatus) to be a pure state (so it is described 629 by a single wave function). Moreover, this should be a specific pure state, where fluctuations of the pointer variable 630 are small. Both of these features are unrealistic. In addition, and most importantly, the von Neumann model does 631 not account for the features of truncation and reduction; it only shows weak reduction (see terminology in § 1.1.2 632 and § 1.3.3). This fact led von Neumann (and later on Wigner [13]) to postulate - on top of the usual Schrödinger 633 evolution – a specific dynamic process that is supposed to achieve the reduction [4]. 634

With all these specific features it is not surprising that the von Neumann model has only one characteristic time driven by the interaction Hamiltonian. Over this time the apparatus variable gets correlated with the initial state of the measured system.

Jauch considers the main problem of the original von Neumann model, i.e. that in the final state it does not ensure 638 specific classical correlations between the apparatus and the system [93]. A solution of this problem is attempted 639 within the lines suggested (using his words) during "the heroic period of quantum mechanics" that is looking for 640 classical features of the apparatus. To this end, Jauch introduces the concept of equivalence between two states (as 641 represented by density matrices): two states are equivalent with respect to a set of observables, if these observables 642 cannot distinguish one of these states from another [93]. Next, he shows that for the von Neumann model there is a 643 natural set of commuting (hence classical) observables, so that with respect to this set the final state of the model is 644 not distinguishable from the one having the needed classical correlations. At the same time Jauch accepts that some 645 other observable of the system and the apparatus can distinguish these states. Next, he makes an attempt to *define* 646 the measurement event via his concept of classical equivalence. In our opinion this attempt is interesting, but not 647 successful. 648

649 2.2. Quantum–classical models: an open issue?

650 651

Gooi geen oude schoenen weg voor je nieuwe hebt¹⁹ Dutch proverb

Following suggestions of Bohr that the proper quantum measurement should imply a classical apparatus [82, 652 83, 84], there were several attempts to work out interaction between a quantum and an explicitly classical system 653 [94, 95, 96, 97, 98, 99, 100, 101, 102, 103, 104, 105, 106, 107, 108]. (Neither Bohr [82, 83], nor Landau and 654 Lifshitz [84] who present Bohr's opinion in quite detail, consider the proper interaction processes.) This subject 655 is referred to as hybrid (quantum-classical) dynamics. Besides the measurement theory it is supposed to apply in 656 quantum chemistry [94, 95] (where the full modeling of quantum degrees of freedom is difficult) and in quantum 657 gravity [109], where the proper quantum dynamics of the gravitational field is not known. There are several versions 658 of the hybrid dynamics. The situation, where the classical degree of freedom is of a mean-field type is especially 659

¹⁹Don't throw away old shoes before you have new ones

well-known [94, 95]. In that case the hybrid dynamics can be derived variationally from a simple combination of quantum and classical Lagrangian. More refined versions of the hybrid dynamics attempt to describe interactions between the classical degree(s) of freedom and quantum fluctuations. Such theories are supposed to be closed and self-consistent, and (if they really exist) they would somehow get the same fundamental status as their limiting cases, i.e., as quantum and classical mechanics. The numerous attempts to formulate such fundamental quantum–classical theories have encountered severe difficulties [97, 98, 99, 100, 101, 102, 103, 104, 105]. There are no–go theorems showing in which specific sense such theories cannot exist [106, 107].

As far as the quantum measurement issues are concerned, the hybrid dynamical models have not received the 667 attention they deserve. This is surprising, because Bohr's insistence on the classicality of the apparatus is widely 668 669 known and frequently repeated. The existing works are summarized as follows. Diosi and co-authors stated that their scheme for the hybrid dynamics is useful for quantum measurements [97, 98], albeit that they did not come with 670 a more or less explicit analysis. Later on Terno has shown that the problem of a quantum measurement cannot be 671 solved via a certain class of hybrid dynamic systems [110]. His arguments rely on the fact that the majority of hybrid 672 system have pathological features in one way or another. Terno also reviews some earlier attempts, in particular by 673 him in collaboration with Peres [105], to describe quantum measurements via hybrid dynamics; see the book of Peres 674 [22] for preliminary ideas within this approach. However, recently Hall and Reginato [108, 111] suggest a scheme 675 for the hybrid dynamics that seems to be free of pathological features. This scheme is based on coupled quantum and classical ensembles. A related set-up of hybrid dynamics is proposed by Elze and coworkers based on a path-677 integral formulation [112], see also [113]. If Hall and Reginato's claim is true that such schemes can circumvent 678 no-go theorems [108, 111], it should be interesting to look again at the features of quantum measurements from 679 the perspective of an explicitly classical apparatus: Bohr's program can still be opened! A modern view on the 680 Copenhagen interpretation developed by (among others) Bohr is presented in Refs. [114, 115]. 681

Everitt, Munro and Spiller discuss a measurement model which, while fully quantum mechanical, makes use of 682 analogy with classical features of the apparatus [116]. The model consists of a two-level system (the measured 683 system), the apparatus, which is a one-dimensional quartic oscillator under external driving, and an environment 684 whose influence on the system + apparatus is described within the Lindblad master-equation approach and its quantum 685 state diffusion unravelling [117]. The main point of this work is that the apparatus can display the chaotic features 686 of a damped forced non-linear oscillator (and is thus not related to Hamiltonian chaos). Everitt, Munro and Spiller 687 make use of this point for the following reason: The feature of chaos allows one to distinguish quantum from classical 688 regimes for the apparatus (this is not fundamental - simply a convenience for demonstrating a quantum to classical 689 transition). The model reproduces certain features expected from individual measurement outcomes, but this happens 690 at the cost of *unravelling* the master equation, a relatively arbitrary procedure of going from density matrices to 691 random wavefunctions. The authors of Ref. [116] are aware of this arbitrariness and attempt to minimize it. It should 692 be noted that, as one would expect, in the classical limit the choice of how to unravel seems to have no effect ons 693 the emergence of a classical dynamic (see, for example, [118]). This implies that the results of [116] may well be 694 independent of the unravelling – but this has yet to be demonstrated. 695

In Ref. [119] Blanchard and Jadczyk discuss a quantum-classical model for measurements. They present it as a 696 minimal phenomenological model for describing quantum measurements within the concept of an explicitly classical 697 apparatus. In contrast to other quantum-classical models, Blanchard and Jadczyk consider a dissipative interaction 698 between the quantum and classical subsystems. This interaction is modeled by a completely positive map. These 699 maps are frequently applied for describing an open-system quantum dynamics, where the target system couples with 700 an external environment; see e.g. Refs. [120, 121, 122]. (However, this is certainly not the only possibility for an 701 open-system quantum dynamics; see in this context Ref. [123].) Blanchard and Jadczyk found a simple form of 702 the completely map that suffices for accounting (phenomenologically) for certain features of quantum measurements, 703 such the response of the pointer classical states to the initial state of the quantum system, as well as the proper final 704 state of the quantum system. 705

This approach is generalized in [124], where Blanchard and Jadczyk account for the emergence of events during the quantum measurements. This is done by introducing an additional phenomenological step thereby the quantumclassical dynamics for the quantum density matrix and classical probability distribution is regarded as the result of averaging over the states of some underlying stochastic process (a procedure akin to unraveling the open-system quantum master equation). The stochastic process – which gives rise to what Blanchard and Jadczyk call eventenhanced quantum theory – is formulated in the tensor product of the classical subsystem's event space and the 712 quantum subsystem's Hilbert space.

727

728

In our opinion this approach to quantum measurements has an extensively phenomenological character, a fact well-admitted by Blanchard and Jadczyk. On the other hand, its central idea that the emergence of measurement events should be related to specific features of the measuring apparatus is certainly valuable and will be developed in the present work.

In closing this subsection we note that the relation between quantum and classical has yet another, geometrical 717 twist, because the pure-state quantum dynamics (described by the Schrödinger equation) can be exactly mapped 718 to a classical Hamiltonian dynamics evolving in a suitable classical symplectic space [125, 126, 127]. Quantum 719 aspects (such as uncertainties and the Planck's constant) are then reflected via a Riemannian metrics in this space 720 [126, 127]; see also [128] for a recent review. This is a geometrical counterpart to the usual algebraic description of 721 quantum mechanics, and is considered to be a potentially rich source for various generalizations of quantum mechanics 722 [128, 129]. A formulation of the quantum measurement problem in this language was attempted in [129]. We note 723 that so far this approach is basically restricted to pure states (see, however, [127] in this context). 724

Further references on crucial aspects of the quantum-to-classical transition are [130, 131, 132].

726 2.3. Explicitly infinite apparatus: Coleman–Hepp and related models

What I cannot build, I cannot understand Richard Feynman

Several authors argued that once the quantum measurement apparatus is supposed to be a macroscopic system, the 729 most natural framework for describing measurements is to assume that it is explicitly infinite; see the review by Bub 730 [133]. C*-algebras is the standard tool for dealing with this situation [134]. Its main pecularity is that there are (many) 731 inequivalent unitary representations of the algebra of observables, i.e., certain superpositions between wavefunctions 732 cannot be physical states (in contrast to finite-dimensional Hilbert spaces) [133]. This is supposed to be helpful in 733 constructing measurement models. Hepp proposed first such models [12]. He starts his investigation by stating some 734 among the goals of quantum measurement models. In particular, he stresses that an important feature of the problem 735 is in getting classical correlations between the measured observable and the pointer variable of the apparatus, and 736 that quantum mechanics is a theory that describes probabilities of certain events. Hepp then argues that the quantum 737 measurement problem can be solved, i.e., the required classical correlations can be established dynamically, if one 738 restricts oneself to macroscopic observables. He then moves to concrete models, which are solved in the C^* -algebraic 739 framework. The infinite system approach is also employed in the quantum measurement model proposed by Whitten-740 Wolfe and Emch [135]. 741

However, working with an infinite measuring apparatus hides the physical meaning of the approach, because some important dynamic scales of the quantum measurement do depend on the number of degrees of freedom of the apparatus [68]. In particular, the truncation time may tend to zero in the limit of an infinite apparatus and cannot then be evaluated. Thus, making the apparatus explicitly infinite (instead of taking it large, but finite) misses an important piece of physics, and does not allow to understand which features of the quantum measurement will survive for a apparatus having a mesoscopic scale.

Hepp also studies several exactly solvable models, which demonstrate various aspects of his proposal. One of 748 them—proposed to Hepp by Coleman and nowadays called the Coleman–Hepp model— describes an ultra-relativistic 749 particle interacting with a linear chain of spins. Hepp analyzes this model in the infinite apparatus situation; this 750 has several drawbacks, e.g., the overall measurement time is obviously infinite. The physical representation of the 751 Coleman-Hepp Hamiltonian is improved by Nakazato and Pascazio [136]. They show that the basic conclusions on 752 the Coleman-Hepp Hamiltonian approach can survive in a more realistic model, where the self-energy of the spin 753 chain is taken into account. Nakazato and Pascazio also discuss subtleties involved in taking the thermodynamic 754 limit for the model [136]. The Coleman–Hepp model with a large but finite number of the apparatus particles is 755 studied by Sewell [137, 138, 139]. He improves on previous treatments by carefully calculating the dependence of 756 the characteristic times of the model on this number, and discusses possible imperfections of the measurement model 757 arising from a finite number of particles. 758

Using the example of the Coleman–Hepp model, Bell demonstrates explicitly [91] that the specific features of the quantum measurement hold only for a certain class of observables, including macroscopic observables [69, 137, 138, 139]. It is then possible to construct an observable for which those features do not hold [91]. We recall that the

same holds in the irreversibility problem: it is always possible to construct an observable of a macroscopic system 762 (having a large, but finite number of particles) that will not show the signs of irreversible dynamics, i.e., it will not 763 be subject to relaxation. Bell takes this aspect as an essential drawback and states that the quantum measurement was 764 not and cannot be solved within a statistical mechanics approach [91]. Our attitude in the present paper is different. 765 We believe that although concrete models of quantum measurements may have various drawbacks, the resolution of 766 the measurement problem is definitely to be sought along the routes of quantum statistical mechanics. The fact that 767 certain restrictions on the set of observables are needed, simply indicates that, similar to irreversibility, a quantum 768 measurement is an emergent phenomenon of a large system – the tested system combined with the apparatus – over 769 some characteristic time. 770

771 2.4. Quantum statistical models

772 773

If I have a thousand ideas and only one turns out to be good, I am satisfied Alfred Bernhard Nobel

Here we describe several models based on quantum statistical mechanics. In contrast to the previous chapter, these
 models do not invoke anything beyond the standard quantum mechanics of finite though large systems.

Green proposed a realistic model of quantum measurement [140]. He emphasizes the necessity of describing the 776 apparatus via a mixed, quasi-equilibrium state and stresses that the initial state of the apparatus should be macroscopic 777 and metastable. The model studied in [140] includes a spin- $\frac{1}{2}$ particle interacting with two thermal baths at different 778 temperatures. The two-temperature situation serves to simulate metastability. The tested particle switches interaction 779 between the baths. By registering the amount of heat flow through the baths (a macroscopic pointer variable), one 780 can draw certain conclusions about the initial state of the spin. Off-diagonal terms of the spin density matrix are 781 suppressed via a mechanism akin to inhomogeneous broadening. However, an explicit analysis of the dynamic regime 782 and its characteristic times is absent. 783

Cini studies a simple model for the quantum measurement process which illustrates some of the aspects related to 784 the macroscopic character of the apparatus [141]. The model is exactly solvable and can be boiled down to a spin- $\frac{1}{2}$ 785 particle (tested spin) interacting with a spin-L particle (apparatus). The interaction Hamiltonian is $\propto \sigma_z L_z$, where σ_z 786 and L_z are, respectively, the third components of the spin- $\frac{1}{2}$ and spin L. Cini shows that in the limit $L \gg 1$ and for a 787 sufficiently long interaction time, the off-diagonal terms introduced by an (arbitrary) initial state of the tested spin give 788 negligible contributions to the observed quantities, i.e., to the variables of the tested spin and the collective variables 789 of the apparatus. The characteristic times of this process are analyzed, as well as the situation with a large but finite 790 value of L. 791

In Refs. [10, 11] Blokhintsev studies, within the statistical interpretation of quantum mechanics, several interesting measurement models with a metastable initial state of the apparatus: an incoming test particle interacting with an apparatus-particle in a metastable potential well, a test neutron triggering a nuclear chain reaction, et cetera. Though the considered models are physically appealing, the involved measurement apparatuses are frequently not really macroscopic. Neither does Blokhintsev pay proper attention to the correlations between the system and the apparatus in the final state.

Requardt studies a quantum measurement model, in which due to collisional interaction with the tested system, 798 the pointer variable of a macroscopic measuring apparatus undergoes a coherent motion, in which the momentum 799 correlates with the values of the measured observable (coordinate) [142]. It is stressed that for the approach to have 800 a proper physical meaning, the apparatus should have a large but finite number of degrees of freedom. However, 801 no detailed account of characteristic measurement times is given. Requardt also assumes that the initial state of the 802 measurement apparatus is described by a wave function, which is merely consistent with the macroscopic information 803 initially available on this apparatus. He focuses on those aspects of the model which will likely survive in a more 804 general theory of quantum measurements; see in this context his later work [69] that is reviewed below. 805

An interesting statistical mechanical model of quantum measurement was proposed and studied in Ref. [143] by Gaveau and Schulman. The role of apparatus is played by a one-dimensional Ising spin model. Two basic energy parameters of the model are an external field and the spin-spin coupling (exchange coupling). An external field is tuned in such a way that a spontaneous flipping of one spin is energetically not beneficial, while the characteristic time of flipping two spins simultaneously is very large. This requirement of metastability puts an upper limit on the number of spins in the apparatus. The tested spin $\frac{1}{2}$ interacts only with one spin of the apparatus; this is definitely an advantage of this model. The spin-apparatus interaction creates a domino effect bringing the apparatus to a unique
ferromagnetic state. This happens for the tested spin pointing up. For the tested spin pointing down nothing happens,
since in this state the tested spin does not interact with the apparatus. Characteristic times of the measurement are
not studied in detail, though Gaveau and Schulman calculate the overall relaxation time and the decay time of the
metastable state. It is unclear whether this model is supposed to work for an arbitrary initial state of the tested spin.

Ref. [144] by Merlin studies a quantum mechanical model for distinguishing two different types of bosonic particles. The model is inspired by Glaser's chamber device, and has the realistic feature that the bosonic particle to be tested interacts only with one particle of the apparatus (which by itself is made out of bosons). The initial state of the apparatus is described by a pure state and it is formally metastable (formally, because this is not a thermodynamic metastability). The relaxation process is not accounted for explicitly; its consequences are simply postulated. No analysis of characteristic relaxation times is presented. Merlin analyses the relation of measurement processes with the phenomenon of spontaneous symmetry breaking.

824 2.4.1. Spontaneous symmetry breaking

825

Bezint eer ge begint²⁰ Dutch proverb

The role of spontaneous symmetry breaking as an essential ingredient of the quantum measurement process is underlined in papers by Grady [145], Fioroni and Immirzi [146] and Pankovic and Predojevic [147]. They stress that superpositions of vacuum states are not allowed in quantum field theory, since these superpositions do not satisfy the cluster property. All three approaches stay mainly at a qualitative level, though Fioroni and Immirzi go somewhat further in relating ideas on quantum measurement process to specific first-order phase transition scenarios. An earlier discussion on symmetry breaking, quantum measurements and geometrical concepts of quantum field theory is given by Ne'eman [148].

Ref. [149] by Zimanyi and Vladar also emphasizes the relevance of phase transitions and symmetry breaking for quantum measurements. They explicitly adopt the statistical interpretation of quantum mechanics. General statements are illustrated via the Caldeira-Leggett model [150, 151, 152, 153]: a two-level system coupled to a bath of harmonic oscillators. This model undergoes a second-order phase transition with relatively weak decay of off-diganonal terms in the thermodynamic limit, provided that the coupling of the two-level system to the bath is sufficiently strong. The authors speculate about extending their results to first-order phase transitions. A dynamical consideration is basically absent and the physical meaning of the pointer variable is not clear.

Thus the concept of spontaneous symmetry breaking is frequently discussed in the context of quantum measure-841 ment models (although it is not anymore strictly spontaneous, but driven by the interaction with the system of which 842 the observable is to be measured). It is also an essential feature of the approach discussed in the present paper. It 843 should however be noted that so far only one scenario of symmetry breaking has been considered in the context of 844 quantum measurements (the one that can be called the classical scenario), where the higher temperature extremum of the free energy becomes unstable (or at least metastable) and the system moves to another, more stable state (with 846 lower free energy). Another scenario is known for certain quantum systems (e.g., quantum antiferromagnets) with a 847 low-temperature spontaneously symmetry broken state; see, e.g., [154]. Here the non-symmetric state is not an eigen-848 state of the Hamiltonian, and (in general) does not have less energy than the unstable ground state. The consequences 849 of this (quantum) scenario for quantum measurements are so far not explored. However, recently van Wezel, van den 850 Brink and Zaanen studied specific decoherence mechanisms that are induced by this scenario of symmetry breaking. 851 [154]. 852

853 2.4.2. System-pointer-bath models

854

855

*Je moet met de juiste wapens ten strijde trekken*²¹ Dutch proverb

²⁰Reflect before you start

²¹You must go into battle with the right weapons

Refs. [155] by Haake and Walls and [156] by Haake and Zukowski study a measurement of a discrete-spectrum 856 variable coupled to a single-particle apparatus (the meter). The latter is a harmonic oscillator, and it interacts with 857 a thermal bath, which is modeled via harmonic oscillators. The interaction between the tested system and the meter 858 is impulsive (it lasts a short time) and involves the tensor product of the measured observable and the momentum 859 of the meter. There are two characteristic times here: on the shorter time, the impulsive interaction correlates the 860 states of the object and of the meter, while on the longer time scale the state of the meter becomes classical under 861 the influence of the thermal bath, and the probability distribution of the meter coordinate is prepared via mixing well-862 localized probability distributions centered at the eigenvalues of the measured quantity, with the weights satisfying 863 the Born rule [1]. (This sequence of processes roughly corresponds to the ideas of decoherence theory; see below for 864 more detail.) At an even longer time scale the meter will completely thermalize and forget about its interaction with 865 the tested system. The authors of [155, 156] also consider a situation where the meter becomes unstable under the 866 influence of the thermal bath, since it now feels an inverted parabolic potential. Then the selection of the concrete 867 branch of instability can be driven by the interaction with the object. Since the initial state of such an unstable 868 oscillator is not properly metastable, one has to select a special regime where the spontaneous instability decay can 869 be neglected. 870

The quantum measurement model studied in [157] by Venugopalan is in many aspects similar to models investigated in [155, 156]. The author stresses relations of the studied model to ideas from the decoherence theory.

Ref. [158] by the present authors investigates a model of quantum measurement where the macroscopic measure-873 ment apparatus is modeled as an ideal Bose gas, in which the amplitude of the condensate is taken as the pointer 874 variable. The model is essentially based on the properties of irreversibility and of ergodicity breaking, which are 875 inherent in the model apparatus. The measurement process takes place in two steps: First, the truncation of the state 876 of the tested system takes place, this process is governed by the apparatus-system interaction. During the second step 877 classical correlations are established between the apparatus and the tested system over the much longer time scale of 878 equilibration of the apparatus. While the model allows to understand some basic features of the quantum measurement 879 as a driven phase-transition, its dynamical treatment contains definite drawbacks. First, the Markov approximation for the apparatus-bath interaction, though correct for large times, is incorrectly employed for very short times, which 881 greatly overestimates the truncation time. Another drawback is that the model is based on the phase transition in an 882 ideal Bose gas. This transition is known to have certain pathological features (as compared to a more realistic phase-883 transition in a weakly interacting Bose gas). Though the authors believe that this fact did not influence the qualitative 884 outcomes of the model, it is certainly desirable to have better models, where the phase transition scenario would be 885 generic and robust. Such models will be considered in later chapters of this work. 886

In Ref. [159, 160] Spehner and Haake present a measurement model that in several aspects improves upon previous 887 models. The model includes the tested system, an oscillator (generally anharmonic), which plays the role of apparatus, 88 and a thermal bath coupled to the oscillator. The time scales of the model are set in such a way that the correlations 889 between the measured observable of the system and the pointer variable of the apparatus (here the momentum of the 890 anharmonic oscillator) and the decay of the off-diagonal terms of the tested system density matrix are established 891 simultaneously. This implies realistically that no macroscopic superpositions are generated. In addition, the initial 892 state of the apparatus and its bath is not assumed to be factorized, which makes it possible to study strong (and also 893 anharmonic) apparatus-bath couplings. 894

Ref. [161] by Mozyrsky and Privman studies a quantum measurement model, which consists of three parts: the tested system, the apparatus and a thermal bath that directly couples to the system (and not to the apparatus). The initial state of the apparatus is not metastable, it is chosen to be an equilibrium state. The dynamics of the measured observable of the system is neglected in the course of measurement. The authors of [161] show that after some decoherence time their model is able to reproduce specific correlations that are expected for a proper quantum measurement.

Omnès recently studied a model for a quantum measurement [162]. The pointer variable of the apparatus is supposed to be its (collective) coordinate. The introduction of the measurement process is accompanied by a discussion on self-organization. For solving this Omnès partially involves the mean-field method, because the many-body apparatus density matrix is substituted by the tensor product of the partial density matrices. The dynamics of the model involves both decoherence and reduction. These two different processes are analysed together and sometimes in rather common terms, which can obscure important physical differences between them. In the second part Omnès studies fluctuations of the observation probabilities for various measurement results. These fluctuations are said to arise due ⁹⁰⁸ to a coupling with an external environment modeled as a phonon bath.

Van Kampen stresses the importance of considering a macroscopic and metastable measuring apparatus and pro-900 poses a model that is supposed to illustrate the main aspects of the measurement process [14]. The model consists of 910 a single atom interacting with a multi-mode electromagnetic field, which is playing the role of apparatus. The emitted 911 photon that is generated correlates with the value of the measured observable. The apparatus can be macroscopic 912 (since the vacuum has many modes), but its (thermodynamically) metastable character is questionable. The model 913 is not solved in detail, and its main dynamical consequences are not analyzed. Nevertheless, van Kampen offers a 914 qualitative analysis of this model, which appears to support the common intuition on quantum measurements. The 915 resulting insights are summarized in his "ten theorems" on quantum measurements. 916

917 2.4.3. Towards model-independent approaches

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9	1	9

Qui se soucie de chaque petite plume ne devrait pas faire le lit²² Swiss proverb

Sewell [137, 138, 139] and independently Requardt [69] attempt to put the results obtained from several models 920 into a single model-independent approach, which presumably may pave a way towards a general theory of quantum 921 measurements. The basic starting point of the approach is that the measuring apparatus, being a many-body quantum 922 system, does have a set of macroscopic, mutually commuting observables $\{A_1, \ldots, A_M\}$ with \mathcal{M} a macroscopic integer. 923 The commutation is approximate for a large, but finite number of reservoir particles, but it becomes exact in the 924 thermodynamic limit for the apparatus. Each A_k is typically a normalized sum over a large number of apparatus 925 particles. The set $\{A_1, \ldots, A_M\}$ is now particled into macroscopic cells; each such cell refers to some subspace in the Hilbert space formed by a common eigenvector. The cells are distinguished from each other by certain combinations of 927 the eigenvalues of $\{A_1, \ldots, A_M\}$. The purpose of partioning into cells is to correlate each eigenvalue of the microscopic 928 observable to be measured with the corresponding cell. In the simplest situation the latter set reduces to just one 929 observable A, while two cells refer to the subspace formed by the eigenvectors of A associated with positive or negative 930 eigenvalues. Further derivations, which so far were carried out on the levels of models only [69, 137, 138, 139], 931 amount to showing that a specific coupling between the system and the apparatus can produce their joint final state, 932 which from the viewpoint of observables $A_k \otimes S$ — where S is any observable of the microscopic measured system 933 does have several features required for a good (or even ideal) quantum measurement. 934

935 2.4.4. Ergodic theory approach

936Wenn I wieder, wieder komm 23937From the German folk song "Muß I denn"

Daneri, Loinger and Prosperi approach quantum measurements via the quantum ergodic theory [21]. Such an 938 approach was anticipated in the late forties by the works of Jordan [163] and Ludwig [164]. Daneri, Loinger and 939 Prosperi model the measuring apparatus as a macroscopic system, which in addition to energy has another conserved 940 quantity, which serves the role of the pointer variable. Under the influence of the system to be measured this conser-941 vation is broken, and there is a possibility to correlate different values of the measured observable with the pointer 942 values. Daneri, Loinger and Prosperi invoke the basic assumption of ergodic theory and treat the overall density 943 matrix via time-averaging [21]. The time-averaged density matrix satisfies the necessary requirements for an ideal measurement. However, the use of the time-averaging does not allow to understand the dynamics of the quantum 945 measurement process, because no information about the actual dynamical time scales is retained in the time-averaged 946 density matrix. Also, although the initial state of the measuring apparatus does have some properties of metastability, 947 it is not really metastable in the thermodynamic sense. 948

The publication of the paper by Daneri, Loinger and Prosperi in early sixties induced a hot debate on the measurement problem; see [165] for a historical outline. We shall not attempt to review this debate here, but only mention one aspect of it: Tausk (see [165] for a description of his unpublished work) and later on Jauch, Wigner and Yanase [166] criticize the approach by Daneri, Loinger and Prosperi via the argument of an interaction free measurement.

27

 $^{^{\}rm 22} \rm Who$ cares about every little feather should not make the bed

²³When I come, come again

This type of measurements is first discussed by Renninger [167]. The argument goes as follows: sometimes one 953 can gather information about the measured system even without any macroscopic process generated in the measuring 95 apparatus. This can happen, for instance, in the double-slit experiment when the apparatus measuring the coordinate 955 of the particle is placed only at one slit. Then the non-detection by this apparatus will — ideally — indicate that the 956 particle passed through the other slit. The argument thus intends to demonstrate that quantum measurements need not 957 be related to macroscopic (or irreversible) processes. This argument however does not present any special difficulty 958 within the statistical interpretation of quantum mechanics, where both the wavefunction and the density matrix refer 959 to an ensemble of identically prepared system. Although it is true that not every single realization of the apparatus-960 particle interaction has to be related to a macroscopic process, the probabilities of getting various measurement results 961 do rely on macroscopic processes in the measuring apparatus. 962

⁹⁶³ 2.5. No-go theorems and small measuring apparatuses

964

965

Non ho l'età, per amarti²⁴ Lyrics by Mario Penzeri, sung by Gigliola Cinquetti

The quantum measurement process is regarded as a fundamental problem, also because over the years several 966 no-go theorems were established showing that the proper conditions for quantum measurement cannot be satisfied 967 if they are demanded as exact features of the final state of the apparatus [13, 168, 169, 170]. The first such theorem was established by Wigner [13]. Then several extensions of this theorem were elaborated by Fine [171] and Shimony 969 with co-authors [168, 169, 170]. The presentation by Fine is especially clear, as it starts from the minimal conditions 970 required from a quantum measurement [171]. After stating the no-go theorem, Fine proceeds to discuss in which 971 sense one should look for approximate schemes that satisfy the measurement conditions, a general program motivating 972 also the present study. The results of Refs. [168, 169, 170] show that even when allowing certain imperfections in 973 the apparatus functioning, the quantum measurement problem remains unsolvable in the sense that the existence of 974 specific classical correlations in the final state of the system + apparatus cannot be ensured; see also in this context 975 the recent review by Bassi and Ghirardi [17]. In our viewpoint, the no-go theorems do not preclude approximate 976 satisfaction of the quantum measurement requirements - owing to a macroscopic size of the apparatus. 977

Turning this point over, one may ask which features of proper quantum measurements (as displayed by successful 978 979 models of this phenomenon) would survive for an apparatus that is not macroscopically large. There are several different ways to pose this question, e.g., below we shall study the measuring apparatus (that already performs well 980 in the macroscopic limit) for a large but finite number of particles. Another approach was recently worked out by 981 Allahverdyan and Hovhannisyan [76]. They assume that the measuring apparatus is a finite system, and study system-982 apparatus interaction setups that lead to transferring certain matrix elements of the unknown density matrix λ of the 983 system into those of the final state \tilde{r} of the apparatus. Such a transfer process represents one essential aspect of the 98 quantum measurement with a macroscopic apparatus. No further limitations on the interaction are introduced, because 985 the purpose is to understand the implications of the transfer on the final state of the system. It is shown that the transfer 986 process eliminates from the final state of the system the memory about the transferred matrix elements (or certain other 987 ones) [76]. In particular, if one diagonal matrix element is transferred, $\tilde{r}_{aa} = \lambda_{aa}$, the memory on all non-diagonal 988 elements $\lambda_{a\neq b}$ or $\lambda_{b\neq a}$ related to this diagonal element is completely eliminated from the final density operator of the 989 system (the memory on other non-diagonal elements λ_{cd} , where $c \neq a$ and $d \neq a$ may be preserved). Thus, the general 990 aspect of state disturbance in quantum measurements is the loss of memory about off-diagonal elements, rather than 99 diagonalization (which means the vanishing of the off-diagonal elements). 992

2.6. An open problem: A model for a non-statistical interpretation of the measurement process.

994 995 We can't go on forever, with suspicious minds Written by Mark James, sung by Elvis Presley

The statistical interpretation together with supporting models does provide a consistent view on measurements within the standard quantum mechanics. However, it should be important to understand whether there are other

²⁴I do not have the age to love you

consistent approaches *from within* the standard quantum formalism that can provide an alternative view on quantum
 measurements. Indeed, it cannot be excluded that the real quantum measurement is a wide notion, which combines
 instances of different interpretations. In the present review we will not cover approaches that introduce additional
 ingredients to the standard quantum theory, and will only mention them in subsection 2.8.

We focus only on one alternative to the statistical interpretation, which is essentially close to the Copenhagen interpretation [82, 83, 84, 172] and is based on effectively non-linear Schrödinger equation. We should however stress that so far the approach did not yet provide a fully consistent and unifying picture of quantum measurements even for one model.

Recently Brox, Olaussen and Nguyen approached quantum measurements via a non-linear Schrödinger equation 1006 [173]. The authors explicitly adhere to a version of the Copenhagen interpretation, where the wave function (the 100 pure quantum state) refers to a single system. They present a model which is able to account for single measurement 1008 events. The model consists of a spin- $\frac{1}{2}$ (the system to be measured), a ferromagnet (the measuring apparatus), and 1009 the apparatus environment. The overall system is described by a pure wavefunction. The ferromagnetic apparatus 1010 is prepared in an (unbiased) initial state with zero magnetization. The two ground states of the ferromagnet have 1011 a lower energy and, respectively, positive and negative magnetization. Moving towards one of these states under 1012 influence of the tested system is supposed to amplify the weak signal coming from this tested system. (The latter 1013 features will also play an important role in the models to be considered in detail later on.) The environment is 101 modeled as a spin-glass: environmental spins interact with random (positive or negative) coupling constants. So far 1015 all these factors are more or less standard, and — as stressed by the authors — these factors alone cannot account 1016 for a solution of the measurement problem within an interpretation that ascribes the wavefunction to a single system. 1017 The new point introduced by Brox, Olaussen and Nguyen is that the effective interaction between the apparatus and 1018 the measured system is non-linear in the wavefunction: it contains an analogue of a self-induced magnetic field [173]. 1019 In contrast to the existing approaches, where non-linearity in the Schrödinger equation are introduced axiomatically, 1020 Brox, Olaussen and Nguyen state that their non-linearity can in fact emerge from the Hartree-Fock approach: it is 1021 known that in certain situations (the Vlasov limit) the many-body Schrödinger equation can be reduced to a non-linear equation for the single-particle wave function [174]. Examples of this are the Gross-Pitaevskii equation for Bose 1023 condensates [174] or the non-linear equation arising during quantum feedback control [175]. However, the statement 1024 by Brox, Olaussen and Nguyen on the emergent non-linearity is not really proven, which is an essential drawback. 1025 Leaving this problem aside, these authors show numerically that the specific nonlinearity in the system-apparatus 1026 interaction may lead to a definite, albeit random, measurement result. The statistics of this randomness approximately 1027 satisfies the Born rule [1], which emerges due to the macroscopic size of the apparatus. The cause of this randomness 1028 is the classical randomness related to the choice of the spin-glass interaction constants in the environment [173], i.e., 1029 for different such choices (each one still ensuring the proper relaxation of the apparatus) one gets different single-103 measurement results. Thus in this approach the cause of the randomness in measurement results is not the irreducible 1031 quantum randomness, but rather the usual classical randomness, which is practically unavoidable in the preparation of 1032 a macroscopic environment. Brox, Olaussen and Nguyen argue that the nonlinearity in the system-bath interaction -1033 which is crucial for obtaining all the above effects — need not be large, since the amplification may be ensured by a 1034 large size of the ferromagnet [173]. Their actual numerical calculations are however carried out only for moderate-size 1035 spin systems. 1036

1037 2.7. Decoherence theory

1038

Coherence is needed under all circumstances Anonymous

Presently it is often believed that decoherence theory solves the quantum measurement problem. So let us introduce this concept. Decoherence refers to a process, where due to coupling with an external environment, off-diagonal elements of the system density matrix decay in time; see [32, 33, 40, 176, 177, 178, 179] for reviews. The basis where this decay happens is selected by the structure of the system-environment coupling. In this way the system acquires some classical features.

Decoherence is well known since the late 40's [181]. One celebrated example is spin relaxation in NMR experiments. The decay of the transverse polarization, perpendicular to the permanently applied field, is in general characterized by the relaxation time T_2 ; it can be viewed as a decoherence of the spin system, since it exhibits the decay of the off-diagonal contributions to the spin density matrix in the representation where the applied Hamiltonian
 is diagonal [182, 183]. Another standard example is related to the Pauli equation for an open quantum system weakly
 coupled to an external thermal bath [184]. This equation can be visualized as a classical stochastic process during
 which the system transits from one energy level to another.

More recently decoherence has attracted attention as a mechanism of quantum-to-classical transition, and was 1052 applied to the quantum measurement problem [32, 33, 40, 176, 177, 178, 179]. The standard pattern of such an appli-1053 cation relies on an initial impulsive interaction of the von Neumann type which correlates (entangles) the measuring 1054 apparatus with the system to be measured. Generally, this step is rather unrealistic, since it realizes macroscopic 1055 superpositions, which were never seen in any realistic measurement or any measurement model. Next, one assumes 1056 a specific environment for the apparatus, with the environment-apparatus interaction Hamiltonian directly related to the variable to be measured. Moreover, within the decoherence approach it is stressed - e.g., by Zurek in [33] and 1058 by Milburn and Walls in [178] – that the observable to-be-measured is determined during the process generated by 1059 the apparatus-environment interaction. The latter is supposed to diagonalize the density matrix of the system plus the 1060 apparatus in a suitable basis. This second step is again unrealistic, since it assumes that the variable to be measured, 1061 which is normally under control of the experimentalist, must somehow correlate with the structure of the system's en-1062 vironment, which – by its very definition – is out of direct control. To put it in metaphoric terms, decoherence theory 1063 asserts that the surrounding air measures a person's size. But without explicit pointer variable that can be read off, this is not what one normally understands under *measuring a person's size*; we consider *measurement without a readable* 1065 pointer variable merely as a linguistic redefinition of the concept, that obscures the real issue. These criticisms of the 1066 decoherence theory approach agree with the recent analysis by Requardt [69]. 1067

One even notes that, as far as the problem of quantum-to-classical transition is concerned, the decoherence cannot be regarded as the only – or even as the basic – mechanism of this transition. As convincingly argued by Wiebe and Ballentine [132] and Ballentine [185], realistic macroscopic Hamiltonian systems can – and sometimes even should – achieve the classical limit without invoking any decoherence effect. This concerns both chaotic and regular Hamiltonian systems, although the concrete scenarios of approaching the classical limit differ for the two cases.

In spite of these caveats that prevent decoherence theory to provide *the* solution, it has been valuable in shaping the ideas on quantum measurement models, In particular, this concerns a recent attempt by Omnès to develop a general theory of decoherence via ideas and methods of non-equilibrium statistical mechanics [88] (see also [162] that we reviewed above). Among the issues addressed in [88] is the generality of the system-environment structure that leads to decoherence, the physical meaning of separating the system from the environment, and the relation of the decoherence theory to the hydrodynamic description.

1079 2.8. Seeking the solution outside quantum mechanics

No, no, you're not thinking; you're just being logical Niels Bohr

Though this review will restrict itself to approaches to quantum measurements within the standard quantum me-1082 chanics, we briefly list for completeness a number of attempts to seek the solution for the quantum measurement 1083 problem beyond it. The de Broglie–Bohm approach [18, 19, 24] is currently one of the most popular alternatives to 1084 the standard quantum mechanics. It introduces an additional set of variables (coordinates of the physical particles) 1085 and represents the Schrödinger equation as an equation of motion for those particles, in addition to the motion of the wavefunction, which keeps the physical meaning of a separate entity (guiding field). Hence in this picture there are 1087 two fundamental and separate entities: particles and fields. Recently Smolin attempted to construct a version of the 1088 de Broglie-Bohm approach, where the wavefunction is substituted by certain phase-variables, which, together with 1089 coordinates, are supposed to be features of particles [186]. In this context see also a related contribution by Schmelzer, 1090 where the fundamental character of the wavefunction is likewise negated [187]. The approach by Smolin is coined in 1091 terms of a real ensemble, which — in contrast to ensembles of non-interacting objects invoked for validation of any 1092 probabilistic theory — does contain highly-nonlocal (distance independent) interactions between its constituents. It is 1093 presently unclear to which extent this substitution of the wavefunction by phase-variables will increase the eligibility 109 of the de Broglie-Bohm approach, while Smolin does not discuss the issues of measurement that are known to be 1095 non-trivial within the approach [18, 19, 189]. 1096

¹⁰⁸⁰

Another popular alternative is the spontaneous localization approach by Ghirardi, Rimini and Weber [190]. This 1097 approach is based on a non-linear and stochastic generalization of the Schrödinger equation such that the collapse 1098 of the wavefunction happens spontaneously (i.e., without any measurement) with a certain rate governed by classical 1099 white noise. Bassi and Ghirardi recently reviewed this and related approaches in full detail [17]; other useful sources 1100 are the book by Adler [191] and the review paper by Pearle [192]. Spontaneous localization models in the energy basis 1101 are especially interesting, since they conserve the average energy of the quantum system; this subject is reviewed by 1102 Brody and Hughston [193]. Non-linear modifications of the Schrödinger equation have by now a long history [89, 1103 90, 194, 195, 196, 197, 198]. All of them in one way or another combine non-linearities with classical randomness. 1104 The first such model was introduced by Bohm and Bub [194] starting from certain hidden-variables assumption. The 1105 approaches that followed were either oriented towards resolving the quantum measurement problem [89, 196, 197] or trying to obtain fundamentally nonlinear generalizations of the Schrödinger equation and quantum mechanics [198]. 1107 Several approaches of the former type were unified within a formalism proposed by Grigorenko [90]. Recently 1108 Svetlichny presented a resource letter on fundamental (i.e., not emerging from the usual, linear theory) non-linearities 1109 in quantum mechanics, where he also discusses their possible origin [199]. Some of those approaches based on 1110 nonlinear generalizations of the Schrödinger equation were confronted to experiments, see e.g. Refs. [200, 201], but 1111 so far with negative result. 1112

A very different approach was taken by De Raedt and Michielsen, who simulate the measurement process by specifying a set of simple rules that mimic the various components of the measurement setup, such as beam splitters, polarizers and detectors. They perform numerical simulations using algorithms that the mimic the underlying events, and are able to reproduce the statistical distributions given by quantum mechanics [202, 203].

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1117 2.9. A short review on consistent histories
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To understand the future you must know the past Tyler Maret

The consistent histories approach negates the fundamental need of measurements for understanding quantum 1120 measurements (quantum mechanics without measurements). It was proposed by Griffiths [204] based on earlier ideas 1121 of Aharonov, Bergmann and Lebowitz [205]. The approach is reviewed, e.g., by Griffiths [206], Gell-Mann [207], 1122 Hohenberg [208], and Omnès [209]. It aims to develop an interpretation of quantum mechanics valid for a closed 1123 system of any size and any number of particles, the evolution of which is governed by the Liouville-von Neumann 1124 (or Heisenberg) equation. Within this approach the notion of an event-together with its probability-is defined from 1125 the outset and "measurements", which do not involve any interaction between the system and some apparatus, simply 1126 reveal the pre-existing values of physical quantities. In particular, it is not necessary to invoke either the micro-1127 macro separation or statistical assumptions on the initial states needed to derive the irreversibility aspect of quantum 1128 measurements. All of these may still be needed to describe concrete measurements, but the fundamental need for 1129 understanding quantum measurements from the viewpoint of statistical mechanics would be gone, if the consistent 1130 histories approach is viable or, at least, will turn out to be really viable in the end. 1131

However, as it stands presently the approach produces results at variance with predictions of the measurementbased quantum mechanics [210] (then it is not important which specific interpretation one prescribes). Hence, within its present status, the consistent histories approach cannot be a substitute for the statistical mechanics-based theory of quantum measurements. Some authors by-pass problems of the consistent histories approach and state that it is useful as a paradox-free reformulation of the standard mecanics; see e.g. the recent review by Hohenberg [208] and the book by Griffiths [206]. In fact the opposite is true: as we explain below, the consistent histories approach adds paradoxes that do not exist within the statistical interpretation of quantum mechanics.

1139 2.9.1. Deeper into consistent histories

The easiest method of introducing the consistent histories approach is to highlight as soon as possible its differences with respect to the standard measurement-based approach. Let us start with the quantum mechanics formula for the probability of two consecutive measurements $\mathcal{M}(t_1)$ and $\mathcal{M}(t_2)$ carried at times t_1 and t_2 ($t_2 > t_1$):

$$p_{t_1,t_2}[i, j|\mathcal{M}(t_1), \mathcal{M}(t_2)] = \text{tr} \left[\Pi_j(t_2) \Pi_i(t_1) \rho \Pi_i(t_1) \Pi_j(t_2) \right],$$
(2.1)

where ρ is the initial state of the system, $\Pi_i(t_1)$ with $\sum_i \Pi_i(t_1) = 1$ and $\Pi_j(t_2)$ with sum $\sum_i \Pi_i(t_2) = 1$ are the projectors for the physical quantities (represented by Hermitean operators) $A(t_1)$ and $B(t_2)$ measured at the times t_1 and t_2 , respectively. For simplicity we assume the Heisenberg representation, and do not write in (2.1) explicit indices for *A* and *B*. What is however *necessary* to do is to indicate that the joint probability in (2.1) is explicitly conditional on the two measurements $\mathcal{M}(t_1)$ and $\mathcal{M}(t_2)$. As expected, the meaning of (2.1) is that the measurement at time t_1 (with probabilities given by Born's rule) is accompanied by selection of the subensemble referring to the result *i*. The members of this subensemble are then measured at the time t_2 . Generalization to *n*-time measurements is obvious.

¹¹⁵⁰ What now the consistent histories approach does is to skip the context-dependence in (2.1) and regard the resulting ¹¹⁵¹ probabilities p[i, j] as a description of events taking place spontaneously, i.e. *without any measurement and without* ¹¹⁵² *any selection of outcome*. The cost to pay is that the initial state ρ and the projectors $\Pi_i(t_1)$ and $\Pi_j(t_2)$ have to satisfy ¹¹⁵³ a special *consistency* condition (without this condition the events are not defined):

$$\operatorname{tr}\left[\Pi_{j}(t_{2})\Pi_{i}(t_{1})\rho\Pi_{i'}(t_{1})\Pi_{j'}(t_{2})\right] = \delta_{ii'}\delta_{jj'}p_{t_{1},t_{2}}[i,j],$$
(2.2)

where $\delta_{ii'}$ is the Kronecker delta. As a consequence of (2.2), one can sum out the first (i. e., the earlier) random variable and and using the completeness relation $\sum_{i} \prod_{i} (t_1) = 1$ get the probability for the second event alone:

$$p_{t_2}[j] = \sum_i p_{t_1, t_2}[i, j] = \operatorname{tr} \left[\Pi_j(t_2) \rho \Pi_j(t_2) \right].$$
(2.3)

Note that without condition (2.2), i.e. just staying within the standard approach (2.1), Eq. (2.3) would not hold, e.g. generally $\sum_{i} p_{t_1,t_2}[i, j|\mathcal{M}(t_1), \mathcal{M}(t_2)]$ still depends on $\mathcal{M}(t_1)$ and is not equal to $p_{t_2}[j|\mathcal{M}(t_2)]$ (probability of the outcome *j* in the second measurement provided that no first measurement was done). This is natural, since quantum measurements generally do perturb the state of the measured system. Hence (2.2) selects only those situations, where this perturbation is seemingly absent.

Any time-ordered sequence of events defines a history. A set of histories satisfying (2.2) is called a *consistent histories* set. Due to (2.2), the overall probability of the consistent histories sums to one.

In effect (2.2) forbids superpositions; hence, it is called decoherence condition [206, 207, 208]. One notes that (2.2) is sufficient, but not necessary for deriving (2.3). Hence, certain weaker conditions instead of (2.2) were also studied [204], but generally they do not satisfy the straightforward statistical independence conditions (independently evolving systems have independent probabilities) [211].

It was however noted that the consistent histories approach can produce predictions at variance with the measurement based quantum mechanics [210]. The simplest example of such a situation is given in [212]. Consider a quantum system with zero Hamiltonian in the pure initial state

$$\rho = |\phi\rangle\langle\phi|, \ |\phi\rangle = \frac{1}{\sqrt{3}}[|a_1\rangle + |a_2\rangle + |a_3\rangle], \tag{2.4}$$

where the vectors $\{|a_k\rangle\}_{k=1}^3$ are orthonormal: $\langle a_k | a_{k'} \rangle = \delta_{kk'}$. Define a two-event history with projectors

$$\Pi_1(t_1) = |a_1\rangle\langle a_1|, \Pi_2(t_1) = 1 - |a_1\rangle\langle a_1|\} \text{ and } \{\Pi_1(t_2) = |\psi\rangle\langle\psi|, \Pi_2(t_2) = 1 - |\psi\rangle\langle\psi|\}, \ t_2 > t_1,$$
(2.5)

1171 where

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p

$$|\psi\rangle = \frac{1}{\sqrt{3}}[|a_1\rangle + |a_2\rangle - |a_3\rangle].$$
 (2.6)

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This history is consistent, since conditions (2.2) hold due to $\langle \phi | \psi \rangle = \langle \phi | a_1 \rangle \langle a_1 | \psi \rangle$. One now calculates

$$t_{1,t_2}[a_1,\psi] = \operatorname{tr}\left[\Pi_1(t_2)\Pi_1(t_1)\rho\Pi_1(t_1)\Pi_1(t_2)\right] = \langle\psi|a_1\rangle\langle a_1|\phi\rangle\langle\phi|a_1\rangle\langle a_1|\psi\rangle = \frac{1}{9},\tag{2.7}$$

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$$p_{t_2}[\psi] = \operatorname{tr}\left[\Pi_1(t_2)\rho\Pi_1(t_2)\right] = |\langle\psi|\phi\rangle|^2 = \frac{1}{9}.$$
(2.8)

Given two probabilities (2.7) and (2.8) one can calculate the following conditional probability:

$$p_{t_1|t_2}[a_1|\psi] = \frac{p_{t_1,t_2}[a_1,\psi]}{p_{t_2}[\psi]} = 1.$$
(2.9)

1175 Yet another two-event consistent history is defined with projectors

$$\{\Pi_1(t_1) = |a_2\rangle\langle a_2|, \Pi_2(t_1) = 1 - |a_2\rangle\langle a_2|\} \text{ and } \{\Pi_1(t_2) = |\psi\rangle\langle\psi|, \Pi_2(t_2) = 1 - |\psi\rangle\langle\psi|\}, \ t_2 > t_1.$$
(2.10)

Comparing (2.10) with (2.5) we note that the first measurement at t_1 is different, i.e. it refers to measuring a different physical observable. Analogously to (2.9) we calculate for the second consistent history

$$p_{t_1|t_2}[a_2|\psi] = 1. \tag{2.11}$$

The consistent histories (2.5) and (2.10) share one event, ψ , at the later time. On the basis of this event (2.9) retrodicts with probability one (i.e., with certainty) that a_1 happened. Likewise, (2.11) retrodicts with certainty that a_2 happened. But the events a_1 and a_2 are mutually incompatible, since their projectors are orthogonal, $\langle a_1 | a_2 \rangle = 0$: if one happened, the other one could not happen.

Note that such an inconsistency is excluded within the measurement-based approach. There the analogues of (2.9) and (2.11) refer to different contexts [different measurements]: they read, respectively, $p[a_1|\psi, \mathcal{M}(t_1), \mathcal{M}(t_2)] = 1$ and $p[a_2|\psi, \widetilde{\mathcal{M}}(t_1), \mathcal{M}(t_2)] = 1$. It is not surprising that different contexts, $\mathcal{M}(t_1) \neq \widetilde{\mathcal{M}}(t_1)$, force conditional probabilities to retrodict incompatible events. Naturally, if within the standard approach one makes the same measurements the incompatible events cannot happen, e.g. $p[a_1|\psi, \mathcal{M}(t_1), \mathcal{M}(t_2)] \times p[a_2|\psi, \mathcal{M}(t_1), \mathcal{M}(t_2)] = 0$, because the second probability is zero.

Following Kent [210] we interpret this effect as a disagreement between the predictions (or more precisely: the retrodictions) of the consistent history approach versus those of the measurement-based quantum mechanics. In response to Kent, Griffiths and Hartle suggested that for avoiding above paradoxes, predictions and retrodictions of the approach are to be restricted to a single consistent history [212, 213]. Conceptually, this seems to betray the very point of the approach, because in effect it brings back the necessity of fixing the context within which a given consistent history takes place. And what fixes this context, once measurements are absent?

Another possible opinion is that condition (2.2) is not strong enough to prevent a disagreement with the measurement based approach, and one should look for a better condition for defining events [214, 215]. To our knowledge, such a condition is so far not found. Bassi and Ghirardi [216] pointed out another logical problem with the consistent histories approach. This produced another debate on the logical consistency of the approach [217, 218], which we will not discuss here.

We hold the opinion that in spite of being certainly thought-provoking and interesting, the consistent histories approach, as it presently stands, cannot be a substitute for the theory of quantum measurements: Both conceptually and practically we still need to understand what is going on in realistic measurements, with their imperfections, and what are the perturbations induced on the system by its interaction with a measuring apparatus.

1203 2.10. What we learned from these models

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Ձկնորսը ձկանը ասեց. "Ի՞նչ կա-չկա ծովում"։ ²⁵ "Ասելիք շատ ունեմ, բայց բերանս լիքը ջուր է"։ Armenian proverb

Each one of the above models enlightens one or another among the many aspects of quantum measurements. 1206 However, none of them reproduces the whole set of desired features: truncation and reduction of S + A, Born's rule, 1207 uniqueness of the outcome of a single process, complete scenario of the joint evolution of S + A, with an evaluation of 1208 its characteristic times, metastability of the initial state of A, amplification within A of the signal, unbiased and robust 1209 registration by A in the final state, accurate establishment between S and the pointer variable of A of the correlations 1210 that characterize an ideal measurement, influence of the parameters of the model on possible imperfections of the 1211 measurement. In particular, permanent registration requires the pointer to be macroscopic. In the following we 1212 study in detail a model, introduced in Refs. [68, 219, 220, 221, 222, 223, 224], which encompasses these various 1213 requirements. 1214

²⁵Fisherman: "What's the news from the sea?" Fish: "I have a lot to say, but my mouth is full of water"

1215 3. A Curie–Weiss model for quantum measurements

1216 3.1. General features

We take for S, the system to be measured, the simplest quantum system, namely a spin $\frac{1}{2}$ (or any two-level system). The observable \hat{s} to be measured is its third Pauli matrix \hat{s}_z , with eigenvalues s_i equal to ± 1 . The statistics of this observable should not change significantly during the measurement process [4, 13, 225]. Hence \hat{s}_z should be *conservative*, i.e., should commute with the Hamiltonian of S + A, at least nearly.

We have stressed at the end of § 1.2.1 that the apparatus A should lie initially in a metastable state [226, 227], and finally in either one of several possible stable states (see section 2 for other models of this type). This suggests to take for A, as in several models described in section 2, a quantum system that may undergo a phase transition with *broken invariance*. The initial state $\hat{R}(0)$ of A is the metastable phase with unbroken invariance. The states \hat{R}_i represent the stable phases with broken invariance, in each of which registration can be permanent. The symmetry between the outcomes prevents any bias.

Here we need two such stable states, in one-to-one correspondence with the two eigenvalues s_i of \hat{s}_z . The simplest system which satisfies these properties is an Ising model [227]. Conciliating mathematical tractability and realistic features, we thus take as apparatus A = M + B, a model that simulates a *magnetic dot*: The magnetic degrees of freedom M consist of $N \gg 1$ spins with Pauli operators $\hat{\sigma}_a^{(n)}$ ($n = 1, 2, \dots, N$; a = x, y, z), which read for each n

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \hat{\sigma}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{3.1}$$

where $\hat{\sigma}_0$ is the corresponding identity matrix; $\hat{\sigma} = (\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z)$ denotes the vector spin operator. The non-magnetic degrees of freedom such as phonons behave as a thermal bath B (Fig. 3.1). Anisotropic interactions between these spins can generate Ising ferromagnetism below the Curie temperature T_c . As pointer variable \hat{A} we take the order parameter, which is the magnetization in the *z*-direction (within normalization), as represented by the quantum observable

$$\hat{m} = \frac{1}{N} \sum_{n=1}^{N} \hat{\sigma}_{z}^{(n)}.$$
(3.2)

We let *N* remain finite, which will allow us to keep control of the equations of motion. It should however be sufficiently large so as to ensure the required properties of phase transitions: The relaxation from $\hat{\mathcal{R}}(0)$ to either one of the two states $\hat{\mathcal{R}}_i$, at the temperature *T* (below T_c) imposed by the bath B, should be irreversible, the fluctuations of the order parameter \hat{m} in each equilibrium state $\hat{\mathcal{R}}_i$ should be weak (as $1/\sqrt{N}$), and the transition between these two states $\hat{\mathcal{R}}_i$ should be nearly forbidden.

The initial state $\hat{\mathcal{R}}(0)$ of A is the metastable paramagnetic state. We expect the final state (1.7) of S + A to involve for A the two stable ferromagnetic states $\hat{\mathcal{R}}_i$, $i = \uparrow$ or \downarrow , that we denote as $\hat{\mathcal{R}}_{\uparrow}$ or $\hat{\mathcal{R}}_{\downarrow}$, respectively²⁶. The equilibrium temperature *T* will be imposed to M by the phonon bath [174, 120] through weak coupling between the magnetic and non-magnetic degrees of freedom. Within small fluctuations, the order parameter (3.2) vanishes in $\hat{\mathcal{R}}(0)$ and takes two opposite values in the states $\hat{\mathcal{R}}_{\uparrow}$ and $\hat{\mathcal{R}}_{\downarrow}$, $A_i \equiv \langle \hat{m} \rangle_i$ equal to $+m_{\rm F}$ for $i = \uparrow$ and to $-m_{\rm F}$ for $i = \downarrow^{27}$. As in real magnetic registration devices [228], information will be stored by A in the form of the sign of the magnetization.

1247 3.2. The Hamiltonian

1248 1249 I ask not for a lighter burden, but for broader shoulders Jewish proverb

 $^{^{26}}$ Here and in the following, single arrows \uparrow, \downarrow will denote the spin S, while double arrows \uparrow, \downarrow denote the magnet M

²⁷Note that the values $A_i = \pm m_F$, which we wish to come out associated with the eigenvalues $s_i = \pm 1$, are determined from equilibrium statistical mechanics; they are not the eigenvalues of $\hat{A} \equiv \hat{m}$, which range from -1 to +1 with spacing 2/N, but thermodynamic expectation values around which small fluctuations of order $1/\sqrt{N}$ occur



Figure 3.1: The Curie-Weiss measurement model and its parameters. The system S is a spin \hat{s} . The apparatus A includes a magnet M and a bath B. The magnet, which acts as a pointer, consists of N spins coupled to one another through an Ising interaction J. The phonon bath B is characterized by its temperature T and a Debye cutoff Γ . It interacts with M through a spin-boson coupling γ . The process is triggered by the interaction g between the measured observable \hat{s}_{z} and the magnetization per spin, \hat{m} , of the pointer.

The full Hamiltonian can be decomposed into terms associated with the system, with the apparatus and with their coupling:

$$\hat{H} = \hat{H}_{\rm S} + \hat{H}_{\rm SA} + \hat{H}_{\rm A}.\tag{3.3}$$

1252 3.2.1. The system

As indicated above, for an ideal measurement the observable \hat{s} should commute with \hat{H} [13, 158, 225]. The simplest self-Hamiltonian that ensures this property (no evolution of S without coupling to A), is the trivial one,

$$\hat{H}_{\rm S} = 0. \tag{3.4}$$

This property is required for ideal measurements, during the process of which the statistics of the tested observable

should not be affected. More generally, in order to describe an imperfect measurement where \hat{s} may move noticeably during the measurement (subsection 8.2), we shall introduce there a magnetic field acting on the tested spin.

¹²⁵⁸ The coupling between the tested system and the apparatus,

$$\hat{H}_{SA} = -g\hat{s}_z \sum_{n=1}^{N} \hat{\sigma}_z^{(n)} = -Ng\hat{s}_z\hat{m},$$
(3.5)

has the usual form of a spin-spin coupling in the z-direction [227], and the constant g > 0 characterizes its strength. 1259 As wished, it commutes with \hat{s}_z . We have assumed that the range of the interaction between the spin S and the N spins 1260 of M is large compared to the size of the magnetic dot, so that we can disregard the space-dependence of the coupling. 1261 The occurrence of the factor N in (3.5) should not worry us, since we will not take the thermodynamic limit $N \to \infty$. 1262 Although sufficiently large to ensure the existence of a clear phase transition, N is finite. We shall resume in § 9.4 the 1263 conditions that N should satisfy. In a realistic setting, the interaction between S and M would first be turned on, then 126 turned off continuously, while the tested spin approaches the dot then moves away. For simplicity we assume \hat{H}_{SA} to 1265 be turned on suddenly at the initial time t = 0, and it will be turned off at a final time t_f , as we discuss below²⁸. 1266

1267 3.2.2. The magnet

The apparatus A consists, as indicated above, of a magnet M and a phonon bath B (Fig. 3.1), and its Hamiltonian can be decomposed into

$$\hat{H}_{\rm A} = \hat{H}_{\rm M} + \hat{H}_{\rm B} + \hat{H}_{\rm MB}.$$
 (3.6)

¹²⁷⁰ The magnetic part is chosen as [228]

$$\hat{H}_{\rm M} = -N \sum_{q=2,4} J_q \frac{\hat{m}^q}{q} = -N J_2 \frac{\hat{m}^2}{2} - N J_4 \frac{\hat{m}^4}{4}, \tag{3.7}$$

 $^{^{28}}$ Contrary to the switching on, this switching off need not be performed suddenly since $m_{\rm F}$ is close to $m_{\rm fl}$

where the magnetization operator \hat{m} was defined by (3.2). It couples all *q*-plets of spins $\hat{\sigma}^{(n)}$ symmetrically, with the same coupling constant $J_q N^{1-q}$ for each *q*-plet. (The factor N^{1-q} is introduced only for convenience.) The space-independence of this coupling is fairly realistic for a small magnetic dot, as in (3.5). The interaction is fully anisotropic, involving only the *z*-components. The exponents *q* are even in order to ensure the up-down symmetry of the apparatus. The term q = 4 describes so-called super-exchange interactions as realized for metamagnets [228]. We shall only consider ferromagnetic interactions ($J_2 > 0$ or $J_4 > 0$ or both).

¹²⁷⁷ We will see in § 3.3.4 that the Hamiltonian (3.6) produces a paramagnetic equilibrium state at high temperature ¹²⁷⁸ and two ferromagnetic states at low temperature, with a transition of second order for $J_2 > 3J_4$, of first order for ¹²⁷⁹ $3J_4 > J_2$. The former case is exemplified by the Curie–Weiss Ising model for an anisotropic magnetic dot [227], with ¹²⁸⁰ pairwise interactions in $\hat{\sigma}_z^{(n)} \hat{\sigma}_z^{(p)}$, recovered here for $J_4 = 0$,

$$\hat{H}_{\rm M} = -\frac{J_2 N}{2} \hat{m}^2 = -\frac{J_2}{2N} \sum_{i,j=1}^N \hat{\sigma}_z^{(i)} \hat{\sigma}_z^{(j)}, \qquad (q=2).$$
(3.8)

Likewise, the first-order case is exemplified by letting $J_2 = 0$, keeping in (3.6) only the quartic "super-exchange" term:

$$\hat{H}_{\rm M} = -\frac{J_4 N}{4} \hat{m}^4 = -\frac{J_4}{4N^3} \sum_{i,j,k,l=1}^N \hat{\sigma}_z^{(i)} \hat{\sigma}_z^{(j)} \hat{\sigma}_z^{(k)} \hat{\sigma}_z^{(l)}, \qquad (q=4).$$
(3.9)

The more physical case (3.6) of mixtures of q = 2 and q = 4 terms will not differ qualitatively from either one of the two pure cases q = 2 or q = 4. It will therefore be sufficient for our purpose, in section 7, to illustrate the two situations $J_2 > 3J_4$ and $J_2 < 3J_4$ by working out the Hamiltonians (3.8) and (3.9), respectively. We may summarize these two cases by $H_{\rm M} = -(NJ/q)\hat{m}^q$ with q = 2 and 4, respectively.

Using A as a measurement apparatus requires the lifetime of the initial state to be larger than the overall measurement time. An advantage of a first-order transition is the local stability of the paramagnetic state, even below the transition temperature, which ensures a much larger lifetime as in the case of a second order transition. We shall see, however (§ 7.3.2), that the required condition can be satisfied even for q = 2 alone (i.e., for $J_4 = 0$).

1291 3.2.3. The phonon bath

The interaction between the magnet and the bath, which drives the apparatus to equilibrium, is taken as a standard spin-boson Hamiltonian [174, 120, 121]

$$\hat{H}_{\rm MB} = \sqrt{\gamma} \sum_{n=1}^{N} \left(\hat{\sigma}_x^{(n)} \hat{B}_x^{(n)} + \hat{\sigma}_y^{(n)} \hat{B}_y^{(n)} + \hat{\sigma}_z^{(n)} \hat{B}_z^{(n)} \right) \equiv \sqrt{\gamma} \sum_{n=1}^{N} \sum_{a=x,y,z} \hat{\sigma}_a^{(n)} \hat{B}_a^{(n)}, \tag{3.10}$$

which couples each component a = x, y, z of each spin $\hat{\sigma}^{(n)}$ with some hermitean linear combination $\hat{B}_a^{(n)}$ of phonon operators. The dimensionless constant $\gamma \ll 1$ characterizes the strength of the thermal coupling between M and B, which is weak.

For simplicity, we require that the bath acts independently for each spin degree of freedom *n*, *a*. (The so-called independent baths approximation.) This can be achieved (i) by introducing Debye phonon modes labelled by the pair of indices *k*, *l*, with eigenfrequencies ω_k depending only on *k*, so that the bath Hamiltonian is

$$\hat{H}_{\rm B} = \sum_{k,l} \hbar \omega_k \hat{b}^{\dagger}_{k,l} \hat{b}_{k,l}, \qquad (3.11)$$

and (ii) by assuming that the coefficients C in

$$\hat{B}_{a}^{(n)} = \sum_{k,l} \left[C(n,a;k,l) \, \hat{b}_{k,l} + C^*(n,a;k,l) \, \hat{b}_{k,l}^{\dagger} \right]$$
(3.12)

1301 are such that

$$\sum_{l} C(n, a; k, l) C^{*}(m, b; k, l) = \delta_{n,m} \delta_{a,b} c(\omega_{k}).$$
(3.13)


Figure 3.2: Notations for the density operators of the system S + A and the subsystems M and B of A. The full density matrix \hat{D} is parametrized by its submatrices \hat{R}_{ij} (with $i, j = \pm 1$ or \uparrow, \downarrow), the density matrix \hat{D} of S + M by its submatrices \hat{R}_{ij} . The marginal density operator of S is denoted as \hat{r} and the one of A as \hat{R} . The marginal density operator of M itself is denoted as \hat{R}_M and the one of B as \hat{R}_B .

This requires the number of values of the index *l* to be at least equal to 3*N*. For instance, we may associate with each component *a* of each spin $\hat{\sigma}^{(n)}$ a different set of phonon modes, labelled by *k*, *n*, *a*, identifying *l* as (*n*, *a*), and thus define \hat{H}_{B} and $\hat{B}_{a}^{(n)}$ as

$$\hat{H}_{\rm B} = \sum_{n=1}^{N} \sum_{a=x,y,z} \sum_{k} \hbar \omega_k \hat{b}_{k,a}^{\dagger(n)} \hat{b}_{k,a}^{(n)}, \qquad (3.14)$$

$$\hat{B}_{a}^{(n)} = \sum_{k} \sqrt{c(\omega_{k})} \left(\hat{b}_{k,a}^{(n)} + \hat{b}_{k,a}^{\dagger(n)} \right).$$
(3.15)

We shall see in § 3.3.2 that the various choices of the phonon set, of the spectrum (3.11) and of the operators (3.12) coupled to the spins are equivalent, in the sense that the joint dynamics of S + M will depend only on the spectrum ω_k and on the coefficients $c(\omega_k)$.

The spin-boson coupling (3.10) between M and B will be sufficient for our purpose up to section 9. This interaction, of the Glauber type, does not commute with \hat{H}_{M} , a property needed for registration, since M has to release energy when relaxing from its initial metastable paramagnetic state to one of its final stable ferromagnetic states at the temperature *T*. However, the complete solution of the measurement problem presented in section 11 will require more complicated interactions. We will therefore introduce in § 11.2.3 a small but random coupling between the spins of M, and in § 11.2.4 a more realistic small coupling between M and B, of the Suzuki type, which produces flip-flops of the spins of M without changing the value of the energy that M would have with only the terms (3.8) and/or (3.9).

1315 3.3. Structure of the states

1316If you do not enter the tiger's cave, you will not catch its cub1317Japanese proverb

1318 3.3.1. Notations

The full state \hat{D} of the system evolves according to the Liouville–von Neumann equation (1.6), which we have to solve. It will be convenient to define through partial traces, at any instant *t*, the following marginal density operators: \hat{r} for the tested system S, $\hat{\mathcal{R}}$ for the apparatus A, $\hat{R}_{\rm M}$ for the magnet M, $\hat{R}_{\rm B}$ for the bath, and \hat{D} for S + M after elimination of the bath (as depicted schematically in Fig. 3.2), according to

$$\hat{r} = \operatorname{tr}_{A}\hat{\mathcal{D}}, \qquad \hat{\mathcal{R}} = \operatorname{tr}_{S}\hat{\mathcal{D}}, \qquad \hat{R}_{M} = \operatorname{tr}_{B}\hat{\mathcal{R}} = \operatorname{tr}_{S,B}\hat{\mathcal{D}}, \qquad \hat{R}_{B} = \operatorname{tr}_{S,M}\hat{\mathcal{D}}, \qquad \hat{D} = \operatorname{tr}_{B}\hat{\mathcal{D}}.$$
 (3.16)

The expectation value of any observable \hat{A} pertaining, for instance, to the subsystem S + M of S + A (including products of spin operators \hat{s}_a and $\hat{\sigma}_a^{(n)}$) can equivalently be evaluated as $\langle \hat{A} \rangle = \text{tr}_{S+A} \hat{D} \hat{A}$ or as $\langle \hat{A} \rangle = \text{tr}_{S+M} \hat{D} \hat{A}$.

As indicated in subsection 1.2, the apparatus A is a large system, treated by methods of statistical mechanics, while we need to follow in detail the microscopic degrees of freedom of the system S and their correlations with A. To this aim, we shall analyze the full state \hat{D} of the system into several sectors, characterized by the eigenvalues of \hat{s}_z . Namely, in the two-dimensional eigenbasis of \hat{s}_z for S, $|\uparrow\rangle$, $|\downarrow\rangle$, with eigenvalues $s_i = +1$ for $i = \uparrow$ and $s_i = -1$ for $i = \downarrow$,

 $\hat{\mathcal{D}}$ can be decomposed into the four blocks

$$\hat{\mathcal{D}} = \begin{pmatrix} \hat{\mathcal{R}}_{\uparrow\uparrow} & \hat{\mathcal{R}}_{\downarrow\downarrow} \\ \hat{\mathcal{R}}_{\downarrow\uparrow} & \hat{\mathcal{R}}_{\downarrow\downarrow} \end{pmatrix}, \qquad (3.17)$$

where each $\hat{\mathcal{R}}_{ij}$ is an operator in the space of the apparatus. We shall also use the partial traces (see again Fig. 3.2)

$$\hat{R}_{ij} = \text{tr}_{\text{B}}\hat{\mathcal{R}}_{ij}, \qquad \hat{D} = \text{tr}_{\text{B}}\hat{\mathcal{D}} = \begin{pmatrix} \hat{R}_{\uparrow\uparrow} & \hat{R}_{\uparrow\downarrow} \\ \hat{R}_{\downarrow\uparrow} & \hat{R}_{\downarrow\downarrow} \end{pmatrix}$$
(3.18)

over the bath; each \hat{R}_{ij} is an operator in the 2^N -dimensional space of the magnet. Indeed, we are not interested in the evolution of the bath variables, and we shall eliminate B by relying on the weakness of its coupling (3.10) with M. The operators \hat{R}_{ij} code our full statistical information about S and M. We shall use the notation \hat{R}_{ij} whenever we refer to S + M and \hat{R}_M when referring to M alone. The magnet M is thus described by $\hat{R}_M = \hat{R}_{\uparrow\uparrow} + \hat{R}_{\downarrow\downarrow}$, the system S alone by the matrix elements $r_{ij} = \text{tr}_M \hat{R}_{ij}$ of \hat{r} . The correlations of \hat{s}_z , \hat{s}_x or \hat{s}_y with and any function of the observables $\hat{\sigma}_a^{(n)}$ (a = x, y, z, n = 1, ...N) are represented by $\hat{R}_{\uparrow\uparrow} - \hat{R}_{\downarrow\downarrow}$, $\hat{R}_{\uparrow\downarrow} + \hat{R}_{\downarrow\uparrow}$, $i\hat{R}_{\uparrow\downarrow} - i\hat{R}_{\downarrow\uparrow}$, respectively. The operators $\hat{R}_{\uparrow\uparrow}$ and $\hat{R}_{\downarrow\downarrow}$ are hermitean positive, but not normalized, whereas $\hat{R}_{\downarrow\uparrow} = \hat{R}_{\uparrow\downarrow}^{\dagger}$. Notice that we now have from (3.16) – (3.18)

$$r_{ij} = \operatorname{tr}_{A}\hat{\mathcal{R}}_{ij} = \operatorname{tr}_{M}\hat{R}_{ij}, \qquad \hat{\mathcal{R}} = \hat{\mathcal{R}}_{\uparrow\uparrow} + \hat{\mathcal{R}}_{\downarrow\downarrow}, \qquad \hat{R}_{M} = \hat{R}_{\uparrow\uparrow} + \hat{\mathcal{R}}_{\downarrow\downarrow}, \qquad \hat{R}_{B} = \operatorname{tr}_{M}(\hat{\mathcal{R}}_{\uparrow\uparrow} + \hat{\mathcal{R}}_{\downarrow\downarrow}).$$
(3.19)

All these elements are functions of the time *t* which elapses from the beginning of the measurement at t = 0 when \hat{H}_{SA} is switched on to the final value t_f that we will evaluate in section 7. To introduce further notation, we mention that the combined system S + A = S + M + B should end up in

$$\hat{\mathcal{D}}(t_{\rm f}) = \begin{pmatrix} p_{\uparrow} \hat{\mathcal{R}}_{\uparrow} & 0\\ 0 & p_{\downarrow} \hat{\mathcal{R}}_{\Downarrow} \end{pmatrix} = p_{\uparrow} |\uparrow\rangle \langle\uparrow| \otimes \hat{\mathcal{R}}_{\uparrow} + p_{\downarrow} |\downarrow\rangle \langle\downarrow| \otimes \hat{\mathcal{R}}_{\Downarrow} = \sum_{i} p_{i} \hat{\mathcal{D}}_{i}, \qquad (3.20)$$

where $\hat{\mathcal{R}}_{\uparrow}(\hat{\mathcal{R}}_{\downarrow})$ is density matrix of the thermodynamically stable state of the magnet and bath, after the measurement, in which the magnetization is up, taking the value $m_{\uparrow}(g)$ (down, taking the value $m_{\downarrow}(g)$); these events should occur with probabilities p_{\uparrow} and p_{\downarrow} , respectively²⁹. The Born rule then predicts that $p_{\uparrow} = \text{tr}_{S}\hat{r}(0)\Pi_{\uparrow} = r_{\uparrow\uparrow}(0)$ and $p_{\downarrow} = r_{\downarrow\downarrow}(0)$. Since no off-diagonal terms occur in (3.20), a point that we wish to explain, and since we expect B to remain nearly in its intial equilibrium state, we may trace out the bath, as is standard in classical and quantum thermal physics, without losing significant information. It will therefore be sufficient for our purpose to show that the final state is

$$\hat{D}(t_{\rm f}) = \begin{pmatrix} p_{\uparrow} \hat{R}_{\rm M\uparrow} & 0\\ 0 & p_{\downarrow} \hat{R}_{\rm M\downarrow} \end{pmatrix} = p_{\uparrow} |\uparrow\rangle\langle\uparrow| \otimes \hat{R}_{\rm M\uparrow} + p_{\downarrow} |\downarrow\rangle\langle\downarrow| \otimes \hat{R}_{\rm M\downarrow},$$
(3.21)

¹³⁴⁸ now referring to the magnet M and tested spin S alone.

Returning to Eq. (3.19), we note that from any density operator \hat{R} of the magnet we can derive the *probabilities* $P_{M}^{\text{dis}}(m)$ for \hat{m} to take the eigenvalues m, where "dis" denotes their discreteness. These N + 1 eigenvalues,

$$m = -1, \qquad -1 + \frac{2}{N}, \qquad \dots, \qquad 1 - \frac{2}{N}, \qquad 1,$$
 (3.22)

have equal spacings $\delta m = 2/N$ and multiplicities

$$G(m) = \frac{N!}{\left[\frac{1}{2}N(1+m)\right]! \left[\frac{1}{2}N(1-m)\right]!} = \sqrt{\frac{2}{\pi N(1-m^2)}} \exp\left[N\left(-\frac{1+m}{2}\ln\frac{1+m}{2} - \frac{1-m}{2}\ln\frac{1-m}{2}\right) + O\left(\frac{1}{N}\right)\right].(3.23)$$

¹³⁵² Denoting by $\delta_{\hat{m},m}$ the projection operator on the subspace *m* of \hat{m} , the dimension of which is *G*(*m*), we have

²⁹Notice that in the final state we denote properties of the tested system by \uparrow , \downarrow and of the apparatus by \uparrow , \downarrow . In sums like (1.7) we will also use $i = \uparrow, \downarrow$, or sometimes $i = \pm 1$

$$P_{\rm M}^{\rm dis}(m,t) = {\rm tr}_{\rm M}\hat{R}_{\rm M}(t)\delta_{\hat{m},m}.$$
(3.24)

In the limit $N \gg 1$, where *m* becomes basically a continuous variable, we shall later work with the functions $P_{\rm M}(m, t)$

$$P_{\rm M}(m,t) = \frac{N}{2} P_{\rm M}^{\rm dis}(m,t), \qquad \int_{-1}^{1} {\rm d}m \, P_{\rm M}(m,t) = \sum_{m} P_{\rm M}^{\rm dis}(m,t) = 1, \qquad (3.25)$$

that have a finite and smooth limit for $N \to \infty$, and use similar relations between the functions P_{ij} and P_{ij}^{dis} , and C_a and C_a^{dis} , introduced next.

In what follows, the density operators $\hat{R}_{\rm M}$ will most often depend only on the observables $\hat{\sigma}_z^{(n)}$ and be symmetric functions of these observables. Hence, $\hat{R}_{\rm M}$ will reduce to a mere function of the operator \hat{m} defined by (3.2). In such a circumstance, eq. (3.24) can be inverted: the knowledge of $P_{\rm M}(m)$ is then sufficient to determine the density operator $\hat{R}_{\rm M}$, through a simple replacement of the scalar *m* by the operator \hat{m} in

$$\hat{R}_{\rm M}(t) = \frac{1}{G(\hat{m})} P_{\rm M}^{\rm dis}(\hat{m}, t) \,. \tag{3.26}$$

The expectation value of any product of operators $\hat{\sigma}_a^{(n)}$ of the magnet can then be expressed in terms of $P_{\rm M}^{\rm dis}(m)$. For instance, the two-spin correlations $(n \neq p)$ are related to the second moment of $P_{\rm M}^{\rm dis}(m)$ by

$$\operatorname{tr}_{S,A}\hat{\mathcal{D}}\hat{\sigma}_{a}^{(n)}\hat{\sigma}_{b}^{(p)} = \operatorname{tr}_{M}\hat{R}_{M}\hat{\sigma}_{a}^{(n)}\hat{\sigma}_{b}^{(p)} = \frac{\delta_{a,z}\delta_{b,z}}{N-1} \left[N\sum_{m}P_{M}^{\mathrm{dis}}(m)\,m^{2}-1\right].$$
(3.27)

Likewise, when the operators \hat{R}_{ij} in (3.18) depend only on \hat{m} , we can parameterize them at each time, according to

$$\hat{R}_{ij}(t) = \frac{1}{G(\hat{m})} P_{ij}^{\text{dis}}(\hat{m}, t), \qquad (3.28)$$

¹³⁶⁴ by functions $P_{ij}^{\text{dis}}(m)$ defined on the set (3.22) of values of *m*, with $[P_{ij}^{\text{dis}}(m)]^* = P_{ji}^{\text{dis}}(m)$. (For the moment we refrain ¹³⁶⁵ from denoting the explicit *t* dependence.) All statistical properties of S + M at the considered time can then be ¹³⁶⁶ expressed in terms of these functions $P_{ij}^{\text{dis}}(m)$. Indeed the combinations

$$C_x^{\rm dis}(m) = P_{\uparrow\downarrow}^{\rm dis}(m) + P_{\downarrow\uparrow}^{\rm dis}(m), \qquad C_y^{\rm dis} = iP_{\uparrow\downarrow}^{\rm dis} - iP_{\downarrow\uparrow}^{\rm dis}, \qquad C_z^{\rm dis} = P_{\uparrow\uparrow}^{\rm dis} - P_{\downarrow\downarrow}^{\rm dis}$$
(3.29)

encompass all the correlations between \hat{s}_x , \hat{s}_y or \hat{s}_z and any number of spins of the apparatus. In particular, the expectation values of the components of \hat{s} are given by

$$tr\hat{D}\hat{s}_{a} = \sum_{m} C_{a}^{dis}(m) = \int_{-1}^{1} dm C_{a}(m), \qquad (3.30)$$

with the continuous functions $C_a(m) = \frac{1}{2}NC_a^{\text{dis}}(m)$ as in (3.25), while the correlations between \hat{s} and one spin of M are

$$\mathrm{tr}\hat{\mathcal{D}}\hat{s}_{a}\hat{\sigma}_{b}^{(n)} = \delta_{b,z} \sum_{m} C_{a}^{\mathrm{dis}}(m) \, m = \delta_{b,z} \int_{-1}^{1} \mathrm{d}m \, C_{a}(m) \, m.$$
(3.31)

¹³⁷⁰ Correlations of $\hat{\mathbf{s}}$ with many spins of M involve higher moments of $C_a^{\text{dis}}(m)$ as in eq. (3.27). We can interpret $P_{\uparrow\uparrow}^{\text{dis}}(m)$ ¹³⁷¹ as the joint probability to find S in $|\uparrow\rangle$ and \hat{m} equal to m, so that $P_{\uparrow\uparrow}^{\text{dis}}(m) + P_{\downarrow\downarrow}^{\text{dis}}(m) = P_M^{\text{dis}}(m)$ reduces to the probability ¹³⁷² $P_M^{\text{dis}}(m)$ which characterizes through (3.26) the marginal state of M.

1373 3.3.2. Equilibrium state of the bath

1374 1375 *C'est si bon*³⁰ Music by Henri Betti, sung by Yves Montand

At the initial time, the bath is set into equilibrium at the temperature $T = 1/\beta^{31}$. The density operator of the bath,

$$\hat{R}_{\rm B}(0) = \frac{1}{Z_{\rm B}} e^{-\beta \hat{H}_{\rm B}},\tag{3.32}$$

when $\hat{H}_{\rm B}$ is given by (3.11), describes the set of phonons at equilibrium in independent modes.

As shown in section 4.2 the bath will be involved in our problem only through its *autocorrelation function* in the equilibrium state (3.32), defined in the Heisenberg picture (see § 10.1.2) by

$$\operatorname{tr}_{B}\left[\hat{R}_{B}\left(0\right)\hat{B}_{a}^{(n)}\left(t\right)\hat{B}_{b}^{(p)}\left(t'\right)\right] = \delta_{n,p}\delta_{a,b}\,K\left(t-t'\right),\tag{3.33}$$

$$\hat{p}_{a}^{(n)}\left(t\right) = \hat{r}_{a}^{\dagger}\left(t\right)\hat{p}_{a}^{(n)}\left(t\right)$$

$$(3.34)$$

$$\hat{B}_{a}^{(n)}(t) \equiv \hat{U}_{\rm B}^{\dagger}(t) \,\hat{B}_{a}^{(n)} \hat{U}_{\rm B}(t) \,, \tag{3.34}$$

$$\hat{U}_{\rm B}\left(t\right) = e^{-i\hat{H}_{\rm B}t/\hbar},\tag{3.35}$$

in terms of the evolution operator $\hat{U}_{\rm B}(t)$ of B alone. The bath operators (3.12) have been defined in such a way that the equilibrium expectation value of $B_a^{(n)}(t)$ vanishes [174, 120, 121]. Moreover, the condition (3.13) ensures that the equilibrium correlations between different operators $\hat{B}_a^{(n)}(t)$ and $\hat{B}_b^{(p)}(t')$ vanish, unless a = b and n = p, and that the autocorrelations for n = p, a = b are all the same, thus defining a unique function K(t) in (3.33). We introduce the Fourier transform and its inverse,

$$\tilde{K}(\omega) = \int_{-\infty}^{+\infty} dt \ e^{-i\omega t} K(t), \qquad K(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega \ e^{i\omega t} \tilde{K}(\omega)$$
(3.36)

and choose for $K(\omega)$ the simplest expression having the required properties, namely the quasi-Ohmic form [151, 152, 174, 120, 121]

$$\tilde{K}(\omega) = \frac{\hbar^2}{4} \frac{\omega e^{-|\omega|/\Gamma}}{e^{\beta\hbar\omega} - 1}.$$
(3.37)

¹³⁸⁷ The temperature dependence accounts for the quantum bosonic nature of the phonons [174, 120, 121]. The Debye ¹³⁸⁸ cutoff Γ characterizes the largest frequencies of the bath, and is assumed to be larger than all other frequencies entering ¹³⁸⁹ our problem. The normalization is fixed so as to let the constant γ entering (3.10) be dimensionless.

The form (3.37) of $\tilde{K}(\omega)$ describes the spectral function of the Nyquist-noise correlator, which is the quantum generalization of the classical white noise. It can be obtained directly through general reasonings based on the detailed balance and the approach to equilibrium [174, 120]. We can also derive it from the expressions (3.11) for $\hat{H}_{\rm B}$, (3.12) and (3.34) for $\hat{B}_a^{(n)}(t)$, and (3.32) for $\hat{R}_{\rm B}(0)$, which under general conditions provide a universal model for the bath [174, 120, 121]. Indeed, by inserting these expressions into the left-hand side of (3.33), we recover the diagonal form of the right-hand side owing to (3.13), which relates $c(\omega)$ to the bath Hamiltonian $\hat{H}_{\rm B}$, with the autocorrelation function K(t) given by

$$K(t) = \sum_{k} c(\omega_{k}) \left(\frac{e^{i\omega_{k}t}}{e^{\beta\hbar\omega_{k}} - 1} + \frac{e^{-i\omega_{k}t}}{1 - e^{-\beta\hbar\omega_{k}}} \right)$$
$$= \int_{0}^{\infty} d\omega \rho(\omega) c(\omega) \left(\frac{e^{i\omega t}}{e^{\beta\hbar\omega} - 1} + \frac{e^{-i\omega t}}{1 - e^{-\beta\hbar\omega}} \right).$$
(3.38)

³⁰It's so good

³¹We use units where Boltzmann's constant is equal to one [226]; otherwise, T and $\beta = 1/T$ should be replaced throughout by k_BT and $1/k_BT$, respectively

¹³⁹⁷ We have expressed above K(t) in terms of the density of modes

$$\rho(\omega) = \sum_{k} \delta(\omega - \omega_{k}), \qquad (3.39)$$

41

1398 and this provides

$$\tilde{K}(\omega) = 2\pi\rho\left(|\omega|\right)c\left(|\omega|\right)\frac{\operatorname{sgn}\omega}{e^{\beta\hbar\omega} - 1}.$$
(3.40)

¹³⁹⁹ In agreement with Kubo's relation, we also find for the dissipative response

$$\int_{-\infty}^{+\infty} \mathrm{d}t e^{-i\omega t} \operatorname{tr}_{\mathrm{B}}\left\{\hat{R}_{\mathrm{B}}\left(0\right)\left[\hat{B}_{a}^{(n)}\left(t\right),\hat{B}_{b}^{(p)}\left(0\right)\right]\right\} = -2\pi\rho\left(|\omega|\right)c\left(|\omega|\right)\operatorname{sgn}\omega.$$
(3.41)

In the limit of a spectrum ω_k of the phonon modes sufficiently dense so that the relevant values of t/\hbar and β are small compared to the inverse level spacing of the phonon modes, we can regard $\rho(\omega) c(\omega)$ as a continuous function. In the quasi-Ohmic regime [151, 152, 153, 174, 120, 121], the dissipative response at low frequencies is proportional to ω , as obvious for a friction-dominated harmonic oscillator. We thus take (for $\omega > 0$)

$$\rho(\omega) c(\omega) = \frac{\hbar^2}{8\pi} \omega e^{-\omega/\Gamma}, \qquad (3.42)$$

In the beginning was the Word

Genesis 1.1

where ω is called the Ohmic factor, and where we include a Debye cutoff Γ on the phonon spectrum and a proper normalization. Then (3.40) reduces to the assumed expression (3.37).

1406 3.3.3. Initial state

1407 1408

In the initial state $\hat{D}(0) = \hat{r}(0) \otimes \hat{\mathcal{R}}(0)$ where S and A are statistically independent, the 2 × 2 density matrix $\hat{r}(0)$ of S is arbitrary, with elements $r_{\uparrow\uparrow}(0)$, $r_{\downarrow\downarrow}(0)$, $r_{\downarrow\downarrow}(0)$ and $r_{\downarrow\downarrow}(0)$ satisfying

$$r_{\uparrow\uparrow}(0) + r_{\downarrow\downarrow}(0) = 1, \qquad r_{\uparrow\downarrow}(0) = r_{\downarrow\uparrow}^*(0), \qquad r_{\uparrow\uparrow}(0) r_{\downarrow\downarrow}(0) \ge r_{\uparrow\downarrow}(0) r_{\downarrow\uparrow}(0).$$
(3.43)

According to the discussion of the section 3.1, the initial density operator $\hat{\mathcal{R}}(0)$ of the apparatus describes the magnetic dot in a metastable paramagnetic state. As justified below, we take for it the factorized form

$$\hat{\mathcal{R}}(0) = \hat{R}_{\mathrm{M}}(0) \otimes \hat{R}_{\mathrm{B}}(0), \qquad (3.44)$$

where the bath is in the equilibrium state (3.32), at the temperature $T = 1/\beta$ lower than the transition temperature of M, while the magnet with Hamiltonian (3.6) is in a paramagnetic equilibrium state at a temperature $T_0 = 1/\beta_0$ higher than its transition temperature:

$$\hat{R}_{\rm M}(0) = \frac{1}{Z_{\rm M}} e^{-\beta_0 \hat{H}_{\rm M}}.$$
(3.45)

How can the apparatus be actually initialized in the non-equilibrium state (3.44) at the time t = 0? This *ini*-1416 *tialization* takes place during the time interval $-\tau_{init} < t < 0$. The apparatus is first set at earlier times into equi-1417 librium at the temperature T_0 . Due to the smallness of γ , its density operator is then factorized and proportional to 1418 $\exp[-\beta_0(\hat{H}_{\rm M}+\hat{H}_{\rm B})]$. At the time $-\tau_{\rm init}$ the phonon bath is suddenly cooled down to T. We shall evaluate in § 7.3.2 1419 the relaxation time of M towards its equilibrium ferromagnetic states under the effect of B at the temperature T. Due 1420 to the weakness of the coupling γ , this time turns out to be much longer than the duration of the experiment. We can 1421 safely assume τ_{init} to be much shorter than this relaxation time so that M remains unaffected by the cooling. On the 1422 other hand, the quasi continuous nature of the spectrum of B can allow the phonon-phonon interactions (which we 1423 have disregarded when writing (3.11) to establish the equilibrium of B at the temperature T within a time shorter 1424 than τ_{init} . It is thus realistic to imagine an initial state of the form (3.44). 1425

An alternative method of initialization consists in applying to the magnetic dot a strong radiofrequency field, which acts on M but not on B. The bath can thus be thermalized at the required temperature, lower than the transition temperature of M, while the populations of spins of M oriented in either direction are equalized. The magnet is then in a paramagnetic state, as if it were thermalized at an *infinite* temperature T_0 in spite of the presence of a cold bath. In that case we have the initial state (see Eq. (3.1))

$$\hat{R}_{\rm M}(0) = \frac{1}{2^N} \prod_{n=1}^N \hat{\sigma}_0^{(n)}.$$
(3.46)

The initial density operator (3.45) of M being simply a function of the operator \hat{m} , we can characterize it as in (3.24) by the probabilities $P_{\rm M}^{\rm dis}(m,0)$ for \hat{m} to take the values (3.22). Those probabilities are the normalized product of the degeneracy (3.23) and the Boltzmann factor,

$$P_{\rm M}^{\rm dis}(m,0) = \frac{1}{Z_0}G(m)\exp\left[\frac{N}{T_0}\left(\frac{J_2}{2}m^2 + \frac{J_4}{4}m^4\right)\right], \qquad Z_0 = \sum_m G(m)\exp\left[\frac{N}{T_0}\left(\frac{J_2}{2}m^2 + \frac{J_4}{4}m^4\right)\right]. \tag{3.47}$$

For sufficiently large N, the distribution $P_{\rm M}(m,0) = \frac{1}{2}NP_{\rm M}^{\rm dis}(m,0)$ is peaked around m = 0, with the Gaussian shape

$$P_{\rm M}(m,0) \simeq \frac{1}{\sqrt{2\pi}\,\Delta m} e^{-m^2/2\Delta m^2} = \sqrt{\frac{N}{2\pi\delta_0^2}} e^{-Nm^2/2\delta_0^2}.$$
(3.48)

¹⁴³⁵ This peak, which has a narrow width of the form

$$\Delta m = \sqrt{\langle m^2 \rangle} = \frac{\delta_0}{\sqrt{N}},\tag{3.49}$$

involves a large number, of order \sqrt{N} , of eigenvalues (3.22), so that the spectrum can be treated as a continuum (except in sections 5.3 and 6). For the Hamiltonian (3.9) with q = 4, only the multiplicity (3.23) contributes to Δm , so that the paramagnetic initial state (3.45) is characterized at any initial temperature T_0 by the distribution $P_M(m, 0)$ equal to

$$P_{\rm M}(m,0) = P_{\rm M0}(m) = \frac{1}{2^N} G(m) \equiv \sqrt{\frac{N}{2\pi}} e^{-Nm^2/2}.$$
 (3.50)

¹⁴⁴⁰ For the general Hamiltonian (3.7), the width is larger, due to correlations between spins, and given by

$$\delta_0 = \sqrt{\frac{T_0}{T_0 - J_2}}, \qquad \Delta m = \sqrt{\frac{T_0}{N(T_0 - J_2)}}.$$
(3.51)

In the pure q = 2 case with Hamiltonian (3.8), and in general in case $J_2 > 0$, the temperature T_0 of initialization should be sufficiently higher than the Curie temperature so that $\delta_0^2 \ll N$, which ensures the narrowness of the peak. For an initialization caused by a radiofrequency, the initial distribution is again (3.50).

1444 3.3.4. Ferromagnetic equilibrium states of the magnet

We expect the final state (1.7) of S + A after measurement to involve the two ferromagnetic equilibrium states $\hat{\mathcal{R}}_i$, 1445 $i = \uparrow$ or \Downarrow . As above these states $\hat{\mathcal{R}}_i$ of the apparatus factorize, in the weak coupling limit ($\gamma \ll 1$), into the product 1446 of (3.32) for the bath and a ferromagnetic equilibrium state \hat{R}_{Mi} for the magnet M. It is tempting to tackle broken 1447 invariance by means of the mean-field approximation, which becomes exact at equilibrium for infinite N owing to 1448 the long range of the interactions [227, 228]. However, we are interested in a finite, though large, value of N, and 1449 the probability distribution $P_{Mi}(m)$ associated with \hat{R}_{Mi} has a significant width around the mean-field value for m. 1450 Moreover, we shall see in subsection 7.3 that, in spite of the constancy of the interaction between all spins, the 1451 time-dependent mean-field approximation may fail even for large N. We will study there the dynamics of the whole 1452 distribution $P_{M}(m, t)$ including the final regime where it is expected to tend to $P_{M\uparrow}(m)$ or $P_{M\downarrow}(m)$, and will determine 1453

in particular the lifetime of the metastable state (3.44). We focus here on equilibrium only. For later convenience we include an external field *h* acting on the spins of the apparatus, so as to arrive from (3.7) at³²

$$\hat{H}_{\rm M} = -Nh\hat{m} - NJ_2\frac{\hat{m}^2}{2} - NJ_4\frac{\hat{m}^4}{4}.$$
(3.52)

As in (3.26) we characterize the canonical equilibrium density operator of the magnet $\hat{R}_{\rm M} = (1/Z_{\rm M}) \exp[-\beta \hat{H}_{\rm M}]$, which depends only on the operator \hat{m} , by the probability distribution

$$P_{\rm M}(m) = \frac{\sqrt{N}}{Z_{\rm M}\sqrt{8\pi}} e^{-\beta F(m)},\tag{3.53}$$

where *m* takes the discrete values m_i given by (3.22); the exponent of (3.53) introduces the free energy function

$$F(m) = -NJ_2\frac{m^2}{2} - NJ_4\frac{m^4}{4} - Nhm + NT\left(\frac{1+m}{2}\ln\frac{1+m}{2} + \frac{1-m}{2}\ln\frac{1-m}{2}\right) + \frac{T}{2}\ln\frac{1-m^2}{4} + O\left(\frac{1}{N}\right), \quad (3.54)$$

which arises from the Hamiltonian (3.52) and from the multiplicity G(m) given by (3.23). The distribution (3.53) displays narrow peaks at the minima of F(m), and the *equilibrium free energy* $-T \ln Z_M$ is equal for large N to the absolute minimum of (3.54). The function F(m) reaches its extrema at values of m given by the self-consistent equation

$$m\left(1-\frac{1}{N}\right) = \tanh\left[\beta\left(h+J_2m+J_4m^3\right)\right],\tag{3.55}$$

which as expected reduces to the mean-field result for large *N*. In the vicinity of a minimum of F(m) at $m = m_i$, the probability $P_M(m)$ presents around each m_i a nearly Gaussian peak, given within normalization by

$$P_{\mathrm{M}i}(m) \propto \exp\left\{-\frac{N}{2}\left[\frac{1}{1-m_i^2} -\beta J_2 - 3\beta J_4 m_i^2\right](m-m_i)^2 - \frac{N}{3}\left[\frac{m_i}{(1-m_i^2)^2} - 3\beta J_4 m_i\right](m-m_i)^3\right\}.$$
 (3.56)

This peak is located at a distance of order 1/N from the mean-field value, it has a width of order $1/\sqrt{N}$ and a weak 1465 asymmetry. The possible values of m are dense within the peak, with equal spacing $\delta m = 2/N$. With each such peak 1466 $P_{Mi}(m)$ is associated through (3.25), (3.26) a density operator \hat{R}_i of the magnet M which may describe a locally stable 1467 equilibrium. Depending on the values of J_2 and J_4 and on the temperature, there may exist one, two or three such 1468 locally stable states. We note the corresponding average magnetizations m_i , for arbitrary h, as m_P for a paramagnetic 1469 state and as m_{\parallel} and m_{\parallel} for the ferromagnetic states, with $m_{\parallel} > 0$, $m_{\parallel} < 0$. We also note as $\pm m_{\rm F}$ the ferromagnetic 1470 magnetizations for h = 0. When h tends to 0 (as happens at the end of the measurement where we set $g \to 0$), $m_{\rm P}$ 1471 tends to 0, m_{\uparrow} to $+m_{\rm F}$ and m_{\downarrow} to $-m_{\rm F}$, namely 1472

$$m_{\uparrow}(h > 0) > 0, \qquad m_{\downarrow}(h > 0) < 0, \qquad m_{\uparrow}(-h) = -m_{\downarrow}(h), \qquad m_{\rm F} = m_{\uparrow}(h \to +0) = -m_{\downarrow}(h \to +0).$$
(3.57)

For h = 0, the system M is invariant under change of sign of m [227]. This invariance is spontaneously broken below some temperature [227]. In the case q = 2 of the Ising interaction (3.8), there is above the Curie temperature $T_c = J_2$ a single paramagnetic peak $P_{M0}(m)$ at $m_P = 0$, given by (3.48), (3.51), and for $T < J_2$ two symmetric ferromagnetic peaks (3.56), $i = \uparrow$ or \downarrow , at the points $m_{\uparrow} = m_F$ and $m_{\downarrow} = -m_F$, given by $m_F = \tanh \beta J_2 m_F$. These peaks are well separated provided

$$\frac{N}{2} \left(\frac{1}{1 - m_{\rm F}^2} - \beta J_2 \right) m_{\rm F}^2 \gg 1, \tag{3.58}$$

 $^{^{32}}$ In section 7 we shall identify *h* with +*g* in the sector $\hat{R}_{\uparrow\uparrow}$ of \hat{D} , or with -*g* in its sector $\hat{R}_{\downarrow\downarrow}$, where *g* is the coupling between S and A, while a true field in the *y*-direction will be introduced in section 8.2 and denoted by *b*, see Eq. (8.46)

in which case they characterize two different equilibrium ferromagnetic states. This condition is satisfied for large *N* and $\beta J_2 - 1$ finite; near $\beta J_2 = 1$, where $m_F^2 \sim 3 (\beta J_2 - 1)$, the two states \hat{R}_{\uparrow} and \hat{R}_{\downarrow} still have no overlap as soon as the temperature differs significantly from the critical temperature, as

$$\frac{J_2 - T}{T} \gg \frac{1}{\sqrt{3N}}.$$
(3.59)

44

This property is needed to ensure a faithful registration by M of the measurement. Little is changed for the Hamiltonian (3.7) with $J_4 > 0$ but still $J_2 > 3J_4$.



Figure 3.3: The free energy F in units of NT for a pure quartic interaction (eq. (3.9), evaluated from Eq. (3.54) with h = 0, as function of the magnetization m at various temperatures. There is always a local paramagnetic minimum at m = 0. A first-order transition occurs at $T_c = 0.363J_4$, below which the ferromagnetic states associated with the minima at $\pm m_F$ near ± 1 become the most stable.

Still for h = 0, but in the case $3J_4 > J_2$ of a first-order transition, F(m) has a minimum at m = 0 if $T > J_2$ 1483 and hence (3.53) has there a peak as (3.50) at m = 0 whatever the temperature, see Fig. 3.3. For the pure quartic 1484 interaction of Eq. (3.9), the two additional ferromagnetic peaks $P_{M\uparrow}(m)$ and $P_{M\downarrow}(m)$ appear around $m_{\uparrow} = m_F = 0.889$ 1485 and $m_{\parallel} = -m_{\rm F}$ when the temperature T is lower than 0.496J₄. As T decreases, $m_{\rm F}$ given by $m_{\rm F} = \tanh\beta J_4 m_{\rm F}^3$ increases 1486 and the value of the minimum $F(m_{\rm F})$ decreases; the weight (3.53) is transferred from $P_{\rm M0}(m)$ to $P_{\rm M\uparrow}(m)$ and $P_{\rm M\downarrow}(m)$. 1487 A first-order transition occurs when $F(m_{\rm F}) = F(0)$, for $T_{\rm c} = 0.363J_4$ and $m_{\rm F} = 0.9906$, from the paramagnetic to the 1488 two ferromagnetic states, although the paramagnetic state remains locally stable. The spontaneous magnetization $m_{\rm F}$ 1489 is always very close to 1, behaving as $1 - m_{\rm F} \sim 2 \exp(-2J_4/T)$. 1490

For the general Hamiltonian (3.7), it is a simple exercise to study the cross-over between first and second-order transitions, which takes place for $m_i \ll 1$. To this aim, the free energy (3.54) is expanded as

$$\frac{F(m) - F(0)}{N} \approx (T - J_2) \frac{m^2}{2} + (T - 3J_4) \frac{m^4}{12} + T \frac{m^6}{30},$$
(3.60)

and its shape and minima are studied as function of J_2 , J_4 and T. This approximation holds for $|T - J_2| \ll J_2$, $|3J_4 - J_2| \ll J_2$. For $J_2 > 3J_4$, the second-order transition takes place at $T_c = J_2$ whatever J_4 . For $3J_4 > J_2$, the first-order transition temperature T_c is given by $T_c - J_2 \sim 5(3J_4 - J_2)^2/48J_2$, and the equilibrium magnetization jumps from 0 to $\pm m_F$, with $m_F^2 \sim 5(3J_4 - J_2)/4J_2$. The paramagnetic state is locally stable down to $T > J_2$, the ferromagnetic states up to $T - J_2 < (4/3)(T_c - J_2)$. When $3J_4 > J_2$, a metastability with a long lifetime of the paramagnetic state is thus ensured if the bath temperature satisfies $T_c > T > J_2$.

Strictly speaking, the canonical equilibrium state of M below the transition temperature, characterized by (3.53), has for h = 0 and finite N the form

$$\hat{R}_{\text{Meq}} = \frac{1}{2} (\hat{R}_{\text{M}\uparrow} + \hat{R}_{\text{M}\Downarrow}).$$
(3.61)

However this state is not necessarily the one reached at the end of a relaxation process governed by the bath B, when 1501 a field h, even weak, is present: this field acts as a source which breaks the invariance. The determination of the 1502 state $\hat{R}_{M}(t_{f})$ reached at the end of a relaxation process involving the thermal bath B and a weak field h requires a 1503 dynamical study which will be worked out in subsection 7.3. This is related to the ergodicity breaking: if a weak field 1504 is applied, then switched off, the full canonical state (3.61) is still recovered, but only after an unrealistically long time 1505 (for $N \gg 1$). For finite times the equilibrium state of the magnet is to be found by restricting the full canonical state 1506 (3.61) to its component having a magnetization with the definite sign determined by the weak external field. This 1507 is the essence of the spontaneous symmetry breaking. However, for our situation this well-known recipe should be 1508 supported by dynamical considerations; see in this respect section 11. 1509

In our model of measurement, the situation is similar, though slightly more complicated. The system-apparatus coupling (3.5) plays the rôle of an operator-valued source, with eigenvalues behaving as a field h = g or h = -g. We shall determine in section 7 towards which state M is driven under the conjugate action of the bath B and of the system S, depending on the parameters of the model.



Figure 3.4: The effect of a positive field h on F(m) for q = 4 at temperature $T = 0.2J_4$. As h increases the paramagnetic minimum m_P shifts towards positive m. At the critical field $h_c = 0.0357J_4$ this local minimum disappears, and the curve has an inflexion point with vanishing slope at $m = m_c = 0.268$. For larger fields, like in the displayed case $g = 0.04J_4$, the locally stable paramagnetic state disappears, and there remain only the two ferromagnetic states, the most stable one with positive magnetization $m_{\rm fl} \simeq 1$ and the metastable one with negative magnetization $m_{\rm fl} \simeq -1$.

As a preliminary step, let us examine here the effect on the free energy (3.54) of a small positive field h. Consider 1514 first the minima of F(m) [226, 227]. The two ferromagnetic minima m_{\uparrow} and m_{\downarrow} given by (3.55) are slightly shifted 1515 away from $m_{\rm F}$ and $-m_{\rm F}$, and $F(m_{\rm f}) - F(m_{\rm F})$ behaves as $-Nhm_{\rm F}$. Hence, as soon as $\exp\{-\beta [F(m_{\rm f}) - F(m_{\rm f})]\}$ 1516 $\exp(2\beta Nhm_{\rm F}) \gg 1$, only the single peak $P_{\rm M\uparrow}(m)$ around $m_{\uparrow\uparrow} \simeq m_{\rm F}$ contributes to (3.53), so that the canonical equi-1517 librium state of M has the form $\hat{R}_{Meq} = \hat{R}_{M\uparrow}$. The shape of F(m) will also be relevant for the dynamics. For 1518 a second order transition, although F(m) has when h = 0 a maximum at m = 0, its stationarity allows the state 1519 $\hat{R}_{M}(m,0) \propto P_{M}(\hat{m},0)$ given by (3.48) to have a long lifetime for $N \gg 1$. The introduction of h produces a negative 1520 slope -Nh at m = 0, which suggests that the dynamics will let $\langle m \rangle$ increase. For a first order transition, the situation is 1521 different (Fig. 3.4). If h is sufficiently small, F(m) retains its paramagnetic minimum, the position of which is shifted 1522 as $m_{\rm P} \sim h/T$; the paramagnetic state $\hat{R}_{\rm M}(0)$ remains locally stable. It may decay towards a stable ferromagnetic state 1523 only through mechanisms of thermal activation or quantum tunnelling, processes with very large characteristic times, 1524 of exponential order in N. However, there is a threshold h_c above which this paramagnetic minimum of F(m), which 1525 then lies at $m = m_c$, disappears. The value of h_c is found by eliminating $m = m_c$ between the equations $d^2 F/dm^2 = 0$ 1526 and dF/dm = 0. In the pure q = 4 case ($J_2 = 0$) on which we focus as an illustration for first order transitions, we find 1527 $2m_c^2 = 1 - \sqrt{1 - 4T/3J_4}, h_c = \frac{1}{2}T \ln[1 + m_c)/(1 - m_c)] - J_4 m_c^3$. At the transition temperature $T_c = 0.363J_4$, we have 1528 $m_{\rm c} = 0.375$ and $h_{\rm c} = 0.0904 J_4$; for $T = 0.2 J_4$, we obtain $m_{\rm c} = 0.268$ and $h_{\rm c} = 0.036 J_4$; for $T \ll J_4$, $m_{\rm c}$ behaves as 1529 $\sqrt{T/3J_4}$ and h_c as $\sqrt{4T^3/27J_4}$. Provided $h > h_c$, F(m) has now a negative slope in the whole interval $0 < m < m_F$. 1530

¹⁵³¹ We can thus expect, in our measurement problem, that the registration will take place in a reasonable delay, either ¹⁵³² for a first order transition if the coupling g is larger than h_c , or for a second order transition. In the latter case, it will ¹⁵³³ be necessary to check, however, that the lifetime of the initial state is larger than the duration of the measurement. This will be done in 7.3.2.

4. Equations of motion

1536 1537 Τὰ πάντα ῥεï³³ Quoted from Heraklitos by Plato and Simplicius

In this technical section, we rewrite the dynamical equations for our model in a form which will help us, in the continuation, to discuss the physical features of the solution. We will make no other approximation than the weak spin-phonon coupling, $\gamma \ll 1$, and will derive the equations up to first order in γ . In subsection 4.5, we take advantage of the large size of the apparatus, $N \gg 1$, to reduce the equations of motion into a pair of partial differential equations.

1542 4.1. A conserved quantity, the measured component of the spin, and the Born rule

1543	All the world's Great Journeys begin with the first step
1544	A 1000 miles journey starts with a single step
1545	Tibetan and Aboriginal Australian proverbs

Since $\hat{H}_{S} = 0$ and since \hat{s}_{x} and \hat{s}_{y} do not occur in the coupling (3.5) between S and A, we can already conclude that \hat{s}_{z} is conserved during the ideal measurement, viz. $i\hbar d\hat{s}_{z}/dt = [\hat{s}_{z}, \hat{H}] = 0$. This implies that the diagonal elements of the density matrix of the spin are conserved, viz. $r_{\uparrow\uparrow}(t_{f}) = r_{\uparrow\uparrow}(t) = r_{\uparrow\uparrow}(0)$ and $r_{\downarrow\downarrow}(t_{f}) = r_{\downarrow\downarrow}(0)$. The result is consistent with *Born's rule*: we expect the probabilities for the possible *outcomes* of an ideal measurement to be given by the diagonal elements of the *initial* density matrix of S. But $r_{\uparrow\downarrow}$ and $r_{\downarrow\uparrow}$, on the other hand, are not conserved (viz. $\hat{s}_{a}, \hat{H}] \neq 0$ for a = x, y), and they will evolve and ultimately vanish³⁴.

1552 4.2. Eliminating the bath variables

1553 1554 Don't call the alligator a big-mouth till you have crossed the river Brazilian proverb

A complete description of the measurement process requires at least the solution, in the Hilbert space of S + A, of the Liouville–von Neumann equation of motion [226]

$$i\hbar\frac{\mathrm{d}\mathcal{D}}{\mathrm{d}t} = \left[\hat{H},\hat{\mathcal{D}}\right],\tag{4.1}$$

1557 with the initial condition

$$\hat{\mathcal{D}}(0) = \hat{r}(0) \otimes \hat{R}_{\rm M}(0) \otimes \hat{R}_{\rm B}(0) = \hat{D}(0) \otimes \hat{R}_{\rm B}(0).$$
(4.2)

We are not interested, however, in the bath variables, and the knowledge of $\hat{D}(t) = \text{tr}_{\text{B}}\hat{D}(t)$ is sufficient for our purpose. As usual in non-equilibrium statistical mechanics [174, 120, 121, 229], we rely on the weakness of the coupling \hat{H}_{MB} between the magnet and the bath, so as to treat perturbatively the dissipative effect of the bath.

Let us therefore split the Hamiltonian (3.3) into $\hat{H} = \hat{H}_0 + \hat{H}_{MB} + \hat{H}_B$ with $\hat{H}_0 = \hat{H}_S + \hat{H}_{SA} + \hat{H}_M$. Regarding the coupling \hat{H}_{MB} as a perturbation, we introduce the unperturbed evolution operators, namely (3.35) for the bath, and

$$\hat{U}_0(t) = e^{-i\hat{H}_0 t/\hbar}, \qquad \hat{H}_0 = -gN\hat{s}_z\hat{m} - N\sum_{q=2,4}\frac{J_q}{q}\hat{m}^q,$$
(4.3)

for S + M. We can then expand the full evolution operator in powers of the coupling $\sqrt{\gamma}$, in the interaction picture, and take the trace over B of eq. (4.1) so as to generate finally an equation of motion for the density operator $\hat{D}(t)$ of S + M. This calculation is worked out in Appendix A.

³³Everything flows

³⁴This has the popular name "decay of Schrödinger cat terms", or "death of Schrödinger cats"

The result involves the *autocorrelation function* K(t) *of the bath*, defined by (3.32) - (3.35) and expressed in our model by (3.36), (3.37). It also involves the operators $\hat{\sigma}_a^{(n)}(u)$ in the space of S + M, defined in terms of the memory time u = t - t' by

$$\hat{\sigma}_{a}^{(n)}(u) \equiv \hat{U}_{0}(t) \hat{U}_{0}^{\dagger}(t') \hat{\sigma}_{a}^{(n)} \hat{U}_{0}(t') \hat{U}_{0}^{\dagger}(t) = \hat{U}_{0}(u) \hat{\sigma}_{a}^{(n)} \hat{U}_{0}^{\dagger}(u).$$
(4.4)

It holds that $\hat{\sigma}_{a}^{(n)}(0) = \hat{\sigma}_{a}^{(n)}$. Altogether we obtain a differential equation for $\hat{D}(t)$, the kernel of which involves times earlier than *t* through K(u) and $\hat{\sigma}_{a}^{(n)}(u)$ [174, 120, 121]:

$$\frac{\mathrm{d}\hat{D}}{\mathrm{d}t} - \frac{1}{i\hbar} \left[\hat{H}_{0}, \hat{D}\right] = \frac{\gamma}{\hbar^{2}} \int_{0}^{t} \mathrm{d}u \sum_{n,a} \left\{ K\left(u\right) \left[\hat{\sigma}_{a}^{(n)}\left(u\right) \hat{D}, \hat{\sigma}_{a}^{(n)}\right] + K\left(-u\right) \left[\hat{\sigma}_{a}^{(n)}, \hat{D}\hat{\sigma}_{a}^{(n)}\left(u\right)\right] \right\} + O\left(\gamma^{2}\right). \tag{4.5}$$

As anticipated in § 3.3.2, the phonon bath occurs in this equation, which governs the dynamics of S + M, only through the function K(t), the memory time being the time-range $\hbar/2\pi T$ of K(t) [174, 120, 121].

1573 4.3. Decoupled equations of motion

In our model, the Hamiltonian commutes with the measured observable \hat{s}_z , hence with the projection operators $\hat{\Pi}_i$ onto the states $|\uparrow\rangle$ and $|\downarrow\rangle$ of S. The equations for the operators $\hat{\Pi}_i \hat{D} \hat{\Pi}_j$ are therefore decoupled. We can replace the equation (4.5) for \hat{D} in the Hilbert space of S + M by a set of four equations for the operators \hat{R}_{ij} defined by (3.18) in the Hilbert space of M. We shall later see (section 8.2) that this simplification underlies the ideality of the measurement process.

The Hamiltonian \hat{H}_0 in the space S + M gives rise to two Hamiltonians \hat{H}_{\uparrow} and \hat{H}_{\downarrow} in the space M, which according to (3.5) and (3.7) are simply two functions of the observable \hat{m} , given by

$$\hat{H}_i = H_i(\hat{m}) = -gNs_i\hat{m} - N\sum_{q=2,4} \frac{J_q}{q}\hat{m}^q, \qquad (i=\uparrow,\downarrow)$$
(4.6)

with $s_i = +1$ (or -1) for $i = \uparrow$ (or \downarrow). These Hamiltonians \hat{H}_i , which describe interacting spins $\hat{\sigma}^{(n)}$ in an external field g_{s_i} , occur in (4.5) both directly and through the operators

$$\hat{\sigma}_{a}^{(n)}(u,i) = e^{-i\hat{H}_{i}u/\hbar}\hat{\sigma}_{a}^{(n)}e^{i\hat{H}_{i}u/\hbar},$$
(4.7)

obtained by projection of (4.4), using (4.3), with $\hat{\Pi}_i$ and reduction to the Hilbert space of M.

The equation (4.5) for $\hat{D}(t)$ which governs the joint dynamics of S+M thus reduces to the four differential equations in the Hilbert space of M (we recall that $i, j = \uparrow, \downarrow$ or ± 1):

$$\frac{\mathrm{d}\hat{R}_{ij}(t)}{\mathrm{d}t} - \frac{\hat{H}_{i}\hat{R}_{ij}(t) - \hat{R}_{ij}(t)\hat{H}_{j}}{i\hbar} = \frac{\gamma}{\hbar^{2}} \int_{0}^{t} \mathrm{d}u \sum_{n,a} \left\{ K(u) \left[\hat{\sigma}_{a}^{(n)}(u,i) \, \hat{R}_{ij}(t), \hat{\sigma}_{a}^{(n)} \right] + K(-u) \left[\hat{\sigma}_{a}^{(n)}, \hat{R}_{ij}(t) \hat{\sigma}_{a}^{(n)}(u,j) \right] \right\}.$$
(4.8)

1586 4.4. Reduction to scalar equations

1587 4.4.1. Representing the pointer by a scalar variable

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Even a small star shines in the darkness Finnish proverb

For each operator \hat{R}_{ij} , the initial conditions are given according to (3.43) and (3.44) by

$$\hat{R}_{ij}(0) = r_{ij}(0)\,\hat{R}_{\rm M}(0)\,,\tag{4.9}$$

and $\hat{R}_{M}(0)$ expressed by the Gibbs state (3.45) is simply a *function of the operator* \hat{m} . We show in Appendix B that this property *is preserved for* $\hat{R}_{ij}(t)$ by the evolution (4.8), owing to the form (4.6) of \hat{H}_{i} and in spite of the occurrence of the separate operators $\hat{\sigma}_{a}^{(n)}$ in the right-hand side.

¹⁵⁹⁴ We can therefore parametrize, as anticipated at the end of § 3.3.1, at each *t*, the operators \hat{R}_{ij} in the form $\hat{R}_{ij} = P_{ij}^{\text{dis}}(\hat{m})/G(\hat{m})$. Their equations of motion (4.8) are then diagonal in the eigenspace of \hat{m} , and are therefore equivalent to scalar equations which govern the functions $P_{ij}(m) = (N/2)P_{ij}^{\text{dis}}(m)$ of the variable *m* taking the discrete values (3.22).

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1598 4.4.2. Equations of motion for $P_{ij}(m, t)$

The equations resulting from this parametrization are derived in Appendix B. The integrals over u entering (4.8) yield the functions

$$\tilde{K}_{t>}(\omega) = \int_{0}^{t} du e^{-i\omega u} K(u) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} d\omega' \tilde{K}(\omega') \frac{e^{i(\omega'-\omega)t} - 1}{\omega' - \omega},$$
(4.10)

1601 and

$$\tilde{K}_{t<}(\omega) = \int_{0}^{t} du e^{i\omega u} K(-u) = \int_{-t}^{0} du e^{-i\omega u} K(u) = \left[\tilde{K}_{t>}(\omega)\right]^{*} = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} d\omega' \tilde{K}(\omega') \frac{1 - e^{i(\omega - \omega')t}}{\omega' - \omega}, \quad (4.11)$$

where ω takes, depending on the considered term, the values Ω^+_{\uparrow} , Ω^-_{\downarrow} , Ω^+_{\downarrow} , Ω^-_{\downarrow} , given by

$$\hbar\Omega_i^{\pm}(m) = H_i(m \pm \delta m) - H_i(m), \qquad (i=\uparrow,\downarrow), \tag{4.12}$$

in terms of the Hamiltonians (4.6) and of the level spacing $\delta m = 2/N$. They satisfy the relations

$$\Omega_i^{\pm}(m \mp \delta m) = -\Omega_i^{\mp}(m). \tag{4.13}$$

The quantities (4.12) are interpreted as excitation energies of the magnet M arising from the flip of one of its spins in the presence of the tested spin S (with value s_i); the sign + (–) refers to a down-up (up-down) spin flip. Their explicit values are:

$$\hbar\Omega_i^{\pm}(m) = \mp 2gs_i + 2J_2(\mp m - \frac{1}{N}) + 2J_4(\mp m^3 - \frac{3m^2}{N} \mp \frac{4m}{N^2} - \frac{2}{N^3}), \tag{4.14}$$

1607 with $s_{\uparrow} = 1, s_{\downarrow} = -1$.

The operators $\hat{\sigma}_x^{(n)}$ and $\hat{\sigma}_y^{(n)}$ which enter (4.8) are shown in Appendix B to produce a flip of the spin $\hat{\sigma}^{(n)}$, that is, a shift of the operator \hat{m} into $\hat{m} \pm \delta m$. We introduce the notations

$$\Delta_{\pm}f(m) = f(m_{\pm}) - f(m), \qquad m_{\pm} = m \pm \delta m, \qquad \delta m = \frac{2}{N}.$$
 (4.15)

The resulting dynamical equations for $P_{ij}(m, t)$ take different forms for the diagonal and for the off-diagonal components. On the one hand, the first *diagonal block* of \hat{D} is parameterized by the *joint probabilities* $P_{\uparrow\uparrow}(m, t)$ to find S in $|\uparrow\rangle$ and \hat{m} equal to *m* at the time *t*. These probabilities evolve according to

$$\frac{\mathrm{d}P_{\uparrow\uparrow}(m,t)}{\mathrm{d}t} = \frac{\gamma N}{\hbar^2} \left\{ \Delta_+ \left[(1+m) \,\tilde{K}_t \left(\Omega^-_{\uparrow}(m) \right) P_{\uparrow\uparrow}(m,t) \right] + \Delta_- \left[(1-m) \,\tilde{K}_t \left(\Omega^+_{\uparrow}(m) \right) P_{\uparrow\uparrow}(m,t) \right] \right\},\tag{4.16}$$

with initial condition $P_{\uparrow\uparrow}(m,0) = r_{\uparrow\uparrow}(0) P_{\rm M}(m,0)$ given by (3.48); likewise for $P_{\downarrow\downarrow}(m)$, which involves the frequencies $\Omega^{\mp}_{\downarrow}(m)$. The factor $\tilde{K}_t(\omega)$ is expressed by the combination of two terms,

$$\tilde{K}_{t}(\omega) \equiv \tilde{K}_{t>}(\omega) + \tilde{K}_{t<}(\omega) = \int_{-t}^{+t} du e^{-i\omega u} K(u) = \int_{-\infty}^{\infty} \frac{d\omega'}{\pi} \frac{\sin(\omega' - \omega) t}{\omega' - \omega} \tilde{K}(\omega').$$
(4.17)

It is real and tends to $\tilde{K}(\omega)$, given in Eq. (3.37), at times *t* larger than the range $\hbar/2\pi T$ of K(t) [174, 120, 121]. This

may be anticipated from the relation $\sin[(\omega' - \omega)t]/\pi(\omega' - \omega) \rightarrow \delta(\omega' - \omega)$ for $t \rightarrow \infty$ and it may be demonstrated with help of the contour integration techniques of Appendix D, which we have as a student everying see 8.0.6.1

with help of the contour integration techniques of Appendix D, which we leave as a student exercise, see § 9.6.1.

On the other hand, the sets $P_{\uparrow\downarrow}(m,t)$ and $P_{\downarrow\uparrow} = P_{\uparrow\downarrow}^{\star}$ which parameterize the *off-diagonal blocks* of \hat{D} , and which are related through (3.29) to the correlations between \hat{s}_x or \hat{s}_y and any number of spins of M, evolve according to

$$\frac{\mathrm{d}}{\mathrm{d}t}P_{\uparrow\downarrow}(m,t) - \frac{2iNgm}{\hbar}P_{\uparrow\downarrow}(m,t) = \frac{\gamma N}{\hbar^2} \left\{ \Delta_+ \left[(1+m)\tilde{K}_-(m,t)P_{\uparrow\downarrow}(m,t) \right] + \Delta_- \left[(1-m)\tilde{K}_+(m,t)P_{\uparrow\downarrow}(m,t) \right] \right\}, \quad (4.18)$$

with initial condition $P_{\uparrow\downarrow}(m, 0) = r_{\uparrow\downarrow}(0) P_{\rm M}(m, 0)$. Here $\tilde{K}_{t>}$ and $\tilde{K}_{t<}$ enter the combination

$$\tilde{K}_{\pm}(m,t) \equiv \tilde{K}_{t>} \left[\Omega^{\pm}_{\uparrow}(m) \right] + \tilde{K}_{t<} \left[\Omega^{\pm}_{\downarrow}(m) \right].$$
(4.19)

1621 4.4.3. Interpretation as quantum balance equations

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Je moet je evenwicht bewaren³⁵ Dutch expression

Our basic equations (4.16) and (4.18) fully describe the dynamics of the measurement. The diagonal equation 1624 (4.16) can be interpreted as a balance equation [174, 120, 121]. Its first term represents elementary processes in 162 which one among the spins $\sigma^{(n)}$ flips from $\sigma_z^{(n)} = +1$ to $\sigma_z^{(n)} = -1$. For the value *m* of the magnetization, a value 1626 taken with probability $P_{\uparrow\uparrow}(m,t)$ at the time t, there are $\frac{1}{2}N(1+m)$ spins pointing upwards, and the probability for 1627 one of these spins to flip down between the times t and t + dt under the effect of the phonon bath can be read off 1628 from (4.18) to be equal to $2\gamma\hbar^{-2}\tilde{K}_t(\Omega_{\uparrow}) dt$. This process produces a decrease of $P_{\uparrow\uparrow}(m)$ and it is accounted for by the 1629 negative contribution (which arises from the second part of Δ_+ and is proportional to $-P_{\uparrow\uparrow}(m, t)$) to the first term in the 1630 right-hand side of (4.16). The coefficient $\tilde{K}_t(\omega)$ depends on the temperature T of the bath B, on the duration t of its 1631 interaction with M, and on the energy $\hbar \omega$ that it has transferred to M; this energy is evaluated for $P_{\uparrow\uparrow}$ (or $P_{\downarrow\downarrow}$) as if the 1632 spins of M were submitted to an external field +g (or -g). The first term in (4.16) also contains a positive contribution 1633 arising from the same process, for which the magnetization decreases from $m + \delta m$ to m, thus raising $P_{\uparrow\uparrow}(m)$ by a term 1634 proportional to $P_{\uparrow\uparrow}(m + \delta m)$. Likewise, the second term in the right-hand side of (4.16) describes the negative and 1635 positive changes of $P_{\uparrow\uparrow}(m)$ arising from the flip of a single spin from $\sigma_z^{(n)} = -1$ to $\sigma_z^{(n)} = +1$. Quantum mechanics 1636 occurs in (4.16) through the expression (3.37) of $\tilde{K}(\omega)$; the flipping probabilities do not depend on the factor \hbar , owing 1637 to the factor \hbar^2 that enters $\tilde{K}(\omega)$ and the fact that we have chosen the dimensionless coupling constant γ , but their 1638 quantum nature is still expressed by the Bose-Einstein occupation number. 1639

The equation (4.18) for $P_{\uparrow\downarrow}$ has additional quantum features. Dealing with an off-diagonal block, it involves simultaneously the two Hamiltonians \hat{H}_{\uparrow} and \hat{H}_{\downarrow} of Eq. (4.6) in the Hilbert space of M, through the expression (4.12) of $\Omega_{\uparrow,\downarrow}^{\pm}$. The quantities $P_{\uparrow\downarrow}$ and $P_{\downarrow\uparrow}$ are complex and cannot be interpreted as probabilities, although we recognize in the right-hand side the same type of balance as in Eq. (4.16). In fact, while $\sum_{m} P_{\uparrow\uparrow}^{\text{dis}}(m) = 1 - \sum_{m} P_{\downarrow\downarrow}^{\text{dis}}(m)$, or in the N $\gg 1$ limit $\int dm P_{\uparrow\uparrow}(m) = 1 - \int dm P_{\downarrow\downarrow}(m)$, remains constant in time because the sum over *m* of the right-hand side of (4.16) vanishes, the term in the left-hand side of (4.18), which arises from $H_i - H_j$, prevents $\sum_m P_{\uparrow\downarrow}^{\text{dis}}(m)$ from being constant; It will, actually, lead to the disappearance of these "Schrödinger cat" terms.

¹⁶⁴⁷ Comparison of the right-hand sides of (4.16) and (4.18) shows moreover that the bath acts in different ways on the ¹⁶⁴⁸ diagonal and off-diagonal blocks of the density operator \hat{D} of S + M.

1649 4.5. Large N expansion

Except in subsection 8.1 we shall deal with a magnetic dot sufficiently large so that $N \gg 1$. The set of values (3.22) on which the distributions $P_{ij}(m, t)$ are defined then become dense on the interval $-1 \le m \le +1$. At the initial time, $P_{ij}(m, 0)$, proportional to (3.48), extends over a range of order $1/\sqrt{N}$ while the spacing of the discrete values of *m* is $\delta m = 2/N$. The initial distributions P_{ij} are thus smooth on the scale δm , and $P_{\uparrow\uparrow}$ and $P_{\downarrow\downarrow}$ will remain smooth at later times. It is therefore legitimate to *interpolate* the set of values of the diagonal quantities $P_{ii}(m, t)$ defined at the discrete points (3.22) into a continuous function of *m*. If we assume the two resulting functions P_{ii} to be several times differentiable with respect to *m*, the discrete equation (4.16) satisfied by the original distributions will give rise

³⁵You have to keep your balance

to continuous equations, which we shall derive below, involving an asymptotic expansion in powers of 1/N. Within exponentially small corrections, the characteristic functions associated with $P_{ii}(m, t)$ then reduce to integrals:

$$\Psi_{ii}(\lambda,t) \equiv \sum_{m} P_{ii}^{\text{dis}}(m,t) e^{i\lambda m} = \int \mathrm{d}m P_{ii}(m,t) e^{i\lambda m}, \qquad (4.20)$$

provided $\lambda \ll N$. The moments of $P_{ii}(m)$ of order less than N can also be evaluated as integrals.

However, the left-hand side of Eq. (4.18) generates for finite times rapid variations of $P_{\uparrow\downarrow}(m, t)$ and $P_{\downarrow\uparrow}(m, t)$ as functions of *m*, and it will be necessary in sections 5 and 6 to account for the discrete nature of *m*. When writing below the equations of motion for these quantities in the large *N* limit, we will take care of this difficulty.

The differences Δ_{\pm} defined by (4.15) satisfy

$$\Delta_{\pm}[f(m)g(m)] = \left[\Delta_{\pm}f(m)\right]g(m) + f(m)\left[\Delta_{\pm}g(m)\right] + \left[\Delta_{\pm}f(m)\right]\left[\Delta_{\pm}g(m)\right],\tag{4.21}$$

and give rise to derivatives with respect to m according to

$$\Delta_{\pm}f(m) \approx \pm \frac{2}{N} \frac{\partial f(m)}{\partial m} + \frac{2}{N^2} \frac{\partial^2 f(m)}{\partial m^2} \pm \frac{4}{3N^3} \frac{\partial^3 f(m)}{\partial m^3}.$$
(4.22)

We can also expand the excitation energies $\hbar \Omega_i^{\pm}$, defined by (4.12) and (4.6), for large N as

$$\Omega_i^{\pm}(m) \approx \mp 2\omega_i - \frac{2}{N} \frac{\mathrm{d}\omega_i}{\mathrm{d}m} = \left(1 \pm \frac{1}{N} \frac{\mathrm{d}}{\mathrm{d}m}\right) (\mp 2\omega_i),\tag{4.23}$$

1666 where we introduced the quantity

$$\hbar\omega_i = -\frac{1}{N} \frac{dH_i}{dm} = gs_i + J_2m + J_4m^3, \qquad (s_i = \pm 1),$$
(4.24)

¹⁶⁶⁷ interpreted as the effective energy of a single spin of M coupled to the other spins of M and to the tested spin S.

The above expansions will allow us to transform, for large *N*, the equations of motion for P_{ij} into partial differential equations. In case $\partial P_{ij}/\partial m$ is finite for large *N*, we can simply replace in (4.16) and (4.18) $N\Delta_{\pm}$ by $\pm 2\partial/\partial m$ and Ω_i^{\pm} by $\pm 2\omega_i$. However, such a situation is exceptional; we shall encounter it only in § 7.3.2. In general P_{ij} will behave for large *N* as *A* exp *NB*. This property, exhibited at t = 0 in §§ 3.3.3 and 3.3.4, is preserved by the dynamics. As $\partial P_{ij}/\partial t$ involves leading contributions of orders *N* and 1, we need to include in the right-hand sides of (4.16) and (4.18) contributions of the same two orders. Let us therefore introduce the functions

$$X_{ij}(m,t) \equiv \frac{1}{N} \frac{\partial \ln P_{ij}}{\partial m} = \frac{1}{NP_{ij}} \frac{\partial P_{ij}}{\partial m},$$
(4.25)

which contain parts of order 1 and 1/N, and their derivatives

$$X'_{ij} \equiv \frac{1}{N} \frac{\partial^2 \ln P_{ij}}{\partial m^2} = \frac{\partial X_{ij}}{\partial m},$$
(4.26)

which can be truncated at finite order in N. The discrete increments of P_{ij} are thus expanded as

$$\Delta_{\pm} P_{ij} = P_{ij} \left[\exp(\Delta_{\pm} \ln P_{ij}) - 1 \right] \approx P_{ij} \left[\exp\left(\pm 2X_{ij} + \frac{2}{N}X'_{ij}\right) - 1 + O\left(\frac{1}{N^2}\right) \right]$$

$$\approx P_{ij} \left[\exp\left(\pm 2X_{ij}\right) - 1 + \frac{2X'_{ij}}{N} \exp\left(\pm 2X_{ij}\right) + O\left(\frac{1}{N^2}\right) \right].$$
(4.27)

We express (4.16) by using the full relation (4.21), with $f = P_{\uparrow\uparrow}$ and $g = (1 \pm m)\tilde{K}_t(\Omega^{\mp}_{\uparrow})$, by evaluating $\Delta_{\pm}f$ from (4.27), and by inserting (4.13) into $\Delta_{\pm}g$. This yields

$$\frac{\partial P_{\uparrow\uparrow}}{\partial t} \approx \frac{2\gamma}{\hbar^2} P_{\uparrow\uparrow} \left\{ N \sinh X_{\uparrow\uparrow} \left[(1+m) \tilde{K}_t(\Omega_{\uparrow}^-) e^{X_{\uparrow\uparrow}} - (1-m) \tilde{K}_t(\Omega_{\uparrow}^+) e^{-X_{\uparrow\uparrow}} \right] \right.$$

$$\left. + e^{X_{\uparrow\uparrow}} \frac{\partial}{\partial m} \left[(1+m) \tilde{K}_t(2\omega_{\uparrow}) e^{X_{\uparrow\uparrow}} \right] - e^{-X_{\uparrow\uparrow}} \frac{\partial}{\partial m} \left[(1-m) \tilde{K}_t(-2\omega_{\uparrow}) e^{-X_{\uparrow\uparrow}} \right] + O\left(\frac{1}{N}\right) \right\}.$$

$$(4.28)$$

The first term on the right-hand side determines the evolution of the exponent of $P_{\uparrow\uparrow}$, which contains parts of order N, but contains also contributions of order 1 arising from the terms of order 1/N of (4.23) and of $X_{\uparrow\uparrow}$. The remaining terms determine the evolution of the amplitude of $P_{\uparrow\uparrow}$. The bath term of the equation (4.18) for $P_{\uparrow\downarrow}(m, t)$ (and for $P_{\downarrow\uparrow} = P_{\uparrow\downarrow}^*$) has a similar form, again obtained from all the terms in (4.21) and (4.27), namely, using the notation (4.19):

$$\frac{\partial P_{\uparrow\downarrow}}{\partial t} - \frac{2iNgm}{\hbar} P_{\uparrow\downarrow} \approx \frac{2\gamma}{\hbar^2} P_{\uparrow\downarrow} \left\{ N \sinh X_{\uparrow\downarrow} \left[(1+m)\tilde{K}_{-}(m,t)e^{X_{\uparrow\downarrow}} - (1-m)\tilde{K}_{+}(m,t)e^{-X_{\uparrow\downarrow}} \right] + e^{X_{\uparrow\downarrow}} \frac{\partial}{\partial m} \left[(1+m)\tilde{K}_{-}(m,t)e^{X_{\uparrow\downarrow}} \right] - e^{-X_{\uparrow\downarrow}} \frac{\partial}{\partial m} \left[(1-m)\tilde{K}_{+}(m,t)e^{-X_{\uparrow\downarrow}} \right] + O\left(\frac{1}{N}\right) \right\}.$$

$$(4.29)$$

A further simplification occurs for large N in the diagonal sector. Then $P_{\uparrow\uparrow}$, which is real, takes significant values only in the vicinity of the maximum of $\ln P_{\uparrow\uparrow}$. This maximum is reached at a point $m = \mu(t)$, and $P_{\uparrow\uparrow}$ is concentrated in a range for $|m - \mu(t)|$ of order $1/\sqrt{N}^{36}$. In this range, $X_{\uparrow\uparrow}$ is proportional to $\mu(t) - m$, and it is therefore of order $1/\sqrt{N}^{37}$. We can therefore expand (4.28) in powers of $X_{\uparrow\uparrow}$, noting also that $X'_{\uparrow\uparrow}$ is finite, and collect the $X_{\uparrow\uparrow}, X'_{\uparrow\uparrow}, X'_{\uparrow\uparrow}$ and $X_{\uparrow\uparrow\uparrow}X'_{\uparrow\uparrow}$ terms. Thus, if we disregard the exponentially small tails of the distribution $P_{\uparrow\uparrow}$, which do not contribute to physical quantities, we find at the considered order, using (4.25) and (4.26),

$$\frac{\partial P_{\uparrow\uparrow}}{\partial t} \approx \frac{\partial}{\partial m} \left[-v(m,t) P_{\uparrow\uparrow} \right] + \frac{1}{N} \frac{\partial^2}{\partial m^2} \left[w(m,t) P_{\uparrow\uparrow} \right], \tag{4.30}$$

1689 where

$$v(m,t) = \frac{2\gamma}{\hbar^2} \left[(1-m) \tilde{K}_t(-2\omega_{\uparrow}) - (1+m) \tilde{K}_t(2\omega_{\uparrow}) \right] + O\left(\frac{1}{N}\right), \tag{4.31}$$

$$w(m,t) = \frac{2\gamma}{\hbar^2} \left[(1-m) \tilde{K}_t \left(-2\omega_{\uparrow} \right) + (1+m) \tilde{K}_t \left(2\omega_{\uparrow} \right) \right] + O\left(\frac{1}{N}\right).$$

$$(4.32)$$

The next contribution to the right hand side of (4.30) would be $-2\nu X_{\uparrow\uparrow} Y_{\uparrow\uparrow}^{} P_{\uparrow\uparrow}$, of order $1/\sqrt{N}$. We have replaced in ν and w the frequencies Ω_{\uparrow}^{\pm} by $\mp 2\omega_{\uparrow}$, which has the sole effect of shifting the position and width of the distribution $P_{\uparrow\uparrow}$ by a quantity of order 1/N. As shown by the original equation (4.28), the two terms of (4.30) have the same order of magnitude (in spite of the presence of the factor 1/N in the second one) when $P_{\uparrow\uparrow}$ has an exponential form in N. Only the first one contributes if $P_{\uparrow\uparrow}$ becomes smooth (§ 7.3.2). The equation for $P_{\downarrow\downarrow}$ is obtained from (4.30) by changing ginto -g.

In the regime where the registration will take place (§ 7.1.1), we shall be allowed to replace $\tilde{K}_t(\pm 2\omega_i)$ by $\tilde{K}(\pm 2\omega_i)$, which according to (3.37) is equal to

$$\tilde{K}(\pm 2\omega_i) = \frac{\hbar^2 \omega_i}{4} \left[\coth(\beta \hbar \omega_i) \mp 1 \right] \exp\left(-\frac{2|\omega_i|}{\Gamma}\right), \qquad (i=\uparrow,\downarrow).$$
(4.33)

³⁶Numerically we find for N = 1000 extended distributions, see Figs. 7.5 and 7.6, since the typical peak width $1/\sqrt{N}$ is still sizable

³⁷This property does not hold for $P_{\uparrow\downarrow}$, since $X_{\uparrow\downarrow}$ contains a term $2igt/\hbar$ arising from the left hand side of (4.29)

 $_{1698}$ Eqs. (4.31) and (4.32) will thereby be simplified.

The final equations (4.29) and (4.30), with the initial conditions $P_{ij}(m, 0) = r_{ij}P_M(m, 0)$ expressed by (3.48), describe the evolution of S + M during the measurement process. We will work them out in sections 5 to 7. The various quantities entering them were defined by (4.25) and (4.26) for X_{ij} and X'_{ij} , by (4.31) and (4.32) for *v* and *w*, by (3.37), (4.10), (4.11), (4.17) and (4.19) for $\tilde{K}_{t>}$, $\tilde{K}_{t<}$, \tilde{K}_t and \tilde{K}_{\pm} , respectively, and by (4.24) for ω_i .

The dynamics of $P_{\uparrow\downarrow}$ has a *purely quantum* nature. The left-hand side of (4.29) governs the evolution of the normalization $\int dm P_{\uparrow\downarrow}(m, t)$, equal to the off-diagonal element $r_{\uparrow\downarrow}(t)$ of the marginal state $\hat{r}(t)$ of S. The bath gives rise on the right-hand side to a non-linear partial differential structure, which arises from the discrete nature of the spectrum of \hat{m} .

The final equation of motion (4.30) for $P_{\uparrow\uparrow}$ has the form of a *Fokker–Planck equation* [229, 230], which describes 1707 a stochastic motion of the variable m. Its coefficient v, which depends on m and t, can be interpreted as a *drift velocity*, 1708 while its coefficient w characterizes a diffusion process. This analogy with a classical diffusion process, should not, 1709 however, hide the *quantum origin of the diffusion term*, which is as sizeable for large N as the drift term. While the 1710 drift term comes out by bluntly taking the continuous limit of (4.16), the diffusion term originates, as shown by the 1711 above derivation, from the conjugate effect of two features: (i) the smallness of the fluctuations of m, and (ii) the 1712 discreteness of the spectrum of the pointer observable \hat{m} . Although the pointer is macroscopic, its quantum nature is 1713 essential, not only in the off-diagonal sector, but also in the diagonal sector which accounts for the registration of the 1714 result. 1715

1716 5. Very short times: truncation

 1717
 Alea iacta est^{38}

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 Julius Caesar

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 Since the coupling γ of the magnet M with the bath B is weak, some time is required before B acts significantly

 1720
 Since the coupling γ of the magnet M with the bath B is weak, some time is required before B acts significantly

on M. In the present section, we therefore study the behavior of S + M at times sufficiently short so that we can neglect the right-hand sides of (4.16) and (4.18). We shall see that the state $\hat{D}(t)$ of S + A is then *truncated*, that is, its off-diagonal blocks $\hat{R}_{\uparrow\downarrow}$ and $\hat{R}_{\downarrow\uparrow}$ rapidly decay, while the diagonal blocks are still unaffected.

- 1724 5.1. The truncation mechanism
- 1725 5.1.1. The truncation time

When their right-hand sides are dropped, the equations (4.16) and (4.18) with the appropriate boundary conditions are readily solved as

$$P_{\uparrow\uparrow}(m,t) = r_{\uparrow\uparrow}(0) P_{M}(m,0), \qquad P_{\downarrow\downarrow}(m,t) = r_{\downarrow\downarrow}(0) P_{M}(m,0), \tag{5.1}$$

$$P_{\uparrow\downarrow}(m,t) = \left[P_{\downarrow\uparrow}(m,t)\right]^* = r_{\uparrow\downarrow}(0) P_{\rm M}(m,0) e^{2iNgmt/\hbar}.$$
(5.2)

From the viewpoint of the tested spin S, these equations describe a Larmor precession around the *z*-axis [60], under the action of an effective magnetic field *Ngm* which depends on the state of M. From the viewpoint of the magnet M, we shall see in § 5.1.3 that the phase occurring in (5.2) generates time-dependent correlations between M and the transverse components of **s**.

The expectation values $\langle \hat{s}_a(t) \rangle$ of the components of **s** are found from (3.29) by summing (5.1) and (5.2) over *m*. These equations are valid for arbitrary *N* and arbitrary time *t* as long as the bath is inactive. If *N* is sufficiently large and *t* sufficiently small so that the summand is a smooth function on the scale $\delta m = 2/N$, that is, if $N \gg 1$ and $t \ll \hbar/g$, we can use (4.20) to replace the summation over *m* by an integration. These conditions will be fulfilled in subsections 5.1 and 5.2; we shall relax the second one in subsection 5.3 where we study the effects of the discreteness of *m*. Using the expression (3.48), (3.49) of $P_M(m, 0)$, we find by integrating (5.2) over *m*:

$$r_{\uparrow\downarrow}(t) = r_{\uparrow\downarrow}(0) e^{-(t/\tau_{\text{trunc}})^2},$$
(5.3)

³⁸The die is cast

Allahverdyan, Balian and Nieuwenhuizen / 00 (2014) 1-187

1738 or equivalently

$$\langle \hat{s}_a(t) \rangle = \langle \hat{s}_a(0) \rangle e^{-(t/\tau_{\text{trunc}})^2}, \qquad (a = x, y), \qquad (5.4)$$

$$\langle \hat{s}_{z}(t) \rangle = \langle \hat{s}_{z}(0) \rangle, \qquad (5.5)$$

¹⁷³⁹ where we introduced the truncation time

$$\tau_{\rm trunc} \equiv \frac{\hbar}{\sqrt{2} Ng\Delta m} = \frac{\hbar}{\sqrt{2N} \,\delta_0 g}.$$
(5.6)

Although $P_{\uparrow\downarrow}(m, t)$ is merely an oscillating function of *t* for each value of *m*, the summation over *m* has given rise to a damping. This property arises from the dephasing that exists between the oscillations for different values of *m*.

In the case $T_0 = \infty$ of a fully disordered initial state, we may solve directly (4.8) (without right-hand side) from the initial condition (4.9). We obtain, for arbitrary N, $\hat{R}_{\uparrow\downarrow}(t) = r_{\uparrow\downarrow}(0)2^{-N} \exp(2iNg\hat{m}t/\hbar)$, whence by using the definition (3.2) of \hat{m} and taking the trace over M, we find the exact result³⁹

$$r_{\uparrow\downarrow}(t) = r_{\uparrow\downarrow}(0) \left(\cos\frac{2gt}{\hbar}\right)^N,\tag{5.7}$$

which reduces to (5.3) for times of order τ_{trunc} .

Thus, over a time scale of order τ_{trunc} , the transverse components of the spin S decay and vanish while the *z*component is unaltered: the off-diagonal elements $r_{\uparrow\downarrow} = r_{\downarrow\uparrow}^*$ of the marginal density matrix of S disappear during the very first stage of the measurement process. It was to be expected that the apparatus, which is a large object, has a rapid and strong effect on the much smaller system S. In the present model, this rapidity arises from the *large number N of spins of the magnet*, which shows up through the factor $1/\sqrt{N}$ in the expression (5.6) of τ_{trunc} .

As we shall see in § 5.1.3, the off-diagonal block $\hat{\mathcal{R}}_{\uparrow\downarrow} = \hat{\mathcal{R}}_{\downarrow\uparrow}^{\dagger}$ of the full density matrix $\hat{\mathcal{D}}$ of S + A is proportional to $\hat{r}_{\uparrow\downarrow}(t)$ and its elements also decrease as $\exp[-(t/\tau_{trunc})^2]$, at least those elements which determine correlations involving a number of spins of M small compared to N. In the vocabulary of 1.3.3, *truncation* therefore takes place for the overall system S + A over the brief initial time lapse τ_{trunc} , while Eq. (5.3) describes *weak truncation* for S.

The quantum nature of the truncation process manifests itself through the occurrence of two different Hamiltonians \hat{H}_{\uparrow} and \hat{H}_{\downarrow} in the Hilbert space of M. Both of them occur in the dynamical equation (4.18) for $P_{\uparrow\downarrow}$, whereas only \hat{H}_{\uparrow} occurs in (4.16) for $P_{\uparrow\uparrow}$ through Ω_{\uparrow}^{\pm} , and likewise only \hat{H}_{\downarrow} for $P_{\downarrow\downarrow}$, through $\Omega_{\downarrow}^{\pm}$.

The truncation time τ_{trunc} is inversely proportional to the coupling *g* between \hat{s}_z and each spin $\hat{\sigma}_z^{(n)}$ of the magnet. It does not depend directly on the couplings J_q (q = 2, 4) between the spins $\hat{\sigma}_z^{(n)}$. Indeed, the dynamical equations (4.16), (4.18) without bath-magnet coupling involve only $H_{\uparrow}(m) - H_{\downarrow}(m)$, so that the interactions \hat{H}_M which are responsible for ferromagnetism cancel out therein. These interactions occur only through the right-hand side which describes the effect of the bath. They also appear indirectly in τ_{trunc} through the factor δ_0 of Δm given by (3.51), in the case q = 2of an Ising magnet M. When $J_2 \neq 0$, the occurrence of $\delta_0 > 1$ thus contributes to accelerate the truncation process.

1764 5.1.2. Truncation versus decoherence: a general phenomenon

It is often said [32, 33, 40, 176, 177, 178, 179] that "von Neumann's reduction is a decoherence effect". (The tra-1765 ditional word "reduction" covers in the literature both concepts of "truncation" and "reduction" as defined in § 1.3.3.) 1766 As is well known, decoherence is the rapid destruction of coherent superpositions of distinct pure states induced by 1767 a random environment, such as a thermal bath. In the latter seminal case, the characteristic decoherence time has 1768 the form of \hbar/T divided by some power of the number of degrees of freedom of the system and by a dimensionless 1769 coupling constant between the system and the bath (see also our discussion of the decoherence approach in section 2). Here, things are different. As we have just seen and as will be studied below in detail, the initial truncation process 1771 involves only the magnet. Although the bath is part of the apparatus, it has no effect here and the characteristic trun-1772 cation time τ_{trunc} does not depend on the bath temperature. Indeed the dimensional factor of (5.6) is \hbar/g , and not \hbar/T . 1773

³⁹An equivalent way to derive this result is to employ (3.28) for making the identification $P_{\uparrow\downarrow}^{\text{dis}}(m,t) = G(m) \times r_{\uparrow\downarrow}(0)2^{-N} \exp(2iNgmt/\hbar)$, and to sum over the values (3.22) of m

The thermal fluctuations are replaced by the fluctuation Δm of the pointer variable, which does not depend on T_0 for q = 4 and which decreases with T_0 as (3.51) for q = 2.

The fact that the truncation is controlled only by the coupling of the pointer variable \hat{m} with S is exhibited by the occurrence, in (5.6), of its number N of degrees of freedom of M. Registration of s_z requires this variable to be *collective*, so that $N \gg 1$. However, long before registration begins to take place in A through the influence on \hat{D} of $r_{\uparrow\uparrow}(0)$ and $r_{\downarrow\downarrow}(0)$, the large size of the detector entails the loss of $r_{\uparrow\downarrow}(0)$ and $r_{\downarrow\uparrow}(0)$.

¹⁷⁸⁰ Moreover, the *basis* in which the truncation takes place is *selected* by the very design of the apparatus. It depends ¹⁷⁸¹ on the observable which is being measured. Had we proceeded to measure \hat{s}_x instead of \hat{s}_z , we would have changed ¹⁷⁸² the orientation of the magnetic dot; the part of the initial state $\hat{r}(0)$ of S that gets lost would have been different. ¹⁷⁸³ Contrary to standard decoherence, truncation is here a controlled effect.

Altogether, it is only the pointer degrees of freedom *directly coupled to* S that are responsible for the rapid truncation. As such, it is a *dephasing*. The effects of the bath are important (sections 6.2 and 7), but do not infer on the initial truncation process, on the time scale τ_{trunc} . We consider it therefore confusing to use the term "decoherence" for the decay of the off-diagonal blocks in a quantum measurement, since its mechanism can be fundamentally different from a standard environment-induced decoherence. Here the truncation is a consequence of dephasing between oscillatory terms which should be summed to generate the physical quantities⁴⁰.

The above considerations hold for the *class of models* of quantum measurements for which *the pointer has many degrees of freedom* directly coupled to S [68, 159, 160] (see also [179] in this context). We have already found for the truncation time a behavior analogous to (5.6) in a model where the detector is a Bose gas [158], with a scaling in $N^{-1/4}$ instead of $N^{-1/2}$. More generally, suppose we wish to measure an arbitrary observable \hat{s} of a microscopic system S, with discrete eigenvalues s_i and corresponding projections $\hat{\Pi}_i$. The result should be registered by some pointer variable \hat{m} of an apparatus A coupled to \hat{s} . The full Hamiltonian has still the form (3.3), and it is natural to assume that the system–apparatus coupling has the same form

$$\hat{H}_{SA} = -Ng\hat{s}\hat{m},$$
 (general operators \hat{s}, \hat{m}) (5.8)

as (3.5). The coupling constant g refers to each one of the N elements of the collective pointer, so that a factor N appears in (5.8) as in (3.5), if \hat{m} is dimensionless and normalized in such a way that the range of its relevant eigenvalues is finite when N becomes large. The truncated density matrix $\hat{r}(t)$ is made of blocks $\langle i\alpha | \hat{r} | j\beta \rangle$ where α takes as many values as the dimension of $\hat{\Pi}_i$. It can be obtained as

$$\langle i\alpha | \hat{r}(t) | j\beta \rangle = \sum_{m} \langle i\alpha | \mathcal{P}(m,t) | j\beta \rangle, \qquad (5.9)$$

where $\langle i\alpha | \mathcal{P}(m,t) | j\beta \rangle$, which generalizes $P_{\uparrow\downarrow}(m,t)$, is defined by

$$\langle i\alpha | \mathcal{P}(m,t) | j\beta \rangle = \langle i\alpha | \operatorname{tr}_{A}\left(\delta_{\hat{m},m}\hat{\mathcal{D}}\right) | j\beta \rangle.$$
(5.10)

We have denoted by *m* the eigenvalues of \hat{m} , and by $\delta_{\hat{m},m}$ the projection operator on *m* in the Hilbert space of A. The quantity (5.10) satisfies an equation of motion dominated by (5.8):

$$\left[i\hbar\frac{\mathrm{d}}{\mathrm{d}t} + Ng(s_i - s_j)m\right]\langle i\alpha|\mathcal{P}(m,t)|j\beta\rangle \simeq 0.$$
(5.11)

In fact, the terms arising from \hat{H}_{S} (which need no longer vanish but only commute with \hat{s}) and from \hat{H}_{A} (which commutes with the initial density operator $\hat{\mathcal{R}}(0)$) are small during the initial instants compared to the term arising from the coupling \hat{H}_{SA} . We therefore find for short times

$$\langle i\alpha | \hat{r}(t) | j\beta \rangle = \langle i\alpha | \hat{r}(0) | j\beta \rangle \operatorname{tr}_{A} \hat{\mathcal{R}}(0) e^{iNg(s_{i}-s_{j})\hat{m}t/\hbar}.$$
(5.12)

The rapidly oscillating terms in the right-hand side interfere destructively as in (5.3) on a short time, if \hat{m} has a dense spectrum and an initial distribution involving many eigenvalues. Each contribution is merely oscillating, but the

⁴⁰In section 6.2 we shall discuss the effects of *decoherence* by the bath, which does take place, but long after the truncation time scale

summation over eigenvalues produces a relaxation. (We come back to this point in subsection 5.2 and in § 12.2.3.) 1809 This decrease takes place on a time scale of order $\hbar/Ng\delta s\Delta m$, where δs is the level spacing of the measured observable 1810 \hat{s} and Δm is the width of the distribution of eigenvalues of \hat{m} in the initial state of the apparatus. Leaving aside the 1811 later stages of the measurement process, we thus acknowledge the generality of the present truncation mechanism, 1812

and that of the expression (5.6) for the truncation time in the spin $\frac{1}{2}$ situation where $\delta s = 2$. 1813

5.1.3. Establishment and disappearance of correlations 181

Let us now examine how the apparatus evolves during this first stage of the measurement process, described by 1815 Eqs. (5.1) and (5.2). The first equation implies that the marginal density operator $\hat{R}_{M}(t) = \hat{R}_{\uparrow\uparrow}(t) + \hat{R}_{\downarrow\downarrow}(t)$ of M 1816 remains unchanged. This property agrees with the idea that M, a large object, has a strong influence on S, a small 1817 object, but that conversely a long time is required before M is affected by its interaction with S. Eqs. (5.1) also imply 1818 that no correlation is created between \hat{s}_z and M. 1819

However, although $\hat{R}_{M}(t) = \hat{R}(0)$, correlations are created between M and the transverse component \hat{s}_{x} (or \hat{s}_{y}) 1820 of S. These correlations are described by the quantities $C_x = P_{\uparrow\downarrow} + P_{\downarrow\uparrow}$ and $C_y = i(P_{\uparrow\downarrow} - P_{\downarrow\uparrow})$ introduced in (3.29). Since $\hat{R}_{\uparrow\downarrow}$ is a function of \hat{m} only, the components $\hat{\sigma}_x^{(n)}$ and $\hat{\sigma}_y^{(n)}$ of the spins of M remain statistically independent, with $\langle \hat{\sigma}_x^{(n)} \rangle = \langle \hat{\sigma}_y^{(n)} \rangle = 0$ and with the quantum fluctuations $\langle \hat{\sigma}_x^{(n)2} \rangle = \langle \hat{\sigma}_y^{(n)2} \rangle = 1$. The correlations between M and S 1821 1822 1823 involve only the z-component of the spins $\hat{\sigma}^{(n)}$ of the magnet and the x- or y-component of the tested spin s. We can 182 derive them as functions of time from the generating function 1825

$$\Psi_{\uparrow\downarrow}(\lambda,t) \equiv \sum_{k=0}^{\infty} \frac{i^k \lambda^k}{k!} \langle \hat{s}_- \hat{m}^k(t) \rangle = \sum_m P_{\uparrow\downarrow}^{\text{dis}}(m,t) e^{i\lambda m} = r_{\uparrow\downarrow}(0) \sum_m P_M^{\text{dis}}(m,0) e^{2iNgmt/\hbar + i\lambda m},$$
(5.13)

where $\hat{s}_{-} = \frac{1}{2}(\hat{s}_x - i\hat{s}_y)$. In fact, whereas $\Psi_{\uparrow\downarrow}(\lambda, t)$ generates the expectation values $\langle \hat{s}_{-}\hat{m}^k \rangle_c$, the correlations $\langle \hat{s}_{-}\hat{m}^k \rangle_c$ 1826 are defined by the cumulant expansion 1827

$$\Psi_{\uparrow\downarrow}(\lambda,t) = \sum_{k=0}^{\infty} \frac{i^k \lambda^k}{k!} \langle \hat{s}_- \hat{m}^k \rangle_c \left(\sum_{k'=0}^{\infty} \frac{i^{k'} \lambda^{k'}}{k'!} \langle \hat{m}^{k'} \rangle \right) = \sum_{k=0}^{\infty} \frac{i^k \lambda^k}{k!} \langle \hat{s}_- \hat{m}^k \rangle_c \exp\left(\sum_{k'=1}^{\infty} \frac{i^{k'} \lambda^{k'}}{k'!} \langle \hat{m}^{k'} \rangle_c \right),$$
(5.14)

which factors out the correlations $\langle \hat{m}^{k'} \rangle_c$ within M. The latter correlations are the same as at the initial time, so that 1828 we shall derive the correlations between S and M from 1829

$$\sum_{k=0}^{\infty} \frac{i^k \lambda^k}{k!} \langle \hat{s}_- \hat{m}^k(t) \rangle_{\rm c} = r_{\uparrow\downarrow}(0) \frac{\Psi_{\uparrow\downarrow}(\lambda, t)}{\Psi_{\uparrow\downarrow}(\lambda, 0)}.$$
(5.15)

For correlations involving not too many spins (we will discuss this point in § 5.3.2), we can again replace the 1830 summation over m in (5.13) by an integral. Since $P_{\rm M}(m,0)$ is a Gaussian, the sole non-trivial cumulant $\langle \hat{m}^k \rangle_{\rm c}$ is 1831 $\langle \hat{m}^2 \rangle = \Delta m^2$, given by (3.48), (3.49), and we get from (5.13) and (5.15) 1832

$$\sum_{k=0}^{\infty} \frac{i^k \lambda^k}{k!} \langle \hat{s}_- \hat{m}^k(t) \rangle_c = r_{\uparrow\downarrow}(0) \exp\left(-\frac{t^2}{\tau_{\text{trunc}}^2} - \sqrt{2} \frac{t}{\tau_{\text{trunc}}} \lambda \Delta m\right) = r_{\uparrow\downarrow}(t) \exp\left(-\sqrt{2} \frac{t}{\tau_{\text{trunc}}} \lambda \Delta m\right).$$
(5.16)

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At first order in
$$\lambda$$
, the correlations between S and any single spin of M are thus expressed by

$$\langle \hat{s}_{x} \hat{\sigma}_{z}^{(n)}(t) \rangle = \langle \hat{s}_{x} \hat{m}(t) \rangle_{c} = \sum_{m} C_{x}^{\text{dis}}(m, t) m = \sqrt{2} \frac{t}{\tau_{\text{trunc}}} \langle \hat{s}_{y}(t) \rangle \Delta m = \sqrt{2} \frac{t}{\tau_{\text{trunc}}} \langle \hat{s}_{y}(0) \rangle e^{-(t/\tau_{\text{trunc}})^{2}} \Delta m,$$

$$\langle \hat{s}_{y} \hat{\sigma}_{z}^{(n)}(t) \rangle = \langle \hat{s}_{y} \hat{m}(t) \rangle_{c} = \sum_{m} C_{y}^{\text{dis}}(m, t) m = -\sqrt{2} \frac{t}{\tau_{\text{trunc}}} \langle \hat{s}_{x}(t) \rangle \Delta m,$$

$$(5.17)$$

where we used (5.4). These correlations first increase, reach a maximum for $t = \tau_{trunc}/\sqrt{2}$, then decrease along with 1834 $\langle \hat{s}_x(t) \rangle$ and $\langle \hat{s}_y(t) \rangle$ (Fig. 5.1). At this maximum, their values satisfy 1835

$$\frac{\langle \hat{s}_x \hat{m}(t) \rangle}{\Delta m} = \langle \hat{s}_y(t) \rangle = \frac{\langle \hat{s}_y(0) \rangle}{\sqrt{e}}, \qquad \qquad \frac{\langle \hat{s}_y \hat{m}(t) \rangle}{\Delta m} = -\frac{\langle \hat{s}_x(0) \rangle}{\sqrt{e}}. \tag{5.18}$$

¹⁸³⁶ They do not lie far below the bound yielded by Heisenberg's inequality

$$|\langle \hat{s}_x \hat{m} \rangle|^2 = \left| \frac{1}{2i} \langle [\hat{s}_y - \langle \hat{s}_y \rangle, \hat{s}_z \hat{m}] \rangle \right|^2 \le \left(1 - \langle \hat{s}_y \rangle^2 \right) \Delta m^2, \tag{5.19}$$

1837 which implies at all times

$$\left(\frac{2t^2}{\tau_{\text{trunc}}^2}+1\right)\langle\hat{s}_y(t)\rangle^2 \le 1,$$
(5.20)

since the left-hand side of (5.20) is 2/e at the maximum of (5.17).

The next order correlations are obtained from (5.16) as (a = x, y)

$$\langle \hat{s}_a \hat{m}^2(t) \rangle_{\rm c} \equiv \langle \hat{s}_a \hat{m}^2(t) \rangle - \langle \hat{s}_a(t) \rangle \langle \hat{m}^2 \rangle = -\frac{2t^2}{\tau_{\rm trunc}^2} \langle \hat{s}_a(t) \rangle \Delta m^2.$$
(5.21)

These correlations again increase, but more slowly than (5.17), reach (in absolute value) a maximum later, at $t = \tau_{\text{trunc}}$, equal to $(-2/e)\langle \hat{s}_a(0)\rangle\Delta m^2$, then decrease together with $\langle \hat{s}_a(t)\rangle$. Accordingly, the correlations between \hat{s}_x and two spins of M, evaluated as in (3.27), are given by

$$\langle \hat{s}_x \hat{\sigma}_a^{(n)} \hat{\sigma}_b^{(p)}(t) \rangle_{\rm c} = \langle \hat{s}_x(t) \rangle \frac{\delta_{a,z} \delta_{b,z}}{N-1} \left(-\frac{2t^2}{\tau_{\rm trunc}^2} N \Delta m^2 - 1 \right), \tag{5.22}$$

which for large N behaves as (5.21).

Likewise, (5.16) together with (5.3) provides the hierarchy of correlations through the real and imaginary parts of

$$\langle (\hat{s}_x - i\hat{s}_y)\hat{m}^k(t)\rangle_{\rm c} = \langle (\hat{s}_x - i\hat{s}_y)(0)\rangle \left(i\sqrt{2}\frac{t}{\tau_{\rm trunc}}\Delta m\right)^k e^{-(t/\tau_{\rm trunc})^2},\tag{5.23}$$

with Δm from (3.51). This expression also holds for more detailed correlations such as $\langle \hat{s}_a \hat{\sigma}_z^{(1)} \hat{\sigma}_z^{(2)} \cdots \hat{\sigma}_z^{(k)}(t) \rangle_c$ within corrections of order 1/N as in eq. (5.22), provided k/N is small.

Altogether (Fig. 5.1) the correlations (5.23) scale as $\Delta m^k = (\delta_0 / \sqrt{N})^k$. If the rank *k* is odd, $\langle \hat{s}_x \hat{m}^k(t) \rangle_c$ is proportional to $\langle \hat{s}_y(0) \rangle$, if *k* is even, it is proportional to $\langle \hat{s}_x(0) \rangle$, with alternating signs. The correlations of rank *k* depend on time as $(t/\tau_{trunc})^k \exp[-(t/\tau_{trunc})^2]$. Hence, correlations of higher and higher rank begin to grow later and later, in agreement with the factor t^k , and they reach a maximum later and later, at the time $t = \tau_{trunc} \sqrt{k/2}$. For even *k*, the maximum of $|\langle \hat{s}_x \hat{m}^k(t) \rangle_c|$ is given by

$$\max\left|\frac{\langle \hat{s}_x \hat{m}^k(t) \rangle_c}{\langle \hat{s}_x(0) \rangle \Delta m^k}\right| = \frac{1}{k!} \left(\frac{2k}{e}\right)^{k/2} \left(\frac{k}{2}\right)! \simeq \frac{1}{\sqrt{2}},\tag{5.24}$$

which is nearly independent of k.

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1853 5.1.4. The truncation, a cascade process

*Het viel in gruzelementen*⁴¹ Dutch saying

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The mechanism of truncation in the present model is therefore comparable to a current mechanism of irreversibility in statistical mechanics (§ 1.2.2). In a classical Boltzmann gas, initially off-equilibrium with a non-uniform density, the relaxation toward uniform density takes place through the establishment of correlations between a larger and larger number of particles, under the effect of successive collisions [54, 55, 56]. Here, similar features occur although quantum dynamics is essential. The relaxation (5.4) of the off-diagonal elements $r_{\uparrow\downarrow} = r_{\downarrow\uparrow}^*$ of the marginal state \hat{r} of S is accompanied by the generation, owing to the coupling \hat{H}_{SA} , of correlations between S and M.

⁴¹It fell and broke into tiny pieces

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Figure 5.1: The relative correlations $corr = \langle (\hat{s}_x - i\hat{s}_y)\hat{m}^k(t) \rangle_c / \langle (\hat{s}_x - i\hat{s}_y)(0) \rangle (i\sqrt{2}\Delta m)^k$ from Eq. (5.23), as function of t/τ_{trunc} . For k = 0 $\langle \hat{s}_x(t) \rangle$ decreases as a Gaussian. The curves for k = 1, 2 and 3 show that the correlations develop, reach a maximum, then disappear later and later.

Such correlations, absent at the initial time, are *built up and fade out in a cascade*, as shown by eq. (5.23) and Fig. 5.1. Let us characterize the state \hat{R} of S + M by the expectation values and correlations of the operators \hat{s}_a and $\hat{\sigma}_a^{(n)}$. The order of S, initially embedded in the expectation values of the transverse components $r_{\uparrow\downarrow}(0)$ of the spin \hat{s} , is progressively transferred to correlations (5.17) between these components and one spin of M, then in turn to correlations (5.22) with two spins, with three spins, and so on. The larger the rank k of the correlations, the smaller they are, as $\Delta m^k \sim 1/N^{k/2}$ (Eq. (5.23)); but the larger their number is, as $N!/k! (N - k)! \approx N^k/k!$. Their time-dependence, in $t^k \exp[-(t/\tau_{\text{trunc}})^2]$, shows how they blow up and blow out successively.

As a specific feature of our model of quantum measurement, the interaction process does not affect the marginal statistical state of M. All the multiple correlations produced by the coupling \hat{H}_{SA} lie astride S and M.

Truncation, defined as the disappearance of the off-diagonal blocks $\hat{\mathcal{R}}_{\uparrow\downarrow} = \hat{\mathcal{R}}_{\downarrow\uparrow}$ of the full density matrix $\hat{\mathcal{D}}$ of S+A, 1871 or equivalently of the expectation values of all operators involving \hat{s}_x or \hat{s}_y , results from the proportionality of $\hat{\mathcal{R}}_{\uparrow\downarrow}(t)$ to 1872 $r_{\uparrow\downarrow}(t)$, within a polynomial coefficient in t associated with the factor t^k in the k-th rank correlations. Initially, only few 1873 among the $2^N \times 2^N$ elements of the matrix $\hat{R}_{\uparrow\downarrow}(0)$ do not vanish, those which correspond to $2r_{\uparrow\downarrow}(0) = \langle \hat{s}_x(0) \rangle - i \langle \hat{s}_y(0) \rangle$ 1874 and to $P_{\rm M}(m,0)$ given by (3.48). The very many elements of $\hat{R}_{\uparrow\downarrow}(0)$ which describe correlations between \hat{s}_x or 1875 \hat{s}_y and the spins of M, absent at the initial time, grow, while an overall factor $\exp[-(t/\tau_{\text{trunc}})^2]$ damps $\hat{R}_{\uparrow\downarrow}(t)$. At 1876 times $\tau_{\text{trunc}} \ll t \ll \hbar/g$, all elements of $\hat{R}_{\uparrow\downarrow}(t)$ and hence of $\hat{R}_{\uparrow\downarrow}(t)$ have become negligibly small⁴². In principle, 1877 no information is lost since the equations of motion are reversible; in particular, the commutation of \hat{H} with the projections $\hat{\Pi}_{\uparrow}$ and $\hat{\Pi}_{\downarrow}$, together with the equation of motion (4.1), implies that $i\hbar d(\hat{R}_{\uparrow\downarrow}\hat{R}_{\downarrow\uparrow})/dt = [\hat{H}, \hat{R}_{\uparrow\downarrow}\hat{R}_{\downarrow\uparrow}]$, and hence that $\text{tr}_A \hat{R}_{\uparrow\downarrow} \hat{R}_{\uparrow\downarrow}^{\dagger}$ is constant in time and remains equal to $|r_{\uparrow\downarrow}(0)|^2 \text{tr}_A [\hat{R}(0)]^2$. However, the initial datum $r_{\uparrow\downarrow}(0)$ 1878 1879 1880 gets spread among very many matrix elements of $\hat{\mathcal{R}}_{\uparrow\downarrow}$ which nearly vanish, exactly as in the irreversibility paradox 1881 (§ 1.2.2). 1882

If N could be made infinite, the progressive creation of correlations would provide a rigorous mathematical charac-1883 terization of the irreversibility of the truncation process, as for relaxation processes in statistical mechanics. Consider, 1884 for some fixed value of K, the set of correlations (5.23) of ranks k such that $0 \le k \le K$, including $\langle \hat{s}_x \rangle$ and $\langle \hat{s}_v \rangle$ for 1885 k = 0. All correlations of this set vanish in the limit $N \to \infty$ for fixed t, since τ_{trunc} then tends to 0. (The coupling 1886 constant g may depend on N, in which case it should satisfy $Ng^2 \rightarrow \infty$.) This property holds even for infinite K, 1887 provided $K \to \infty$ after $N \to \infty$, a limit which characterizes the irreversibility. However, such a limit is not uni-1888 form: the reversibility of the underlying dynamics manifests itself through the finiteness of high-order correlations for 1889 sufficiently large t (§ 5.3.2). 1890

Anyhow *N* is not allowed in physics to go to infinity, since the time τ_{trunc} would unrealistically vanish. For large but finite *N*, there is no rigorous qualitative characterization of irreversibility, neither in this model of measurement nor in statistical mechanics, but the above discussion remains relevant. In fact, physically, it is legitimate to regard as equal to zero a quantity which is less than some small bound, and to regard as unobservable and irrelevant all

 $^{^{42}}$ The latter implication follows because the bath contributions cannot raise the S + A correlations

correlations which involve a number k of spins exceeding some bound K much smaller than N. We shall return to this issue in § 12.2.3.

1897 5.2. Randomness of the initial state of the magnet

Initial states $\hat{\mathcal{R}}(0)$ that can actually be prepared at least in a thought experiment, such as the paramagnetic canonical equilibrium distribution of § 3.3.3, involve large randomness. In particular, if the initialization temperature T_0 is sufficiently large, the state (3.46), i. e., $\hat{\mathcal{R}}_{M}(0) = 2^{-N} \prod_{n} \hat{\sigma}_{0}^{(n)}$, is the most disordered statistical state of M; in such a case, $P_{M}(m, 0)$ is given by (3.50). We explore in this subsection how the truncation process is modified for other, less random, initial states of M.

1903 5.2.1. Arbitrary initial states

The derivations of the equations of motion in subsections 4.1 and 4.2 were general, irrespective of the initial state. However, in subsections 4.3 and 5.1 we have relied on the fact that $\hat{R}_{M}(0)$ depends only on \hat{m} . In order to deal with an arbitrary initial state $\hat{R}_{M}(0)$, we return to eq. (4.8), where we can as above neglect for very short times the coupling with the bath. The operators $\hat{R}_{ij}(t)$ and \hat{H}_{i} in the Hilbert space of M no longer commute because \hat{R}_{ij} now involves spin operators other than \hat{m} . However, the probabilities and correlations $P_{ij}(m, t)$ defined by (3.24) still satisfy Eqs. (B.13) of Appendix B without right-hand side. Hence the expressions (5.1) and (5.2) for $P_{ij}(m)$ at short times hold for any initial state $\hat{R}_{M}(0)$, with $P_{M}(m, 0)$ given by $\operatorname{tr}_{M}\hat{R}_{M}(0) \delta_{\hat{m},m}$.

The various expressions (5.4), (5.5), (5.6), (5.17), (5.21), (5.23) relied only on the Gaussian shape of the probability distribution $P_{\rm M}(0,m)$ associated with the initial state. They will therefore remain valid for any initial state $\hat{R}_{\rm M}(0)$ that provides a narrow distribution $P_{\rm M}(m, 0)$, centered at m = 0 and having a width Δm small ($\Delta m \ll 1$) though large compared to the level spacing, viz. $\Delta m \gg 2/N$. Indeed, within corrections of relative order 1/N, such distributions are equivalent to a Gaussian. The second condition ($\Delta m \gg 2/N$) ensures that $\tau_{\rm trunc}$ is much shorter than \hbar/g , another characteristic time that we shall introduce in § 5.3.1.

In fact, the behavior in $1/\sqrt{N}$ for Δm is generic, so that the truncation time has in general the same expression (5.6) as for a paramagnetic canonical equilibrium state, with δ_0 defined by $\delta_0^2 = N \text{tr}_M \hat{R}_M(0) \hat{m}^2$. The dynamics of the truncation process described above holds for most possible initial states of the apparatus: decay of $\langle \hat{s}_x(t) \rangle$ and $\langle \hat{s}_y(t) \rangle$; generation of a cascade of correlations $\langle \hat{s}_a \hat{m}^k(t) \rangle$ of order Δm^k between the transverse components of the spin s and the pointer variable \hat{m} ; increase, then decay of the very many matrix elements of $\hat{R}_{\uparrow\downarrow}(t)$, which are small as

¹⁹²²
$$\left(\sqrt{2}\Delta m t/\tau_{\text{trunc}}\right)^{\sim} \exp\left[-\left(t/\tau_{\text{trunc}}\right)^2\right]$$
 for $t \ll \hbar/g$

In case the initial density operator $\hat{R}_{M}(0)$ is a symmetric function of the *N* spins, the correlations between \hat{s}_{x} or \hat{s}_{y} and the *z*-components of the individual spins of M are still given by expressions such as (5.22). However, in general, $\hat{R}_{M}(0)$ no longer depends on the operator \hat{m} only; it involves transverse components $\hat{\sigma}_{x}^{(n)}$ or $\hat{\sigma}_{y}^{(n)}$, and so does $\hat{R}_{\uparrow\downarrow}(t)$, which now includes correlations of \hat{s}_{x} or \hat{s}_{y} with *x*- or *y*-components of the spins $\hat{\sigma}^{(n)}$. The knowledge of $P_{\uparrow\downarrow}(m, t)$ is in this case not sufficient to fully determine $\hat{R}_{\uparrow\downarrow}(t)$, since (3.24) holds but not (3.26).

The proportionality of the truncation time $\tau_{trunc} = \hbar / \sqrt{2} Ng\Delta m$ to the inverse of the fluctuation Δm shows that the truncation is a disorder effect, since Δm measures the randomness of the pointer variable in the initial state. This is easy to understand: S sees an effective magnetic field Ngm which is random through m, and it is this very randomness which causes the relaxation. The existence of such a randomness in the initial state, even though it is small as $1/\sqrt{N}$, is necessary to ensure the transfer of the initial order embodied in $r_{\uparrow\downarrow}$ (0) into the cascade of correlations between S and M and to entail a brief truncation time τ_{trunc} . Boltzmann's elucidation of the irreversibility paradox also relied on statistical considerations about the initial state of a classical gas which will relax to equilibrium.

1935 5.2.2. Pure versus mixed initial state

It is therefore natural to wonder whether the truncation of the state would still take place for pure initial states of M, which are the least random ones in quantum physics, in contrast to the paramagnetic state (3.46) or (3.50) which is the most random one. To answer this question, we first consider the pure state with density operator

$$\hat{R}_{\rm M}(0) = \prod_{n=1}^{N} \frac{1}{2} \left(1 + \hat{\sigma}_x^{(n)} \right), \tag{5.25}$$

in which all spins $\hat{\sigma}^{(n)}$ point in the x-direction. This initialization may be achieved by submitting M to a strong field in 1939 the x-direction and letting it thermalize with a cold bath B for a long duration before the beginning of the measurement. 1940 The fluctuation of \hat{m} in the state (5.25) is $1/\sqrt{N}$. Hence, for this pure initial state of M, the truncation takes place 194 exactly as for the fully disordered initial paramagnetic state, since both yield the same probability distribution (3.50) 1942 for *m*. 1943

A similar conclusion holds for the most general factorized pure state, with density operator 1944

$$\hat{R}_{\mathrm{M}}(0) = \prod_{n=1}^{N} \frac{1}{2} \left(1 + \mathbf{u}^{(n)} \cdot \hat{\boldsymbol{\sigma}}^{(n)} \right), \qquad (5.26)$$

where the $\mathbf{u}^{(n)}$ are arbitrary unit vectors pointing in different directions⁴³. The fluctuation Δm , then given by 1945

$$\delta_0^2 = N\Delta m^2 = \frac{1}{N} \sum_{n=1}^N \left[1 - \left(u_z^{(n)} \right)^2 \right], \tag{5.27}$$

is in general sufficiently large to ensure again the properties of subsection 5.1, which depend on $\hat{R}_{M}(0)$ only through 1946 Λm 1947

Incoherent or coherent superpositions of such pure states will yield the same effects. We will return to this point 1948 in § 12.1.4, noting conversely that an irreversibility which occurs for a mixed state is also statistically present in most 1949 of the pure states that underlie it. 1950

Quantum mechanics brings in another feature: a given mixed state can be regarded as a superposition of pure states 1951 in many different ways. For instance, the completely disordered paramagnetic state (3.46), $\hat{R}_{\rm M}(0) = 2^{-N} \prod_n \hat{\sigma}_0^{(n)}$, can 1952 be described by saying that each spin points at random in the +z or in the -z-direction; it can also be described as 1953 an incoherent superposition of the pure states (5.26) with randomly oriented vectors $\mathbf{u}^{(n)}$. This ambiguity makes the 1954 analysis into pure components of a quantum mixed state unphysical (§ 10.2.3). 1955

Let us stress that the statistical or quantum nature of the fluctuations Δm of the pointer variable in the initial state 1956 is irrelevant as regards the truncation process. In the most random state (3.46) this fluctuation $1/\sqrt{N}$ appears as purely 1957 statistical; it would be just the same for "classical spins" having only a z-component with random values ± 1 . In the 1958 pure state (5.25), it is merely quantal; indeed, its value $1/\sqrt{N}$ is the lower bound provided by Heisenberg's inequality 1959

$$\Delta m_y^2 \Delta m_z^2 \ge \frac{1}{4} \left| \left\langle \left[\hat{m}_y, \hat{m}_z \right] \right\rangle \right|^2 = \frac{1}{N^2} \left\langle \hat{m}_x \right\rangle^2 \tag{5.28}$$

for the operators $\hat{m}_a = N^{-1} \sum_n \hat{\sigma}_a^{(n)}$ (a = x, y, z), with here $\Delta m_y = \Delta m_z = 1/\sqrt{N}$, $\langle \hat{m}_x \rangle = 1$. Differences between these 1960 two situations arise only at later times, through the coupling \hat{H}_{MB} with the bath. 196

5.2.3. Squeezed initial states 1962

196

There exist states $\hat{R}_{M}(0)$, which we will term as "squeezed", for which the fluctuation Δm is of smaller order than 1965 $1/\sqrt{N}$. An extreme case in which $\Delta m = 0$ is, for even N, a pure state in which N/2 spins point in the +z-direction, 1966 N/2 in the -z-direction; then $P_{\rm M}(m,0) = \delta_{m,0}$. Coherent or incoherent superpositions of such states yield the same 1967 distribution $P_{\rm M}(m,0) = \delta_{m,0}$, in particular the microcanonical paramagnetic state $\hat{R}_{\rm M}(0) = \delta_{\hat{m},0}[(N/2)!]^2/N!$. In all 1968 such cases, m and Δm exactly vanish so that the Hamiltonian and the initial state of S + M satisfy $(\hat{H}_{SA} + \hat{H}_M)\hat{D}(0) = 0$, 1969 $\hat{D}(0)(\hat{H}_{SA} + \hat{H}_M) = 0$. According to Eq. (4.8), nothing will happen, both in the diagonal and off-diagonal sectors, until 1970 the bath begins to act through the weak terms of the right-hand side. The above mechanism of truncation based on 1971 the coupling between S and M thus fails for the states $\hat{D}(0)$ such that $P_M(m,0) = \delta_{m,0}$, whether these states are pure 1972 or not. 1973

⁴³The consideration of such a state is academic since it would be impossible, even in a thought experiment, to set M in it ⁴⁴You should not begin in this way

The situation is similar for slightly less squeezed states in which the fluctuation Δm is of the order of the level spacing $\delta m = 2/N$, with about half of the spins nearly oriented in the +*z*-direction and half in the -*z*-direction. When the bath B is disregarded, the off-diagonal block $\hat{R}_{\uparrow\downarrow}(t)$ then evolves, as shown by (5.2), on a time scale of order $\hbar/2g$ instead of the much smaller truncation time (5.6), of order $1/\sqrt{N}$.

In such cases the truncation will appear (contrary to our discussion of § 5.1.2) as a phenomenon of the decoherence type, governed indirectly by B through \hat{H}_{SA} and \hat{H}_{MB} , and taking place on a time scale much longer than τ_{trunc} . This circumstance occurs in many models of measurement, see section 2, in particular those for which S is not coupled with many degrees of freedom of the pointer. It is clearly the large size of M which is responsible here for the fast truncation. We return to this point in § 8.1.4.

Here again, we recover ideas that were introduced to elucidate the irreversibility paradox. In a Boltzmann gas, one 1983 can theoretically imagine initial states with a uniform density which would give rise after some time to a macroscopic 1984 inhomogeneity [55, 56]. But such states are extremely scarce and involve subtle specific correlations. Producing one 1985 of them would involve the impossible task of handling the particles one by one. However, for the present truncation 1986 mechanism, the initial states of M such that the off-diagonal blocks of the density matrix fail to decay irreversibily 1987 are much less exceptional. While the simplest types of preparation of the apparatus, such as setting M in a canonical 1988 paramagnetic state through interaction with a warm bath (§ 3.3.3), yield a fluctuation Δm of order $1/\sqrt{N}$, we can 198 imagine producing squeezed states even through macroscopic means. For instance, a microcanonical paramagnetic 1990 type of initial state of M could be obtained by separating the sample of N spins into two equal pieces, by setting 1991 them (using a cold bath and opposite magnetic fields) into ferromagnetic states with opposite magnetizations, and by 1992 mixing them again. Some spin-conserving interaction can then randomize the orientations before the initial time of 1993 the measurement process. We can also imagine, as in modern experiments on optical lattices, switching on and off a 1994 strong antiferromagnetic interaction to equalize the numbers of spins pointing up and down. 1995

¹⁹⁹⁶ 5.3. Consequences of discreteness

1998

Somewhat surprisingly since N is large, it appears that the discreteness of the pointer variable m has specific implications in the off-diagonal blocks of the density matrix. We shall later see that such effects do not occur in the diagonal sectors related to registration.

2002 5.3.1. The recurrence time

2003 2004 *De klok is rond gegaan*⁴⁶ Dutch expression

Dutch proverb

Hij keek of hij water zag branden⁴⁵

Although we have displayed the truncation of the state as an irreversible process on the time scale τ_{trunc} , the dynamics of our model without the bath is so simple that we expect the reversibility of the equations of motion to manifest itself for finite *N*. As a matter of fact, the irreversibility arises as usual (§ 1.2.2) from an approximate treatment, justified only under the conditions considered above: large *N*, short time, correlations of finite order. This approximation, which underlined the results (5.4) and (5.16) of subsections 5.1 and 5.2, consisted in treating *m* as a continuous variable. We now go beyond it by returning to the expression (5.13), which is exact if the bath is inactive $(\gamma = 0)$, and by taking into account the discreteness of the spectrum of \hat{m} .

For $N \gg 1$, we can still use for $P_{\rm M}(m,0)$ the Gaussian form (3.48) based on (3.23). The generating function (5.13) then reads

$$\Psi_{\uparrow\downarrow}(\lambda,t) = r_{\uparrow\downarrow}(0) \sqrt{\frac{2}{\pi}} \frac{1}{N\Delta m} \sum_{m} \exp\left[-\frac{m^2}{2\Delta m^2} + i\pi Nm \frac{t}{\tau_{\text{recur}}} + i\lambda m\right],$$
(5.29)

²⁰¹⁴ where we have introduced the recurrence time

$$\tau_{\rm recur} \equiv \frac{\pi\hbar}{2g} = \pi \sqrt{2} \frac{\Delta m}{\delta m} \tau_{\rm trunc}.$$
(5.30)

⁴⁵He looked as if he saw water burn, i.e., he was very surprised

⁴⁶The clock has made a turn

The values (3.22) of *m* that contribute to the sum (5.28) are equally spaced, at distances $\delta m = 2/N$. The replacement of this sum by an integral, which was performed in § 5.1.3, is legitimate only if $t \ll \tau_{\text{recur}}$ and $|\lambda| \ll N$. When the time *t* increases and begins to approach τ_{recur} within a delay of order τ_{trunc} , the correlations undergo an *inverse cascade*: Simpler and simpler correlations are gradually generated from correlations involving a huge number of spins of M. This process is the time-reversed of the one described in § 5.1.3. When *t* reaches τ_{recur} , or a multiple of it, the various terms of (5.29) add up, instead of interfering destructively as when *t* is of order of τ_{trunc} . In fact, the generating function (5.29) satisfies

$$\Psi_{\uparrow\downarrow}(\lambda, t + \tau_{\text{recur}}) = (-1)^N \Psi_{\uparrow\downarrow}(\lambda, t), \qquad (5.31)$$

so that without the bath the state $\hat{D}(t)$ of S + M *evolves periodically*, returning to its initial expression $\hat{r}(0) \otimes \hat{R}(0)$ at equally spaced times: the Schrödinger cat terms revive.

This recurrence is a quantum phenomenon [55, 56]. It arises from the discreteness and regularity of the spectrum of the pointer variable operator \hat{m} , and from the oversimplified nature of the model solved in the present section, which includes only the part (3.5) of the Hamiltonian. We will exhibit in section 6 two mechanisms which, in less crude models, modify the dynamics on time scales larger than τ_{trunc} and prevent recurrences from occurring.

The recurrence time (5.30) is much longer than the truncation time, since $\Delta m/\delta m = \frac{1}{2}\delta_0 \sqrt{N}$. Thus, long after the initial order carried by the transverse components $\langle \hat{s}_x \rangle$ and $\langle \hat{s}_y \rangle$ of the spin S has dissolved into numerous and weak correlations, *this order revives* through an inverse cascade. At the time τ_{recur} , S gets decorrelated from M, with $r_{\uparrow\downarrow}(\tau_{\text{recur}}) = (-1)^N r_{\uparrow\downarrow}(0)$. The memory of the off-diagonal elements, which was hidden in correlations, was only dephased, it was not lost for good, and it emerges back. Such a behavior of the transverse components of the spin S is reminiscent of the behavior of the transverse magnetization in spin echo experiments [60, 61, 62, 63, 64, 65]. By itself it is a dephasing which can cohere again, and will do so unless other mechanisms (see section 6) prevent this.

2035 5.3.2. High-order correlations

2036 2037

We can write $\Psi_{\uparrow\downarrow}(\lambda, t)$ given by (5.29) more explicitly, for large *N*, by formally extending the summation over *m* beyond -1 and +1, which is innocuous, and by using Poisson's summation formula, which reads

$$\sum_{m} f(m) = \frac{N}{2} \sum_{p=-\infty}^{+\infty} (-1)^{pN} \int dm \, e^{-i\pi N m p} f(m) \,.$$
(5.32)

As a result, we get

$$\Psi_{\uparrow\downarrow}(\lambda,t) = r_{\uparrow\downarrow}(0) \sum_{p=-\infty}^{+\infty} (-1)^{pN} \exp\left(\frac{i\lambda\Delta m}{\sqrt{2}} + i\frac{t-p\tau_{\text{recur}}}{\tau_{\text{trunc}}}\right)^2,$$
(5.33)

which is nothing but a sum of contributions deduced from (5.15), (5.16) and (5.3) by repeated shifts of t (with alternating signs for odd N). This obviously periodic expression exhibits the recurrences and the corrections to the results of subsections 5.1 and 5.2 due to the discreteness of m.

In fact, $\Psi_{\uparrow\downarrow}(\lambda, t)$ is related to the elliptic function θ_3 [231] through

$$\frac{\Psi_{\uparrow\downarrow}(\lambda,t)}{r_{\uparrow\downarrow}(0)} = \exp\left(\frac{i\lambda\Delta m}{\sqrt{2}} + \frac{it}{\tau_{\text{trunc}}}\right)^2 \theta_3 \left[\frac{1}{2}\left(i\lambda\delta_0^2 + \eta + i\pi N^2 \Delta m^2 \frac{t}{\tau_{\text{recur}}}\right), \frac{N^2 \Delta m^2}{2}\right]$$
$$= \sqrt{\frac{2}{\pi}} \frac{1}{N\Delta m} \exp\left[-\eta\left(\frac{i\pi t}{\tau_{\text{recur}}} + \frac{i\lambda}{N} + \frac{1}{2N\delta_0^2}\right)\right] \theta_3 \left[\frac{t}{\tau_{\text{recur}}} - \frac{i}{N\pi}\left(i\lambda + \frac{\eta}{\delta_0^2}\right), \frac{2}{\pi^2 N^2 \Delta m^2}\right], \quad (5.34)$$

with $\eta = 0$ for even N, $\eta = 1$ for odd N. It satisfies two periodicity properties, (5.31) and

⁴⁷It can always be better

Het kan altijd beter⁴⁷

Anonymous (Dutch)

$$\Psi_{\uparrow\downarrow}\left(\lambda - \frac{2i}{\delta_0^2}, t\right) = \exp\left(\frac{2\pi it}{\tau_{\text{recur}}} + \frac{2i\lambda}{N} + \frac{2}{N\delta_0^2}\right)\Psi_{\uparrow\downarrow}\left(\lambda, t\right).$$
(5.35)

According to (5.15) and (5.33) the dominant corrections to the results of § 5.1.3 are given for $t \ll \tau_{\text{recur}}$ by the terms $p = \pm 1$ in $\Psi_{\uparrow\downarrow}(\lambda, t)$ and $\Psi_{\uparrow\downarrow}(\lambda, 0)$, that is,

$$\langle \hat{s}_{-} \hat{m}^{k} \rangle_{c} = r_{\uparrow\downarrow} (0) \exp\left(-\frac{t^{2}}{\tau_{trunc}^{2}}\right) \left(i\sqrt{2}\Delta m\right)^{k} \left[\left(\frac{t}{\tau_{trunc}}\right)^{k} + (-1)^{N} A_{k}(t) \exp\left(-\frac{\tau_{recur}^{2}}{\tau_{trunc}^{2}}\right)\right],$$

$$A_{k}(t) \equiv \left(\frac{t - \tau_{recur}}{\tau_{trunc}}\right)^{k} \exp\left(\frac{2t\tau_{recur}}{\tau_{trunc}^{2}}\right) + \left(\frac{t + \tau_{recur}}{\tau_{trunc}}\right)^{k} \exp\left(\frac{-2t\tau_{recur}}{\tau_{trunc}^{2}}\right) + \left[(-1)^{k+1} - 1\right]\left(\frac{\tau_{recur}}{\tau_{trunc}}\right)^{k}.$$

$$(5.36)$$

For $t \to 0$, the correction behaves as t^2 or t depending on whether k is even or odd, whereas the main contribution behaves as t^k . However the coefficient is so small that this correction is negligible as soon as $t > \tau_{\text{trunc}} \exp(-\pi^2 N \delta_0^2/2k)$, an extremely short time for $k \ll N$.

We expected the expression (5.23) for the correlations to become invalid for large k. In fact, the values of interest for t are of order τ_{trunc} , or of $\tau_{\text{trunc}} \sqrt{k}$ for large k, since the correlations reach their maximum at $t = \tau_{\text{trunc}} \sqrt{k/2}$. In this range, the correction in (5.36) is dominated by the first term of $A_k(t)$, which is negligibly small provided

$$\left(\frac{t}{\tau_{\text{trunc}}}\right)^k \gg \left(\frac{\tau_{\text{recur}}}{\tau_{\text{trunc}}}\right)^k \exp\left[-\frac{\tau_{\text{recur}}(\tau_{\text{recur}}-2t)}{\tau_{\text{trunc}}^2}\right].$$
(5.37)

Hence, in the relevant range $t \sim \tau_{\text{trunc}} \sqrt{k}$, the expression (5.23) for the correlations of rank k is valid provided

$$k \ll \frac{\pi^2 N \delta_0^2}{2 \ln \left(\tau_{\text{recur}}/t\right)},\tag{5.38}$$

but the simple shape (5.23) does not hold for correlations between a number k of particles violating (5.38).

In fact, when *t* becomes sizeable compared to τ_{recur} , the generating function (5.33) is dominated by the terms p = 0and p = 1. The correlations take, for arbitrary *k*, the form

$$\langle \hat{s}_{-}\hat{m}^{k}\rangle_{c} = r_{\uparrow\downarrow}(0)\left(i\pi\delta_{0}^{2}\right)^{k}\left\{\left(\frac{t}{\tau_{\text{recur}}}\right)^{k}\exp\left(-\frac{t^{2}}{\tau_{\text{trunc}}^{2}}\right) + \left(\frac{\tau_{\text{recur}}-t}{\tau_{\text{recur}}}\right)^{k}\exp\left[-\frac{(\tau_{\text{recur}}-t)^{2}}{\tau_{\text{trunc}}^{2}}\right]\right\}.$$
(5.39)

They are all exponentially small for $N \gg 1$ since $\tau_{\text{recur}}^2/\tau_{\text{trunc}}^2$ is large as *N*. The large rank correlations dominate. If for instance *t* is half the recurrence time, both terms of (5.39) have the same size, and apart from the overall exponential exp $(-N\pi^2\delta_0^2/8)$ the correlations increase with *k* by the factor $(\pi\delta_0^2/2)^k$, where $\delta_0 \ge 1$.

2061 6. Irreversibility of the truncation

2062 2063

Quare fremuerunt gentes, et populi meditati sunt inania?⁴⁸ Psalm 2

The sole consideration of the interaction between the tested spin S and the pointer M has been sufficient to explain and analyze the truncation of the state, which takes place on the time scale τ_{trunc} , at the very early stage of the measurement process. However this Hamiltonian (Eq. (3.5)) is so simple that if it were alone it would give rise to recurrences around the times τ_{recur} , $2\tau_{recur}$, ... In fact the evolution is modified by other processes, which as we shall see hinder the possibility of recurrence and render the truncation irreversible on any reachable time scale.

⁴⁸Why do the heathen rage, and the people imagine a vain thing?

2069 6.1. Destructive interferences

207

207

*Bis repetita (non) placent*⁴⁹ diverted from Horace

We still neglect in this subsection the effects of the phonon bath (keeping $\gamma = 0$), but will show that the recurrent behavior exhibited in § 5.3.1 is suppressed by a small change in the model, which makes it a little less idealized.

2074 6.1.1. Spread of the coupling constants

When we introduced the interaction (3.5) between S and A, we assumed that the coupling constants between the tested spin \hat{s} and each of the spins $\hat{\sigma}^{(n)}$ of the apparatus were all the same. However, even though the range of the forces is long compared to the size of the magnetic dot, these forces can be different, at least slightly. This is similar to the inhomogeneous broadening effect well known in NMR physics [60, 61, 62, 63, 64, 65]. We thus replace here \hat{H}_{SA} by the more general interaction

$$\hat{H}'_{SA} = -\hat{s}_z \sum_{n=1}^{N} (g + \delta g_n) \,\hat{\sigma}_z^{(n)},\tag{6.1}$$

where the couplings $g + \delta g_n$ are constant in time and have the small dispersion

$$\delta g^{2} = \frac{1}{N} \sum_{n=1}^{N} \delta g_{n}^{2}, \qquad \sum_{n=1}^{N} \delta g_{n} = 0.$$
(6.2)

The equations of motion (4.8) for \hat{D} , the right-hand side of which we disregard, remain valid, with the effective Hamiltonian

$$\hat{H}_i = -s_i \sum_n \left(g + \delta g_n\right) \hat{\sigma}_z^{(n)} - \sum_q \frac{NJ_q}{q} \hat{m}^q$$
(6.3)

instead of (4.6). This Hamiltonian, as well as the initial conditions $\hat{R}_{ij}(0) = r_{ij}(0) \hat{R}_{M}(0)$, depends only on the commuting observables $\hat{\sigma}_{z}^{(n)}$. Hence the latter property is also satisfied by the operators $\hat{R}_{ij}(t)$ at all times. Accordingly, $\hat{R}_{\uparrow\uparrow}(t)$ and $\hat{R}_{\downarrow\downarrow}(t)$ remain constant, and the part \hat{H}_{M} of \hat{H}_{i} does not contribute to the equation for $\hat{R}_{\uparrow\downarrow}(t)$, which is readily solved as

$$\hat{R}_{\uparrow\downarrow}(t) = r_{\uparrow\downarrow}(0)\,\hat{R}_{\rm M}(0)\exp\frac{2i}{\hbar}\left(Ng\hat{m}t + \sum_{n=1}^N \delta g_n\hat{\sigma}_z^{(n)}t\right),\tag{6.4}$$

with $\hat{R}_{\rm M}(0)$ given in terms of \hat{m} by (3.45). Notice that here the operator $\hat{R}_{\uparrow\downarrow}$ does not depend only on \hat{m} .

If $\hat{R}_{M}(0)$ is the most random paramagnetic state (3.46), produced for q = 2 by initializing the apparatus with $T_{0} \gg J$ or with a strong RF field, or for q = 4 with any temperature higher than the transition, (6.4) takes the form

$$\hat{R}_{\uparrow\downarrow}(t) = r_{\uparrow\downarrow}(0) \prod_{n=1}^{N} \frac{1}{2} \left[\hat{\sigma}_{0}^{(n)} \cos \frac{2(g+\delta g_{n})t}{\hbar} + i\hat{\sigma}_{z}^{(n)} \sin \frac{2(g+\delta g_{n})t}{\hbar} \right].$$
(6.5)

²⁰⁹⁰ The off-diagonal elements of the state of S thus evolve according to

$$r_{\uparrow\downarrow}(t) = r_{\uparrow\downarrow}(0) \prod_{n=1}^{N} \cos \frac{2(g + \delta g_n)t}{\hbar}.$$
(6.6)

The right-hand side behaves as (5.4) for $\delta g \ll g$ as long as *t* is of order τ_{trunc} . However, it is expected to remain extremely small at later times since the factors of (6.6) interfere destructively unless *t* is close to a multiple of $\pi \hbar/2 (g + \delta g_n)$ for most *n*. In particular, the successive recurrences which occurred in § 4.4.1 at the times τ_{recur} , $2\tau_{\text{recur}}$, ... for $\delta g = 0$ and $\gamma = 0$ are now absent provided the deviations δg_n are sufficiently large. We thus obtain a permanent truncation if we have at the time $t = \tau_{\text{recur}}$

⁴⁹Repetitions are (not) appreciated

$$1 \gg \prod_{n=1}^{N} \cos \frac{\pi \delta g_n}{g} \approx \prod_{n=1}^{N} e^{-\pi^2 \delta g_n^2 / 2g^2} = e^{-\pi^2 \sum_n \delta g_n^2 / 2g^2} = e^{-N\pi^2 \delta g^2 / 2g^2},$$
(6.7)

2096 that is,

$$\frac{\delta g}{g} \gg \frac{1}{\pi} \sqrt{\frac{2}{N}}.$$
(6.8)

Provided this condition is satisfied, all results of subsections 5.1 and 5.2 hold, even for large times. The whole set of correlations $\langle \hat{s}_{-}\hat{m}^{k} \rangle_{c}$, first created by the coupling (6.1), disappear for not too large *k* after a time of order $\tau_{trunc} \sqrt{k}$, and *do not revive* as *t* becomes larger. As in usual irreversible processes of statistical mechanics, it is mathematically not excluded that (6.6) takes significant values around some values of *t*, if *N* is not too large and if many deviations δg_n are arithmetically related to one another; but this can occur only for extremely large times, physically out of reach, as shown in § 6.1.2.

These conclusions hold for an arbitrary initial state (3.45). The expression (6.4) is the product of $\hat{R}_{\uparrow\downarrow}(t)$, as evaluated in section 5 for $\delta g = 0$, by the phase factor

$$\prod_{n=1}^{N} \exp\left(\frac{2i\delta g_n \sigma_z^{(n)} t}{\hbar}\right).$$
(6.9)

A generic set of coupling constants satisfying (6.2) provides the same results as if they were chosen at random, with a narrow gaussian distribution of width δg . Replacing then (6.9) by its expectation value, we find that the whole statistics of S + M (without the bath) is governed by the product of the generating function (5.33) by [60]

$$\prod_{n=1}^{N} \overline{\exp\left(2i\delta g_n \sigma_z^{(n)} t/\hbar\right)} = e^{-\left(t/\tau_{\text{irrev}}^{\text{M}}\right)^2},\tag{6.10}$$

2108 which introduces a characteristic decay time

$$\tau_{\rm irrev}^{\rm M} = \frac{\hbar}{\sqrt{2N}\delta g}.$$
(6.11)

This damping factor suppresses all the recurrent terms with $p \neq 0$ in (5.33) if δg satisfies the condition (6.8). Since the exponent of (6.10) is $(\delta g/g \delta_0)^2 (t/\tau_{\text{trunc}})^2$, the first correlations $\langle \hat{s}_- \hat{m}^k(t) \rangle_c$ are left unchanged if $\delta g \ll g$, while those of higher order are overdamped as $\exp(-k\delta g/2g \delta_0)$ for large k since $(t/\tau_{\text{trunc}})^2 = k/2$ at their maximum.

Thus, the truncation of the state produced on the time scale τ_{trunc} by the coupling \hat{H}'_{SA} of eq. (6.1), characterized by the decay (5.4) of $\langle \hat{s}_x(t) \rangle$ and $\langle \hat{s}_y(t) \rangle$ and by the time dependence (5.23) of $\langle \hat{s}_-\hat{m}^k(t) \rangle_c$, is fully irreversible. The time $\tau^{\text{M}}_{\text{irrev}}$ characterizes this *irreversibility induced by the magnet* M alone, caused by the dispersion of the constants $g + \delta g_n$ which couple \hat{s} with the elements $\hat{\sigma}_z^{(n)}$ of the pointer variable. If $\tau^{\text{M}}_{\text{irrev}}$ is such that $\tau_{\text{trunc}} \ll \tau^{\text{M}}_{\text{irrev}} \ll \tau_{\text{recur}}$, that is, when (6.8) is satisfied, the off-diagonal blocks $\hat{R}_{\uparrow\downarrow}(t)$ of $\hat{D}(t)$ remain negligible on time scales of order τ_{recur} . We will show in § 6.1.2 that recurrences might still occur, but at inaccessibly large times.

2118 6.1.2. Generality of the direct damping mechanism

We have just seen that a modification of the direct coupling between the tested spin S and the magnet M, without any intervention of the bath, is sufficient to prevent the existence of recurrences after the initial damping of the offdiagonal blocks of \hat{D} . In fact, recurrences took place in § 5.3.1 only because our original model was peculiar, involving a complete symmetry between the N spins which constitute the pointer. We will now show that the mechanism of irreversibility of § 6.1.1, based merely on the direct coupling between the tested system and the pointer of the apparatus, is quite general: it occurs as soon as the pointer presents no regularity.

Let us therefore return to the wide class of models introduced in § 5.1.2, characterized by a coupling

$$\hat{H}_{SA} = -Ng\hat{s}\hat{m},$$
 (general operators \hat{s}, \hat{m}) (6.12)

between the measured observable \hat{s} of the system S and the pointer observable \hat{m} of the apparatus A. We assume that the pointer, which has N degrees of freedom, has no symmetry feature, so that the spectrum of \hat{m} displays neither systematic degeneracies nor arithmetic properties. We disregard the other degrees of freedom of A, in particular the indirect coupling with the bath. The model considered above in § 6.1.1 enters this general frame, since its Hamiltonian (6.1) takes the form (6.12) if we identify our \hat{s}_z with the general \hat{s} and if we redefine \hat{m} as

$$\hat{m} = \frac{1}{N} \sum_{n=1}^{N} \left(1 + \frac{\delta g_n}{g} \right) \hat{\sigma}_z^{(n)}.$$
(6.13)

Indeed, provided the condition (6.8) is satisfied, the 2^N eigenvalues of (6.13) are randomly distributed over the interval (-1, 1) instead of occurring at the values (3.22) with the huge multiplicities (3.23).

In all such models governed by the Hamiltonian (6.12), the off-diagonal elements of \hat{r} behave as (5.12) so that their time-dependence, and more generally that of the off-diagonal blocks of \hat{R} , has the form

$$F(t) = \frac{1}{Q} \sum_{q=1}^{Q} e^{i\omega_q t}.$$
(6.14)

²¹³⁵ Indeed, the matrix element (5.11) is a sum of exponentials involving the eigenfrequencies

$$\omega_q \equiv \frac{Ng(s_i - s_j)m_q}{\hbar},\tag{6.15}$$

where m_q are the eigenvalues of \hat{m} . The number Q of these eigenfrequencies is large as an exponential of the number N of microscopic degrees of freedom of the pointer, for instance $Q = 2^N$ for (6.13). To study a generic situation, we can regard the eigenvalues m_q or the set ω_q as independent random variables. Their distribution is governed by the density of eigenvalues of \hat{m} and by the initial density operator $\hat{R}(0)$ of the apparatus which enters (5.12) and which describes a metastable equilibrium. For sufficiently large N, we can take for each dimensionless m_q a narrow symmetric gaussian distribution, with width of relative order $1/\sqrt{N}$. The statistics of F(t) that we will study then follows from the probability distribution for the frequencies ω_q ,

$$p(\omega_q) = \frac{1}{\sqrt{2\pi}\Delta\omega} \exp\left(-\frac{\omega_q^2}{2\Delta\omega^2}\right),\tag{6.16}$$

where $\Delta \omega$ is of order \sqrt{N} due to the factor N entering the definition (6.15) of ω_q . This problem has been tackled long ago by Kac [232].

 $_{2145}$ We first note that the expectation value of F(t) for this random distribution of frequencies,

$$\overline{F(t)} = e^{-\Delta\omega^2 t^2/2},\tag{6.17}$$

decays exactly, for all times, as the Gaussian (5.3) with a truncation time $\tau_{\text{trunc}} = \sqrt{2}/\Delta\omega$, encompassing the expression (5.6) that we found for short times in our original model. This result holds for most sets ω_q , since the statistical fluctuations and correlations of F(t), given by

$$\overline{F(t)F(t')} - \overline{F(t)}\overline{F(t')} = \frac{1}{O} \left(e^{-\Delta\omega^2 (t+t')^2/2} - e^{-\Delta\omega^2 (t^2+t'^2)/2} \right),$$
(6.18)

$$\overline{F(t) F^*(t')} - \overline{F(t)} \overline{F^*(t')} = \frac{1}{Q} \left(e^{-\Delta \omega^2 (t-t')^2/2} - e^{-\Delta \omega^2 (t^2+t'^2)/2} \right),$$
(6.19)

²¹⁴⁹ are small for large Q.

Nevertheless, for any specific choice of the set ω_q , nothing prevents the real part of F(t) from reaching significant values at some times t large as $t \gg \Delta \omega$, due to the tail of its probability distribution. Given some positive number f(less than 1), say f = 0.2, we define the *recurrence time* τ_{recur} as the typical delay we have to wait on average before $\Re F(t)$ rises back up to f. We evaluate this time in Appendix C. For f sufficiently small so that $\ln[I_0(2f)] \simeq f^2$, a property which holds for f = 0.2, we find

$$\tau_{\rm recur} = \frac{2\pi}{\Delta\omega} \exp\left(Qf^2\right) = \pi \sqrt{2}\tau_{\rm trunc} \exp\left(Qf^2\right). \tag{6.20}$$

As *Q* behaves as an exponential of *N*, this generic recurrence time is *inaccessibly large*. Even for a pointer involving only N = 10 spins, in which case $Q = 2^N = 2^{10}$, and for f = 0.2, we have $\tau_{\text{recur}}/\tau_{\text{trunc}} = 2.7 \cdot 10^{18}$. The destructive interferences taking place between the various terms of (5.12) explain not only the truncation of the state (§ 5.1.2) but also, owing to the randomness of the coupling, the irretrievable nature of this decay process over any reasonable time lapse, in spite of the unitarity of the evolution.

Although we expect the eigenfrequencies ω_q associated with a large pointer to be distributed irregularly, the 2160 distribution (6.16) chosen above, for which they are completely random and uncorrelated, is not generic. Indeed, 2161 according to (6.15), these eigenfrequencies are quantum objects, directly related to the eigenvalues m_q of the operator 2162 \hat{m} . A more realistic model should therefore rely on the idea that \hat{m} is a complicated operator, which is reasonably 2163 represented by a random matrix. As well known, the eigenvalues of a random matrix are correlated: they repell 2164 according to Wigner's law. The above study should therefore be extended to random matrices \hat{m} instead of random 2165 uncorrelated frequencies ω_q , using the techniques of the random matrix theory [233]. We expect the recurrence time 2166 thus obtained to be shorter than above, due to the correlations among the set ω_a , but still to remain considerably longer 2167 than with the regular specrum of \S 5.3.1. 2168

2169 6.2. Effect of the bath on the initial truncation

You can't fight City Hall Saying

Returning to our original model of subsection 3.2 with a uniform coupling *g* between S and the spins of M, we now take into account the effect, on the off-diagonal blocks of \mathcal{D} , of the coupling γ between M and B. We thus start from eq. (4.29), to be solved for times of the order of the recurrence time. We will show that the damping due to the bath can prevent $P_{\uparrow\downarrow}$ and hence $\hat{R}_{\uparrow\downarrow}$ from becoming significant at all times *t* larger than τ_{trunc} , in spite of the regularity of the spectrum of \hat{m} which leads to the anomalously short recurrence time $\pi\hbar/2g$ of (5.30)⁵⁰.

Readers interested mainly in the physics of the truncation may jump to § 9.6.1, where the mathematics is simplified using insights gained about the behavior of the equation of motion for $t \gg \hbar/T$ through the rigorous approach of § 6.2.1 and of appendix D.

2180 6.2.1. Determination of $P_{\uparrow\downarrow}(t)$

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We have found recurrences in $P_{\uparrow\downarrow}(m, t)$ by solving (4.18) without right-hand side and by taking into account the discreteness of m (§ 5.3.1). The terms arising from the bath will modify for each m the modulus and the phase of $P_{\uparrow\downarrow}^{\text{dis}}(m, t) = (2/N)P_{\uparrow\downarrow}(m, t)$.

²¹⁸⁴ In order to study these changes, we rely on the equation of motion (4.18), the right-hand side of which has been obtained in the large *N* limit while keeping however the values of *m* discrete as in § 5.3.1. Note first that the functions $\tilde{K}_{t>}(\omega)$ and $\tilde{K}_{t<}(\omega)$ defined by Eqs. (4.10) and (4.11), respectively, are complex conjugate for the same value of ω . It then results from Eq. (4.18) together with its initial condition that ⁵¹

$$P_{\uparrow\downarrow}(-m,t) = P_{\uparrow\downarrow}^*(m,t) = P_{\downarrow\uparrow}(m,t).$$
(6.21)

For $\gamma = 0$, the solution of (4.18) with the initial condition (3.46) is given by (5.2). Starting from this expression, we parametrize $P_{\uparrow\downarrow}(m, t)$ as

$$P_{\uparrow\downarrow}(m,t) = r_{\uparrow\downarrow}(0) \sqrt{\frac{N}{2\pi\delta_0^2}} \exp\left[-\frac{Nm^2}{2\delta_0^2} + \frac{2iNgmt}{\hbar} - NA(m,t)\right], \tag{6.22}$$

⁵⁰For the related, effective decay of $\mathcal{R}_{\uparrow\downarrow}(t)$ and $\mathcal{R}_{\downarrow\uparrow}(t)$, see § 12.2.3

⁵¹Changing g into -g would also change $P_{\uparrow\downarrow}(m,t)$ into $P_{\uparrow\uparrow}^*(m,t)$, but we shall stick to the ferromagnetic interaction g > 0



Figure 6.1: The damping function B(t) issued from the interaction of the magnet with the bath. This function is measured in units of the dimensionless magnet-bath coupling constant γ , and the time is measured in units of the recurrence time $\tau_{\text{recur}} = \pi \hbar/2g$. The parameters are T = 0.2J and g = 0.045J and $\hbar\Gamma = 50 \sqrt{\pi/2} J$. After an initial t^4 growth, the curve is quasi linear with periodic oscillations. "Anti-damping" with dB/dt < 0 occurs during the delay $\alpha \tau_{\text{recur}}$ before each recurrence (Eq.(6.33)). The condition $NB(\tau_{\text{recur}}) \gg 1$ entails the irreversible suppression of all the recurrences. Bullets denote the local maxima (see (6.36)) and the local minima at integer values of t/τ_{recur} .

- in terms of the function A(m, t), to be determined at first order in γ from Eq. (4.29) with the initial condition A(m, 0) = 0. For large *N*, A(m, t) contains contributions of orders 1 and 1/*N*. Its complete expression is exhibited in Appendix D in terms of the autocorrelation function K(t) of the bath (Eq. (D.3)).
- The distribution $P_{\uparrow\downarrow}(m, t)$ takes significant values only within a sharp peak centered at m = 0 with a width of order $1/\sqrt{N}$. We can therefore consistently expand A(m, t) in powers of m up to second order, according to

$$A(m,t) \approx B(t) - i\Theta(t)m + \frac{1}{2}D(t)m^2, \qquad (6.23)$$

so that we can write from (6.22) and (6.23) the expression for $P_{\uparrow\downarrow}^{\text{dis}} = (2/N)P_{\uparrow\downarrow}$ in the form

$$P_{\uparrow\downarrow}^{\rm dis}(m,t) = r_{\uparrow\downarrow}(0) \sqrt{\frac{2}{\pi N \delta_0^2}} \exp\left\{-NB(t) + iN\left[\frac{2gt}{\hbar} + \Theta(t)\right]m - N\left[\frac{1}{\delta_0^2} + D(t)\right]\frac{m^2}{2}\right\}.$$
(6.24)

The functions B(t), $\Theta(t)$ and D(t), proportional to γ , describe the effect of the bath on the off-diagonal blocks of the density matrix of S + M. They are real on account of (6.21). The overall factor $\exp[-NB(t)]$ governs the amplitude of $P_{\uparrow\downarrow}^{\text{dis}}$. The term $\Theta(t)$ modifies the oscillations which arose from the coupling between S and M. The term D(t) modifies the width of the peak of $P_{\uparrow\downarrow}^{\text{dis}}$. The explicit expressions of these functions, given by (D.15) for B(t), (D.26) for $\Theta(t)$ and (D.29) for D(t), are derived in appendix D from the equation of motion (D.3) for A(m, t), which itself results directly from Eq. (4.29) for $P_{\uparrow\downarrow}^{\text{dis}}$. We analyze them below.

2202 6.2.2. The damping function

The main effect of the bath is the introduction in (6.23) of the overall factor $\exp[-NB(t)]$, which produces a damping of the off-diagonal blocks $\hat{R}_{\uparrow\downarrow}$ and $\hat{R}_{\downarrow\uparrow}$ of the density matrix \hat{D} of S + M. The expression for B(t) derives from Eq. (D.8) and is given explicitly by

$$B(t) = \gamma \int_0^\infty \frac{d\omega \,\omega}{\pi} \coth \frac{\hbar\omega}{2T} \,\exp\left(-\frac{\omega}{\Gamma}\right) \left\{ \frac{\sin^2 \Omega t}{2(\omega^2 - \Omega^2)} + \frac{\Omega^2 (1 - \cos \omega t \,\cos \Omega t) - \omega \Omega \sin \omega t \,\sin \Omega t}{(\omega^2 - \Omega^2)^2} \right\},\tag{6.25}$$

with $\Omega = 2g/\hbar$. The ω -integral can be carried out analytically if one replaces in the spectrum of phonon modes (3.37) the Debye cutoff by a quasi Lorentzian one, see Eq. (D.10) and the connection (D.11) between the cutoff parameters; the result for *B* is given in (D.15). The function B(t) of Eq. (D.15), or, nearly equivalently, Eq. (6.25), is illustrated by fig. 6.1. We discuss here its main features in the limiting cases of interest.

²²¹⁰ Consider first the short times $t \ll 1/\Gamma$. This range covers the delay τ_{trunc} during which the truncation takes place, ²²¹¹ but it is much shorter than the recurrence time. We have shown in Appendix D that B(t) behaves for $t \ll 1/\Gamma$ as

$$B(t) \sim \frac{\gamma \Gamma^2 g^2}{2\pi \hbar^2} t^4, \tag{6.26}$$

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increasing slowly as shown by fig. 6.1. If NB(t) remains sufficiently small during the whole truncation process so that exp[-NB(t)] remains close to 1, the bath is ineffective over the delay τ_{trunc} . This takes place under the condition

$$NB(\tau_{\rm trunc}) = N \frac{\gamma \Gamma^2 g^2}{2\pi \hbar^2} \tau_{\rm trunc}^4 = \frac{\gamma \hbar^2 \Gamma^2}{8\pi N \delta_0^4 g^2} \ll 1, \tag{6.27}$$

which is easily satisfied in spite of the large value of $\hbar\Gamma/g$, since $\gamma \ll 1$ and $N \gg 1$. Then the coupling with the bath does not interfere with the truncation by the magnet studied in section 5. Otherwise, if $NB(\tau_{\text{trunc}})$ is finite, the damping by *B*, which behaves as an exponential of $-t^4$, enhances the truncation effect in $\exp[-(t/\tau_{\text{trunc}})^2]$ of M, and reduces the tails of the curves of fig. 5.1.

²²¹⁸ Consider now the times t larger than $\hbar/2\pi T$, which is the memory time of the kernel K(t). We are then in the Markovian regime. This range of times encompasses the recurrences which in the absence of the bath occur periodically at the times $t = p\tau_{\text{recur}}$, with $\tau_{\text{recur}} = \pi\hbar/2g$. Under the condition $t \gg \hbar/2\pi T$, we show in Appendix D Eq. (D.18)), that B(t) has the form

$$B(t) = \frac{\gamma\pi}{4} \coth \frac{g}{T} \left(\frac{t}{\tau_{\text{recur}}} - \frac{1}{2\pi} \sin \frac{2\pi t}{\tau_{\text{recur}}} \right) + \frac{\gamma}{4\pi} \ln \frac{\hbar\Gamma}{2\pi T} \left(1 - \cos \frac{2\pi t}{\tau_{\text{recur}}} \right).$$
(6.28)

On average, B(t) thus increases linearly along with the first term of (6.28), as exhibited by fig. 6.1. Hence, the bath generates in this region $t \gg \hbar/2\pi T$ the exponential damping

$$\exp[-NB(t)] \sim \exp\left(-\frac{t}{\tau_{\text{irrev}}^{\text{B}}}\right),\tag{6.29}$$

where the decay is characterized by the bath-induced irreversibility time

$$\tau_{\rm irrev}^{\rm B} = \frac{2\hbar \tanh g/T}{N\gamma g}.$$
(6.30)

The recurrences, at $t = p\tau_{recur}$, are therefore attenuated by the factor

$$\exp\left(-\frac{p\tau_{\text{recur}}}{\tau_{\text{irrev}}^{\text{B}}}\right) = \exp\left(-\frac{p\pi N\gamma}{4\tanh g/T}\right).$$
(6.31)

Thus, all recurrences are irreversibly suppressed, so that the initial truncation becomes definitive, provided the coupling between M and B is sufficiently strong so as to satisfy $NB(\tau_{recur}) \gg 1$, or equivalently $\tau_{irrev}^{B} \ll \tau_{recur}$, that is:

$$\gamma \gg \frac{4 \tanh g/T}{\pi N}.$$
(6.32)

In case $T \gg g$, the irreversibility time

$$\tau_{\rm irrev}^{\rm B} \sim \frac{2\hbar}{N\gamma T} \tag{6.33}$$

depends only on the temperature of the bath, on the number of spins of the magnet, and on the magnet-bath coupling, irrespective of the system-magnet coupling.

In spite of the smallness of γ , the large value of *N* makes the condition (6.32) easy to satisfy. In fact, if the hardly more stringent condition $NB(\hbar/2\pi T) \gg 1$, that is, $N\gamma \gg 4\pi$, is satisfied, we have $NB(t) \gg 1$ in the region $t \gg \hbar/2\pi T$ where the approximation (6.28) holds. Thus, although B(t) is quasi linear in this region, the exponential shape of the decay (6.29), with its characteristic time τ_{irrev}^{B} , loses physical relevance since $\exp[-NB(t)]$ is there practically zero.

In this same region $t \gg \hbar/2\pi T$, the expression (6.28) of B(t) involves oscillatory contributions superimposed to the linear increase considered above (fig. 6.1). In fact, the time derivative

$$\frac{\tau_{\text{recur}}}{\gamma} \frac{\mathrm{d}B}{\mathrm{d}t} = \left(\frac{\pi}{2} \coth \frac{g}{T} \sin \frac{\pi t}{\tau_{\text{recur}}} + \ln \frac{\hbar\Gamma}{2\pi T} \cos \frac{\pi t}{\tau_{\text{recur}}}\right) \sin \frac{\pi t}{\tau_{\text{recur}}}.$$
(6.34)

of B(t) is periodic, with period τ_{recur} , and it vanishes at the times t such that

$$\sin\frac{\pi t}{\tau_{\text{recur}}} = 0 \quad \text{or} \quad \tan\frac{\pi t}{\tau_{\text{recur}}} = -\frac{2}{\pi}\ln\frac{\hbar\Gamma}{2\pi T} \tanh\frac{g}{T}.$$
(6.35)

The first set of zeros occur at the recurrence times $p\tau_{\text{recur}}$, which are local minima of B(t). The second set provide local maxima, which occur somewhat earlier than the recurrences (fig. 6.1), at the times

$$t = (p - \alpha)\tau_{\text{recur}}, \qquad \alpha = \frac{1}{\pi}\arctan\left(\frac{2}{\pi}\ln\frac{\hbar\Gamma}{2\pi T}\tanh\frac{g}{T}\right).$$
 (6.36)

An unexpected quantum effect thus takes place in the off-diagonal blocks of the density matrix of S + M. Usually, a bath produces a monotonous relaxation. Here, the damping factor $\exp[-NB(t)]$, which results from the coupling of M with the bath, *increases* between the times $(p - \alpha)\tau_{recur}$ and $p\tau_{recur}$. During these periods, the system S + M undergoes an "*anti-damping*". This has no incidence on our measurement process, since the recurrences are anyhow killed under the condition (6.29) and since their duration, τ_{trunc} , is short compared to the delay $\alpha \tau_{recur}$. One may imagine, however, other processes that would exhibit a similar effect.

2247 6.2.3. Time-dependence of physical quantities

All the off-diagonal physical quantities, to wit, the expectation values $\langle \hat{s}_x(t) \rangle$, $\langle \hat{s}_y(t) \rangle$, and the correlations between \hat{s}_x or \hat{s}_y and any number of spins of the apparatus are embedded in the generating function $\Psi_{\uparrow\downarrow}(\lambda, t)$ defined as in (5.13). As we recalled in § 6.2.1, we must sum over the *discrete values* (3.22) of *m*, rather than integrate over *m*; the distinction between summation and integration becomes crucial when the time *t* reaches τ_{recur} , since then the period in *m* of the oscillations of $P_{\uparrow\downarrow}^{\text{dis}}(m, t)$ becomes as small as the level spacing. From (6.23), we see that the characteristic function, modified by the bath terms, has the same form as in § 5.3.2 within multiplication by $\exp[-NB(t)]$ and within modification of the phase and of the width of $P_{\uparrow\downarrow}^{\text{dis}}(m, t)$.

Let us first consider the effect of $\Theta(t)$. Its introduction changes the phase of $P_{\uparrow\downarrow}$ according to

$$\frac{2iNgmt}{\hbar} \mapsto \frac{2iNgmt}{\hbar} + iN\Theta(t)m.$$
(6.37)

Hence, the occurrence of the term $\Theta(t)$ might shift the recurrences, which take place when

$$\frac{2gt}{\hbar} + \Theta(t) = p\pi. \tag{6.38}$$

However, the expression of $\Theta(t)$ derived in the appendix D, Eq. (D.26),

$$\Theta(t) \sim -\frac{\gamma}{8g} \left[\left(\frac{2}{\delta_0^2} - 1 \right) T + J_2 \right] \left[1 - \cos \frac{2\pi t}{\tau_{\text{recur}}} \right].$$
(6.39)

vanishes for $t = p\tau_{\text{recur}} = p\pi\hbar/2g$, so that the replacement (6.34) does not affect the values of the recurrence times. Between these recurrence times, the truncation makes all correlations of finite rank negligible even in the absence of the bath, as if $P_{\uparrow\downarrow}^{\text{dis}}$ did vanish; then, the phase of $P_{\uparrow\downarrow}^{\text{dis}}$ is irrelevant. Altogether, $\Theta(t)$ is completely ineffective. Likewise, the term D(t) is relevant only at the recurrence times. We evaluate it in Eq. (D.29) as

$$D(p\tau_{\text{recur}}) \simeq p\eta, \qquad \eta = \frac{\pi\gamma}{2} \frac{J_2}{g} \left(\frac{J_2}{3T} - 1\right). \tag{6.40}$$

This term changes the width of the distribution $P_{\uparrow\downarrow}^{\text{dis}}(m, t)$ by a small relative amount of order $\gamma \ll 1$, according to

$$\Delta m = \frac{\delta_0}{\sqrt{N}} \mapsto \Delta m_p = \frac{\delta_0}{\sqrt{N(1 + p\eta\delta_0^2)}} = \Delta m(1 - \frac{1}{2}p\eta\delta_0^2). \tag{6.41}$$

The width therefore increases if $J_2 < 3T$, or decreases if $J_2 > 3T$, but this effect is significant only if the recurrences are still visible, that is, if the condition (6.32) is not satisfied.

The expression (5.33) of the generating function is thus modified into

$$\Psi_{\uparrow\downarrow}(\lambda,t) = r_{\uparrow\downarrow}(0)e^{-NB(t)}\sum_{p=-\infty}^{\infty}(-1)^{pN}\exp\left(\frac{i\lambda\Delta m_p}{\sqrt{2}} + i\frac{t-p\tau_{\rm recur}}{\tau_{\rm trunc}}\right)^2.$$
(6.42)

The crucial change is the presence of the damping factor $\exp[-NB(t)]$, which invalidates the periodicity (5.30) of $\Psi_{\uparrow\downarrow}(\lambda, t)$ and which inhibits the recurrences. Moreover, for any t > 0, the terms p < 0 in (6.42) are negligible, since they involve (for t = 0) the factor $\exp[-(p\tau_{\text{recur}}/\tau_{\text{trunc}})^2]$. Thus, under the conditions (6.27) and (6.32), the sum (6.42) reduces at all times to its term p = 0. Accordingly, it is legitimate to express for arbitrary times $P_{\uparrow\downarrow}$ as

$$P_{\uparrow\downarrow}(m,t) = P_{\uparrow\downarrow}(m,0) \exp\left[\frac{2iNgmt}{\hbar} - NB(t)\right],$$
(6.43)

and to treat *m* as a continuous variable. As a consequence, the full density matrix of S + M, which results from (3.26), has off-diagonal blocks given by

$$\hat{R}_{\uparrow\downarrow}(t) = r_{\uparrow\downarrow}(0)\hat{R}_{\rm M}(0)\exp\left[\frac{2iNgmt}{\hbar} - NB(t)\right],\tag{6.44}$$

where we recall the expressions (D.15), (6.26) and (6.28) for B(t).

Altogether, as regards the evolution of the physical quantities $\langle \hat{s}_a \hat{m}^k(t) \rangle$ (*a* = *x* or *y*), nothing is changed in the results of § 5.1.3 on the scale $t \ll \tau_{\text{recur}}$; these results are summarized by Eq. (5.22) and illustrated by fig. 5.1. For $t \gg \tau_{\text{irrev}}^{\text{B}}$, the factor exp[-NB(t)] makes all these off-diagonal quantities *vanish irremediably*, including the high-rank correlations of § 5.3.2.

In spite of the simplicity of this result, our derivation was heavy because we wanted to produce a rigorous proof. 2277 It turned out that the interaction between the spins of M, which occurs both through δ_0 in the initial state of M and 2278 through J_2 in the dynamics generated by the bath, has a negligible effect. Taking this property for granted, treating 2279 M as a set of independent spins and admitting that for $t \gg \hbar/2\pi T$ the autocorrelation function of the bath enters the 2280 dynamical equation through (D.21), we present in § 9.6.1 a simpler derivation, which may be used for tutorial purposes 2281 and which has an intuitive interpretation: Both the precession of \hat{s} and the damping of $\hat{R}_{\uparrow\downarrow}(t)$ by the bath arise from a 2282 dynamical process in which each spin of M is independently driven by its interaction with S and independently relaxes 2283 under the effect of the bath B. 2284

6.2.4. The off-diagonal bath effect, an ongoing decoherence process regulated by the tested observable

Het houdt maar niet op⁵²

Dutch expression

The damping described above has two unusual features: on the one hand (fig. 6.1), its coefficient does not monotonically decrease; on the other hand, it is governed by a resonance effect. However, it has also clearly the features of a standard decoherence [32, 33, 40, 176, 177, 178, 179]. It takes place in the compound system S + M under the influence of B which plays the role of an environment. The decay (6.29) is quasi-exponential, apart from non-essential oscillations. The expression (6.33) of the irreversibility time $\tau^{\rm B}_{\rm irrev} = 2\hbar/N\gamma T$ (for $T \gg g$) is typical of a bath-induced decoherence: It is inversely proportional to the *temperature T* of B, to the number N which characterizes the *size* of the system S + M, and to the *coupling* γ of this system with its environment, which is here the bath.

Nevertheless, we have stressed (§ 5.1.2) that the fundamental mechanism of the initial truncation of the state $\hat{D}(t)$ of S + M has not such a status of decoherence. It takes place in the brief delay $\tau_{trunc} = \hbar/\sqrt{N}\delta_0 g$, during which the bath does not yet have any effect. Contrary to *decoherence*, this *dephasing* process is internal to the system S + M, and does not involve its environment B. It is governed by the direct coupling g between S and the pointer M, as shown by the expression of the truncation time. It is during delays of order τ_{trunc} that the phenomena described in section 5 occur – decay of the average transverse components of the spin S, creation then disappearance of correlations with higher and higher rank (§ 5.1.3 and fig. 5.1). The bath has no effect on this truncation proper.

When the bath begins to act, that is, when NB(t) becomes significant, the truncation can be considered as prac-2302 *tically achieved* since Eq (6.27) is easily satisfied. The only tracks that remain from the original blocks $\hat{R}_{\uparrow\downarrow}(0)$ and 2303 $\hat{R}_{\downarrow\uparrow}(0)$ of $\hat{D}(0)$ are correlations of very high rank (§ 5.3.2), so that the state $\hat{D}(t)$ cannot be distinguished at such 230 times from a state without off-diagonal blocks. However, if the Hamiltonian did reduce solely to \hat{H}_{SA} (Eq. (3.5)), the 2305 simplicity of the dynamics would produce, from these hidden correlations, a revival of the initial state $\hat{D}(0)$, taking 2306 place just before τ_{recur} , during a delay of order τ_{trunc} . The weak interaction γ with the bath wipes out the high rank 2307 *correlations*, at times t such that $\tau_{\text{trunc}} \ll t \ll \tau_{\text{recur}}$ for which they are the only remainder of $r_{\uparrow \downarrow}(0)$. Their destruction 230 prevents the inverse cascade from taking place and thus suppresses all recurrences. 2309

The interaction between S and M does not only produce the initial truncation of \hat{D} described in section 5. It is also an essential ingredient in the very mechanism of decoherence by the bath B. Indeed, the interaction (3.10) between M and B is isotropic, so that it is the coupling between S and M which should govern the selection of the basis in which the suppression of recurrences will occur after the initial truncation. To understand how this ongoing *preferred basis problem* is solved, let us return to the derivation of the expression (6.25) for the damping term B(t), valid in the time range of the bath-induced irreversibility. This expression arose from the integral (D.8), to wit,

$$\frac{\mathrm{d}B}{\mathrm{d}t} = \frac{4\gamma\sin\Omega t}{\pi\hbar^2} \int_{-\infty}^{\infty} \mathrm{d}\omega\,\tilde{K}(\omega)\,\frac{\Omega(\cos\Omega t - \cos\omega t)}{\omega^2 - \Omega^2} \tag{6.45}$$

which analyzes the influence, on the damping, of the various frequencies ω of the autocorrelation function $\tilde{K}(\omega)$ of the phonon bath. The effect of the system-magnet interaction g is embedded in the frequency $\Omega = 2g/\hbar = \pi/\tau_{\text{recur}}$, directly related to the period of the recurrences. In appendix D, we have shown that the quasi-linear behaviour of B(t) results from the approximation (D.20) for the last factor of (6.45): This factor is peaked around $\omega = \pm \Omega$ for $t \gg \hbar/2\pi T$. The integral (6.45) then reduces to

$$\frac{\mathrm{d}B(t)}{\mathrm{d}t} = \frac{\gamma}{\hbar^2} \left[\tilde{K}(\Omega) + \tilde{K}(-\Omega) \right] (1 - \cos 2\Omega t),\tag{6.46}$$

the constant part of which produces the dominant, linear term $B \propto t$ of (6.28). In the autocorrelation function $\tilde{K}(\omega)$ which controls the damping by B in the equation of motion of S + M, $\hbar\omega$ is the energy of the phonon that is created or annihilated by interaction with a spin of the magnet (§ 3.2.2). Thus, through a resonance effect arising from the peak of the integrand in (6.44), the frequency ω of the phonons that contribute to the damping adjusts itself onto the

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⁵²It keeps going on

frequency $\Omega = 2g/\hbar$ associated with the precession of the spins of the magnet under the influence of the tested spin. Owing to this *resonance effect*, the bath acts mainly through the frequency of the recurrences. Accordingly, phonons with energy $\hbar\omega$ close to the energy $\hbar\Omega = 2g$ of a spin flip in M (see Eq. (3.5)) are continuously absorbed and emitted, and this produces the shrinking of the off-diagonal blocks $\hat{R}_{\uparrow\downarrow}$ and $\hat{R}_{\downarrow\uparrow}$. The effect is cumulative, since $B \propto t$. The *decoherence by the bath is thus continuously piloted by the coupling of the magnet with* S.

In conclusion, the initial truncation and its further consolidation are in the present model the results of an interplay 2330 between the three interacting objects, S, M and B. The main effect, on the time scale τ_{trunc} , arises from the coupling 2331 between S and the many degrees of freedom of M, and it should not be regarded as decoherence. Rather, it is 2332 a dephasing effect as known in nuclear magnetic resonance. Viewing the magnet M as "some kind of bath or of 2333 environment", as is often done, disregards the essential role of M: to act as the pointer that indicates the outcome 233 of the quantum measurement. Such an idea also confers too much extension to the concept of bath or environment. 2335 Decoherence usually requires some randomness of the environment, and we have seen (§ 5.2.2) that truncation may 2336 occur even if the initial state of M is pure. 2337

The mechanisms that warrant, on a longer time scale τ_{irrev}^{M} or τ_{irrev}^{B} , the permanence of the truncation can be regarded as *adjuvants of the main initial truncation process*, since they become active after all accessible off-diagonal expectation values and correlations have (provisionally) disappeared. We saw in subsection 6.1 that the intervention of B is not necessary to entail this irreversibility, which can result from a dispersion of the coupling constants g_n . For the more efficient mechanism of suppression of recurrences of subsection 6.2, we have just stressed that it is a decoherence process arising from the phonon thermal bath but *steered by the spin-magnet coupling*.

In section 7, we turn to the most essential role of the bath B in the measurement, to allow the registration of the outcome by the pointer.

2346 7. Registration

2347	Wie schrijft, die blijft ³³
2348	Les paroles s'envolent, les écrits restent ⁵⁴
2349	Dutch and French proverbs

The main issue in a measurement process is the establishment of correlations between S and A, which will allow 2350 us to gain information on S through observation of A [10, 11, 31, 48, 84]. As shown in § 5.1.3, the process creates 2351 correlations in the off-diagonal blocks $\hat{\mathcal{R}}_{\uparrow\downarrow}(t)$ and $\hat{\mathcal{R}}_{\downarrow\uparrow}(t)$ of the density matrix $\hat{\mathcal{D}}(t)$ of S + A, but those which survive 235 after the brief truncation time $au_{ ext{trunc}}$ involve a large number of spins $\hat{\sigma}^{(n)}$ of M and are inobservable. The considered 2353 quantum measurement thus cannot provide information on the off-diagonal elements $r_{\uparrow\downarrow}(0)$ of the density matrix $\hat{r}(0)$ 2354 of S. We now show, by studying the dynamics of the diagonal blocks of $\hat{D}(t)$, how M can register the statistical 2355 information embedded in $r_{\uparrow\uparrow}(0)$ and $r_{\downarrow\downarrow}(0)$ through creation of system-apparatus correlations. If we can select the 2356 outcome, a question discussed in section 11, the process can be used as a preparation of S in the pure state $|\uparrow\rangle$ or $|\downarrow\rangle$. 2357

The registration process presents two qualitatively different behaviours, depending on the nature of the phase transition of the magnet, of second order if the parameters of its Hamiltonian (3.7) satisfy $J_2 > 3J_4$, of first order if they satisfy $3J_4 > J_2$. Recalling our discussion in § 3.3.2, we will exemplify these two situations with the two pure cases q = 2 and q = 4. In the former case, for $J_2 \equiv J$ and $J_4 = 0$, the Hamiltonian is expressed by (3.8); in the latter case, for $J_4 \equiv J$ and $J_2 = 0$, it is expressed by (3.9). We summarize these two cases by $H_M = -(NJ/q)\hat{m}^q$ with q = 2and 4, respectively.

2364 7.1. Properties of the dynamical equations

The dynamics of the diagonal blocks $\hat{\mathcal{R}}_{\uparrow\uparrow}(t)$ of $\hat{\mathcal{D}}(t)$ results for large *N* from the equation (4.30) for the scalar function $P_{\uparrow\uparrow}(t)$, with initial condition $P_{\uparrow\uparrow}(0) = r_{\uparrow\uparrow}(0)P_{M}(m,0)$. The initial distribution $P_{M}(m,0)$ for the magnetization of M, given by (3.48), is a Gaussian, peaked around m = 0 with the small width δ_0/\sqrt{N} . We have noted (subsection

⁵³Who writes, stays

⁵⁴ Words fly away, writings stay
4.4) the analogy of the equation of motion (4.30) with a Fokker-Planck equation [230] for the random variable *m* submitted to the effects of the thermal bath B. In this equation, which reads

$$\frac{\partial P_{\uparrow\uparrow}}{\partial t} = \frac{\partial}{\partial m} \left(-vP_{\uparrow\uparrow} \right) + \frac{1}{N} \frac{\partial^2}{\partial m^2} \left(wP_{\uparrow\uparrow} \right), \tag{7.1}$$

the first term describes a *drift*, the second one a *diffusion* [230]. The drift velocity v(m, t) is a function of m and tdefined by (4.31), whereas the diffusion coefficient $w_{\uparrow\uparrow}(m, t)$ is defined by (4.32). The normalization of $P_{\uparrow\uparrow}$ remains unchanged in time:

$$\int \mathrm{d}m P_{\uparrow\uparrow}(m,t) = \int \mathrm{d}m P_{\uparrow\uparrow}(m,0) = r_{\uparrow\uparrow}(0), \tag{7.2}$$

so that the ratio $P(m, t) = P_{\uparrow\uparrow}(m, t) / r_{\uparrow\uparrow}(0)$ can be interpreted as a conditional probability of *m* if $s_z = 1$.

2374 7.1.1. Initial and Markovian regimes

For very *short times* such that $t \ll 1/\Gamma$, we have

$$\tilde{K}_t(\omega) \sim 2tK(0) = \frac{\hbar^2}{4\pi} \Gamma^2 t, \qquad (7.3)$$

2376 and hence

$$v \sim -\frac{\gamma}{\pi}\Gamma^2 mt, \qquad w \sim \frac{\gamma}{\pi}\Gamma^2 t.$$
 (7.4)

The solution of (7.1) then provides a Gaussian which remains centered at m = 0. Its width $\sqrt{D/N}$ decays for q = 2, $\delta_0 > 1$ as

$$D(t) = \delta_0^2 - (\delta_0^2 - 1) \left[1 - \exp\left(-\frac{\gamma}{\pi}\Gamma^2 t^2\right) \right],$$
(7.5)

and is constant ($D = \delta_0^2 = 1$) for q = 4. Anyhow, on the considered time scale, the change in $P_{\uparrow\uparrow}(m, t)$ is not perceptible since $\gamma \ll 1$. The registration may begin to take place only for larger times.

The weakness of the magnet-bath coupling γ implies that the time scale of the registration is larger than the memory time $\hbar/2\pi T$ of K(t). Then $\tilde{K}_t(\omega)$ defined by (4.17) reduces to $\tilde{K}(\omega)$, that is, to (3.37). The equation of motion (7.1) for $P_{\uparrow\uparrow}$ becomes Markovian [120, 121, 174], with v and w depending only on m and not on t. As soon as $t \gg \hbar/2\pi T$, $P_{\uparrow\uparrow}$ thus evolves in a short-memory regime. Its equation of motion is invariant under time translation.

The explicit expressions (4.31) and (4.32) of $v_{\uparrow\uparrow}$ and w become in this regime

$$v(m) = \gamma \omega_{\uparrow} (1 - m \coth \beta \hbar \omega_{\uparrow}), \tag{7.6}$$

$$w(m) = \gamma \omega_{\uparrow} (\coth \beta \hbar \omega_{\uparrow} - m), \qquad (7.7)$$

where $\hbar \omega_{\uparrow} = g + J_2 m + J_4 m^3$ (including both q = 2 and q = 4) from the definition (4.24). These functions contain in fact an extra factor $\exp(-2|\omega_{\uparrow}|/\Gamma)$, which we disregard since the Debye cutoff is large:

$$\hbar\Gamma \gg g, \qquad \hbar\Gamma \gg J.$$
 (7.8)

While the diffusion coefficient w(m) is everywhere positive, the drift velocity v(m) changes sign at the values $m = m_i$ that are solutions of (3.55). We illustrate the behavior of v(m) in Figs. 7.1 for q = 2 and 7.2 for q = 4.

2390 7.1.2. Classical features

We have stressed (subsection 4.4) that the drift term in (7.1) is "classical", in the sense that it comes out for large N by taking the continuous limit of the spectrum of \hat{m} , and that the diffusion term, although relevant in this large N limit, results from the discreteness of the spectrum of \hat{m} and has therefore a quantum origin. We can, however, forget this origin and regard this diffusion term as a "classical" stochastic effect. As a preliminary exercise, we show below



Figure 7.1: The drift velocity field $V(m) = \hbar v(m)/\gamma T = \beta (Jm + g)[1 - m \coth \beta (Jm + g)]$ for second-order transitions (q = 2, i. e., $J_2 = J$, $J_4 = 0$), at the temperature T = 0.65J. The fixed points, the zeroes of V(m), are the extrema of the free energy F(m). For g=0, the attractive fixed points lie at $\pm m_F = \pm 0.87$. For g=0.05J, the two attractive fixed points lie at $m_{\parallel} = 0.90$ and $m_{\parallel} = 0.84$, and the repulsive bifurcation lies at $m = -m_B = -0.14$. For g = 0 the attractive fixed points lie at $\pm m_F = \pm 0.91$.



Figure 7.2: The drift velocity field $V(m) = \hbar v(m)/\gamma T = \beta (Jm^3 + g)[1 - m \coth \beta (Jm^3 + g)]$ for first-order transitions (q = 4, i. e., $J_2 = 0$, $J_4 = J$) at T = 0.2J and for various couplings g. The zeroes of V(m) are the extrema of the free energy F(m) (see Figs. 3.3 and 3.4). For g = 0 there are three attractive fixed points, $m_P = 0$ and $\pm m_F$ with $m_F = 1 - 9.1 \cdot 10^{-5}$ and two repulsive fixed points, at ± 0.465 , close to $\pm \sqrt{T/J} = \pm 0.447$. For increasing g, m_P increases up to $m_c = 0.268$ until g reaches $h_c = 0.0357 J$. For larger g, the paramagnetic fixed point m_P disappears together with the positive repulsive point, and, since V is positive for all m > 0, the distribution of m can easily move from values near 0 to values near m_F , "rolling down the hill" of F(m). If g is too small, V(m) vanishes with a negative slope at the attractive paramagnetic fixed point m_P near the origin; the distribution of m then ends up around m_P and the apparatus returns to its paramagnetic state when g is switched off so that the registration fails.

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that an empirical classical approach of the registration provides us at least with a drift, similar to the one occurring in eq. (7.1).

For times $t \gg \tau_{\text{trunc}}$ it is legitimate to disregard the off-diagonal blocks $\hat{\mathcal{R}}_{\uparrow\downarrow}$ and $\hat{\mathcal{R}}_{\downarrow\uparrow}$ of $\hat{\mathcal{D}}$, and the process that takes place later on involves only $P_{\uparrow\uparrow}$ and $P_{\downarrow\downarrow}$. (In our present model the blocks evolve independently anyhow). This process looks like the measurement of a "classical discrete spin" which would take only two values +1 and -1 with respective probabilities $r_{\uparrow\uparrow}(0)$ and $r_{\downarrow\downarrow}(0)$; the *x*- and *y*-components play no role. The magnet M also behaves, in the present diagonal sectors, as a collection of *N* classical spins $\sigma_z^{(n)}$, the *x*- and *y*-components of which can be disregarded. The dynamics of M is governed by its coupling with the thermal bath B. If this coupling is treated classically, we recover a standard problem in classical statistical mechanics [60, 234, 235, 236, 237]. Indeed, the dynamics of

$$P(m,t) = \frac{P_{\uparrow\uparrow}(m,t)}{r_{\uparrow\uparrow}(0)}$$
(7.9)

is the same as the relaxation of the random order parameter *m* of an Ising magnet, submitted to a magnetic field h = gand weakly coupled to the bath B at a temperature lower than the transition temperature. Likewise, $P_{\downarrow\downarrow}(m,t)/r_{\downarrow\downarrow}(0)$ behaves as the time-dependent probability distribution for *m* in a magnetic field h = -g.

Such dynamics have been considered long since, see e.g. [60, 61, 62, 63, 64, 65, 234, 235, 236, 237]. The variables $\sigma_z^{(n)}$ are regarded as *c*-numbers, which can take the two values ±1. Due to the presence of transverse spin components at the quantum level they may flip with a transition rate imposed by the bath. Since *N* is large, it seems natural to assume that the variance of *m* remains weak at all times, as D(t)/N. (In fact, this property fails in circumstances that we shall discuss in subsection 7.3.) The probability distribution *P* is then equivalent to a Gaussian,

$$P(m,t) = \sqrt{\frac{N}{2\pi D(t)}} \exp\left\{-\frac{N[m-\mu(t)]^2}{2D(t)}\right\},$$
(7.10)

In the present classical approximation we neglect *D*, assuming that *m* is nearly equal to the expectation value $\mu(t)$. This quantity is expected to evolve according to an equation of the form

$$\frac{\mathrm{d}\mu(t)}{\mathrm{d}t} = v(\mu(t)). \tag{7.11}$$

In our case *v* is given by Eq. (7.6). This type of evolution has been considered many times in the literature. In order to establish this law and to determine the form of the function *v*, most authors start from a balance equation governing the probability that each spin $\sigma_z^{(n)}$ takes the values $\sigma_i = \pm 1$ (with $i = \uparrow$ or \downarrow). The bath induces a transition probability $W_i(m)$ per unit time, which governs the possible flip of each spin from σ_i to $-\sigma_i$, in a configuration where the total spin is $\sum_i \sigma_i = Nm$. A detailed balance property must be satisfied, relating two inverse processes, that is, relating W_i and W_{-i} ; it ensures that the Boltzmann-Gibbs distribution for the magnet at the temperature of the bath is stationary, to wit,

$$\frac{W_{-i}[m - (2/N)\sigma_i]}{W_i(m)} = \exp[-\beta \Delta E_i(m)], \qquad (7.12)$$

where $\Delta E_i(m)$ is the energy brought in by one spin flip from σ_i to $-\sigma_i$. For large *N*, we have $\Delta E_i = 2\sigma_i(h + Jm^{q-1})$ (which reads for general couplings $\Delta E_i = 2\sigma_i(h + J_2m + J_4m^3)$), so that $W_i(m)$ depends on σ_i as

$$W_{i}(m) = \frac{1}{2\theta(m)} [1 + \tanh\beta\sigma_{i}(h + Jm^{q-1})],$$
(7.13)

including a transition time $\theta(m)$ which may depend on *m* and on the temperature $T = \beta^{-1}$ of B. (Indeed, $W_{-i}(m)$, obtained from W_i by changing σ_i into $\sigma_{-i} = -\sigma_i$, satisfies (7.12).) As explained in § 4.4.3, a balance provides the variation during the time dt of the probabilities $P^{\text{dis}}(m, t)$ as function of the flipping probability $W_i(m)dt$ of each spin. The continuous limit then generates, as in the derivation of Eq. (4.30), the drift coefficient

$$v(m) = \frac{1}{\theta(m)} [\tanh \beta (h + Jm^{q-1}) - m].$$
(7.14)

Various forms for $\theta(m)$ can be found in the works devoted to this subject; they are based either on phenomenology or on an approximate solution of models [234, 235, 236, 237]. In all cases the stable fixed points of the motion (7.11), at which v(m) vanishes, are the values m_i given for large N by (3.55), where the free energy (3.54) is minimal. However, the time-dependence of $\mu(t) = \langle m \rangle$ as well as the behavior of higher order cumulants of m depend on the coefficient $\theta(m)$. For instance, while θ is a constant in [234], it is proportional to $\tanh \beta(h + Jm^{q-1})$ in [236] and [237]; it still has another form if v(m) is taken to be proportional to -dF/dm.

In the present, fully quantum approach, which relies on the Hamiltonian introduced in subsection 3.2, the drift velocity v(m) has been found to take the specific form (7.6) in the Markovian regime $t \gg \hbar/2\pi T$. We can then identify the coefficient $\theta(m)$ of (7.14) with

$$\theta(m) = \frac{\hbar \tanh \beta(h + Jm^{q-1})}{\gamma(h + Jm^{q-1})}.$$
(7.15)

With this form of $\theta(m)$, which arises from a quantum microscopic theory, the dynamical equation (7.11) keeps a satisfactory behavior when *h* or *m* becomes negative, contrary to the ad hoc choice $\theta(m) \propto \tanh \beta(h + Jm^{q-1})$. It provides, for q = 2, as shown in § 7.3.2, a long lifetime for the paramagnetic state, and better low temperature features than for $\theta(m)$ = constant.

Altogether, our final equations for the evolution of the diagonal blocks of \hat{D} are, at least in the Markovian regime, similar to equations readily found from a classical phenomenology. However, the quantum starting point and the rather realistic features of our model provide us unambiguously with the form (7.6) for the drift velocity, which meets several natural requirements in limiting cases. The occurrence of Planck's constant in (7.15) reveals the quantum origin of our classical-like equation. Moreover, quantum mechanics is also at the origin of the diffusion term and it provides the explicit form (7.7) for w. Finally, by varying the parameters of the model, we can discuss the validity of this equation and explore other regimes.

2447 7.1.3. H-theorem and dissipation

In order to exhibit the dissipative nature of our quantum equations of motion for $P_{\uparrow\uparrow}$ and $P_{\downarrow\downarrow}$ in the Markovian regime, we establish here an associated *H*-theorem [230]. This theorem holds for any Markovian dynamics, with or without detailed balance. We start from the general, discrete equation (4.16), valid even for small *N*, where $\tilde{K}_t(\omega)$ is replaced by $\tilde{K}(\omega)$. We consider the probability $P^{\text{dis}}(m,t) = (2/N)P(m,t)$, normalized under summation, which encompasses $P_{\uparrow\uparrow}^{\text{dis}}(m,t)/r_{\uparrow\uparrow}(0)$ for h = g > 0 and $P_{\downarrow\downarrow}^{\text{dis}}(m,t)/r_{\downarrow\downarrow}(0)$ for h = -g < 0, and denote as E(m) = $-hNm - JNq^{-1}m^q$ the Hamiltonian (4.6) with $h = \pm g$. We associate with $P^{\text{dis}}(m,t)$ the time-dependent entropy

$$S(t) = -\sum_{m} P^{\text{dis}}(m, t) \ln \frac{P^{\text{dis}}(m, t)}{G(m)},$$
(7.16)

where the denominator G(m) accounts for the multiplicity (3.23) of *m*, and the average energy

$$U(t) = \sum_{m} P^{\text{dis}}(m, t) E(m).$$
(7.17)

The time-dependence of the *dynamical free energy* $F_{dyn}(t) = U(t) - TS(t)$ is found by inserting the equations of motion (4.16) for the set P(m, t) into

$$\frac{\mathrm{d}F_{\mathrm{dyn}}}{\mathrm{d}t} = \sum_{m} \frac{\mathrm{d}P^{\mathrm{dis}}(m,t)}{\mathrm{d}t} \left[E(m) + T \ln \frac{P^{\mathrm{dis}}(m,t)}{G(m)} \right].$$
(7.18)

²⁴⁵⁷ The resulting expression is simplified through summation by parts, using

$$\sum_{m} \left[\Delta_{+} f_{1}(m) \right] f_{2}(m) = \sum_{m} f_{1}(m) \left[\Delta_{-} f_{2}(m) \right] = -\sum_{m} f_{1}(m_{+}) \left[\Delta_{+} f_{2}(m) \right], \tag{7.19}$$

with the notations (4.15). (No boundary term arises here.) This yields

$$\frac{\mathrm{d}F_{\mathrm{dyn}}(t)}{\mathrm{d}t} = -\frac{N\gamma}{\beta\hbar^2} \sum_{m} \left[(1+m_+)e^{\beta\Delta_+ E(m)}P^{\mathrm{dis}}(m_+,t) - (1-m)P^{\mathrm{dis}}(m,t) \right] \tilde{K}[\hbar^{-1}\Delta_+ E(m)] \Delta_+ \left[\ln \frac{P^{\mathrm{dis}}(m,t)e^{\beta E(m)}}{G(m)} \right], (7.20)$$

where we used $\tilde{K}(-\omega) = \tilde{K}(\omega) \exp \beta \hbar \omega$. Noting that $(1 - m)G(m) = (1 + m_+)G(m_+)$, we find

$$\frac{\mathrm{d}F_{\mathrm{dyn}}(t)}{\mathrm{d}t} = -\frac{\gamma N}{4\beta\hbar} \sum_{m} (1-m)G(m) \frac{\Delta_{+}E(m)}{\Delta_{+}\exp\beta E(m)} e^{-|\Delta_{+}E(m)|/\hbar\Gamma} \Delta_{+} \left[\frac{P^{\mathrm{dis}}(m,t)e^{\beta E(m)}}{G(m)}\right] \Delta_{+} \left[\ln\frac{P^{\mathrm{dis}}(m,t)e^{\beta E(m)}}{G(m)}\right].$$
(7.21)

The last two factors in (7.21) have the same sign, while the previous ones are positive, so that each term in the sum is negative. Thus the dynamical free energy is a decreasing function of time. The quantity $-\beta dF_{dyn}/dt$ can be interpreted as the *dissipation rate* (or the entropy production) of the compound system M+B, that is, the increase per unit time of the entropy (7.16) of the magnet plus the increase $-\beta dU/dt$ of the entropy of the bath. In fact the entropy of M is lower in the final state than in the initial state, but the increase of entropy of B associated with the energy dumping dominates the balance. The negativity of (7.21) characterizes the irreversibility of the registration.

The right-hand side of (7.21) vanishes only if all its terms vanish, that is, if $P^{\text{dis}}(m,t) \exp[\beta E(m)]/G(m)$ does 2466 not depend on m. This takes place for large times, when the dynamical free energy F(t) has decreased down to the 2467 minimum allowed by the definitions (7.16), (7.17). We then reach the limit $P^{\text{dis}}(m) \propto G(m) \exp[-\beta E(m)]$, which 2468 is the distribution associated with the canonical equilibrium of M for the Hamiltonian $E(\hat{m})$, that is, with the *static* 2469 free energy⁵⁵. We have thus proven for our model the following property, often encountered in statistical physics 2470 [174, 230]. The same probability distribution for *m* arises in two different circumstances. (i) In equilibrium statistical 247 mechanics, (§ 3.3.4), $P^{\text{dis}}(m)$ follows from the Boltzmann-Gibbs distribution $\hat{R}_{\text{M}} \propto \exp[-\beta \hat{H}_{\text{M}}]$ for the magnet alone. 2472 (ii) In non-equilibrium statistical mechanics, it comes out as the asymptotic distribution reached in the long time limit 2473 when M is weakly coupled to the bath. 247

It is only in the Markovian regime that the dynamical free energy is ensured to decrease. Consider in particular, for the quadratic coupling q = 2, the evolution of $P^{\text{dis}}(m, t)$ on very short times, which involves the narrowing (7.5) of the initial peak. The free energy associated with a Gaussian distribution centered at m = 0, with a time-dependent variance D(t)/N, is

$$F_{\rm dyn}(t) = \sum_{m} P^{\rm dis}(m,t) \left[-gNm - \frac{1}{2}JNm^2 + T\ln\frac{P^{\rm dis}(m,t)}{G(m)} \right] = -\frac{1}{2}(JD + T - TD + T\ln D).$$
(7.22)

The time-dependence of *D* is expressed for short times $t \ll \Gamma^{-1}$ by (7.5). The initial value δ_0^2 of D(t) being given by (3.51), we find

$$\frac{\mathrm{d}F_{\mathrm{dyn}}}{\mathrm{d}t} = \frac{\gamma\Gamma^2 t}{\pi} \frac{J^2(T_0 - T)}{T_0(T_0 - J)}.$$
(7.23)

Thus at the very beginning of the evolution, F_{dyn} slightly increases, whereas for $t \gg \hbar/2\pi T$ it steadily decreases according to (7.21). In fact, the negative sign of v in the initial non-Markovian regime (7.4) indicates that, for very short times, the fixed point near m = 0 is stable although the bath temperature is lower than J.

2484 7.1.4. Approach to quasi-equilibrium

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Անձրևոտ օրը շատերը կասեն. "Ջուր տար, քո հավերին լողացրու":⁵⁶ Armenian proverb

The above proof that the system eventually reaches the canonical equilibrium state $\hat{R}_{M} \propto \exp(-\beta \hat{H}_{M})$ is mathematically correct for finite *N* and $t \rightarrow \infty$. However, this result is not completely relevant physically in the large *N* limit. Indeed, the times that we consider should be attainable in practice, and "large times" does not mean "infinite times" in the mathematical sense [55, 56].

⁵⁵The notions of dynamical (moderate time) and static (infinite time) free energy are well known in the theory of glasses and spin glasses, see e.g. [238, 239, 240]. In corresponding mean field models, they differ strongly; here, however, the dynamical free energy simply refers to processes close to equilibrium and decreases down to the static equilibrium free energy in agreement with the macroscopic Clausius–Duhem inequality [56, 73]

⁵⁶On a rainy day many people offer to bathe your chickens

In order to analyze this situation, we note that the summand of (7.21) contains a factor $P^{\text{dis}}(m, t)$; thus the ranges 2491 of m over which $P^{dis}(m, t)$ is not sizeable should be disregarded. When the time has become sufficiently large so that 2492 the rate of decrease of F(t) has slowed down, a regime is reached where $P^{\text{dis}}(m, t) \exp[\beta E(m)]/G(m)$ is nearly time-2493 independent and nearly constant (as function of m) in any interval where $P^{\text{dis}}(m, t)$ is not small. Within a multiplicative 2494 factor, $P^{\text{dis}}(m, t)$ is then locally close to $\exp[-\beta F(m)]$ where $F(m) = U(m) - T \ln G(m)$ is given by (3.54). It is thus 2495 concentrated in peaks, narrow as $1/\sqrt{N}$ and located in the vicinity of points m_i where F(m) has a local minimum. 249 Above the transition temperature, or when the field $h = \pm g$ is sufficiently large, there is only one such peak, and the 249 asymptotic form of $P^{\text{dis}}(m,t)$ is unique. However, below the critical temperature, two separate peaks may occur for 2498 q = 2, and two or three peaks for q = 4, depending on the size of h. 249

In such a case, $P^{\text{dis}}(m, t)$ can be split into a sum of non-overlapping contributions $P^{\text{dis}}_{\text{Mi}}(m, t)$, located respectively 2500 near m_i and expected to evolve towards the equilibrium distributions $P_{M_i}^{dis}(m)$ expressed by (3.56). Since for sufficiently 2501 long times $P^{\text{dis}}(m, t)$ is concentrated around its maxima m_i with a shape approaching the Gaussian (3.56), its equation 2502 of motion (7.1) does not allow for transfers from one peak to another over any reasonable delay. (Delays exponentially 2503 large with N are physically inaccessible.) Once such a regime has been attained, each term $P_{Mi}^{dis}(m,t)$ evolves inde-250 pendently according to (7.1). Its normalization remains constant, and its shape tends asymptotically to (3.56). Hence, 2505 below the transition temperature, ergodicity is broken in the physical sense. (A breaking of ergodicity may occur in 2506 a mathematically rigorous sense only for infinite N or zero noise.) If the system starts from a configuration close to 250 some m_i , it explores, during a physically large time, only the configurations for which m lies around m_i . Configura-250 tions with the same energy but with values of m around other minima of F(m) remain out of reach. This phenomenon 2509 is essential if we want to use M as the pointer of a measurement apparatus. If the spin S lies upwards, its interaction 2510 with A should lead to values of m that fluctuate weakly around $+m_{\rm F}$, not around $-m_{\rm F}$. Ergodicity would imply that 251 A spends the same average time in all configurations having the same energy, whatever the sign of m [55, 56], once 2512 the interaction \hat{H}_{SA} is turned off. The breaking of invariance is thus implemented through the dynamics: unphysical 2513 times, exponentially large with N, would be needed to reach the symmetric state $\exp(-\beta \hat{H}_{\rm M})$. 251

In analogy with what happens in glasses and spin glasses [238, 239, 240], for physical large times *t*, the asymptotic value of $F_{dyn}(t)$ is not necessarily the absolute minimum of F(m). It is a weighted average of the free energies of the stable and metastable states, with magnetizations m_i . The weights, that is, the normalizations of the contributions $P_{Mi}^{dis}(m, t)$ to $P^{dis}(m, t)$ are determined by the initial distribution $P^{dis}(m, 0)$, and they depend on N and on the couplings g and J which enter the equations of motion. For an ideal measurement, we require the process to end up at a single peak, $+m_F$ for $P_{\uparrow\uparrow}^{dis}$, $-m_F$ for $P_{\downarrow\downarrow}^{dis}$ (subsection 7.2). Otherwise, if M may reach either one of the two ferromagnetic states $\pm m_F$, the measurement is not faithful; we will determine in § 7.3.3 its probability of failure.

In the present regime where the variations with *m* of $Pe^{\beta E}/G$ are slow, we can safely write the continuous limit of the *H*-theorem (7.21) by expressing the discrete variations Δ_+ over the interval $\delta m = 2/N$ as derivatives. We then find the dissipation rate as (we switch to the function $P(m) = (N/2)P^{\text{dis}}(m)$ and to an integral over *m*)

$$-\frac{1}{T} \frac{\mathrm{d}F_{\mathrm{dyn}}}{\mathrm{d}t} = \frac{\gamma NT}{\hbar} \int \mathrm{d}m \, P(m,t)\phi(m) [\coth\phi(m) - m] \\ \times \left[\frac{1}{NP} \frac{\partial P}{\partial m} - \frac{\tanh\phi(m) - m}{1 - m\tanh\phi(m)}\right] \left[\frac{1}{NP} \frac{\partial P}{\partial m} - \phi(m) + \frac{1}{2}\ln\frac{1 + m}{1 - m}\right], \tag{7.24}$$

2525 where we use the notation

$$\phi(m) = \beta(h + Jm^{q-1}), \qquad h = \pm g. \tag{7.25}$$

For large *N*, the term (1/NP)dP/dm is not negligible in case ln *P* is proportional to *N*, that is, in the vicinity of a narrow peak with width $1/\sqrt{N}$. The expression (7.24) is not obviously positive. However, once $P(m, t) = \sum_{i=\pm 1} P_{Mi}(m, t)$ has evolved into a sum of separate terms represented by peaks around the values m_i , we can write the dissipation as a sum of contributions, each of which we expand around m_i . The last two brackets of (7.24) then differ only at order $(m - m_i)^3$, and we get the obviously positive integrand

$$-\frac{1}{T}\frac{\mathrm{d}F_{\mathrm{dyn}}}{\mathrm{d}t} = \frac{\gamma NT}{\hbar}\sum_{i=\pm 1}\int \mathrm{d}m \,P_{\mathrm{M}i}(m,t)\phi(m)[\coth\phi(m)-m] \tag{7.26}$$

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$$\times \quad \left\{\frac{1}{NP_{\mathrm{M}i}}\frac{\partial P_{\mathrm{M}i}}{\partial m} + \left[\frac{1}{1-m_{i}^{2}} - (q-1)\beta Jm_{i}^{q-2}\right](m-m_{i}) + \left[\frac{m_{i}}{(1-m_{i}^{2})^{2}} - \frac{(q-1)(q-2)}{2}\beta Jm_{i}^{q-3}\right](m-m_{i})^{2}\right\}^{2}.$$

We thus check that $F_{dyn}(t)$ decreases, down to the weighted sum of free energies associated with the stable or metastable equilibrium distributions (3.56). In fact, among the stationary solutions of (7.1), those which satisfy

$$vP - \frac{1}{N}\frac{d(wP)}{dm} = 0,$$
 (7.27)

with *v* and *w* given by (7.6) and (7.7), coincide with (3.56) around the values of m_i given by (3.55), not only in the mean-field approximation but also including the corrections that we retained in those formulae.

2535 7.2. Registration times

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Quid est ergo tempus? Si nemo ex me quaerat, scio; si quaerenti explicare velim, nescio 57 Saint Augustine

In the present subsection, we study the evolution of the distribution $P_{\uparrow\uparrow}(m, t)$, which is such that $(1/N) \ln P_{\uparrow\uparrow}$ is finite for large *N*. This property holds at t = 0 and hence at all times. As a consequence, $P_{\uparrow\uparrow}$ presents a narrow peak with width of order $1/\sqrt{N}$, and it is equivalent to a Gaussian. We first note that the evolution (7.1) conserves its normalization $r_{\uparrow\uparrow}(0)$. The ratio (7.9) can then be parametrized as in (7.10) by the position $\mu(t)$ of the peak and by its width parameter D(t), which are both finite for large *N*.

2543 7.2.1. Motion of a single narrow peak

The equations of motion for $\mu(t)$ and D(t) are derived by taking the first moments of the equation (7.1) for $P_{\uparrow\uparrow}(m, t)$. Integration over *m* of (7.1) first entails the conservation in time of the normalization $r_{\uparrow\uparrow}(0)$ of $\int dm P_{\uparrow\uparrow}(m, t)$. We then integrate (7.1) over *m* after multiplication, first by $m - \mu(t)$, second by $N[m - \mu(t)]^2 - D(t)$, using on the right-hand side an integration by parts and the steepest descents method. To wit, expanding $\nu(m, t)$ and w(m, t) in powers of $m - \mu(t)$, we rely on the vanishing of the integrals of $m - \mu(t)$ and of $N[m - \mu(t)]^2 - D(t)$ when weighted by $P_{\uparrow\downarrow}(m, t)$, and we neglect for k > 1 the integrals of $[m - \mu(t)]^{2k}$, which are small as N^{-k} . This yields for sufficiently large N

$$\frac{\mathrm{d}\mu(t)}{\mathrm{d}t} = v[\mu(t), t], \tag{7.28}$$

$$\frac{1}{2}\frac{\mathrm{d}D(t)}{\mathrm{d}t} = \frac{\partial v[\mu(t), t]}{\partial \mu}D(t) + w[\mu(t), t].$$
(7.29)

At the very beginning of the evolution, when *t* is not yet large compared to $\hbar/2\pi T$, Eqs. (7.28) and (7.29) should be solved self-consistently, using the expressions (4.31) for *v* and (4.32) for *w*. However, if the coupling γ is weak, the Markovian regime is reached before the shape of $P_{\uparrow\uparrow}$ is significantly changed. We can thus solve (7.28) and (7.29) with the time-independent forms (7.6) and (7.7) for *v* and *w*, the initial conditions being $\mu(0) = 0$, $D(0) = \delta_0^2$.

The solution of (7.28) is then, for $t \gg \hbar/2\pi T$,

$$t = \int_0^{\mu} \frac{d\mu'}{\nu(\mu')} = \frac{\hbar}{\gamma T} \int_0^{\mu} \frac{d\mu'}{\phi(\mu')[1 - \mu' \coth \phi(\mu')]},$$
(7.30)

where the function ϕ is defined by (7.25) with h = +g. Inversion of (7.30) provides the motion $\mu(t)$ of the peak of $P_{\uparrow\uparrow}(m, t)$. For $P_{\downarrow\downarrow}$, we have to change g into -g in (7.25), and $\mu(t)$ expressed by (7.30) is then negative.

If *N* is very large, the probabilistic nature of the registration process fades out and the magnetization is located at $\mu(t)$ with near certainty. The evaluation of the time dependence of $\mu(t)$ may be proposed to students as an exercise (§ 9.6.2). Results for quadratic coupling (q = 2) and for quartic coupling (q = 4), which exemplify second and first-order transitions, respectively, are illustrated by Fig. 7.3 and by Fig. 7.4, respectively. The evolution from the initial paramagnetic state to the final ferromagnetic state exhibits several stages, which will be studied in § 7.2.3 for q = 2 and in § 7.2.4 for q = 4.

⁵⁷What then is time? If no one asks me, I know what it is; if I wish to explain it to him who asks, I do not know



Figure 7.3: The average magnetization $\mu(t)$ for a quadratic interaction (q = 2) goes from zero to m_{fl} . The time dependence, given by (7.30), results from the local velocity of Fig. 7.1. The parameters are T = 0.65J and g = 0.05J, while the time scale is $\tau_J = \hbar/\gamma J$. One can distinguish the three stages of §7.2.3, characterized by the first registration time $\tau_{\text{reg}} = [J/(J-T)]\tau_J = 2.86\tau_J$ (eq. (7.44)) and the second registration time $\tau_{\text{reg}} = 8.4\tau_J$ (eq. (7.48)): (i) Increase, first linearly as $(g/J)(t/\tau_J) = 0.05t/\tau_J$, then exponentially according to (7.42), with a coefficient $m_{\text{B}} = g/(J-T) = 0.143$ and a time scale τ_{reg} . After a delay of a few τ_{reg} , the coupling may be switched off without spoiling the registration. (ii) Rise, according to (7.47), up to $m_{\text{F}} - \frac{1}{2}m_{\text{B}} = 0.80$ reached at the second registration time τ'_{reg} . (iii) Exponential relaxation towards $m_{\text{fl}} = 0.90$ (or $m_{\text{F}} = 0.87$ if g is switched off) according to (7.49) with the time scale $1.6\tau_J$.



Figure 7.4: The average magnetization $\mu(t)$ for a quartic interaction (q = 4) goes from zero to $m_{\parallel} \simeq 1$. The time dependence, given by (7.30), results from the local velocity of Fig. 7.2. The parameters are T = 0.2J and g = 0.045J while the time scale is $\tau_J = \hbar/\gamma J$. The characteristic registration time $\tau_{\text{reg}} = 38\tau_J$ is now given by (7.52). (Note that it is much larger than for a quadratic interaction.) The initial increase of $\mu(t)$ takes place, first linearly as $(g/J) t/\tau_J = 0.045t/\tau_J$, then slows down according to (7.51), with a coefficient g/J = 0.045 and a time scale $\tau_1 = (g/J)\tau_J$. The region of $m_c = 0.268$, where the drift velocity is small, is a bottleneck: around this point, reached at the time $t = \frac{1}{2}\tau_{\text{reg}}$, the average magnetization $\mu(t)$ lingers according to (7.53) where $\delta m_c = 0.11$. It then increases rapidly so as to reach at the time τ_{reg} a value close to $m_F \simeq 1$, and finally reaches m_F exponentially on the time scale τ_J .

The width of the peak is obtained by regarding *D* as a function of $\mu(t)$ and by solving the equation for $dD/d\mu$ that results from (7.28) and (7.29). This yields

$$D(\mu) = v^{2}(\mu) \left[\frac{\delta_{0}^{2}}{v^{2}(0)} + \int_{0}^{\mu} \frac{d\mu' \, 2w(\mu')}{v^{3}(\mu')} \right] = \phi^{2}(\mu) [1 - \mu \, \coth \phi(\mu)]^{2} \left\{ \frac{\delta_{0}^{2}}{\beta^{2}g^{2}} + \int_{0}^{\mu} \frac{d\mu' \, 2[\coth \phi(\mu') - \mu']}{\phi^{2}(\mu')[1 - \mu' \, \coth \phi(\mu')]^{3}} \right\}.$$
(7.31)

To analyze this evolution of D(t), we first drop the term in *w* from the equation of motion (7.1) of $P_{\rm M}(m, t)$. This simplified equation describes a deterministic flow in the space of *m*, with a local drift velocity v(m). For any initial condition, its solution is the mapping

$$P_{\rm M}(m,t) = \frac{1}{v(m)} \int {\rm d}m' P_{\rm M}(m',0) \,\delta\left(t - \int_{m'}^{m} \frac{{\rm d}m''}{v(m'')}\right), \tag{7.32}$$

where m' is the initial point of the trajectory that reaches m at the time t. For a distribution (7.10) peaked at all times, 256 we recover from (7.32) the motion (7.30) of the maximum of $P_{\uparrow\uparrow}(m, t)$ and the first term of the variance (7.31). If only 2569 the drift term were present, the width of the peak would vary as $v(\mu)$: Indeed, in a range of m where the drift velocity 2570 increases with *m*, the front of the peak progresses more rapidly than its tail so that the width increases, and conversely. 2571 The second term of (7.31) arises from the term in w. Since w(m) is positive, it describes a diffusion which widens 2572 the distribution. This effect of w is enhanced when v is small. In particular, by the end of the evolution when $\mu(t)$ 2573 tends to a zero m_i of v(m) with $\partial v/\partial m < 0$, the competition between the narrowing through v and the widening 2574 through w leads to the equilibrium variance $D = -(dv/d\mu)^{-1}w$, irrespective of the initial width. This value is given by 2575 $D^{-1} = (1 - m_i^2)^{-1} - (q - 1)\beta J m_i^{q-2}$, in agreement with (3.56) and with (7.27). 257

²⁵⁷⁷ We have noted that the drift velocity v(m) has at each point the same sign as -dF/dm, where *F* is the free energy ²⁵⁷⁸ (3.54), and that the zeroes m_i of v(m), which are the fixed points of the drift motion, coincide with the extrema of *F*. ²⁵⁷⁹ At such an extremum, given by (3.55), we have

$$-\frac{\mathrm{d}v}{\mathrm{d}m} = \frac{\gamma}{N\hbar} \frac{2\phi(m_i)}{\sinh 2\phi(m_i)} \frac{\mathrm{d}^2 F}{\mathrm{d}m^2}.$$
(7.33)

The minima of *F* correspond to *attractive* fixed points, with negative slope of v(m), its maxima to repulsive points, that is, *bifurcations*. In the present case of a narrow distribution, $\mu(t)$ thus increases from $\mu(0) = 0$ to the smallest positive minimum m_i of F(m), which is reached asymptotically for large times. However, the present hypothesis of a single narrow peak is valid only if P(m, t) lies entirely and at all times in a region of *m* free of bifurcations. We will discuss in subsection 7.3 the situation where *P* lies astride a bifurcation, either at the initial time or a little later on, if a tail due to diffusion crosses the bifurcation.

²⁵⁸⁶ 7.2.2. Threshold for the system-apparatus coupling; possibilities of failure

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If you are not big enough to lose, you are not big enough to win Walter Reuther

The measurement is successful only if $P(m, t) \equiv P_{\uparrow\uparrow}(m, t)/r_{\uparrow\uparrow}(0)$, which is interpreted as the conditional probability distribution for *m* if $s_z = +1$, approaches for large times the narrow normalized peak (3.56) located at the positive ferromagnetic solution m_{\uparrow} of (3.55) with h = +g, close to m_F for $g \ll T^{58}$. This goal can be achieved only if (*i*) the center $\mu(t)$ of the peak approaches m_{\uparrow} ; (*ii*) its width remains small at all times so that the above derivation is valid.

(*i*) The first condition is relevant only for a first-order transition (q = 4), since $m_{\uparrow\uparrow}$ is anyhow the only attractive fixed point in the region m > 0 for a second-order transition (q = 2). For quartic interactions, the first minimum of F(m) that occurs for increasing m is not necessarily $m_{\uparrow\uparrow}$ (Fig. 7.2). Indeed, we have seen (end of § 3.3.4 and Fig. 3.4) that for a field lower than

$$h_{\rm c} = T \arctan m_{\rm c} - J m_{\rm c}^3 \approx \frac{2}{3} T m_{\rm c}, \qquad m_{\rm c}^2 = \frac{1}{2} - \frac{1}{2} \sqrt{1 - \frac{4T}{3J}} \approx \frac{T}{3J} + \frac{T^2}{9J^2},$$
 (7.34)

⁵⁸We recall Eq. (3.57) where $m_{\uparrow\uparrow} \simeq +1$ and $m_{\downarrow\downarrow} \simeq -1$ are defined as the fixed points at finite g, and $m_{\rm F}$ and $-m_{\rm F}$ as their $g \to 0$ limits, respectively

the free energy F(m) has not only a ferromagnetic minimum at $m_{\uparrow\uparrow}$, but also a local paramagnetic minimum m_P at a smaller value of *m*. Hence, if the spin-apparatus coupling *g* is smaller than h_c , $\mu(t)$ reaches for large times the locally stable point m_P in the sector $\uparrow\uparrow$. It reaches $-m_P$ in the sector $\downarrow\downarrow$, so that the apparatus seems to distinguish the values $s_z = \pm 1$ of S. However, if the coupling is switched off at the end of the process, the magnetization *m* of M returns to 0 in both cases. The result of the measurement thus cannot be registered robustly for $g < h_c$.

The center $\mu(t)$ of the peak may escape the region of the origin only if $g > h_c$ (Fig. 7.2). Relying on the smallness of T/3J (equal to 0.121 at the transition temperature), we can simplify the expression of h_c as in (7.34), so that this threshold for g is (q = 4):

$$g > h_{\rm c} \simeq \frac{2T}{3} \sqrt{\frac{T}{3J}}.\tag{7.35}$$

Under this condition, the peak $\mu(t)$ of $P_{\uparrow\uparrow}(m,t)$ reaches for large times $m_{\uparrow\uparrow}$, close to the magnetization $m_{\rm F}$ of the ferromagnetic state. If the coupling g is removed sufficiently after $\mu(t)$ has passed the maximum of F(m), the peak is expected to end up at $m_{\rm F}$. Likewise, the peak of $P_{\downarrow\downarrow}(m,t)$ reaches $-m_{\rm F}$ at the end of the same process. The apparatus is non-ergodic and the memory of its triggering by S may be kept forever under the necessary (but not sufficient) condition (7.35).

(*ii*) The second requirement involves the width of the distribution $P_{\uparrow\uparrow}(m,t)$ and the location $-m_{\rm B} < 0$ of the 2610 repulsive fixed point, at which F(m) is maximum. Consider first the pure drift flow (7.32) without diffusion, for which 2611 $-m_{\rm B}$ is a bifurcation. The part $m > -m_{\rm B}$ of $P_{\uparrow\uparrow}(m, 0)$ is properly shifted upwards so as to reach eventually the vicinity 2612 of the positive ferromagnetic value $+m_{\rm F}$; however its tail $m < -m_{\rm B}$ is pushed towards the negative magnetization 2613 $-m_{\rm F}$. If the relative weight of this tail is not negligible, *false measurements*, for which the value $-m_{\rm F}$ is registered 2614 by A although s_z equals +1, can occur with a sizeable probability. Such a failure is excluded for q = 4, because m_B 2615 is then much larger than the width $1/\sqrt{N}$ of $P_{\uparrow\uparrow}(m,0)$; for instance, in the case q = 4 we have $m_{\rm B} = 0.544$ for the 2616 parameters T = 0.2J and g = 0.045J (which satisfy (7.35)). However, in the case q = 2 and $g \ll J - T$, the point 2617 $-m_{\rm B}$ with 2618

$$m_{\rm B} \simeq \frac{g}{J-T},\tag{7.36}$$

lies close to the origin (Fig. 7.1), and a risk exists that the initial Gaussian distribution in $\exp(-Nm^2/2\delta_0^2)$ extends below $-m_{\rm B}$ if g is too small. The probability of getting a wrong result is significant if the condition $\delta_0 \ll m_{\rm B}\sqrt{N}$ is not fulfilled. We return to this point in § 7.3.3.

Moreover, in this case q = 2, the lower bound thus guessed for the coupling,

$$g = (J - T)m_{\rm B} \gg \frac{(J - T)\delta_0}{\sqrt{N}},\tag{7.37}$$

is not sufficient to ensure a faithful registration. The diffusive process, which tends to increase D(t) and thus to thicken the dangerous tail $m < -m_{\rm B}$ of the probability distribution $P_{\uparrow\uparrow}(m, t)$, raises the probability of a false registration towards $-m_{\rm F}$ instead of $+m_{\rm F}$. In order to trust the Ansatz (7.10) and the ensuing solution (7.30), (7.31) for $P_{\uparrow\uparrow}(m, t)$, we need D(t) to remain at all times sufficiently small so that $P_{\uparrow\uparrow}(m, t)$ is negligible for $m < -m_{\rm B}$. This is expressed, when taking $\mu(t)$ as a variable instead of t, as

$$\frac{D(\mu)}{N(m_{\rm B}+\mu)^2} \ll 1$$
(7.38)

for any μ between 0 and $m_{\rm F}$: The width $\sqrt{D/N}$ of the peak of $P_{\uparrow\uparrow}(m, t)$ should not increase much faster than its position μ . For sufficiently small g, we have $m_{\rm B} \ll m_{\rm F}$, and we only need to impose (7.38) for times such that $\mu(t)$ lies in an interval $0 < \mu(t) < \mu_{\rm max}$ such that $m_{\rm B} \ll \mu_{\rm max} \ll T/J$. In this range we can evaluate $D(\mu)$ from (7.31) by

simplifying $\tanh \phi(\mu)$ into $\phi(\mu)$, which yields

$$\frac{D(\mu)}{(m_{\rm B}+\mu)^2} = \frac{\delta_0^2}{m_{\rm B}^2} + \frac{T}{J-T} \left[\frac{1}{m_{\rm B}^2} - \frac{1}{(m_{\rm B}+\mu)^2} \right].$$
(7.39)

This ratio increases in time from $\delta_0^2/m_{\rm B}^2$ to $\delta_1^2/m_{\rm B}^2$, where

$$\delta_1^2 = \delta_0^2 + \frac{T}{J - T} = \frac{T_0}{T_0 - J} + \frac{T}{J - T},$$
(7.40)

so that the left-hand side of (7.38) remains at all times smaller than δ_1^2 / Nm_B^2 . The lower bound on *g* required to exclude false registrations is therefore (*q* = 2)

$$g \gg \frac{(J-T)\delta_1}{\sqrt{N}},\tag{7.41}$$

a condition more stringent than (7.37) if $J - T \ll J$. Altogether, for q = 2 the system-apparatus coupling may for large N be small, for instance as $N^{1/3}$, provided it satisfies (7.41).

For q = 4, and more generally for a first-order transition $(3J_4 > J_2)$, the lower bound found as (7.35) remains finite for large *N*: A free energy barrier of order *N* has to be overpassed. Moreover, the diffusion hinders the trend of *m* to increase and may push part of the distribution $P_{\uparrow\uparrow}(m, t)$ leftwards, especially its left tail, while its peak moves rightwards. The widening of $P_{\uparrow\uparrow}(m, t)$ when the barrier is being reached should not be too large, and this effect raises further the threshold for *g*. We shall show in § 7.2.4 that the condition (7.35) should thus be strengthened into (7.57). Another difference between first- and second order transitions lies in the possible values of the temperature. For

Another difference between first- and second order transitions lies in the possible values of the temperature. For q = 2, if *T* lies near the critical temperature *J*, the minima m_i of F(m) are very sensitive to *g* and the ferromagnetic value m_F in the absence of a field is small as $\sqrt{3(J - T)/J}$. Using M as the pointer of a measurement apparatus requires the temperature to lie sufficiently below *J*. For q = 4, registration is still possible if *T* lies near the transition temperature, and even above, although in this case the ferromagnetic states are not the most stable ones for h = 0. However, the coupling *g* should then be sufficiently strong.

²⁶⁴⁸ 7.2.3. The registration process for a second-order transition

Assuming *g* to satisfy (7.41) and m_F to be significantly large, we resume the dynamics of $P_{\uparrow\uparrow}(m, t)$ for q = 2 so as to exhibit its characteristic times. After a short delay of order \hbar/T , most of the process takes place in the Markovian regime, and the Gaussian Ansatz (7.10) is justified. We can distinguish three stages in the evolution of $P_{\uparrow\uparrow}(m, t)$, which are exhibited on the example of Figs. 7.3 and 7.5.

(*i*) During the first stage, as long as $\mu(t) \ll m_{\rm F}$, we can replace $\phi(m) \coth \phi(m)$ by 1 in v and w, so that the drift velocity v behaves (Fig. 7.1) as

$$v(m) \approx \frac{\gamma T}{\hbar} \left[\frac{g + Jm}{T} - m \right] = \frac{\gamma (J - T)(m_{\rm B} + m)}{\hbar}, \tag{7.42}$$

and the diffusion coefficient as $w \approx \gamma T/\hbar$. Integration of (7.30) then yields the motion

$$\mu(t) \sim m_{\rm B}(e^{t/\tau_{\rm reg}} - 1) = \frac{g}{J - T}(e^{t/\tau_{\rm reg}} - 1)$$
(7.43)

²⁶⁵⁶ for the center of the peak, with the characteristic time

$$\tau_{\rm reg} = \frac{\hbar}{\gamma (J-T)}.\tag{7.44}$$

After beginning to move as $\mu \sim \gamma gt/\hbar$, the distribution shifts away from the origin faster and faster. Once μ has reached values of the order of several times $m_{\rm B}$, $(J - T)\mu$ becomes larger than g, so that $\nu(\mu)$ does not depend much on g. It little matters for the subsequent evolution whether the coupling g is present or not. Thus, after $t/\tau_{\rm reg}$ reaches 2 or 3, *the spin-apparatus coupling may be switched off* and the increase of μ goes on nearly unchanged. In fact, the distribution moves towards $m_{\rm F}$ rather than m_{\uparrow} , but $m_{\rm F}-m_{\uparrow}$ is small, less than g/J. We shall call $\tau_{\rm reg}$ the *first registration time*. After it, M will necessarily reach the ferromagnetic state $+m_{\rm F}$, independent of S, although the evolution is not achieved yet.

We have seen that during this first stage the width (7.39) is governed both by the drift which yields the factor $(m_{\rm B} + \mu)^2$, increasing as $e^{2t/\tau_{\rm reg}}$, and by the diffusion which raises δ_0 up to δ_1 .

(*ii*) During the second stage $\mu(t)$ rises rapidly from $m_{\rm B}$ to $m_{\rm F}$, since the drift velocity $v(\mu)$ is no longer small. The distribution has become wide, and its width is now governed mainly by the drift term. Matching $D(\mu)$ with (7.39) for μ larger than $m_{\rm B}$ yields the width

$$\sqrt{\frac{D(\mu)}{N}} \sim \frac{\tau_{\text{reg}}\delta_1}{m_{\text{B}}\sqrt{N}}v(\mu) = \frac{\hbar\delta_1}{\gamma g\sqrt{N}}v(\mu),\tag{7.45}$$

which varies proportionally to $v(\mu)$. The drift velocity v(m) first increases and then decreases as function of m (Fig. 7.1), down to 0 for $m = m_{\uparrow} \simeq m_{\rm F}$. Accordingly, the width D(t) increases as function of time, then decreases (Fig. 7.5). The time dependence (7.30) of $\mu(t)$ and hence of D(t) is evaluated explicitly in the Appendix E.1, where μ is related to t through Eq. (E.2), that is,

$$\frac{t}{\tau_{\rm reg}} = \ln \frac{m_{\rm B} + \mu}{m_{\rm B}} + a \ln \frac{m_{\rm F}^2}{m_{\rm F}^2 - \mu^2},\tag{7.46}$$

where the coefficient a, given by

$$a = \frac{T(J-T)}{J[T-J(1-m_{\rm F}^2)]},\tag{7.47}$$

lies between $\frac{1}{2}$ and 1.

We define the *second registration time* τ'_{reg} as the delay taken by the average magnetization $\mu(t)$ to go from the paramagnetic value $\mu = 0$ to the value $m_{\rm F} - \frac{1}{2}m_{\rm B}$ close to $m_{\rm F}$. From the equation (7.46) that relates μ to *t*, we find this second registration time, the duration of the second stage, much longer than the first, as

$$\tau'_{\rm reg} = \tau_{\rm reg} (1+a) \ln \frac{m_{\rm F}}{m_{\rm B}},$$
 (7.48)

(*iii*) The third stage of the registration, the *establishment of thermal equilibrium*, has been studied in § 7.1.3 and § 7.1.4. While $\mu(t)$ tends exponentially to m_{\uparrow} (or to $m_{\rm F}$ if the coupling g has been switched off), we saw that the equilibrium width of $P_{\uparrow\uparrow}(m, t)$ is reached as a result of competition between the drift, which according to (7.45) narrows the distribution, and the diffusion which becomes again relevant and tends to widen it. It is shown in the Appendix E.1 that the final relaxation takes place, for times $t - \tau'_{\rm reg} \sim \tau_{\rm reg}$, according to

$$\mu(t) = m_{\rm F} \left[1 - \frac{1}{2} \left(\frac{m_{\rm F}}{m_{\rm B}} \right)^{1/a} \exp\left(-\frac{t}{a \tau_{\rm reg}} \right) \right]. \tag{7.49}$$

At low temperatures, $T \ll J$, we have $m_{\rm F} \sim 1$, $m_{\rm B} \sim g/J$, $a \sim 1$. If T lies close to the transition temperature, $J - T \ll J$, we have $m_{\rm F}^2 \sim 3(J - T)/J$, $m_{\rm B} = g/(J - T)$ and $a \sim \frac{1}{2}$.

The above scenario for the registration process is illustrated by Fig. 7.5 which represents a numerical solution of 2685 the equation for $P(m, t) = P_{\uparrow\uparrow}(m, t)/r_{\uparrow\uparrow}(0)$. The curves exhibit the motion from 0 to $m_{\rm F}$ of the center $\mu(t)$ of the peak 2686 (also shown by Fig. 7.3), its large initial widening, the intermediate regime where the width $\sqrt{D(t)/N}$ is proportional 2687 to $\mu(t)$, and the final adjustment of μ and D to their equilibrium values in the ferromagnetic state. Except near the 2688 initial and final state, the width is not small although we have taken a fairly large value N = 1000, but one can see that 2689 the Gaussian approximation used for $P_{\uparrow\uparrow}(m,t)$ is sufficient and that the resulting formulae given above for $\mu(t)$ and 2690 D(t) fit the curves. While a mean-field theory neglecting the fluctuations is satisfactory at equilibrium, the dynamics 2691 entails large fluctuations of *m* at intermediate times. 269

2693 7.2.4. The registration process for a first-order transition

The process is different when the interaction is quartic (q = 4), a case that we chose to exemplify the first-order transitions which occur when $3J_4 > J_2$. The spin-apparatus coupling g must then at least be larger than the threshold (7.35) to ensure that v(m) remains positive up to m_{\uparrow} , which now lies near $m_{\rm F} \simeq 1$ (Figs. 3.3 and 7.2). At the beginning of the evolution, we find from $v(m) \approx (\gamma/\hbar)(g - Tm)$, using $g \ll T$, the motion



Figure 7.5: The registration process for a quadratic interaction (q = 2). The probability density $P(m, t) = P_{\uparrow\uparrow}(m, t)/r_{\uparrow\uparrow}(0)$ for the magnetization m of M is represented at different times. The parameters were chosen as N = 1000, T = 0.65J and g = 0.05J as in Figs. 7.1 and 7.3. The time scale is here the registration time $\tau_{reg} = \hbar/\gamma(J - T) = 2.86 \tau_J$. After a few times τ_{reg} the evolution is no longer sensitive to the system-apparatus coupling g. In the initial fully disordered paramagnetic state ($T_0 = \infty$), P(m, 0) is a Gaussian centered at m = 0 with width $1/\sqrt{N}$. In the course of time, the peak of P considerably widens, then narrows and reaches eventually the equilibrium ferromagnetic distribution with positive magnetization $m_{\uparrow} = 0.90$, which is given by (3.56). The repulsive fixed point lies at $-m_{\rm B}$ with $m_{\rm B} = 0.14$ and no weight is found below this. The second registration time, at which $\mu(t)$ reaches 0.80, is $\tau'_{\rm reg} = 3\tau_{\rm reg}$. It is seen that beyond this, the peak at m_{\uparrow} quickly builds up.

$$\mu(t) \approx \frac{g}{T} (1 - e^{-t/\tau_1}), \qquad \tau_1 = \frac{\hbar}{\gamma T}.$$
(7.50)

Like for q = 2, the peak shifts first as $\mu \sim \gamma gt/\hbar$, but here its motion slows down as t increases, instead of escaping 2698 more and more rapidly off the paramagnetic region, as exhibited on the example of Figs. 7.4 and 7.6. Extrapolation 2699 of (7.50) towards times larger than τ_1 is not possible, since μ would then not go beyond g/T, and could not reach $m_{\rm F}$. 2700 In fact, v(m) does not vanish at m = g/T as implied by the above approximation but only decreases down to a positive 270 minimum near $m_c \simeq 3h_c/2T$ according to (7.34). The vicinity of m_c is thus a *bottleneck* for the motion from $\mu = 0$ to 2702 $\mu = 1$ of the peak of $P_{\uparrow\uparrow}(m, t)$: This motion is the slowest around m_c . The determination of the evolution of $P_{\uparrow\uparrow}(m, t)$, 2703 embedded in $\mu(t)$ and D(t), and the evaluation of the registration time thus require a control of the shape of v(m), not 2704 only near its zeroes, but also near its minimum (Fig. 7.2). 2705

Let us recall the parameters which characterize v(m). For g = 0, it has 5 zeroes. Three of them correspond to the attractive fixed points $\pm m_{\rm F} \simeq \pm 1$ and 0 associated with the ferromagnetic and paramagnetic states. The other two are repulsive, producing a bifurcation in the flow of P(m, t); they are located at $m \simeq \pm \sqrt{T/J}$, that is, at $m \simeq \pm m_c \sqrt{3}$ according to (7.34). When g increases and becomes larger than h_c , there remain the two ferromagnetic points, while the repulsive point $-m_c \sqrt{3}$ is shifted towards $-m_{\rm B} \simeq -2m_c$. The paramagnetic point and the repulsive point $m_c \sqrt{3}$ converge towards each other, giving rise to the minimum of v(m) near $m = m_c$. The value of v(m) at this minimum is expressed by

$$\frac{\hbar}{\gamma T} v(m_{\rm c}) \simeq \frac{\delta m_{\rm c}^2}{m_{\rm c}}, \qquad \delta m_{\rm c} \simeq \sqrt{\frac{(g-h_{\rm c})m_{\rm c}}{T}}, \qquad h_{\rm c} \simeq \frac{2}{3}Tm_{\rm c}, \qquad m_{\rm c} = \sqrt{\frac{T}{3J}}.$$
(7.51)

We construct in Appendix E.2, for $\delta m_c \ll m_c$ and m_c small, a parametrization of v(m) which reproduces all these features, so as to derive an algebraic approximation (E.12) which expresses the time dependence of $\mu(t)$ over all times. After the initial evolution (7.50) of $\mu(t)$ for $t \ll \tau_1 = \hbar/\gamma T$, the motion of the peak $P_{\uparrow\uparrow}(m, t)$ is characterized by a much larger time scale. We define the *registration time* as

$$\tau_{\rm reg} = \frac{\pi\hbar}{\gamma T} \sqrt{\frac{m_{\rm c}T}{g - h_{\rm c}}}.$$
(7.52)

The bottleneck stage takes place around $\frac{1}{2}\tau_{reg}$. Between the times $t = \frac{1}{4}\tau_{reg}$ and $t = \frac{3}{4}\tau_{reg}$, the average magnetization $\mu(t)$ lingers in the narrow range $m_c \pm \delta m_c$, according to (Fig. 7.4)

$$\mu(t) = m_{\rm c} - \delta m_{\rm c} \cot n \frac{\pi t}{\tau_{\rm reg}}.$$
(7.53)

It is shown in Appendix E.2 that, under the considered conditions on the parameters, $\mu(t)$ rises thereafter rapidly according to (E.15), and that the full time taken by the peak $\mu(t)$ of $P_{\uparrow\uparrow}(m, t)$ to go from 0 to the close vicinity of 1 is τ_{reg} (Eq. (7.52)). It is also shown in Appendix E.2 that the final relaxation takes place on the short time scale $\hbar/\gamma J$.

We have focused on the location of the peak of $P_{\uparrow\uparrow}(m, t)$. The consideration of its width D(t) is essential to determine when S and A may be decoupled. During the bottleneck stage, the sole drift effect would produce a narrowing of D(t) around $t = \frac{1}{2}\tau_{reg}$ expressed by the first term of (7.31), but the smallness of v(m) enhances the second term, so that the diffusion acts during a long time and produces a large widening of D(t). By using the parabolic approximation for v(m), which is represented by the first term of (E.11), and by replacing w(m) by $\gamma T/\hbar$, we obtain, with $\mu(t)$ expressed by (7.53),

$$D(\mu) \sim 2m_{\rm c} \left[(\mu - m_{\rm c})^2 + \delta m_{\rm c}^2 \right] \int_0^{\mu} \frac{\mathrm{d}\mu'}{[(\mu' - m_{\rm c})^2 + \delta m_{\rm c}^2]^3}.$$
 (7.54)

After the bottleneck has been passed, the diffusion may again be neglected. From (7.31) and (7.54), we find for all values of $\mu(t)$ such that $\mu - m_c \gg \delta m_c$

$$D(\mu) \sim \frac{3\pi\hbar^2 m_{\rm c}^3}{4\gamma^2 T^2 \delta m_{\rm c}^5} v^2(\mu) = \frac{3\pi \sqrt{Tm_{\rm c}}}{4(g-h_{\rm c})^{5/2}} (J\mu^3 + g)^2 [1 - \mu \coth\beta (J\mu^3 + g)]^2,$$
(7.55)

where we used (7.6), (7.51) and (4.24). Without any diffusion, the coefficient of $v^2(\mu)$ would have been $1/v^2(0) = \frac{9\hbar^2}{4\gamma^2 T^2 m_c^2}$; both factors $v(\mu)$ are multiplied by the large factor $\sqrt{\pi/3}(m_c/\delta m_c)^{5/2}$ due to diffusion.

The distribution $P_{\uparrow\uparrow}(m, t)$ thus extends, at times larger than $\frac{3}{4}\tau_{reg}$, over the region $\mu(t) \pm \sqrt{D(t)/N}$. The *first registration time* has been defined in § 7.2.3 as the time after which S and A can be decoupled without affecting the process. When g is switched off $(g \rightarrow 0)$, a repulsive fixed point appears at the zero $m = m_c \sqrt{3}$ of $\nu(m)$. In order to ensure a proper registration we need this decoupling to take place after the whole distribution $P_{\uparrow\uparrow}(m, t)$ has passed this bifurcation, that is, at a time t_{off} such that

$$\mu(t_{\rm off}) - \sqrt{D(t_{\rm off})/N} > m_{\rm c} \sqrt{3}.$$
(7.56)

The time dependence (E.15) of μ shows that the lower bound of t_{off} is equal to τ_{reg} within a correction of order $\tau_1 \ll \tau_{\text{reg}}$. Moreover, we need the distribution to be sufficiently narrow so that (7.56) is satisfied after g is switched off. Taking for instance $\mu(t_{\text{off}}) = 2m_c$, which according to (E.15) is reached at the time $t_{\text{off}} = \tau_{\text{reg}}(1 - 0.25 \, \delta m_c/m_c)$, we thus find, by inserting (7.55) with $\mu = 2m_c$ and $g \simeq h_c$ into (7.56), by using (7.51) and evaluating the last bracket of (7.55) for $m_c = 0.268$, a *further lower bound* for the coupling g in our first order case q = 4:

$$\frac{g - h_c}{h_c} \gg 8 \left(\frac{J}{NT}\right)^{2/5}.$$
(7.57)

The first registration time, which governs the possibility of decoupling, and the second one, which is the delay after which the pointer variable approaches the equilibrium value, are therefore nearly the same, namely τ_{reg} , contrary to the case q = 2 of a second order transition (§ 7.2.3).

The registration process for q = 4 is illustrated by Figs. 7.4 and 7.6, obtained through numerical integration. The time dependence of $\mu(t)$ as well as the widening of the distribution are influenced by the existence of the minimum for the drift velocity. Although in this example *g* lies above the threshold h_c , *N* is not sufficiently large to fulfil the condition (7.57). The widening is so large that a significant part of the weight P(m, t) remains for a long time below the bifurcation $m_c \sqrt{3}$ which appears when *g* is switched off. The bound (7.57) was evaluated by requiring that such a switching off takes place after the average magnetization μ passes $2m_c = 0.54$. Here however, for N = 1000, T = 0.2J and g = 0.045J, the bound is very stringent, since we cannot switch off g before μ has reached (at the time $1.09\tau_{reg}$ found from (E.15)) the value $1 - 13 \cdot 10^{-5}$, close to the equilibrium value $m_{\rm F} = 1 - 9 \cdot 10^{-5}$.

Altogether, for q = 2 as well as for q = 4, we can check that the approximate algebraic treatment of §§ 7.2.3 and 7.2.4 fits the numerical solution of Eq. (7.1) exemplified by the figures 7.3 to 7.6. In both cases, the registration times (7.44) and (7.48) for q = 2 or (7.52) for q = 4, which characterize the evolution of the diagonal blocks of the density matrix of the total system \hat{D} , are much longer than the truncation time (5.6) over which the off-diagonal blocks decay. Two reasons conspire to ensure this large ratio: the weakness of the coupling γ between magnet and bath, which makes τ_{reg} large; and the large value of *N*, which makes τ_{trunc} small.



Figure 7.6: The registration process for quartic interactions (q = 4). The probability density $P(m, t) = P_{\uparrow\uparrow}(m, t)/r_{\uparrow\uparrow}(0)$ as function of *m* is represented at different times up to $t = 1.5 \tau_{reg}$. The parameters are chosen as N = 1000, T = 0.2J and g = 0.045J as in Fig 7.4. The time scale is here the registration time $\tau_{reg} = 38\tau_J = 38\hbar/\gamma J$, which is large due to the existence of a bottleneck around $m_c = 0.268$. The coupling *g* exceeds the critical value $h_c = 0.0357J$ needed for proper registration, but since $(g - h_c)/h_c$ is small, the drift velocity has a low positive minimum at 0.270 near m_c (Fig. 7.2). Around this minimum, reached at the time $\frac{1}{2}\tau_{reg}$, the peak shifts slowly and widens much. Then, the motion fastens and the peak narrows rapidly, coming close to ferromagnetism around the time τ_{reg} , after which equilibrium is exponentially reached.

2759 7.3. Giant fluctuations of the magnetization

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We have studied in subsection 7.2 the evolution of the probability distribution $P(m,t) = P_{\uparrow\uparrow}(m,t)/r_{\uparrow\uparrow}(0)$ of the 2762 magnetization of M in case this distribution presents a single peak (7.10) at all times. This occurs when P(m, t) always 2763 remains entirely located, except for negligible tails, on a single side of the bifurcation $-m_{\rm B}$ of the drift flow v(m). We 2764 will now consider the case of an active bifurcation [241, 242, 243, 244, 245]: The initial distribution is split during 2765 the evolution into two parts evolving towards $+m_{\rm F}$ and $-m_{\rm F}$. This situation is relevant to our measurement process for 2766 q = 2 in regard to two questions: (i) How fast should one perform the cooling of the bath before the initial time, and 2767 the switching on of the system-apparatus interaction around the initial time? (ii) What is the percentage of errors of 276 registration if the coupling g is so small that it violates the condition (7.41)? 2769

2770 7.3.1. Dynamics of the invariance breaking

In order to answer the above two questions, we first determine the Green's function for the equation of motion (7.1) which governs P(m, t) for q = 2 in the Markovian regime. This will allow us to deal with an arbitrary initial condition. The *Green's function* G(m, m', t - t') is characterized by the equation

$$\frac{\partial}{\partial t}G(m,m',t-t') + \frac{\partial}{\partial m}[v(m)G(m,m',t-t')] - \frac{1}{N}\frac{\partial^2}{\partial m^2}[w(m)G(m,m',t-t')] = \delta(m-m')\delta(t-t'),$$
(7.58)

88

with G(m, m', t-t') = 0 for t < t'. We have replaced the initial time 0 by a running time t' in order to take advantage of the convolution property of *G*. The functions v(m) and w(m) defined by (7.6) and (7.7) involve a field *h* which stands either for an applied external field if A = M + B evolves alone (a case that could appear but which we do not consider), or for $\pm g$ if we consider $P_{\uparrow\uparrow}$ or $P_{\downarrow\downarrow}$ if A is coupled to S during the measurement. We wish to face the situation in which P(m, t) lies, at least after some time, astride the bifurcation point $-m_{\rm B} = -h/(J - T)$. Such a situation has extensively been studied [241, 242, 243, 244, 245], and we adapt the existing methods to the present problem which is similar to Suzuki's model.

We first note that the initial distribution P(m, t' = 0) is concentrated near the origin, a property thus satisfied by the variable m' in G(m, m', t). In this region, it is legitimate to simplify v(m') and w(m') into

$$v(m') \approx \frac{\gamma}{\hbar} [h + (J - T)m'], \qquad w(m') \approx \frac{\gamma T}{\hbar},$$
(7.59)

where we also used $h \ll T$. In order to implement this simplification which holds only for $m' \ll 1$, we replace the forward equation (7.58) in terms of *t* which characterizes G(m, m', t - t') by the equivalent *backward equation*, for $\partial G(m, m', t - t')/\partial t'$, in terms of the initial time *t'* which runs down from *t* to 0. This equation is written and solved in Appendix F. The distribution P(m, t) is then given by

$$P(m,t) = \int dm' G(m,m',t) P(m',0).$$
(7.60)

²⁷⁸⁷ We derive below several approximations for P(m, t), which are valid in limiting cases. These various results are ²⁷⁸⁸ encompassed by the general expression (F.13)–(F.15) for P(m, t), obtained through the less elementary approach of ²⁷⁸⁹ Appendix F.

As in § 7.2.3, the evolution takes place in three stages [241, 242, 243, 244, 245]: (i) *widening* of the initial distribution, which here takes place over the bifurcation $-m_{\rm B}$; (ii) *drift* on both sides of $-m_{\rm B}$ towards $+m_{\rm F}$ and $-m_{\rm F}$; (iii) narrowing around $+m_{\rm F}$ and $-m_{\rm F}$ of the two final peaks, which evolve *separately towards equilibrium*. We shall not need to consider here the last stage, the approach to quasi-equilibrium, that we studied in § 7.1.4.

The probability distribution P(m, t) is thus expressed in terms of the initial distribution P(m, 0) by (7.60), at all times, except during the final equilibration. If P(m, 0) is a narrow Gaussian peak centered at $m = \mu_0$ with a width δ_0 / \sqrt{N} , we can use the expression (F.10) of *G*, which yields

$$P(m,t) = \frac{v(\mu')}{v(m)} \sqrt{\frac{N}{2\pi}} \frac{1}{\delta_1(t)} \exp\left[-\frac{N}{2} \frac{(\mu'-\mu_0)^2}{\delta_1^2(t)}\right].$$
(7.61)

²⁷⁹⁷ The function $\mu'(m, t)$ is defined for arbitrary values of *m* by

$$t = \int_{\mu'(m,t)}^{m} \frac{\mathrm{d}m''}{v(m'')},\tag{7.62}$$

²⁷⁹⁸ while the variance that enters (7.61) is determined by

$$\delta_1^2(t) \equiv \delta_0^2 + \frac{T}{J - T} (1 - e^{-2t/\tau_{\text{reg}}}) \equiv \delta_1^2 - \frac{T}{J - T} e^{-2t/\tau_{\text{reg}}}, \qquad \delta_1^2 \equiv \frac{T_0}{T_0 - J} + \frac{T}{J - T}.$$
(7.63)

²⁷⁹⁹ With time, it increases from δ_0^2/N to δ_1^2/N .

2800 7.3.2. Spontaneous relaxation of the initial paramagnetic state

The registration process that we studied in § 7.2.3 is the same as the relaxation, for q = 2 and T < J, of the initial paramagnetic state (3.48) towards the positive ferromagnetic state $+m_{\rm F}$ in the presence of a sufficiently large positive external field *h*. We now consider the situation in which A evolves *in the absence of a field*. The process will describe the *dynamics of the spontaneous symmetry breaking*, which leads from the unstable symmetric paramagnetic distribution $P_{\rm M}(m, 0)$ to the ferromagnetic distribution (3.56) for $+m_{\rm F}$ and $-m_{\rm F}$, occurring with equal probabilities.

We present below an approximate analytic solution, and illustrate it by Fig. 7.7 which relies on a numerical solution.



Figure 7.7: Relaxation of an unstable paramagnetic state (q = 2) in the absence of a field (g = 0). The probability distribution $P_{\rm M}(m, t)$ is represented at several times. As in Figs. 7.3 and 7.5 the parameters are N = 1000 and T = 0.65J. First the Gaussian paramagnetic peak around m = 0 with width $1/\sqrt{N}$ widens considerably. Around $t = \tau_{\rm flat} = 2.2\tau_{\rm reg}$, the distribution extends over most of the interval $-m_{\rm F}$, $+m_{\rm F}$ ($m_{\rm F} = 0.872$) and is nearly flat. Then, two peaks progressively build up, moving towards $-m_{\rm F}$ and $+m_{\rm F}$. Finally each peak tends to the Gaussian ferromagnetic equilibrium shape, the curves at $t = 10\tau_{\rm reg}$ basically coincide.

Apart from the final stage, the result is given by (7.61) with $\mu_0 = 0$, $\delta_0^2 = T_0/(T_0 - J)$, and $v(m) = (\gamma/\hbar)Jm[1 - m \coth(Jm/T)]$. During the first stage, we have $v(m) \sim m/\tau_{\text{reg}}$ and hence $\mu' \sim me^{-t/\tau_{\text{reg}}}$, so that (7.61) reduces to

$$P_{\rm M}(m,t) = \sqrt{\frac{N}{2\pi}} \frac{e^{-t/\tau_{\rm reg}}}{\delta_1(t)} \exp\left[-\frac{Nm^2 e^{-2t/\tau_{\rm reg}}}{2\delta_1^2(t)}\right].$$
 (7.64)

On the time scale $\tau_{reg} = \hbar/\gamma (J - T)$, this distribution widens exponentially, with the variance

$$\frac{1}{N} \left[\delta_1^2 e^{2t/\tau_{\rm reg}} - \frac{T}{J-T} \right], \qquad \delta_1^2 \equiv \frac{T_0}{T_0 - J} + \frac{T}{J-T}.$$
(7.65)

As in § 7.2.3, the widening is first induced by the diffusion term, which is then relayed by the gradient of the drift velocity v(m). However, the effect is much stronger here because the distribution remains centered around m = 0.

In fact, at times of order $\tau_{reg} \ln \sqrt{N}$, the width of the peak of $P_M(m, t)$ is no longer of order $1/\sqrt{N}$, but it is *finite* for large N. If we define the lifetime τ_{para} of the paramagnetic state as the delay during which this width is less that α , say $\alpha = 1/10$, it equals (for $\alpha \sqrt{N} \gg 1$)

$$\tau_{\text{para}} = \tau_{\text{reg}} \ln \alpha \sqrt{N} = \frac{\hbar}{\gamma (J - T)} \ln \alpha \sqrt{N}.$$
(7.66)

The second stage of the evolution is then reached (Fig. 7.7). An analytic expression of $P_{\rm M}(m, t)$ can then be found by using the Mittag-Leffler approximation (E.1) for v(m) (with h = 0). The relation between μ' , *m* and *t* becomes

$$\frac{t}{\tau_{\rm reg}} = \ln \frac{m}{\mu'} + a \ln \frac{m_{\rm F}^2 - {\mu'}^2}{m_{\rm F}^2 - m^2},\tag{7.67}$$

where the coefficient *a*, defined by (7.47), lies between $\frac{1}{2}$ for $J - T \ll J$ and 1 for $T \ll J$. In this stage, when $t \gg \tau_{\text{reg}}$, the distribution

$$P_{\rm M}(m,t) = \frac{\mu'(m_{\rm F}^2 - \mu'^2)}{m(m_{\rm F}^2 - m^2)} \frac{m_{\rm F}^2 + (2a-1)m^2}{m_{\rm F}^2 + (2a-1)\mu'^2} \sqrt{\frac{N}{2\pi\delta_1^2}} \exp\left(-\frac{N\mu'^2}{2\delta_1^2}\right),\tag{7.68}$$

depends on time only through μ' . It flattens while widening. In particular, around the time

$$\tau_{\rm flat} = \tau_{\rm reg} \ln \left(\frac{m_{\rm F}}{\delta_1} \sqrt{\frac{N}{6a}} \right), \tag{7.69}$$

2820 it behaves for small *m* as

$$P_{\rm M}(m,t) \approx \frac{1}{m_{\rm F}} \sqrt{\frac{3}{\pi}} e^{-(t-\tau_{\rm flat})/\tau_{\rm reg}} \left\{ 1 + \frac{3am^2}{m_{\rm F}^2} \left[1 - e^{-2(t-\tau_{\rm flat})/\tau_{\rm reg}} \right] + O\left(\frac{m^4}{m_{\rm F}^4}\right) \right\}.$$
(7.70)

When *t* reaches τ_{flat} , the distribution $P_{\text{M}}(m, \tau_{\text{flat}})$ has widened so much that it has become *nearly flat*: The probabilities of the possible values (3.22) of *m* are nearly the same on a range which extends over most of the interval $-m_{\text{F}}, +m_{\text{F}}$. This property agrees with the value of $\frac{1}{2}NP_{\text{M}}(0, \tau_{\text{flat}}) = 0.98/m_{\text{F}}$; the coefficient of the term in $(m/m_{\text{F}})^4$, equal to $-a(8a - \frac{5}{2})$, yields a correction $-(0.93 m/m_{\text{F}})^4$ for small $m_{\text{F}}, -(1.53 m/m_{\text{F}})^4$ for large m_{F} .

When *t* increases beyond τ_{flat} , the distribution begins to deplete near m = 0 and two originally not pronounced maxima appear there (Fig. 7.7), which move apart as

$$m = \pm m_{\rm F} \sqrt{\frac{6(t - \tau_{\rm flat})}{(16a - 5)\tau_{\rm reg}}}.$$
 (7.71)

They then become sharper and sharper as they move towards $\pm m_{\rm F}$. When they get well separated, $P_{\rm M}(m, t)$ is concentrated in two symmetric regions, below $m_{\rm F}$ and above $-m_{\rm F}$, and it reaches a *scaling regime* [241, 242, 243, 244, 245] in which (for m > 0)

$$\mu'(m,t) \sim m_{\rm F} e^{-t/\tau_{\rm reg}} \left[\frac{m_{\rm F}}{2(m_{\rm F}-m)} \right]^2 \tag{7.72}$$

is small, of order $1/\sqrt{N}$. If we define, with $a(\frac{1}{2} < a < 1)$ given by Eq. (7.47),

$$\xi(m,t) \equiv \sqrt{\frac{N}{2}} \frac{\mu'(m,t)}{\delta_1} = \sqrt{3a} \left[\frac{m_{\rm F}}{2(m_{\rm F}-m)} \right]^a e^{-(t-\tau_{\rm flat})/\tau_{\rm reg}},\tag{7.73}$$

²⁸³¹ $P_{\rm M}(m,t)$ takes in the region $m > 0, \xi > 0$, the form

$$P_{\rm M}(m,t) \approx \frac{1}{\sqrt{\pi}} \frac{\partial \xi}{\partial m} e^{-\xi^2}.$$
(7.74)

²⁸³² Its maximum lies at the point m_{max} given by

$$\xi(m,t) = \sqrt{\frac{a+1}{2a}}, \qquad \frac{m_{\rm F} - m_{\rm max}}{m_{\rm F}} = \frac{1}{2} \left(\frac{6a^2}{a+1}\right)^{1/(2a)} e^{-(t-\tau_{\rm flat})/a\tau_{\rm reg}},\tag{7.75}$$

which approaches $m_{\rm F}$ exponentially, and its shape is strongly asymmetric. In particular, its tail above $m_{\rm max}$ is short, whereas its tail below $m_{\rm max}$ extends far as $1/(m_{\rm max} - m)^{a+1}$; only moments $\langle (m_{\rm F} - m)^k \rangle$ with k < a exist.

After a delay of order $a\tau_{reg} \ln \sqrt{N}$, the width of the peaks of $P_M(m, t)$ and their distance to $\pm m_F$ reach an order of magnitude $1/\sqrt{N}$. The diffusion term becomes active, and each peak tends to the Gaussian shape (3.56) as in § 7.1.4. This crossover could be expressed explicitly by writing the Green's function for *m* and *m'* near m_F (as we did near 0 in § 7.3.1) and by taking (7.74) as initial condition. All the above features fit the numerical solution shown by Fig. 7.7.

In our measurement problem, q = 2, the above evolution begins to take place at the time $-\tau_{\text{init}}$ at which the apparatus is initialized (§ 3.3.3). Before $t = -\tau_{\text{init}}$, paramagnetic equilibrium has been reached at the temperature $T_0 > J$, and the initial distribution of *m* is given by (3.48), (3.49) (3.51). The sudden cooling of the bath down to the temperature T < J lets the evolution (7.64) start at the time $-\tau_{\text{init}}$. We wish that, at the time t = 0 when the coupling g is switched on and the measurement begins, the distribution $P_{\text{M}}(m, 0)$ is still narrow, close to (3.48). We thus need $\delta_1(\tau_{\text{init}})$ to be of the order of δ_0 , that is,

$$\frac{2\tau_{\text{init}}}{\tau_{\text{reg}}} < \delta_0^2 \frac{J-T}{T} = \frac{T_0}{T_0 - J} \frac{J-T}{T}.$$
(7.76)

The bath should be cooled down and the system-apparatus interaction \hat{H}_{SA} should be switched on *over a delay* τ_{init} *not larger than the registration time* $\tau_{reg} = \hbar/\gamma (J - T)$. The situation is more favourable in case the initial depolarized state of the spins of M is generated by a radiofrequency field rather than through equilibration with the phonon bath at a high but finite temperature T_0 . In this case, a sudden cooling of the bath at the time $-\tau_{init}$ is not needed. The bath can beforehand be cooled at the required temperature T lower than $T_c = J$. At the time $-\tau_{init}$, the spins are suddenly set by the field into their most disordered state, a process which hardly affects the bath since $\gamma \ll 1$. The above discussion then holds as if T_0 were infinite.

If a weak field h_0 is accidentally present during the preparation by thermalization of the initial paramagnetic state, it should not produce a bias in the measurement. This field shifts the initial expectation value $\langle m \rangle$ of m from 0 to $\mu_0 = h_0/(T_0 - J)$, which enters (7.61). At the time 0, $\langle m \rangle$ has become $\mu_0 \exp(\tau_{\text{init}}/\tau_{\text{reg}})$, so that the residual field h_0 is ineffective provided $\mu_0 < \delta_0$, that is for

$$h_0 < \sqrt{\frac{T_0(T_0 - J)}{N}}.$$
(7.77)

The success of the measurement process thus requires the conditions (7.76) and (7.77) on the parameters τ_{init} , T_0 , h_0 that characterize the preparation of the initial state of the apparatus.

For a quartic interaction (q = 4), the initial paramagnetic state is metastable rather than unstable. Its spontaneous decay in the absence of a field requires *m* to cross the potential barrier of the free energy which ensures metastability, as shown by Fig. 3.3. At temperatures *T* below the transition point but not too low, the dynamics is governed by an activation process, with a characteristic duration of order $(\hbar/\gamma J) \exp(\Delta F/T)$, where ΔF is the height of the barrier, for instance $\Delta F = 0.054NT$ for T = 0.2J. The lifetime of the paramagnetic state is thus exponentially larger than the registration time for large *N*, so that there is no hurry in performing the measurement after preparation of the initial state.

2865 7.3.3. Probability of wrong registrations for second order phase transitions of the magnet

We have seen (§ 7.2.3) how the magnet M, under the conjugate effect of B and S, reaches quasi certainly the final magnetization $+m_F$ in the sector $\uparrow\uparrow$ where $s_z = +1$, provided g is not too small. We expect that if the condition (7.41) on g is violated, the apparatus will indicate, with some probability \mathcal{P}_- , the wrong magnetization $-m_F$, although $s_z = +1$. The evolution of $P_{\uparrow\uparrow}(m, t)$ in such a situation is illustrated by Fig. 7.8. A similar failure may occur if the average magnetization μ_0 in the initial state is not 0 but takes a negative value due to a *biased preparation*.



Figure 7.8: Wrong registration for quadratic interactions (q = 2). The probability distribution P(m, t) is represented at different times for the same parameters N = 1000 and T = 0.65 as in Fig. 7.5, but the coupling g = 0.03J is now sufficiently weak so that the apparatus registers the magnetization $-m_{\rm F}$ with a significant probability \mathcal{P}_{-} , although the system has a spin $s_z = +1$. Like in Fig. 7.7, the probability distribution flattens before the two ferromagnetic peaks emerge (with weights \mathcal{P}_{+} and \mathcal{P}_{-}).

The probability \mathcal{P}_{-} of a wrong registration $-m_{\rm F}$ for $s_z = +1$ arises from values $m < m_{\rm B} < 0$ and reads

$$\mathcal{P}_{-} = \int_{-1}^{m_{\rm B}} \mathrm{d}m \, \frac{P_{\uparrow\uparrow}(m,t)}{r_{\uparrow\uparrow}(0)} \equiv \int_{-1}^{m_{\rm B}} \mathrm{d}m \, P(m,t) \,, \tag{7.78}$$

where the time *t* is in principle such that $P_{\uparrow\uparrow}(m, t)$ has reached its equilibrium shape, with two peaks around $+m_{\rm F}$ and $-m_{\rm F}$. In fact, we do not need the final equilibrium to have been reached since (7.78) remains constant after $P_{\uparrow\uparrow}$ has split into two separate parts. And even the latter condition is not necessary: After the time $\tau_{\rm reg}$ the diffusion term becomes inactive and the evolution of $P_{\uparrow\uparrow}(m, t)$ is governed by the pure drift Green's function (F.7); then there is no longer any transfer of weight across the bifurcation $-m_{\rm B} = -g/(J - T)$. We can therefore evaluate (7.78) at the rather early stage when the distribution has not yet spread out beyond the small *m* region where (7.59) holds, provided we take $t \gg \tau_{\rm reg}$.

We thus use the expression (7.61) of $P_{\uparrow\uparrow}(m, t)$ valid during the first stage of the process, which reads

$$P(m,t) = e^{-t/\tau_{\rm reg}} \sqrt{\frac{N}{2\pi}} \frac{1}{\delta_1(t)} \exp\left\{-\frac{N}{2\delta_1^2(t)} \left[(m+m_{\rm B})e^{-t/\tau_{\rm reg}} - m_{\rm B} - \mu_0\right]^2\right\}.$$
(7.79)

By taking $(m + m_B)e^{-t/\tau_{reg}}$ as variable we check that the integral (7.78) depends on time only through the exponential in (7.63), so that it remains constant as soon as $t \gg \tau_{reg}$, when the second stage of the evolution is reached. We eventually find:

$$\mathcal{P}_{-} = \frac{1}{2} \operatorname{erfc} \lambda, \qquad \lambda \equiv \sqrt{\frac{N}{2}} \frac{1}{\delta_1} (m_{\rm B} + \mu_0),$$
(7.80)

²⁸⁸³ where the error function, defined by

$$\operatorname{erfc} \lambda = \frac{2}{\sqrt{\pi}} \int_{\lambda}^{\infty} \mathrm{d}\xi e^{-\xi^2},\tag{7.81}$$

²⁸⁸⁴ behaves for $\lambda \gg 1$ as

$$\operatorname{erfc} \lambda \sim \frac{1}{\sqrt{\pi}\lambda} e^{-\lambda^2}.$$
 (7.82)

The diffusion which takes place during the first stage of the evolution has changed in (7.80) the initial width δ_0 into δ_1 , given by (7.40).

For $\mu_0 = 0$, the probability of error becomes sizeable when \sqrt{Ng}/J is not sufficiently large. For example, for T = 0.65J and g = 0.03J, we find numerically $\mathcal{P}_{-} = 21\%$, 13%, 5.4%, 1.15% and 0.065% for N = 250, 500, 1000, 2000 and 4000, respectively. These data are reasonably fitted by the approximation $\mathcal{P}_{-}(N) = 1.2 N^{-1/4} \exp(-0.0014N)$ for (7.80). The result for N = 1000 is illustrated by the weight of the peak near $-m_{\rm F}$ in Fig. 7.8. False registrations were also present with the data of Fig. 7.5 (N = 1000, T = 0.65J, g = 0.05J), with a probability $\mathcal{P}_{-} = 0.36\%$, but the effect is too small to be visible on the scale of the figure.

The occurrence of a negative μ_0 increases \mathcal{P}_- , an effect which, with the above data, becomes sizeable for $|\mu_0| \sim 0.05$. For $P_{\downarrow\downarrow}$ the percentage of errors is given by (7.80) with μ_0 changed into $-\mu_0$ in λ .

We write for completeness in Appendix F the evolution of the shape of P(m, t). This is not crucial for the measurement problem (for which $P_{\uparrow\uparrow}(m, t) = r_{\uparrow\uparrow}(0) P(m, t)$), but it is relevant for the dynamics of the phase transition, depending on the initial conditions and on the presence of a parasite field. Here again, Suzuki's regime [241, 242, 243, 244, 245], where the distribution is no longer peaked, is reached for $t \gg \tau_{reg}$. Now P(m, t) is asymmetric, but it still has a quasi linear behavior in a wide range around m = 0 when $\tau \simeq \tau_{flat}$ (see Eqs. (7.69), (7.70)).

2901 7.3.4. Possible failure of registration for first order transitions

Quem não tem cão, caça com gato ⁵⁹ Portuguese proverb

The situation is quite different for first-order transitions (q = 4) as regards the possibility of wrong registrations. Note first that F(m) has a high maximum for negative m between 0 and $m_{\downarrow} < 0$ (Figs. 3.3 and 3.4), which constitutes a practically impassable barrier that diffusion is not sufficient to overcome. Accordingly, the zero of v(m) at $m = -m_B \approx$ $-2m_c$ is a repulsive fixed point (Fig.7.2 and § 7.2.4), which prevents the distribution from developing a tail below it. We shall therefore never find any registration with negative ferromagnetic magnetization in the sector $s_z = +1$.

²⁹⁰² 2903

⁵⁹If you don't have a dog, hunt with a cat



Figure 7.9: Failure of measurement for quartic interactions (q = 4). The probability distribution P(m, t) is represented at times up to $10 \tau_J$, where $\tau_J = \hbar/\gamma J$. The parameters are N = 1000 and T = 0.2 as in Figs. 7.4 and 7.6, but here g = 0.01J lies below the threshold h_c . The peak evolves towards metastable paramagnetic equilibrium in the presence of the field g, but g is too small to allow crossing the barrier and reaching the more stable ferromagnetic equilibrium around $m_F \simeq 1$. Switching off the coupling g brings back the distribution to its original place around 0, so that no proper registration is achieved.

Nevertheless, we have seen (§ 7.2.2) that registration is possible only if the coupling g exceeds h_c . For $g < h_c$, the peak of P(m) initially at m = 0 moves upwards in the sector $\uparrow\uparrow$ associated with $s_z = +1$ (Fig. 7.9), and ends up by stabilizing at the first attractive point encountered, at $m = m_P$ (Fig. 7.2). Symmetrically, the distribution of $P_{\downarrow\downarrow}(m)$ ends up at $-m_P$. However this difference between the two values of s_z cannot be regarded as a registration since switching off the coupling g between S and A brings back both distributions $P_{\uparrow\uparrow}$ and $P_{\downarrow\downarrow}$ to the initial Gaussian shape around m = 0. The apparatus A then always relaxes back to the locally stable paramagnetic state.

Finally, if the coupling g, although larger than the threshold h_c , is close to it, the registration takes place correctly provided this coupling remains active until the distribution $P_{\uparrow\uparrow}(m)$ has completely passed the bifurcation $m_c \sqrt{3}$ occurring for g = 0 (Eq. (7.56)). The lower bound t_{off} of the time when g can thus be safely switched off is close to τ_{reg} (which is also close to the time needed to reach ferromagnetic equilibrium).

In case S and A are decoupled too early, so that the condition (7.56) is violated, the tail of $P_{\uparrow\uparrow}(m, t)$ lying below the bifurcation $m = m_c \sqrt{3}$ is pushed back towards the paramagnetic region $m \approx 0$. If the decoupling $g \to 0$ is made suddenly at the time t_{off} , the probability \mathcal{P}_0 of such events can be evaluated as in § 7.3.3 in terms of the error function by integration of $P_{\uparrow\uparrow}(m, t_{\text{off}})$ from m = -1 up to m_c . It represents the *probability of aborted measurement processes*, for which the apparatus returns to its neutral paramagnetic state without giving any indication, while S is left in the state $|\uparrow\rangle$. In a set of repeated measurements, a proportion \mathcal{P}_0 of runs are not registered at all, the other ones being registered correctly.

2926 7.3.5. Erasure of the pointer indication

As shown in §§ 7.2.3 and 7.2.4, the registration is achieved at a time t_f sufficiently larger than the delay τ_{reg} after 2927 which S and M have been decoupled. The state $\hat{D}(t_f)$ of S + A is then given by the expected expression (1.7). Within 2928 the considered approximations, the distributions $P_{\uparrow\uparrow}(m, t)$ and $P_{\downarrow\downarrow}(m, t)$ no longer evolve for $t > t_f$, and remain fully 2929 concentrated near $m_{\rm F}$ and $-m_{\rm F}$, respectively, so that the results can be read out or processed at any observation time 2930 $t_{\rm obs} > t_{\rm f}$. However, the breaking of invariance, on which we rely to assert that the two ferromagnetic states of the 2931 pointer are stationary, is rigorous only in the large N limit. Strictly speaking, for finite N, the states $\hat{R}_{M\parallel}$ and $\hat{R}_{M\parallel}$ 2932 reached by M at this stage in each sector are not in equilibrium (though they may have a long lifetime). Indeed, 2933 in the Markovian regime, we have shown in § 7.1.3 that the evolution of M under the influence of the thermal bath 293 cannot stop until $P_{\rm M}(m,t)$ becomes proportional to $G(m) \exp[-\beta E(m)]$, with $E(m) = -JNq^{-1}m^q$. Otherwise, the time-2935 derivative (7.21) of the free energy F(m) of the state $\hat{R}_{\rm M}(t)$ cannot vanish. The limit reached by $\hat{R}_{\uparrow\uparrow}(t)/r_{\uparrow\uparrow}(0)$ (and of 2936 $\hat{R}_{\perp\downarrow}(t)/r_{\perp\downarrow}(0)$ is then $\frac{1}{2}(\hat{R}_{M\uparrow}+\hat{R}_{M\downarrow})$. Hence, when the latter true equilibrium state for finite N is attained, the indication 2937 of the pointer is completely random. We have lost all information about the initial state of S, and the spin S has been 2938 completely depolarized whatever its initial state: the result of the measurement has been washed out. We denote as 2939

 $\tau_{\rm eras}$ the characteristic time which governs this erasure of the indication of the pointer.

It is therefore essential to read or process the registered data before such a loss of memory begins to occur⁶⁰. The observation must take place at a time t_{obs} much shorter than the erasure time:

$$\tau_{\rm reg} < t_{\rm f} < t_{\rm obs} \ll \tau_{\rm eras}. \tag{7.83}$$

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The dynamics of the erasure, a process leading M from $\hat{R}_{M\uparrow}$ or $\hat{R}_{M\downarrow}$ to the state $\frac{1}{2}(\hat{R}_{M\uparrow} + \hat{R}_{M\downarrow})$ of complete 2943 equilibrium, is governed by the Eq. (4.16) for $P_{M}(m,t)$ (with $\tilde{K}_{t}(\omega)$ replaced by $\tilde{K}(\omega)$ and g = 0), which retains the 294 quantum character of the apparatus. We will rely on this equation in subsection 8.1 where studying the Curie-Weiss 2945 model in the extreme case of N = 2. For the larger values of N and the temperatures considered here, we can use its 2946 continuous semi-classical limit (7.1), to be solved for an initial condition expressed by (3.54) with $m_i = m_F$ or $-m_F$. 2947 Here we have to deal with the progressive, very slow leakage of the distribution $P_{M}(m, t)$ from one of the ferromagnetic 29 states to the other through the free energy barrier that separates them. This mechanism, disregarded in §§ 7.2.3, 7.2.4, 294 7.3.3 and 7.3.4, is controlled by the weak tail of the distribution $P_{M}(m, t)$ which extends into the regions of m where 2950 F(m) is largest. The drift term of (7.1) alone would repel the distribution $P_{\uparrow\uparrow}(m, t)$ and keep it concentrated near $m_{\rm F}$. 2951 An essential role is now played by the diffusion term, which tends to flatten this distribution over the whole range of 2952 m, and thus allows the leak towards $-m_{\rm F}$. Rather than solving this equation, it will be sufficient for our purpose to 2953 rely on a semi-phenomenological argument: Under the considered conditions, the full equilibration is an activation 2954 process governed by the height of the free energy barrier. Denoting as ΔF the difference between the maximum of 2955 F(m) and its minimum, F_{ferro} , we thus estimate the time scale of erasure as: 2956

$$\tau_{\rm eras} \sim \frac{\hbar}{\gamma J} \exp \frac{\Delta F}{T},$$
(7.84)

which is large as an exponential of N. In order to use the process as a measurement, we need this time to be much larger than the registration time so that we are able to satisfy (7.83), which yields

$$\frac{J}{J-T} \ll \exp\frac{\Delta F}{T}, \quad (q=2); \qquad \frac{J}{T}\sqrt{\frac{m_{\rm c}T}{g-h_{\rm c}}} \ll \exp\frac{\Delta F}{T}, \quad (q=4).$$
(7.85)

From (3.54) (taken for h = 0), we find the numerical value of $\Delta F/T$ for the examples of figs. 7.5 and 7.6, namely 0.130N for q = 2, T = 0.065J, and 0.607N for q = 4, T = 0.2J (see fig. 3.3). The condition (7.85) sets again a lower bound on N to allow successful measurements, $N \gg 25$ for the example with quadratic interactions, $N \gg 7$ for the example with quartic interactions. Such a condition is violated for a non-macroscopic apparatus, in particular in the model with N = 2 treated below in subsection 8.1 which will require special care to ensure registration.

2964 7.3.6. "Buridan's ass" effect: hesitation

In the case of a second-order transition (q = 2), the subsections 7.2 and 7.3, illustrated by Figs. 7.5, 7.7 and 7.8, 2965 show off the occurrence, for the evolution of the probability distribution P(m, t), of two contrasted regimes, depending 2966 whether the bifurcation $-m_{\rm B}$ is active or not. The mathematical problem is the same as for many problems of statistical 2967 mechanics involving dynamics of instabilities, such as directed Brownian motion near an unstable fixed point, and it 2968 has been extensively studied [241, 242, 243, 244, 245]. The most remarkable feature is the behavior exemplified by 2969 Fig. 7.7: For a long duration, the random magnetization m hesitates so much between the two stable values $+m_{\rm F}$ and 2970 $-m_{\rm F}$ that a wide range of values of m in the interval $-m_{\rm F}$, $+m_{\rm F}$ have nearly equal probabilities. We have proposed to 2971 term this anomalous situation Buridan's ass effect [222], referring to the celebrated argument attributed to Buridan, 2972 a dialectician of the first half of the XIVth century: An ass placed just half way between two identical bales of hay 2973 would theoretically stay there indefinitely and starve to death, because the absence of causal reason to choose one bale 2974 or the other would let it hesitate for ever, at least according to Buridan⁶¹. 2975

⁶⁰Photographs on film or paper fade out after some time

⁶¹The effect was never observed, though, at the farm where the last author of this work grew up

In fact, major qualitative differences distinguish the situation in which the final state $+m_F$ is reached with probability $\mathcal{P}_+ = 1$ (subsection 7.2) from the situation in which significant probabilities \mathcal{P}_+ and \mathcal{P}_- to reach either $+m_F$ or $-m_F$ exist (subsection 7.3). In the first case, the peak of P(m, t) moves simply from 0 to $+m_F$; the *fluctuation of m remains* of order $1/\sqrt{N}$ at all times, even when it is largest, at the time when the average drift velocity $v(\mu)$ is maximum (Eq. (7.45)). In the second case, the exponential rise of the fluctuations of *m* leads, during a long period, to a broad and flat distribution P(m, t), with a *shape independent of N*.

In both cases, we encountered (for q = 2) the same *time scale* $\tau_{reg} = \hbar/\gamma(J - T)$, which characterizes the first stage of the motion described by either (7.43), (7.39) or (7.79). However, in the first case, $\mathcal{P}_+ \simeq 1$, the duration (7.48) of the whole process is just the product of τ_{reg} by a factor independent of N, of order $2 \ln[m_F(J - T)/g]$, as also shown by (E.6), whereas in the second case, $\mathcal{P}_+ < 1$, the dynamics becomes infinitely slow in the large N limit. The characteristic time τ_{flat} at which the distribution is flat, given by (7.69), is of order $\tau_{reg} \ln[\sqrt{N}m_F(J - T)/J]$. Suzuki's scaling regime [241, 242, 243, 244, 245] is attained over times of order τ_{flat} . Then P(m, t) does not depend on N for $N \to \infty$, but the duration of the relaxation process is *large as* $\ln \sqrt{N}$. It is this long delay which allows the initial distribution, narrow as $1/\sqrt{N}$, to broaden enormously instead of being shifted towards one side.

Buridan's argument has been regarded as a forerunner of the idea of probability. The infinite time during which the ass remains at m = 0 is recovered here for $N \to \infty$. An infinite duration of the process is also found in the absence of diffusion in the limit of a narrow initial distribution ($\delta_0 \to 0$). The flatness of P(m, t) at times of order τ_{flat} means that at such times we cannot predict at all where the ass will be on the interval $-m_F$, $+m_F$, an idea that Buridan could not emit before the elaboration of the concept of probability. The counterpart of the field *h*, for Buridan's ass, would be a strong wind which pushes it; the counterpart of μ_0 would be a different distance from the two bales of hay; in both of these cases, the behavior of the ass becomes predictable within small fluctuations.

Since the slowing factor which distinguishes the time scales in the two regimes is *logarithmic*, very large values of *N* are required to exhibit a large ratio for the relaxation times. In Figs. 7.5, 7.7 and 7.8 we have taken N = 1000 so as to make the fluctuations in $1/\sqrt{N}$ visible. As a consequence, the duration of the registration is hardly larger in Fig. 7.7 than in Fig. 7.5.

Except during the final equilibration, the magnet keeps during its evolution some memory of its initial state through δ_1 (Eq. (7.40)). If the bifurcation is inactive (§ 7.2.3), this quantity occurs through the variance (7.45) of the distribution. If it is active (§ 7.3.2), it occurs through the time scale τ_{flat} , but not through the shape of P(m, t).

Our model of the ferromagnet is well-known for being exactly solvable at equilibrium in the large *N* limit by means of a static mean-field approach. In the single peak regime, the dynamics expressed by (7.30) is also the same as the outcome of a time-dependent mean-field approach. However, in the regime leading to two peaks at $+m_F$ and $-m_F$, *no mean-field approximation can describe the dynamics* even for large *N*, due to the giant fluctuations. The intuitive idea that the variable *m*, because it is macroscopic, should display fluctuations small as $1/\sqrt{N}$ is then wrong, except near the initial time or for each peak of P(m, t) near the final equilibrium.

The giant fluctuations of m which occur in Buridan's ass regime may be regarded as a dynamic counterpart of 3010 the fluctuations that occur at equilibrium at the critical point T = J [226, 227]. In both cases, the order parameter, 301 although macroscopic, presents large fluctuations in the large N limit, so that its treatment requires statistical mechan-3012 ics. Although no temperature can be associated with M during the relaxation process, the transition from $T_0 > J$ to 3013 T < J involves intermediate states which behave as in the critical region. The well-known critical fluctuations and 301 critical slowing down manifest themselves here by the large uncertainty on m displayed during a long delay by P(m, t). 3015 Suzuki's slowing down and flattening [241, 242, 243, 244, 245] take place not only in the symmetric case (§ 7.3.2), 3016 but also in the asymmetric case (§ 7.3.3), provided \mathcal{P}_{-} is sizeable. Thus the occurrence of Buridan's ass effect 3017 is governed by the non vanishing of the probabilities \mathcal{P}_+ and \mathcal{P}_- of $+m_{\rm F}$ and $-m_{\rm F}$ in the final state. Everything 3018 takes place as if the behavior were governed by final causes: The process is deterministic if the target is unique; it 3019 displays large uncertainties and is slow if hesitation may lead to one target or to the other. These features reflect in a 3020 probabilistic language, first, the slowness of the pure drift motion near the bifurcation which implies a long random 3021 delay to set *m* into motion, and, second, the importance of the diffusion term there. 3022

3023 8. Imperfect measurements, failures and multiple measurements

3024Niet al wat blinkt is goud 623025Tout ce qui brille n'est pas or 623026Dutch and french proverbs

In sections 5 to 7, we have solved our model under conditions on the various parameters which ensure that the 302 measurement is ideal. We will resume these conditions in section 9.4. We explore beforehand some situations in 3028 which they may be violated, so as to set forth how each violation prevents the dynamical process from being usable as 3029 a quantum measurement. We have already seen that, in case the spin-apparatus interaction presents no randomness, 3030 the magnet-bath interaction should not be too small; otherwise, recurrence would occur in the off-diagonal blocks 303 of the density operator, and would thus prevent their truncation (§ 5.1). We have also shown how a spin-apparatus 3032 coupling that is too weak may prevent the registration to take place for q = 4 (§ 7.2.2 and § 7.2.4), or may lead to 3033 wrong results for q = 2 (§ 7.3.3). We study below what happens if the number of *degrees of freedom of the pointer* 3034 is small, by letting N = 2 (subsection 8.1); see [141, 143] for model studies along this line. We then examine the 3035 importance of the commutation [4, 13, 225, 246, 247, 248] of the measured observable with the Hamiltonian of the 3036 system (subsection 8.2). Finally we exhibit a process which might allow imperfect simultaneous measurements of 3037 non-commuting observables (subsection 8.3) [249, 250, 251, 252, 253, 254]. 3038

The solution of these extensions of the Curie–Weiss model involves many technicalities that we could not skip. The reader interested only in the results will find them in subsection 9.5.

3041 8.1. Microscopic pointer

3042Ce que je sais le mieux, c'est mon commencement633043Jean Racine, Les Plaideurs

In the above sections, we have relied on the large number N of degrees of freedom of the magnet M. As the statis-3044 tical fluctuations of the magnetization m are then weak, the magnet can behave as a macroscopic pointer with classical 3045 features. Moreover the truncation time τ_{trunc} is the shortest among all the characteristic times (section 5) because it 304 behaves as $1/\sqrt{N}$. The large value of N was also used (section 7) to describe the registration process by means of a 3047 partial differential equation. It is natural to wonder whether a small value of N can preserve the characteristic proper-3048 ties of a quantum measurement. Actually the irreversibility of any measurement process (subsection 6.2) requires the 3049 apparatus to be large. In subsection 6.1, we showed that the irreversibility of the truncation can be ensured by a large 3050 value of N and a randomness in the couplings g_n , $n = 1, \dots, N$ (subsection 6.2); but this irreversibility, as well as that 3051 of the registration (section 7), can also be caused by the large size of the bath. For small N, the irreversibility of both the truncation and the registration should be ensured by the bath. We now study the extreme situation in which N = 2. 305

3054 8.1.1. Need for a low temperature

For N = 2 the magnetization \hat{m} has the eigenvalue m = 0 with multiplicity 2, regarded as "paramagnetic", and two non-degenerate eigenvalues m = +1 and m = -1 regarded as "ferromagnetic". Since $\hat{m}^4 = \hat{m}^2$, we may set $J_4 = 0$ and denote $J_2 = J$. The corresponding eigenenergies of \hat{H}_M are 0 and -J, and those of the Hamiltonian \hat{H}_i of Eq. (4.6) are $-2gs_im - Jm^2$.

The equations of motion of § 4.4.2 involve only the two frequencies ω_{\pm} , defined by

$$\hbar\omega_{\pm} \equiv J \pm 2g,\tag{8.1}$$

and they have the detailed form (notice that $P_{ij} \equiv \frac{1}{2}NP_{ij}^{\text{dis}} = P_{ij}^{\text{dis}}$ for N = 2)

⁶²All that glitters is not gold

⁶³What I know the best I shall begin with

$$\frac{\mathrm{d}P_{\uparrow\uparrow}(0,t)}{\mathrm{d}t} = \frac{2\gamma}{\hbar^2} \left\{ 2P_{\uparrow\uparrow}(1,t)\tilde{K}_t(\omega_+) + 2P_{\uparrow\uparrow}(-1,t)\tilde{K}_t(\omega_-) - P_{\uparrow\uparrow}(0,t) \left[\tilde{K}_t(-\omega_+) + \tilde{K}_t(-\omega_-) \right] \right\},\tag{8.2}$$

$$\frac{P_{\uparrow\uparrow}(\pm 1, t)}{dt} = \frac{2\gamma}{\hbar^2} \Big[P_{\uparrow\uparrow}(0, t) \tilde{K}_t(-\omega_{\pm}) - 2P_{\uparrow\uparrow}(\pm 1, t) \tilde{K}_t(\omega_{\pm}) \Big],$$
(8.3)

$$\frac{\mathrm{d}P_{\uparrow\downarrow}(0,t)}{\mathrm{d}t} = \frac{2\gamma}{\hbar^2} \left\{ 2P_{\uparrow\downarrow}(1,t) \left[\tilde{K}_{t>}(\omega_+) + \tilde{K}_{t<}(\omega_-) \right] + 2P_{\uparrow\downarrow}(-1,t) \left[\tilde{K}_{t>}(\omega_-) + \tilde{K}_{t<}(\omega_+) \right] - P_{\uparrow\downarrow}(0,t) \left[\tilde{K}_t(-\omega_+) + \tilde{K}_t(-\omega_-) \right] \right\},$$
(8.4)

$$\frac{\mathrm{d}P_{\uparrow\downarrow}(\pm 1,t)}{\mathrm{d}t} = \frac{4ig}{\hbar}P_{\uparrow\downarrow}(\pm 1,t) = \frac{2\gamma}{\hbar^2} \left\{ P_{\uparrow\downarrow}(0,t) \left[\tilde{K}_{t>}(-\omega_{\pm}) + \tilde{K}_{t<}(-\omega_{\mp}) \right] - 2P_{\uparrow\downarrow}(\pm 1,t) \left[\tilde{K}_{t>}(\omega_{\pm}) + \tilde{K}_{t<}(\omega_{\mp}) \right] \right\} (8.5)$$

As initial state for M we take the "paramagnetic" one, $P_M(0) = 1$, $P_M(\pm 1) = 0$, prepared by letting $T_0 \gg J$ or with a 306 radiofrequency field as in § 3.3.3. (We recall that $P_{\rm M} = P_{\uparrow\uparrow} + P_{\downarrow\downarrow}$.) The initial conditions are thus $P_{ij}(m, 0) = r_{ij}(0)\delta_{m,0}$. 3062 In order to identify the process with an ideal measurement, we need at least to find at sufficiently large times 3063 (i) the truncation, expressed by $P_{\uparrow\downarrow}(m,t) \rightarrow 0$, and (ii) the system-pointer correlations expressed by $P_{\uparrow\uparrow}(m,t) \rightarrow 0$ 3064 $r_{\uparrow\uparrow}(0)\delta_{m,1}$ and $P_{\downarrow\downarrow}(m,t) \rightarrow r_{\downarrow\downarrow}(0)\delta_{m,-1}$. This requires, for the magnet in contact with the bath, a long lifetime for 3065 the "ferromagnetic" states m = +1 and m = -1. However, the breaking of invariance, which for large N allows 3066 the ferromagnetic state where m is concentrated near $+m_{\rm F}$ to be stable, cannot occur here: Nothing hinders here the 3067 coupling with the bath to induce transitions from m = +1 to m = -1 through m = 0, so that for large times P(+1, t)3068 and P(-1, t), where $P(m, t) \equiv P_{\uparrow\uparrow}(m, t)/r_{\uparrow\uparrow}(0)$, tend to a common value close to $\frac{1}{2}$ for $T \ll J$. 3069

This is made obvious by the expression of (7.21) of the *H*-theorem. The dissipation in the Markovian regime [174, 120, 121] reads here

$$\frac{\mathrm{d}F(t)}{\mathrm{d}t} = -\frac{\gamma}{2\beta} \frac{\omega_{+} e^{-|\omega_{+}|/\Gamma}}{e^{\beta\hbar\omega_{+}} - 1} \left[P(0,t) e^{\beta\hbar\omega_{+}} - 2P(1,t) \right] \ln \frac{P(0,t) e^{\beta\hbar\omega_{+}}}{2P(1,t)} + \left[\omega_{+} \mapsto \omega_{-}, \quad P(1,t) \mapsto P(-1,t) \right], \quad (8.6)$$

and the free energy decreases until the equilibrium $2P_{\rm M}(\pm 1) = P_{\rm M}(0) \exp \beta \hbar \omega_{\pm}$ is reached. The only possibility to preserve a long lifetime for the state m = +1 is to have a low transition rate from m = +1 to m = 0, that is, according to (8.2), a small $\tilde{K}_t(\omega_+)$. This quantity is dominated in the Markovian regime by a factor $\exp(-\beta \hbar \omega_+)$. Hence, unless $T \ll J$, the apparatus cannot keep the result of the measurement registered during a significant time, after the interaction with S has been switched off. If this condition is satisfied, we may expect to reach for some lapse of time a state where $P(1, t) = P_{\uparrow\uparrow}(1, t)/r_{\uparrow\uparrow}(0)$ remains close to 1 while P(0, t) is small as P(-1, t).

Moreover, a faithful registration requires that the coupling *g* with S is sufficiently large so that the final state, in the evolution of $P_{\uparrow\uparrow}(m, t)$, has a very small probability to yield m = -1. Since in the Markovian regime the transition probabilities in (8.2) and (8.3) depend on *g* through $\omega_{\pm} = J \pm 2g$ in $\tilde{K}(\omega_{\pm})$ and $\tilde{K}(-\omega_{\pm})$ [174, 120, 121], and since this dependence arises mainly from $\exp\beta\hbar\omega_{\pm}$, we must have $\exp 4\beta g \gg 1$. The coupling *g* should moreover not modify much the spectrum, so that we are led to impose the conditions

$$T \ll 4g \ll J. \tag{8.7}$$

3083 8.1.2. Relaxation of the apparatus alone

3084 3085 Laat hem maar met rust⁶⁴ Dutch expression

As we did in § 7.3.2 for large *N*, we focus here on the evolution of the probabilities $P(m, t) \equiv P_{\uparrow\uparrow}(m, t)/r_{\uparrow\uparrow}(0)$ for the apparatus alone. It is governed by equations (8.2) and (8.3) in which $\omega_+ = \omega_- = J/\hbar$. For a weak coupling γ we expect that the Markovian regime, where $\tilde{K}_t(\omega) = \tilde{K}(\omega)$ will be reached before the probabilities have deviated much from their initial value. The equations of motion then reduce to

$$\tau \frac{\mathrm{d}P(0,t)}{\mathrm{d}t} = e^{-J/T} \left[P(1,t) + P(-1,t) \right] - P(0,t), \qquad \tau \frac{\mathrm{d}P(\pm 1,t)}{\mathrm{d}t} = \frac{1}{2} P(0,t) - e^{-J/T} P(\pm 1,t), \tag{8.8}$$

⁶⁴Better leave him alone

3090 where we made use of

$$\tilde{K}\left(\frac{J}{\hbar}\right) = e^{-J/T}\tilde{K}\left(-\frac{J}{\hbar}\right) = \frac{\hbar J}{4}\frac{e^{-J/\hbar\Gamma}}{e^{J/T} - 1},$$
(8.9)

as well as $J/T \gg 1$ and $J/\hbar\Gamma \ll 1$, and where we defined a characteristic time related to the spin-spin coupling as

$$\tau \equiv \tau_J = \frac{\hbar}{\gamma J}.\tag{8.10}$$

The Markovian approximation is justified provided this characteristic time scale τ is longer than the time *t* after which $\tilde{K}_t(\omega) = \tilde{K}(\omega)$, that is, for

$$\gamma \ll \frac{T}{J}.\tag{8.11}$$

The general solution of (8.8), obtained by diagonalization, is expressed by

$$P(0,t) + P(1,t) + P(-1,t) = 1,$$

$$P(0,t) - e^{-J/T} \left[P(1,t) + P(-1,t) \right] \propto \exp\left[-\frac{t}{\tau} (1 + e^{-J/T}) \right] \approx \exp\left(-\frac{t}{\tau} \right),$$

$$P(1,t) - P(-1,t) \propto \exp\left(-\frac{t}{\tau} e^{-J/T} \right).$$
(8.12)

Let us first consider the *relaxation of the initial paramagnetic state*, for which P(0,0) = 1 and $P(\pm 1,0) = 0$. We find from the above equations

$$P(0,t) = \frac{e^{-t/\tau} + e^{-J/T}}{1 + e^{-J/T}}, \qquad P(1,t) = P(-1,t) = \frac{1 - e^{-t/\tau}}{2(1 + e^{-J/T})}.$$
(8.13)

The lifetime of this initial unstable state is therefore $\tau = \hbar/\gamma J$. In a measurement, the interaction *g* between S and A must thus be switched on rapidly after the preparation (§ 3.3.3), in a delay $\tau_{\text{init}} \ll \tau$ so that *P*(0) is still close to 1 when the measurement process begins.

We now evaluate the delay τ_{obs} during which the pointer keeps its value and can be observed, after the measurement is achieved and after the coupling with S is switched off. If in the sector $\uparrow\uparrow$ the value m = 1 is reached at some time t_1 with a near certainty, the probabilities evolve later on, according to the above equations, as

$$P(0,t_1+t) = \frac{(1-e^{-t/\tau})e^{-J/T}}{1+e^{-J/T}}, \qquad P(\pm 1,t_1+t) = \frac{1}{2} \left[\frac{1+e^{-t/\tau}e^{-J/T}}{1+e^{-J/T}} \pm \exp\left(-\frac{t}{\tau}e^{-J/T}\right) \right].$$
(8.14)

As expected, the information is lost for $t \to \infty$, or, more precisely, for $t \gg \tau \exp(J/T)$, since P(1, t) and P(-1, t) then tend to $\frac{1}{2}$. However, during the time lapse $\tau \ll t \ll \tau \exp(J/T)$, P(1, t) retains a value $1 - \frac{1}{2}\exp(-J/T)$ close to 1, so that the probability of a false registration is then weak. Although microscopic, the pointer is a rather robust and reliable device provided $T \ll J$, on the time scale $t \ll \tau_{obs}$ where the *observation time* is

$$\tau_{\rm obs} = \tau e^{J/T} = \frac{\hbar}{\gamma J} e^{J/T}.$$
(8.15)

3107 8.1.3. Registration

We now study the time-dependence of the registration process, and determine the probability to reach a false result, that is, to find m = -1 in the sector $\uparrow\uparrow$. In the Markovian regime and under the conditions (8.7), the equations of motion (8.2), (8.3) for the probabilities $P(m, t) = P_{\uparrow\uparrow}(m, t)/r_{\uparrow\uparrow}(0)$ read

$$\tau \frac{\mathrm{d}P(0,t)}{\mathrm{d}t} = e^{-(J+2g)/T} P(1,t) + e^{-(J-2g)/T} P(-1,t) - P(0,t), \tag{8.16}$$

$$\tau \frac{\mathrm{d}P(\pm 1,t)}{\mathrm{d}t} = \frac{1}{2}P(0,t) - e^{-(J\pm 2g)/T}P(\pm 1,t).$$
(8.17)

We have disregarded in each term contributions of relative order $\exp(-J/T)$ and 2g/J. The general solution of Eqs. (8.16), (8.17) is obtained by diagonalizing their 3 × 3 matrix. Its three eigenvalues -z are the solutions of

$$z^{3} - z^{2} \left(1 + 2e^{-J/T} \cosh \frac{2g}{T} \right) + ze^{-J/T} \left(\cosh \frac{2g}{T} + e^{-J/T} \right) = 0,$$
(8.18)

3113 that is, apart from z = 0,

$$z = \frac{1}{2} + e^{-J/T} \cosh \frac{2g}{T} \pm \frac{1}{2} \sqrt{1 + 4e^{-2J/T} \sinh^2 \frac{2g}{T}},$$
(8.19)

which under the conditions (8.7) reduce to $z \simeq 1$ and $z \simeq \exp(-J/T) \cosh 2g/T \simeq \frac{1}{2} \exp[-(J-2g)/T]$. The corresponding characteristic times τ/z are therefore $\tau = \hbar/\gamma J$ and

$$\tau_{\rm reg} = 2\tau e^{(J-2g)/T} = \frac{2\hbar}{\gamma J} e^{(J-2g)/T}.$$
(8.20)

The solutions of (8.16) and (8.17) are then given by

$$P(0,t) + P(1,t) + P(-1,t) = 1,$$
(8.21)

$$P(0,t) - e^{-(J+2g)/T} P(1,t) - e^{-(J-2g)/T} P(-1,t) \propto e^{-t/\tau},$$
(8.22)

$$P(1,t) - P(-1,t) - \tanh \frac{2g}{T} \propto e^{-t/\tau_{\text{reg}}}.$$
 (8.23)

The decay time τ associated with the combination (8.22) is much shorter than the time τ_{reg} which occurs in (8.23).

With the initial condition P(0,0) = 1 we obtain, dropping contributions small as $\exp(-J/T)$,

$$P(0,t) = e^{-t/\tau}, (8.24)$$

$$P(1,t) = \frac{1}{2} \left[\left(1 - e^{-t/\tau} \right) + \tanh \frac{2g}{T} \left(1 - e^{-t/\tau_{\text{reg}}} \right) \right],$$
(8.25)

$$P(-1,t) = \frac{1}{2} \left[\left(1 - e^{-t/\tau} \right) - \tanh \frac{2g}{T} \left(1 - e^{-t/\tau_{\text{reg}}} \right) \right].$$
(8.26)

The evolution takes place in two stages, first on the time scale $\tau = \hbar/\gamma J$, then on the much larger time scale $\tau_{reg} = 2\tau \exp[(J-2g)/T]$.

During the first stage, M relaxes from the paramagnetic initial state m = 0 to both "ferromagnetic" states m = +13121 and m = -1, with equal probabilities, as in the spontaneous process where g = 0. At the end of this stage, at times 3122 $\tau \ll t \ll \tau_{reg}$ we reach a nearly stationary situation in which P(0, t) is small as $2 \exp(-J/T)$, while P(1, t) and P(-1, t)3123 are close to $\frac{1}{2}$. Unexpectedly, in spite of the presence of the coupling g which is large compared to T, the magnet 3124 M remains for a long time in a state close to the equilibrium state which would be associated to g = 0, without any 3125 *invariance breaking.* This behavior arises from the large value of the transition probabilities from m = 0 to $m = \pm 1$, 3126 which are proportional to $\tilde{K}(-\omega_{\pm})$. For $J \pm 2g \gg T$, the latter quantity reduces to $\hbar(J \pm 2g)/4$, which is not sensitive 3127 to g for $2g \ll J$. 3128

In contrast to the situation for large *N*, the magnet thus begins to *lose memory* of its initial state. For $N \gg 1$, it was the coupling *g* which triggered the evolution of M, inducing the motion of the peak of $P_{\uparrow\uparrow}(m, t)$, initially at m = 0, towards larger and larger values of *m*. Only an initial state involving values $m < -m_B$ led to false results at the end of the process. Here, rather surprisingly, the two possible results m = +1 and m = -1 come out nearly symmetrically after the first stage of the process, for $\tau \ll t \ll \tau_{reg}$. In fact we do not even need the initial state to be "paramagnetic". On this time scale, any initial state for which P(1,0) = P(-1,0) leads to $P(1,t) = P(-1,t) \simeq \frac{1}{2}$. (An arbitrary initial condition would lead to $P(\pm 1, t) = P(\pm 1, 0) + \frac{1}{2}P(0, 0)$.)

Fortunately, when *t* approaches τ_{reg} the effect of *g* is felt. For $t \gg \tau_{reg}$ the probabilities P(m, t) reach the values

$$P(1,t) = \frac{1}{1 + e^{-4g/T}}, \qquad P(-1,t) = \frac{e^{-4g/T}}{1 + e^{-4g/T}}, \qquad P(0,t) = 2e^{-J/T}, \tag{8.27}$$

which correspond to the thermal equilibrium of M in the field g. Thus, the probability of a false measurement is here

$$\mathcal{P}_{-} = e^{-4g/T},\tag{8.28}$$

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and it is small if the conditions (8.7) are satisfied. On the other hand, the *registration time* is τ_{reg} , and the registration can be achieved only if the interaction \hat{H}_{SA} remains switched on during a delay larger than τ_{reg} . After this delay, if we switch off the coupling *g*, the result remains registered for a time which allows observation, since τ_{obs} , determined in \$141 § 8.1.2, is much larger than τ_{trunc} .

Thus, not only the first stage of the registration process is odd, but also the second one. The mechanism at play in section 7 was a *dynamical breaking of invariance* whereas here we have to rely on the *establishment of thermal equilibrium in the presence of g*. The coupling should be kept active for a long time until the values (8.27) are reached, whereas for $N \gg 1$, only the beginning of the evolution of $P_{\uparrow\uparrow}(m, t)$ required the presence of the coupling g; afterwards $P_{\uparrow\uparrow}$ reached the ferromagnetic peak at $m = m_{\rm F}$, and remained there stably.

For N = 2 the possibility of registration on the time scale τ_{reg} relies on the form of the transition probabilities 3147 from $m = \pm 1$ to m = 0, which are proportional to $\tilde{K}(\omega_{\pm})$. Although small as $\exp(-\beta\hbar\omega_{\pm})\tilde{K}(-\omega_{\pm})$, these transition 3148 probabilities contain a factor $\exp(-\beta\hbar\omega_{\pm}) \propto \exp(\pm 2g/T)$ which, since $2g \gg T$, strongly distinguishes +1 from -1, 3149 whereas $\hat{K}(-\omega_{+}) \simeq \hat{K}(-\omega_{-})$. Hence the transition rate from m = -1 to m = 0, behaving as $\exp[-(J-2g)/T]$, allows 3150 P(0) to slowly increase at the expense of P(-1), then to rapidly decay symmetrically. Since the transition rate from 3151 m = +1 to m = 0, behaving as $\exp[-(J + 2g)/T]$, is much weaker, the resulting increase of P(1) remains gained. 3152 Altogether P(1, t) rises in two steps, from 0 to $\frac{1}{2}$ on the time scale τ , then from $\frac{1}{2}$ to nearly 1 on the time scale τ_{reg} , as 3153 shown by (8.25). Meanwhile, P(-1, t) rises from 0 to $\frac{1}{2}$, then decreases back to 0, ensuring a correct registration only 3154 at the end of the process, while P(0, t) remains nearly 0 between τ and τ_{reg} . 3155

3156 8.1.4. Truncation

It remains to study the evolution of the off-diagonal blocks of the density operator \hat{D} , which are characterized by the three functions of time $P_{\uparrow\downarrow}(m,t)$. Their equations of motion (8.4), (8.5) involve oscillations in $P_{\uparrow\downarrow}(\pm 1, t)$ with frequency $2g/\pi\hbar$ generated by the coupling g with S and by a relaxation process generated by the bath. Since the oscillations are not necessarily rapid, and since γ is small, the damping effect of the bath is expected to occur over times large compared to \hbar/T , so that we can again work in the Markovian regime. Moreover, since $g \ll J$, we are led to replace ω_+ and ω_- in $\tilde{K}_{t<}$ by $J\hbar$. Hence, we can replace, for instance, $\tilde{K}_{t>}(\omega_+) + \tilde{K}_{t<}(\omega_-)$ by $\tilde{K}(J/\hbar)$.

The equations of motion for the set
$$P_{\uparrow\downarrow}(m, t)$$
 are thus simplified into

τ

$$\tau \frac{dP_{\uparrow\downarrow}(0,t)}{dt} = \varepsilon \left[P_{\uparrow\downarrow}(1,t) + P_{\uparrow\downarrow}(-1,t) \right] - P_{\uparrow\downarrow}(0,t), \tag{8.29}$$

$$\frac{\mathrm{d}P_{\uparrow\downarrow}(\pm 1,t)}{\mathrm{d}t} = \pm i\lambda P_{\uparrow\downarrow}(\pm 1,t) + \frac{1}{2}P_{\uparrow\downarrow}(0,t) - \varepsilon P_{\uparrow\downarrow}(\pm 1,t), \qquad (8.30)$$

3164 where ε and λ are defined by

$$\varepsilon = e^{-J/T}, \qquad \lambda = \frac{4g}{\gamma J},$$
(8.31)

with $\gamma \ll 1$, $g \ll T \ll J$. The truncation process is governed by the interplay between the oscillations in $P(\pm 1, t)$, generated by the coupling *g* between M and S, and the damping due to the bath. The two dimensionless parameters λ and ε characterize these effects.

The eigenvalues of the matrix relating $-\tau dP_{\uparrow\downarrow}(m,t)/dt$ to $P_{\uparrow\downarrow}(m,t)$ are the solutions of the equation

$$(z-1)[(z-\varepsilon)^2+\lambda^2]-\varepsilon(z-\varepsilon)=0.$$
(8.32)

The largest eigenvalue behaves for $T \ll J$ as

$$z_0 \approx 1 + \frac{\varepsilon}{1+\lambda^2} + \frac{\varepsilon^2 \lambda^2 (1-\lambda^2)}{(1+\lambda^2)^3},\tag{8.33}$$

whereas the other two eigenvalues z_1 and z_2 , obtained from

$$z^{2} - z\varepsilon \left(\frac{1+2\lambda^{2}}{1+\lambda^{2}} + \frac{\varepsilon^{2}\lambda^{2}(\lambda^{2}-1)}{(1+\lambda^{2})^{3}}\right) + \lambda^{2} \left(1 - \frac{\varepsilon}{1+\lambda^{2}} + \frac{\varepsilon^{2}(1+\lambda^{4})}{(1+\lambda^{2})^{3}}\right) = 0,$$
(8.34)

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have a real part small as ε . The solution of (8.29), (8.30), with the initial condition $P_{\uparrow\downarrow}(m, 0) = r_{\uparrow\downarrow}(0)\delta_{m,0}$ is given by

$$P_{\uparrow\downarrow}(0,t) = r_{\uparrow\downarrow}(0) \left[e^{-z_0 t/\tau} - \frac{(z_1 - \varepsilon)^2 + \lambda^2}{(z_0 - z_1)(z_1 - z_2)} (e^{-z_1 t/\tau} - e^{-z_0 t/\tau}) - \frac{(z_2 - \varepsilon)^2 + \lambda^2}{(z_0 - z_2)(z_2 - z_1)} (e^{-z_2 t/\tau} - e^{-z_0 t/\tau}) \right], \quad (8.35)$$

$$P_{\uparrow\downarrow}(\pm 1, t) = r_{\uparrow\downarrow}(0) \left[\frac{z_1 - \varepsilon \mp i\lambda}{2(z_0 - z_1)(z_1 - z_2)} \left(e^{-z_1 t/\tau} - e^{-z_0 t/\tau} \right) + \frac{z_2 - \varepsilon \mp i\lambda}{2(z_0 - z_2)(z_2 - z_1)} \left(e^{-z_2 t/\tau} - e^{-z_0 t/\tau} \right) \right].$$
(8.36)

According to (8.35), the first term of $P_{\uparrow\downarrow}(0, t)$ is damped for $\varepsilon \ll 1$ over the time scale $\tau = \hbar/\gamma J$, just as $P_{\uparrow\uparrow}(0, t)$ in the registration process. However, here again, the other two quantities $|P_{\uparrow\downarrow}(\pm 1, t)|$ increase in the meanwhile and the truncation of the state is far from being achieved after the time τ . In fact, all three components $P_{\uparrow\downarrow}(m, t)$ survive over a much longer delay, which depends on the ratio $2\lambda/\varepsilon$.

In the overdamped situation $2\lambda < \varepsilon$ or $8g < \gamma J \exp(-J/T)$, the eigenvalues

$$z_{1,2} = \frac{1}{2}\varepsilon \pm \frac{1}{2}\sqrt{\varepsilon^2 - 4\lambda^2}$$
(8.37)

are real, so that we get, in addition to the relaxation time τ , two much longer off-diagonal relaxation times, $\tau_{1,2} = \tau/z_{1,2}$. The long-time behavior of $P_{\uparrow\downarrow}(m, t)$, governed by z_2 , is

$$P_{\uparrow\downarrow}(0,t) \sim r_{\uparrow\downarrow}(0) \frac{\varepsilon(\varepsilon + \sqrt{\varepsilon^2 - 4\lambda^2})}{2\sqrt{\varepsilon^2 - 4\lambda^2}} e^{-t/\tau_{\rm trunc}}, \qquad P_{\uparrow\downarrow}(\pm 1,t) \sim r_{\uparrow\downarrow}(0) \frac{\varepsilon \pm 2i\lambda + \sqrt{\varepsilon^2 - 4\lambda^2}}{4\sqrt{\varepsilon^2 - 4\lambda^2}} e^{-t/\tau_{\rm trunc}}.$$
(8.38)

3179 The truncation time

$$\tau_{\rm trunc} = \frac{\tau}{2\lambda^2} \left(\varepsilon + \sqrt{\varepsilon^2 - 4\lambda^2} \right) = \frac{\hbar\gamma J}{32g^2} \left(e^{-J/T} + \sqrt{e^{-2J/T} - \frac{64g^2}{\gamma^2 J^2}} \right),\tag{8.39}$$

which characterizes the decay of $\langle \hat{s}_{x} \rangle$, $\langle \hat{s}_{x} \rangle$, and of their correlations with \hat{m} , is here much longer than the registration time (10), since $\tau_{\text{trunc}}/\tau_{\text{reg}}$ is of order $(\varepsilon/2\lambda)^2 \exp(2g/T)$, and even larger than τ_{obs} . The quantities $P_{\uparrow\downarrow}(m, t)$ remain for a long time proportional to $r_{\uparrow\downarrow}(0)$, with a coefficient of order 1 for $P_{\uparrow\downarrow}(\pm 1, t)$, of order ε for $P_{\uparrow\downarrow}(0, t)$. Truncation is thus here a much slower process than registration: equilibrium is reached much faster for the diagonal elements (8.27) than for the off-diagonal ones which are long to disappear. Let us stress that for the present case of a small apparatus, they disappear due to the bath ("environment-induced decoherence" [32, 33, 40, 176, 177, 178, 179]) rather than, as in our previous discussion of a large apparatus, due to fast dephasing caused by the large size of M.

For $2\lambda > \varepsilon$, we are in an oscillatory situation, where the eigenvalues

$$z_{1,2} = \frac{\varepsilon}{2} \frac{1+2\lambda^2}{1+\lambda^2} \pm i \sqrt{\lambda^2 - \frac{\varepsilon^2}{4} - \frac{\varepsilon\lambda^2}{1+\lambda^2}}$$
(8.40)

are complex conjugate. (Nothing prevents $\lambda = 4g/\gamma J$ from being large.) The long-time behavior is given by

$$P_{\uparrow\downarrow}(0,t) \sim \frac{\varepsilon r_{\uparrow\downarrow}(0)}{(1+\lambda^2)^2} e^{-t/\tau_{\rm trunc}} \left[(1-\lambda^2) \cos \frac{2\pi t}{\theta} + \frac{2\lambda^2}{\sqrt{\lambda^2 - \varepsilon^2/4}} \sin \frac{2\pi t}{\theta} \right],$$

$$P_{\uparrow\downarrow}(\pm 1,t) \sim \frac{r_{\uparrow\downarrow}(0)}{2(1\pm i\lambda)} e^{-t/\tau_{\rm trunc}} \left[\cos \frac{2\pi t}{\theta} \pm \frac{i\lambda}{\sqrt{\lambda^2 - \varepsilon^2/4}} \sin \frac{2\pi t}{\theta} \right],$$
(8.41)

3189 with a truncation time

$$\tau_{\rm trunc} = \frac{2(1+\lambda^2)}{\varepsilon(1+2\lambda^2)}\tau = \frac{2\hbar e^{J/T}(1+\lambda^2)}{\gamma J(1+2\lambda^2)} = \frac{1+\lambda^2}{1+2\lambda^2}\tau_{\rm reg}e^{2g/T},\tag{8.42}$$

again much larger than the registration time. While being damped, these functions oscillate with a period

$$\theta = \frac{2\pi\tau}{\sqrt{\lambda^2 - \varepsilon^2/4}} \tag{8.43}$$

shorter than τ_{trunc} if $2\lambda > \varepsilon \sqrt{4\pi^2 + 1}$. The *truncation time* (8.42) practically does not depend on *g* (within a factor 2 when 2λ varies from ε to ∞), in contrast to both the truncation time of section 5 and the irreversibility time of section 6. The present truncation time is comparable to the lifetime τ_{obs} of an initial pure state m = +1 when it spontaneously decays towards $m = \pm 1$ with equal probabilities (§ 8.1.2). Hence in both cases the truncation takes place over the delay during which the result of the measurement can be observed.

For $\lambda \gg \varepsilon$ and $t \gg \tau$, the off-diagonal contributions (8.41) to \hat{D} are governed by

$$P_{\uparrow\downarrow}(\pm 1, t) \sim \frac{r_{\uparrow\downarrow}(0)}{2(1\pm i\lambda)} e^{-t/\tau_{\rm red} \pm i\lambda t/\tau}.$$
(8.44)

The effects on M of S and B are well separated: the oscillations are the same as for $\gamma = 0$, while the decay, with characteristic time $\tau/\varepsilon = (\hbar/\gamma J) \exp(J/T)$, is a pure effect of the bath. The amplitude becomes small for $\lambda \gg 1$, that is, $g \gg \gamma J$.

3200 8.1.5. Is this process with bath-induced decoherence a measurement?

When the number N of degrees of freedom of the pointer is small as here, the present model appears as a specific example among the general class of models considered by Spehner and Haake [159, 160]. As shown by these authors, the truncation is then governed by the large number of degrees of freedom *of the bath, not of the pointer*; the truncation is then not faster than the registration. Our detailed study allows us to compare the mechanisms of two types of processes, for large N and for small N.

We have seen (§ 8.1.3) that for N = 2 as for $N \gg 1$ both couplings g and γ between S, M and B establish 3206 the diagonal correlations between \hat{s}_z and \hat{m} needed to establish Born's rule. This result is embedded in the values 3207 reached by $P_{\uparrow\uparrow}$ and $P_{\downarrow\downarrow}$ after the time $\tau_{\text{reg}} = (2\hbar/\gamma J) \exp[(J-2g)/T]$, much longer than the lifetime $\tau = \hbar/\gamma J$ of 3208 the initial state in the absence of a field or a coupling. Although this property is one important feature of a quantum 3209 measurement, its mechanism is here only a relaxation towards thermal equilibrium. The registration is fragile and 3210 does not survive beyond a delay $\tau_{obs} = (\hbar/\gamma J) \exp(J/T)$ once the coupling with S is switched off. For larger N, the 3211 existence of a spontaneously broken invariance ensured the long lifetime of the ferromagnetic states, and hence the 3212 robust registration of the measurement. 3213

Another feature of a quantum measurement, the truncation of the state that represents a large set of runs, has also been recovered for N = 2, but with an unsatisfactorily long time scale. For large N, the truncation process took place rapidly and was achieved before the registration in the apparatus really began, but here, whatever the parameters ε and λ , the expectation values $\langle \hat{s}_x \rangle$, $\langle \hat{s}_y \rangle$ and the off-diagonal correlations embedded in $P_{\uparrow\downarrow}$ and $P_{\downarrow\uparrow}$ fade out over a truncation time τ_{trunc} given by (8.39) or (8.42), which is longer than the registration time and even than the observation time if $2\lambda \ll \varepsilon$. It is difficult to regard such a slow decay as the "collapse" of the state.

By studying the case N = 2, we wished to test whether an *environment-induced decoherence* [32, 33, 40, 176, 177, 178, 179] might cause truncation. Here the "environment" is the bath B, which is the source of irreversibility. It imposes thermal equilibrium to S + M, hence suppressing gradually the off-diagonal elements of \hat{D} which vanish at equilibrium, a suppression that we defined as "truncation". However, usually, decoherence time scales are the shortest of all; here, for N = 2, contrary to what happened for $N \gg 1$, the truncation time is not shorter than the registration time.

The effect of the bath is therefore quite different for large and for small *N*. For $N \gg 1$, we have seen in §§ 5.1.2 and 6.2.4 that the rapid initial truncation was ensured by the large size of the pointer M, whereas bath-induced decoherence played only a minor role, being only one among the two possible mechanisms of suppression of recurrences. For N =2, the truncation itself is caused by the bath, but we cannot really distinguish decoherence from thermal equilibration: Although the dynamics of the diagonal and off-diagonal blocks of \hat{D} are decoupled, there is no neat separation of time scales for the truncation and the registration.

A last feature of measurements, the uniqueness of the outcome of individual runs, is essential as it conditions both Born's rule and von Neumann's reduction. We have stressed (§§ 1.1.2 and 1.3.3) that truncation, which concerns the

large set of runs of the measurement, does not imply reduction, which concerns individual runs. The latter property will be proven in section 11 for the Curie–Weiss model; its explanation will rely on a coupling between the large number of eigenstates of M involved for $N \gg 1$ in each ferromagnetic equilibrium state. Here, for N = 2, the "ferromagnetic" state is non degenerate, and that mechanism cannot be invoked.

Anyhow, the process that we described cannot be regarded for N = 2 as a full measurement. Being microscopic, the pair of spins M is not a "pointer" that can be observed directly. In order to get a stable signal, which provides us with information and which we may use at a macroscopic level, we need to couple M to a genuine macroscopic apparatus. This should be done after the time $\tau_{reg} = (2\hbar/\gamma J) \exp[(J - 2g)/T]$ when the correlations $P_{\uparrow\uparrow}(m, t) =$ $r_{\uparrow\uparrow}(0)\delta_{m,1}$ and $P_{\downarrow\downarrow}(m, t) = r_{\downarrow\downarrow}(0)\delta_{m,-1}$ have been created between S and M. Then, S and M should be decoupled, and the measurement of *m* should be performed in the delay $\tau_{obs} = (\hbar/\gamma J) \exp(J/T)$. In this hypothetical process, the decoupling of S and M will entail truncation, the correlations which survive for the duration $\tau_{red} = 2\tau_{obs}$ in $P_{\uparrow\downarrow}$ and $P_{\downarrow\uparrow}$ being destroyed.

Altogether, it is not legitimate for small *N* to regard M + B as a "measurement apparatus", since nothing can be said about individual runs. Anyhow, registering robustly the outcomes of the process so as to read them during a long delay requires a *further apparatus* involving a *macroscopic pointer*. The system M, even accompanied with its bath, is not more than a quantum device coupled to S. However, its marginal state is represented by a diagonal density matrix, in the basis which diagonalizes \hat{m} , so that the respective probabilities of m = 0, m = +1 and m = -1, from which we may infer $r_{\uparrow\uparrow}(0)$ and $r_{\downarrow\downarrow}(0)$, can be determined by means of an apparatus with classical features.

3252 8.1.6. Can one simultaneously "measure" non-commuting variables?

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Je kunt niet alles tegelijk doen⁶⁵ Dutch saying

Although the process described above cannot be regarded as an ideal measurement, we have seen that it allows us to determine the diagonal elements $r_{\uparrow\uparrow}(0)$ and $r_{\downarrow\downarrow}(0)$ of the density matrix of S at the initial time. Surprisingly, the same device may also give us access to the off-diagonal elements, owing to the pathologically slow truncation. Imagine S and M are decoupled at some time τ_{dec} of order τ_{red} . For $2\lambda \ll \varepsilon$, this time can be shorter than the observation time, so that a rapid measurement of *m* will inform us statistically on $r_{\uparrow\uparrow}(0)$ and $r_{\downarrow\downarrow}(0)$. However, the transverse components of the spin S have not disapppeared on average, and $r_{\uparrow\downarrow}(\tau_{dec})$ is given at the decoupling time and later on by

$$r_{\uparrow\downarrow}(\tau_{\rm dec}) = \sum_{m} P_{\uparrow\downarrow}(m, \tau_{\rm dec}) = r_{\uparrow\downarrow}(0) \frac{\varepsilon + \sqrt{\varepsilon^2 - 4\lambda^2}}{2\sqrt{\varepsilon^2 - 4\lambda^2}} e^{-\tau_{\rm dec}/\tau_{\rm red}}.$$
(8.45)

A subsequent measurement on S in the *x*-direction at a time $t > \tau_{dec}$ will then provide $r_{\uparrow\downarrow}(t) + r_{\downarrow\uparrow}(t) = 2\Re r_{\downarrow\uparrow}(\tau_{dec})$. If the various parameters entering (8.45) are well controlled, we can thus, through repeated measurements, determine indirectly $r_{\uparrow\downarrow}(0) + r_{\downarrow\uparrow}(0)$, as well as $r_{\uparrow\uparrow}(0)$ and $r_{\downarrow\downarrow}(0)$.

Note, however, that such a procedure gives us access only to the *statistical* properties of the *initial state* of S, and that von Neumann's reduction is precluded.

Thus a unique experimental setting may be used to determine the statistics of the non-commuting observables \hat{s}_x 3266 and \hat{s}_z . This possibility is reminiscent of a general result [255]; see also [256, 257]. Suppose we wish to determine all 3267 the matrix elements r_{ij} of the unknown $n \times n$ density matrix of a system S at the initial time. Coupling during some 3268 delay S with a similar auxiliary system S', the initial state of which is known, leads to some density matrix for the 3269 compound system. The set r_{ij} is thus mapped onto the n^2 diagonal elements of the latter. These diagonal elements 3270 may be measured simultaneously by means of a single apparatus, and inversion of the mapping yields the whole set 3271 r_{ii} . Here the magnet M plays the role of the auxiliary system S'; we can thus understand the paradoxical possibility 3272 of determining the statistics of both \hat{s}_x and \hat{s}_z with a single device. 3273

In this context we note that the simultaneous measurement of non-commuting observables is an important chapter of modern quantum mechanics. Its recent developments are given in [249, 250, 251, 252, 253] (among other references) and reviewed in [254].

With this setup we can also repeat measurements in the *z*-direction and see how much lapse should be in between to avoid non-idealities.

⁶⁵You can't do everything at the same time

3279 8.2. Measuring a non-conserved quantity

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L'homme est plein d'imperfections, mais ce n'est pas étonnant si l'on songe à l'époque où il a été créé⁶⁶ Alphonse Allais

It has been stressed by Wigner [258] that an observable that does not commute with some conserved quantity of 3283 the total system (tested system S plus apparatus A) cannot be measured exactly, and the probability of unsuccessful 3284 experiments has been estimated by Araki and Yanase [225, 259]. (Modern developments of this Wigner-Araki-Yanase 3285 limitation are given in [246, 247, 248].) However, neither the irreversibility nor the dynamics of the measurement 3286 process were considered. We focus here on the extreme case in which Wigner's conserved quantity is the energy 3287 itself. We have assumed till now that the measured observable \hat{s}_z commuted with the full Hamiltonian of S + A. This 3288 has allowed us to split the dynamical analysis into two separate parts: The diagonal blocks $R_{\uparrow\uparrow}$, $R_{\downarrow\downarrow}$ of the full density 3289 matrix of S + A are not coupled to the off-diagonal blocks $R_{\uparrow\downarrow}$, $R_{\downarrow\uparrow}$. This gives rise, for $N \gg 1$, to a large ratio between 3290 the time scales that characterize the truncation and the registration. 3291

We will discuss, by solving a slightly modified version of our model, under which conditions one can still measure a quantity which is not conserved. We allow therefore transitions between different eigenvalues of \hat{s}_z , by introducing a magnetic field that acts on S. The part \hat{H}_S of the Hamiltonian, instead of vanishing as in (3.4), is taken as

$$\hat{H}_{\rm S} = -b\hat{s}_{\rm y}.\tag{8.46}$$

Plus ça change, plus c'est la même chose⁶⁷

French saying

We take as measuring device a large, Ising magnet, with q = 2 and $N \gg 1$.

We wish to study how the additional field affects the dynamics of the measurement. We shall therefore work out the equations at lowest order in *b*, which however need not be finite as $N \to \infty$. In fact, a crucial parameter turns out to be the combination $b/g \sqrt{N}$.

8.2.1. The changes in the dynamics
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The formalism of subsection 4.2 remains unchanged, the unperturbed Hamiltonian being now

$$\hat{H}_0 = \hat{H}_{\rm S} + \hat{H}_{\rm SA} + \hat{H}_{\rm M} = -b\hat{s}_y - Ng\hat{m}\hat{s}_z - \frac{1}{2}JN\hat{m}^2.$$
(8.47)

The additional contribution (8.46) enters the basic equation (4.5) in two different ways.

(*i*) On the left-hand side, the term $-|\hat{H}_{\rm S}, \hat{D}|/i\hbar$ yields a contribution

$$\frac{b}{\hbar} \begin{pmatrix} \hat{R}_{\uparrow\downarrow} + \hat{R}_{\downarrow\uparrow} & \hat{R}_{\downarrow\downarrow} - \hat{R}_{\uparrow\uparrow} \\ \hat{R}_{\downarrow\downarrow} - \hat{R}_{\uparrow\uparrow} & -\hat{R}_{\uparrow\downarrow} - \hat{R}_{\downarrow\uparrow} \end{pmatrix}$$
(8.48)

to $d\hat{D}/dt$ which couples the diagonal and off-diagonal sectors of (3.18). Accordingly, we must add to the right-hand side of the equation of motion (4.16) for $dP_{\uparrow\uparrow}/dt$ the term $\hbar^{-1}b(P_{\uparrow\downarrow} + P_{\downarrow\uparrow})$, and subtract it from the equation for $dP_{\downarrow\downarrow}/dt$; we should add to the equations (4.18) for $dP_{\uparrow\downarrow}/dt$ and $dP_{\downarrow\uparrow}/dt$ the term $\hbar^{-1}b(P_{\downarrow\downarrow} - P_{\uparrow\uparrow})$.

(*ii*) The presence of $\hat{H}_{\rm S}$ in \hat{H}_0 has another, indirect effect. The operators $\hat{\sigma}_a^{(n)}(u)$ defined by (4.4), which enter the right-hand side of eq. (4.5), no longer commute with \hat{s}_z . In fact, while $\hat{\sigma}_z^{(n)}(u)$ still equals $\hat{\sigma}_z^{(n)}$, the operators

$$\hat{\sigma}_{+}^{(n)}(u) = \left[\hat{\sigma}_{-}^{(n)}(u)\right]^{\dagger} = \hat{\sigma}_{+}^{(n)} e^{-i\hat{H}_{0}(\hat{m}+\delta m)u/\hbar} e^{i\hat{H}_{0}(\hat{m})u/\hbar} = e^{-i\hat{H}_{0}(\hat{m})u/\hbar} e^{i\hat{H}_{0}(\hat{m}-\delta m)u/\hbar} \hat{\sigma}_{+}^{(n)}$$
(8.49)

now contain contributions in \hat{s}_x and \hat{s}_y , which can be found by using the expression (8.47) of \hat{H}_0 and the identity exp $(i \mathbf{a} \cdot \hat{\mathbf{s}}) = \cos(a) + i \sin(a) \mathbf{a} \cdot \hat{\mathbf{s}}/a$. For $N \gg 1$ and arbitrary *b*, we should therefore modify the bath terms in dP_{ij}/dt by using the expression

⁶⁶Man is full of imperfections, but this is not surprising if one considers when he was created

⁶⁷The more it changes, the more it remains the same

$$\hat{\sigma}_{+}^{(n)}(u) = \hat{\sigma}^{(n)} \exp\left[\frac{2i\hat{m}u}{\hbar} \left(J + Ng^2 \frac{Ng\hat{m}\hat{s}_z + b\hat{s}_y}{N^2 g^2 \hat{m}^2 + b^2}\right)\right],\tag{8.50}$$

instead of (B.7); we have dropped in the square bracket contributions that oscillate rapidly as $\exp\left(2iu\sqrt{N^2g^2\hat{m}^2+b^2}/\hbar\right)$ with factors \hat{s}_x and coefficients of order 1/N.

Except in § 8.2.5 we assume that S and A remain coupled at all times. Their joint distribution $\hat{D}(t)$ is then expected 3316 to be driven by the bath B to an equilibrium $\hat{D}(t_f) \propto \exp(-\hat{H}_0/T)$ at large times. The temperature T is imposed by the 3317 factor K(u) that enters the equation of motion (4.5), while \hat{H}_0 is imposed by the form of $\hat{\sigma}_a^{(n)}(u)$. The additional terms 3318 in (8.50) are needed to ensure that S + M reaches the required equilibrium state. As discussed in § 7.1.4, invariance 3319 is broken in the final state. Its density operator involves two incoherent contributions, for which the magnetization of 3320 M lies either close to $+m_{\rm F}$ or close to $-m_{\rm F}$. In the first one, the marginal state of S is $\hat{r}(t_{\rm f}) \propto \exp\left[\left(b\hat{s}_{y} + Ngm_{\rm F}\hat{s}_{z}\right)/T\right]$. 3321 If $b \ll Ng$, a condition that we will impose from now on, this state cannot be distinguished from the projection 3322 on $s_z = \pm 1$. As when b = 0, the sign of the observed magnetization $\pm m_{\rm F}$ of M is fully correlated with that of the 3323 z-component of the spin S in the final state, while $\langle \hat{s}_x(t_f) \rangle = \langle \hat{s}_y(t_f) \rangle = 0$. If dynamical stability of subensembles is 3324 ensured as in § 11.2.3, the process is consistent with von Neumann's reduction, and it can be used as a preparation. 3325

Nevertheless, nothing warrants the weights of the two possible outcomes, $+m_F$, $s_z = +1$ and $-m_F$, $s_z = -1$, to be equal to the diagonal elements $r_{\uparrow\uparrow}(0)$ and $r_{\downarrow\downarrow}(0)$ of the *initial density matrix: Born's rule may be violated*. A full study of the dynamics is required to evaluate these weights, so as to determine whether the process is still a faithful measurement.

This study will be simplified by noting that the expression (8.50) depends on *b* only through the ratio $b/Ng\hat{m}$. Once the registration has been established, at times of order τ_{reg} , the relevant eigenvalues of \hat{m} , of order m_B , are finite for large *N* and the field *b* does not contribute to $\hat{\sigma}_a^{(n)}(u)$ since $b \ll Ng$. For short times, during the measurement process, the distribution of *m* is Gaussian, with a width of order $1/\sqrt{N}$, so that *b* may contribute significantly to $\hat{\sigma}_a^{(n)}(u)$ if *b* is of order $g\sqrt{N}$. However, we have shown (section 6) that for the off-diagonal blocks the bath terms in (4.29) have the sole effect of inhibiting the recurrences in $P_{\uparrow\downarrow}(m, t)$. Anyhow, such recurrences are not seen when *m* is treated as a continuous variable. We shall therefore rely on the simplified equations of motion

$$\frac{\partial P_{\uparrow\downarrow}}{\partial t} - \frac{2iNgm}{\hbar}P_{\uparrow\downarrow} = \frac{\partial P_{\downarrow\uparrow}}{\partial t} + \frac{2iNgm}{\hbar}P_{\downarrow\uparrow} = \frac{b}{\hbar}\left(P_{\downarrow\downarrow} - P_{\uparrow\uparrow}\right). \tag{8.51}$$

As regards the diagonal blocks we shall disregard *b* not only at times of order τ_{reg} , but even earlier. This is legitimate if $b \ll g \sqrt{N}$; if *b* is of order $g \sqrt{N}$, such an approximation retains the main effects of the bath, driving the distributions $P_{\uparrow\uparrow}(m, t)$ and $P_{\downarrow\downarrow}(m, t)$ apart from $-m_{\rm B}$ and $+m_{\rm B}$, respectively, and widening them. We write therefore:

$$\frac{\partial P_{\uparrow\uparrow}}{\partial t} + \frac{\partial}{\partial m} \left(v_{\uparrow\uparrow} P_{\uparrow\uparrow} \right) - \frac{1}{N} \frac{\partial^2}{\partial m^2} \left(w P_{\uparrow\uparrow} \right) = \frac{b}{\hbar} \left(P_{\uparrow\downarrow} + P_{\downarrow\uparrow} \right), \tag{8.52}$$

$$\frac{\partial P_{\downarrow\downarrow}}{\partial t} + \frac{\partial}{\partial m} \left(v_{\downarrow\downarrow} P_{\downarrow\downarrow} \right) - \frac{1}{N} \frac{\partial^2}{\partial m^2} \left(w P_{\downarrow\downarrow} \right) = -\frac{b}{\hbar} \left(P_{\uparrow\downarrow} + P_{\downarrow\uparrow} \right). \tag{8.53}$$

(Here we should distinguish the drift velocities $v_{\uparrow\uparrow}$ and $v_{\downarrow\downarrow}$, but the diffusion coefficients are equal.) Since the outcome of the registration is governed by the first stage studied in § 7.2.3(*i*), and since the Markovian regime (§ 7.1.1) is reached nearly from the outset, we shall use the simplified forms

$$v_{\uparrow\uparrow} = \frac{\gamma}{\hbar} [g + (J - T)m] = \frac{1}{\tau_{\rm reg}} (m_{\rm B} + m), \qquad w = \frac{\gamma T}{\hbar}, \tag{8.54}$$

for the drift velocity and the diffusion coefficient; $v_{\downarrow\downarrow}$ follows from $v_{\uparrow\uparrow}$ by changing g into -g.

We have to solve (8.51), (8.52) and (8.53) with initial conditions $P_{ij}(m, 0)/r_{ij}(0) = P_{\rm M}(m, 0)$ expressed by (3.48). The drift and diffusion induced by the bath terms are slow since $\gamma \ll 1$, and the distribution $P_{\rm M}(m, t) = P_{\uparrow\uparrow}(m, t) + P_{\downarrow\downarrow}(m, t)$ of the magnetization of M can be regarded as constant on the time scales $\tau_{\rm red}$ and $\tau_{\rm Larmor} = \pi\hbar/b$, which is the period of the precession of the spin S when it does not interact with A. Over a short lapse around any time *t*, the coupled equations for $C_x = P_{\uparrow\downarrow} + P_{\downarrow\uparrow}$, $C_y = iP_{\uparrow\downarrow} - iP_{\downarrow\uparrow}$ and $C_z = P_{\uparrow\uparrow} - P_{\downarrow\downarrow}$ simply describe, for each *m*, a Larmor precession of S [60, 61, 62, 63, 64, 65] submitted to the field *b* along \hat{y} and to the field *Ngm* along \hat{z} , where *m* is a classical random variable governed by the probability distribution $P_M(m, t)$. The slow evolution of $P_M(m, t)$ is coupled to this rapid precession through (8.52) and (8.53).

3352 8.2.2. Ongoing truncation

We first eliminate the off-diagonal contributions by formally solving (8.51) as

$$P_{\uparrow\downarrow}(m,t) = P_{\downarrow\uparrow}^* = r_{\uparrow\downarrow}(0)e^{2iNgmt/\hbar}P_{\rm M}(m,0) - \frac{b}{\hbar}\int_0^t {\rm d}t' \ e^{2iNgm(t-t')/\hbar} \left[P_{\uparrow\uparrow}(m,t') - P_{\downarrow\downarrow}(m,t')\right]. \tag{8.55}$$

The physical quantities (except for correlations involving a large number of spins of M, see § 5.1.3) are obtained by summing over *m* with a weight smooth on the scale $1/\sqrt{N}$. The first term of (8.55), the same as in section 5 then yields a factor decaying as $\exp[-(t/\tau_{red})^2]$, with $\tau_{red} = \hbar/g\delta_0 \sqrt{2N}$, due to destructive interferences.

However, the second term survives much later because the precession induced by the field *b* along \hat{y} couples $2P_{\uparrow\downarrow} = C_x - iC_y$ to $C_z = P_{\uparrow\uparrow} - P_{\downarrow\downarrow}$ at all times *t*. truncation takes place through the oscillatory factor in the integral, which hinders the effect of precession except at times *t'* just before *t*. Truncation is an *ongoing process*, which may take place (if *b* is sufficiently large) for $t \gg \tau_{red}$: The non-conservation of the measured quantity s_z tends to *feed up the off-diagonal components* $\hat{R}_{\uparrow\downarrow}$ and $\hat{R}_{\downarrow\uparrow}$ of the density matrix \hat{D} of S + M. In compensation, $\hat{R}_{\uparrow\uparrow}$ and $\hat{R}_{\downarrow\downarrow}$ may be progressively eroded through the right-hand side of (8.52) and (8.53).

At lowest order in b, we can rewrite explicitly the second term of (8.55) by replacing $P_{\uparrow\uparrow}$ by

$$P_{\uparrow\uparrow}^{(0)}(m,t) = r_{\uparrow\uparrow}(0) \sqrt{\frac{N}{2\pi D(t)}} \exp\left[-\frac{N}{2D(t)} \left(m + m_{\rm B} - m_{\rm B} e^{t/\tau_{\rm reg}}\right)^2\right],$$

$$D(t) = \delta_0^2 e^{2t/\tau_{\rm reg}} + \frac{T}{J-T} \left(e^{2t/\tau_{\rm reg}} - 1\right), \qquad \tau_{\rm reg} = \frac{\hbar}{\gamma(J-T)},$$
(8.56)

that we evaluated for b = 0 in section 7. We have simplified the general expression (7.61) by noting that the final outcome will depend only on the first stage of the registration, when *t* is of order τ_{reg} . For $P_{\downarrow\downarrow}^{(0)}$ we have to change $\tau_{\uparrow\uparrow}(0)$ into $r_{\downarrow\downarrow}(0)$ and $m_{\text{B}} = g/(J - T)$ into $-m_{\text{B}}$.

3367 8.2.3. Leakage

The expectation values of \hat{s}_x or \hat{s}_y and their correlations with the pointer variable \hat{m} are now found as in § 5.1.3 through summation over *m* of $P_{\uparrow\downarrow}(m, t)e^{i\lambda m}$. At times *t* long compared to τ_{red} and short compared to τ_{reg} , we find the characteristic function

$$\Psi_{\uparrow\downarrow}(\lambda,t) \equiv \langle \hat{s}_{-}e^{i\lambda\hat{m}}(t)\rangle = \int \mathrm{d}m \, P_{\uparrow\downarrow}(m,t)e^{i\lambda m} \simeq -\frac{b}{\hbar} \int \mathrm{d}m \int_{0}^{t} \mathrm{d}t' e^{2iNgm(t-t')/\hbar + i\lambda m} \left[P_{\uparrow\uparrow}^{(0)}(m,t') - P_{\downarrow\downarrow}^{(0)}(m,t') \right] (8.57)$$

$$= -\frac{b}{\hbar} \int_{0}^{t} \mathrm{d}t' r_{\uparrow\uparrow}(0) \exp \left[-\left(\frac{t-t'}{\tau_{\rm red}} + \frac{\lambda\delta_{0}}{\sqrt{2N}}\right)^{2} + \frac{2it'}{\tau_{\rm leak}} \left(\frac{t-t'}{\tau_{\rm red}} + \frac{\lambda\delta_{0}}{\sqrt{2N}}\right) \right] - \left\{ r_{\uparrow\uparrow} \mapsto r_{\downarrow\downarrow}, \tau_{\rm leak} \mapsto -\tau_{\rm leak} \right\}.$$

We have recombined the parameters so as to express the exponent in terms of two characteristic times, the truncation time $\tau_{red} = \hbar/g\delta_0 \sqrt{2N}$ introduced in (5.6) and the *leakage time*

$$\tau_{\text{leak}} = \sqrt{\frac{2}{N}} \frac{\hbar \delta_0}{\gamma g} = \sqrt{\frac{2}{N}} \frac{\tau_{\text{red}} \delta_0}{m_{\text{B}}} = \frac{2\tau_{\text{red}} \delta_0^2}{\gamma}.$$
(8.58)

Integration over t' can be performed in the limit $\tau_{\text{leak}} \gg \tau_{\text{red}}$, by noting that the dominant contribution arises from the region $t - t' \ll t$, which yields in terms of the error function (7.81)

$$\Psi_{\uparrow\downarrow}(\lambda,t) = -\frac{b}{2g\delta_0} \sqrt{\frac{\pi}{2N}} e^{-(t/\tau_{\text{leak}})^2} \left[r_{\uparrow\uparrow}(0) \operatorname{erfc}\left(-\frac{it}{\tau_{\text{leak}}} + \frac{\lambda\delta_0}{\sqrt{2N}}\right) - r_{\downarrow\downarrow}(0) \operatorname{erfc}\left(\frac{it}{\tau_{\text{leak}}} + \frac{\lambda\delta_0}{\sqrt{2N}}\right) \right].$$
(8.59)

The leakage time characterizes the dynamics of the transfer of polarization from the z-direction towards the x- and 3375 y-directions. It is much shorter than the registration time, since $N \gg 1$ and $\gamma \ll 1$. It also characterizes the delay over 337 which the distribution $P_{\uparrow\uparrow}^{(0)}(m,t)$ keeps a significant value at the origin: The peak of $P_{\uparrow\uparrow}^{(0)}$ with width δ_0/\sqrt{N} , moves as $m_{\rm B}(e^{t/\tau_{\rm reg}}-1) \sim m_{\rm B}t/\tau_{\rm reg}$, and at the time $t = \tau_{\rm leak}$ we have $P_{\uparrow\uparrow}^{(0)}(0, \tau_{\rm leak})/P_{\uparrow\uparrow}^{(0)}(0, 0) = 1/e$. Using the properties of the error function we can derive from Eq. (8.59), which is valid at times $t \gg \tau_{\rm red}$ such that 3377 3378

3379 the memory of $2r_{\uparrow\downarrow}(0) = \langle \hat{s}_x(0) \rangle - i \langle \hat{s}_y(0) \rangle$ is lost, by expanding the first equality of (8.57) in powers of λ , the results 3380

$$\langle \hat{s}_x(t) \rangle = -\frac{b}{g\delta_0} \sqrt{\frac{\pi}{2N}} \langle \hat{s}_z(0) \rangle \exp\left[-\left(\frac{t}{\tau_{\text{leak}}}\right)^2\right],\tag{8.60}$$

$$\langle \hat{s}_{y}(t) \rangle \approx \frac{b}{g\delta_{0}} \sqrt{\frac{2}{N}} \frac{t}{\tau_{\text{leak}}} \left[1 - \frac{2}{3} \left(\frac{t}{\tau_{\text{leak}}} \right)^{2} \right], \quad t \ll \tau_{\text{leak}},$$
(8.61)

$$\langle \hat{s}_{y}(t) \rangle \sim \frac{b}{g\delta_{0}} \frac{1}{\sqrt{2N}} \frac{\tau_{\text{leak}}}{t}, \qquad t \gg \tau_{\text{leak}}.$$
 (8.62)

where we also used that $r_{\uparrow\uparrow}(0) - r_{\downarrow\downarrow}(0) = \langle \hat{s}_z(0) \rangle$ and $r_{\uparrow\uparrow}(0) + r_{\downarrow\downarrow}(0) = 1$. For t of order τ_{leak} these results are of order 3381 $b\Delta m/g\delta_0^2$, with $\Delta m = \delta_0/\sqrt{N}$ (see Eq. (3.49)). Because $1 - \operatorname{erfc}(z) = \operatorname{erf}(z)$ is imaginary for imaginary values of z, the 3382 correlations $\langle \hat{s}_{x} \hat{m}^{k}(t) \rangle$, $k \ge 1$, vanish in this approximation, while $\langle \hat{s}_{y} \hat{m}^{k}(t) \rangle$ involves an extra factor Δm^{k} , for instance: 3383

$$\langle \hat{s}_{y}\hat{m}(t)\rangle = \frac{b}{gN}\langle \hat{s}_{z}(0)\rangle = \frac{b\,\Delta m^{2}}{g\delta_{0}^{2}}\langle \hat{s}_{z}(0)\rangle, \qquad \langle \hat{s}_{y}\hat{m}^{2}(t)\rangle = \frac{b\delta_{0}\sqrt{2}}{gN^{3/2}}\frac{t}{\tau_{\text{leak}}} = \frac{b\,\sqrt{2}\,\Delta m^{3}}{g\delta_{0}^{2}}\frac{t}{\tau_{\text{leak}}}.$$
(8.63)

To understand these behaviors, we remember that the spin S is submitted to the field b in the y-direction and to the 3384 random field Ngm in the z-direction, where m has a fluctuation δ_0/\sqrt{N} and an expectation value which varies as 3385 $\pm m_{\rm B} t / \tau_{\rm reg} = \pm \sqrt{2/N} \delta_0 t / \tau_{\rm leak}$ if the spin S is polarized in the $\pm z$ -direction. The stationary value of $\langle \hat{s}_{\rm v} \hat{m}(t) \rangle$ agrees 3386 with the value of the random field applied to S. The precession around \hat{y} explains the factor $-b\langle \hat{s}_z(0)\rangle$ in $\langle \hat{s}_x(t)\rangle$. The 3387 rotation around z hinders $\langle \hat{s}_x(t) \rangle$ through randomness of m, its effects are characterized by the parameter Ngm, of order 338 $g\delta_0 \sqrt{N}$. This explains the occurrence of this parameter in the denominator. Moreover, this same rotation around \hat{z} 3389 feeds up $\langle \hat{s}_v(t) \rangle$ from $\langle \hat{s}_x(t) \rangle$, and it takes place in a direction depending on the sign of m; as soon as registration begins, 3390 this sign of m is on average positive for $s_z = +1$, negative for $s_z = -1$. Thus the two rotations around \hat{y} and \hat{z} yield 3391 a polarization along \hat{x} with a sign opposite to that along \hat{z} , whereas the polarization along \hat{y} is positive whatever that 3392 along \hat{z} . When $t \gg \tau_{\text{leak}}$, the random values of *m* are all positive (for $P_{\uparrow\uparrow}$) or all negative (for $P_{\downarrow\downarrow}$), with a modulus 3393 larger than $1/\sqrt{N}$. Hence S precesses around an axis close to $+\hat{z}$ or $-\hat{z}$, even if b is of order $g\sqrt{N}$, so that the leakage 3394 from C_z towards C_x and C_y is inhibited for such times. Altogether, the duration of the effect is τ_{leak} , and its size is 339 characterized by the dimensionless parameter $b/g\sqrt{N}$. 3396

8.2.4. Possibility of an ideal measurement 3397

We wish to find an upper bound on the field b such that the process can be used as a measurement. Obviously, if 339 the Larmor period $\tau_{\text{Larmor}} = \pi \hbar/b$ is longer than the registration time $\tau_{\text{reg}} = \hbar/\gamma (J - T)$, we can completely disregard 3399 the field. However, we shall see that this condition, $b \ll \pi \gamma (J - T)$, is too stringent and that even large violations of 3400 the conservation law of the measured quantity \hat{s}_z do not prevent an ideal measurement. 3401

We therefore turn to the registration, still assuming that S and A remain coupled till the end of the process. At 3402 lowest order in b, the right-hand side of (8.52) and (8.53) is expressed by (8.55) with (8.56). The Green's functions 3403 G_{\uparrow} and G_{\downarrow} of the left-hand sides are given by (F.10) with h = +g and h = -g, respectively. We thus find $P_{\uparrow\uparrow}(m, t)$ 3404 through convolution of $G_{\uparrow}(m, m', t - t')$ with the initial condition $\delta(t')P_{\uparrow\uparrow}^{(0)}(m, t')$, with 3405

$$\frac{b}{\hbar}C_x^{(0)}(m',t') = \frac{b}{\hbar} \Big[P_{\uparrow\downarrow}^{(0)}(m',t') + P_{\downarrow\uparrow}^{(0)}(m',t') \Big],$$
(8.64)

and with 3406

$$\frac{b}{\hbar}C_x^{(1)}(m',t') = -\frac{2b^2}{\hbar^2} \int_0^{t'} dt'' \cos\left[2Ngm'(t'-t'')/\hbar\right] \left[P_{\uparrow\uparrow}^{(0)}(m',t') - P_{\downarrow\downarrow}^{(0)}(m',t')\right].$$
(8.65)

For $P_{\downarrow\downarrow}$ we change G_{\uparrow} into G_{\downarrow} and C_x into $-C_x$. The zeroth-order contribution, evaluated in section 7, corresponds to an ideal measurement. The first-order correction in b, $P_{\uparrow\uparrow}^{(1)}$ issued from $C_x^{(0)}$, depends on the transverse initial conditions $r_{\uparrow\downarrow}(0)$, while the second-order correction, $P_{\uparrow\uparrow}^{(2)}$ issued from $C_x^{(1)}$, depends, as the main term, on $r_{\uparrow\uparrow}(0) = 1 - r_{\downarrow\downarrow}(0)$. Performing the Gaussian integrals on m', we find:

$$\begin{split} P_{\uparrow\uparrow}^{(1)}(m,t) &= \frac{2b}{\hbar} \Re \int dm' dt' G_{\uparrow}(m,m',t-t') P_{\uparrow\downarrow}^{(0)}(m',t') \\ &= \frac{2b}{\hbar} \Re \int_{0}^{t'} dt' r_{\uparrow\downarrow}(0) \sqrt{\frac{N}{2\pi}} \frac{e^{-(t-t')/\tau_{\text{reg}}}}{\delta_{1}(t-t')} \exp\left\{-\frac{N}{2\delta_{1}^{2}(t-t')} \left[\mu'^{2} + 4g^{2}\delta_{0}^{2}\delta_{2}^{2}\frac{t'^{2}}{\hbar^{2}} - 4ig\delta_{0}^{2}\mu'\frac{t'}{\hbar}\right]\right\}, \end{split}$$
(8.66)
$$P_{\uparrow\uparrow}^{(2)}(m,t) &= -\frac{2b^{2}}{\hbar^{2}} \Re \int_{0}^{t} dt' \int_{0}^{t'} dt'' r_{\uparrow\uparrow}(0) \sqrt{\frac{N}{2\pi}} \frac{e^{-(t-t'+t'')/\tau_{\text{reg}}}}{\delta_{1}(t-t'+t'')} \\ &\times \exp\left\{-\frac{Ne^{-2t''/\tau_{\text{reg}}}}{2\delta_{1}^{2}(t-t'+t'')} \left[(\mu'-\mu'')^{2} + 4g^{2}e^{2t''/\tau_{\text{reg}}}\delta_{1}^{2}(t'')\delta_{2}^{2}\frac{(t-t'')^{2}}{\hbar^{2}} - 4ig\left(e^{2t''/\tau_{\text{reg}}}\delta_{1}^{2}(t'')\mu' + \delta_{2}^{2}\mu''\right)\frac{t'-t''}{\hbar}\right]\right\}, \\ &-\{r_{\uparrow\uparrow} \mapsto r_{\downarrow\downarrow}; \quad \mu'' \mapsto -\mu''\}, \end{aligned}$$

where $\delta_1(t)$ was defined by (7.63), where $\delta_2^2 \equiv \delta_1^2(t-t') - \delta_0^2 = T[1 - e^{-2(t-t')/\tau_{\text{reg}}}]/(J-T)$, where $\mu' \equiv -m_{\text{B}} + (m + m_{\text{B}}) \exp[-(t-t')/\tau_{\text{reg}}]$ and where $\mu'' \equiv m_{\text{B}}[\exp(t''/\tau_{\text{reg}}) - 1]$. These expressions hold for times *t* of order τ_{reg} . For later times, the part of $P_{\uparrow\uparrow}(m,t)$ for which *m* is above (below) the bifurcation $-m_{\text{B}}$ (with $m_{\text{B}} = g/(J-T)$) develops a peak around $+m_{\text{F}}(-m_{\text{F}})$. For $P_{\downarrow\downarrow}$, we have to change the sign in $P_{\uparrow\uparrow}^{(1)}$ and $P_{\uparrow\uparrow}^{(2)}$ and to replace m_{B} by $-m_{\text{B}}$ in μ' ; the bifurcation $in + m_{\text{B}}$. The probability of finding $s_z = +1$ and $m \simeq m_{\text{F}}$ at the end of the measurement is thus $\int_{-m_{\text{B}}}^{1} dm P_{\uparrow\uparrow}(m,t)$, while $\int_{-1}^{-m_{\text{B}}} dm P_{\uparrow\uparrow}(m,t)$ corresponds to $s_z = 1$ and $m \simeq -m_{\text{F}}$. Since $\int_{-m_{\text{B}}}^{1} dm P_{\uparrow\uparrow}^{(0)}(m,t) = r_{\uparrow\uparrow}(0)$ and $\int_{-1}^{-m_{\text{B}}} dm P_{\uparrow\uparrow}^{(0)}(m,t) = 0$, the contributions $P_{\uparrow\uparrow}^{(0)}$ to $P_{\uparrow\uparrow}$ and $P_{\downarrow\downarrow}^{(0)}$ to $P_{\downarrow\downarrow}$ correspond to an ideal measurement. The corrections of order *b* and *b*² to $P_{\uparrow\uparrow}$ and $P_{\downarrow\downarrow}$ give thus rise to violations of Born's rule, governed at first order in *b* by the off-diagonal elements $r_{\uparrow\downarrow}(0)$. For instance, $\int_{-m_{\text{B}}}^{1} dm P_{\uparrow\uparrow}^{(2)}(m,t)$ and $\int_{-1}^{-m_{\text{B}}} dm P_{\uparrow\uparrow}^{(2)}(m,t)$ are the contributions of these initial diagonal elements to the wrong counts $+m_{\text{F}}$ and $-m_{\text{F}}$, respectively, associated with $s_z = +1$ in the final state of S.

In order to estimate these deviations due to non-conservation of \hat{s}_z , we evaluate, as we did for the transverse quantities (8.60-8.62), the expectation values $\langle \hat{s}_z(t) \rangle$, $\langle \hat{m}(t) \rangle$, $\langle \hat{s}_z \hat{m}(t) \rangle$ issued from (8.66) and (8.67). For times $t \gg \tau_{\text{red}}$ and *t* not much longer than τ_{reg} , we find

$$\langle \hat{s}_{z}(t) \rangle = \int dm \left[P_{\uparrow\uparrow}(m,t) - P_{\downarrow\downarrow}(m,t) \right] = r_{\uparrow\uparrow}(0) - r_{\downarrow\downarrow}(0) + \frac{4b}{\hbar} \Re \int_{0}^{t} dt' r_{\uparrow\downarrow}(0) \exp\left[-\left(\frac{t'}{\tau_{red}}\right)^{2} \right]$$

$$- \frac{4b^{2}}{\hbar^{2}} \Re \int_{0}^{t} dt' \int_{0}^{t'} dt'' \left[r_{\uparrow\uparrow}(0) - r_{\downarrow\downarrow}(0) \right] \exp\left[-\left(\frac{t'-t''}{\tau_{red}}\right)^{2} + 2i\frac{t''}{\tau_{leak}} \left(\frac{t'-t''}{\tau_{red}}\right) \right]$$

$$= \langle \hat{s}_{z}(0) \rangle + \frac{b}{g\delta_{0}} \sqrt{\frac{\pi}{2N}} \langle \hat{s}_{x}(0) \rangle - \frac{b^{2}}{2N\gamma g^{2}} \langle \hat{s}_{z}(0) \rangle \left[1 - \operatorname{erfc}\left(\frac{t}{\tau_{red}}\right) \right];$$

$$(8.68)$$

we noted that only short times t', t'' and t' - t'' contribute. A similar calculation provides

$$\langle \hat{m}(t) \rangle = \int \mathrm{d}m \, m \left[P_{\uparrow\uparrow}(m,t) + P_{\downarrow\downarrow}(m,t) \right] = \langle \hat{s}_z(t) \rangle m_\mathrm{B} \left(e^{t/\tau_{\mathrm{reg}}} - 1 \right). \tag{8.69}$$

For $t \gg \tau_{\text{leak}}$, $\langle \hat{s}_z(t) \rangle$ tends to a constant which differs from the value $\langle \hat{s}_z(0) \rangle$ expected for an ideal measurement. The ratio $\langle \hat{m}(t) \rangle / \langle \hat{s}_z(t) \rangle$ is, however, the same as in section 7 where b = 0. Finally the correlation is obtained as
$$\langle \hat{s}_{z}\hat{m}(t)\rangle = \int dm \, m \left[P_{\uparrow\uparrow}(m,t) - P_{\downarrow\downarrow}(m,t)\right] = m_{\rm B} \left(e^{t/\tau_{\rm reg}} - 1\right) + \frac{4b}{\hbar} \Re \int_{0}^{t} dt' r_{\uparrow\downarrow}(0) 2ig \delta_{0}^{2} \frac{t'}{\hbar} e^{t/\tau_{\rm reg}} \exp\left[-\left(\frac{t'}{\tau_{\rm red}}\right)^{2}\right] - \frac{4b^{2}}{\hbar^{2}} \Re \int_{0}^{t} dt' \int_{0}^{t'} dt'' \left(\mu'' + 2ig \delta_{0}^{2} \frac{t'-t''}{\hbar} e^{t/\tau_{\rm reg}}\right) \exp\left[-\left(\frac{t'-t''}{\tau_{\rm red}}\right)^{2} + 2i \frac{t''}{\tau_{\rm leak}} \left(\frac{t'-t''}{\tau_{\rm red}}\right)\right] = m_{\rm B} \left(e^{t/\tau_{\rm reg}} - 1\right) + \frac{b}{Ng} \langle \hat{s}_{y}(0) \rangle e^{t/\tau_{\rm red}};$$

$$(8.70)$$

the terms in b^2 cancel out. As in (8.61), (8.62) the correlation $\langle \hat{s}_z \hat{m}(t) \rangle$ is weaker by a factor \sqrt{N} than the expectation value $\langle \hat{s}_z(t) \rangle$.

Altogether, the field *b* enters all the results (8.50), (8.60-8.62) and (8.68-8.70) through the combination $b/g\sqrt{N}$. However, the dominant deviation from Born's rule, arising from the last term of (8.68), also involves the coupling γ of M with B. The process can therefore be regarded as an ideal measurement provided

$$b \ll g \sqrt{N\gamma}.\tag{8.71}$$

Contrary to the probability of an unsuccessful measurement found in [225], which depended solely on the size of the apparatus, the present condition involves *b*, which characterizes the magnitude of the violation, as well as the couplings, *g* between S and M, and γ between M and B, which characterize the dynamics of the process. A large number *N* of degrees of freedom of the pointer and/or a large coupling *g* inhibit the transitions between $s_z = +1$ and $s_z = -1$ induced by \hat{H}_S , making the leakage time short and rendering the field *b* ineffective. If *g* is small, approaching the lower bound (7.41), the constraint (8.71) becomes stringent, since $\gamma \ll 1$. Too weak a coupling γ with the bath makes the registration so slow that *b* has time to spoil the measurement during the leakage delay.

³⁴⁴⁰ 8.2.5. Switching on and off the system-apparatus interaction

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Haastige spoed is zelden goed⁶⁸ Dutch proverb

The condition (8.71), which ensures that the process behaves as an ideal measurement although \hat{s}_z is not conserved, has been established by assuming that S and A interact from the time t = 0 to the time $t = t_f$ at which the pointer has reached $\pm m_F$. However, in a realistic ideal measurement, S and A should be decoupled both before t = 0 and after some time larger than τ_{reg} . At such times, the observable \hat{s}_z to be tested suffers oscillations with period $\tau_{Larmor} = \pi b/\hbar$, which may be rapid. Two problems then arise.

(*i*) The repeated process informs us through reading of M about the diagonal elements of the density matrix \hat{r} of S, not at any time, but *at the time when the coupling g is switched on*, that we took as the origin of time t = 0. Before this time, the diagonal elements $r_{\uparrow\uparrow}(t)$ and $r_{\downarrow\downarrow}(t)$ oscillate freely with the period τ_{Larmor} . If we wish the outcomes of M to be meaningful, we need to control, within a latitude small compared to τ_{Larmor} , the time at which the interaction is turned on. Moreover, this coupling must occur suddenly: The time during which g rises from 0 to its actual value should be short, much shorter than the leakage time.

(*ii*) Suppose that the coupling g is switched off at some time τ_{dec} larger than τ_{reg} , the condition (8.71) being 3454 satisfied. At this decoupling time $P_{\uparrow\uparrow}(m, \tau_{dec})$ presents a peak for m > 0, with weight $\int dm m P_{\uparrow\uparrow}(m, \tau_{dec}) = r_{\uparrow\uparrow}(0)$, 3455 $P_{\downarrow\downarrow}(m, \tau_{dec})$ a peak for m < 0 with weight $r_{\downarrow\downarrow}(0)$, while $P_{\uparrow\downarrow}(m, \tau_{dec})$ vanishes. Afterwards the system and the apparatus 3456 evolve independently. The Larmor precession of S [60, 61, 62, 63, 64, 65] manifests itself through oscillations of 3457 $\int dm \left[P_{\uparrow\uparrow}(m,t) - P_{\downarrow\downarrow}(m,t) \right]$ and of $\int dm \left[P_{\uparrow\downarrow}(m,t) + P_{\downarrow\uparrow}(m,t) \right]$, while M relaxes under the influence of the bath B. 3458 The two peaks of the probability distribution $P_{\rm M}(m,t) = P_{\uparrow\uparrow}(m,t) + P_{\downarrow\downarrow}(m,t)$ move apart, towards $+m_{\rm F}$ and $-m_{\rm F}$, 3459 respectively. At the final time $t_{\rm f}$, once the apparatus has reached equilibrium with broken invariance, we can observe 3460 on the pointer the outcomes $+m_F$ with probability $r_{\uparrow\uparrow}(0)$, or $-m_F$ with probability $r_{\downarrow\downarrow}(0)$. Thus the counting rate agrees 3461 with Born's rule. However the process is not an ideal measurement in von Neumann's sense: Even if the outcome of 3462

⁶⁸Being quick is hardly ever good

A is well-defined at each run (section 11), it is correlated not with the state of S at the final reading time, but only with its state $\hat{r}(\tau_{dec})$ at the decoupling time, a state which has been kept unchanged since the truncation owing to the interaction of S with M. Selecting the events with $+m_F$ at the time t_f cannot be used as a preparation of S in the state $|\uparrow\rangle$, since $\hat{r}(t)$ has evolved after the decoupling.

3467 8.3. Attempt to simultaneously measure non-commutative variables

3468Je moet niet teveel hooi op je vork nemen693469Qui trop embrasse mal étreint 703470Dutch and French proverbs

Books of quantum mechanics tell that a *precise* simultaneous measurement of non-commuting variables is impossible [10, 11, 31, 48, 84]. It is, however, physically sensible to imagine a setting with which we would try to perform such a measurement approximately [249, 250, 251, 252, 253, 254, 260]. It is interesting to analyze the corresponding dynamical process so as to understand how it differs from a standard measurement.

Consider first successive measurements. In a first stage the component \hat{s}_z of the spin S is tested by coupling S to 3475 A between the time t = 0 and some time τ_{dec} at which \hat{H}_{SA} is switched off. If τ_{dec} is larger than the registration time 3476 τ_{reg} , the apparatus A produces $m = m_{\text{F}}$ with probability $r_{\uparrow\uparrow}(0)$ and $m = -m_{\text{F}}$ with probability $r_{\downarrow\downarrow}(0)$. An interaction 3477 $\hat{H}_{SA'}$ is then switched on between S and a second apparatus A', analogous to A but coupled to the component \hat{s}_v of 347 $\hat{\mathbf{s}}$ in some v-direction. It is the new diagonal marginal state $\hat{r}(\tau_{dec})$, equal to the diagonal part of $\hat{r}(0)$, which is then 3479 tested by A'. In this measurement of \hat{s}_v the probability of reading $m' = +m_F$ on A' and finding \hat{s} in the v-direction 3480 is $r_{\uparrow\uparrow}(0)\cos^2\frac{1}{2}\theta + r_{\downarrow\downarrow}(0)\sin^2\frac{1}{2}\theta$, where θ and ϕ are the Euler angles of v. The measurement of \hat{s}_v alone would have 3481 provided the additional contribution $\Re r_{\uparrow\downarrow}(0) \sin \theta e^{i\phi}$. We therefore recover dynamically all the standard predictions 3482 of quantum mechanics. 3483

Things will be different if the second apparatus is switched on too soon after the first one or at the same time.

3485 8.3.1. A model with two apparatuses

3486 3487 Life is really simple, but we insist on making it complicated Confucius

Let us imagine we attempt to measure simultaneously the non-commuting components \hat{s}_z and \hat{s}_x of the spin \hat{s} . To this aim we extend our model by assuming that, starting from the time t = 0, S is coupled with two apparatuses A and A' of the same type as above, A' being suited to the measurement of \hat{s}_x . We denote by γ' , g', N', J', T', ..., the parameters of the second apparatus. The overall Hamiltonian $\hat{H} = \hat{H}_{SA} + \hat{H}_{SA'} + \hat{H}_A + \hat{H}_{A'}$ thus involves, in addition to the contributions defined in subsection 3.2, the Hamiltonian $\hat{H}_{A'}$ of the second apparatus A', analogous to $\hat{H}_A = \hat{H}_M + \hat{H}_B + \hat{H}_{MB}$, with magnetization $m' = (1/N') \sum_{n=1}^{N'} \hat{\sigma}_x'^{(n)}$, and the coupling term

$$\hat{H}_{\mathrm{SA}'} = -N'g'\hat{s}_x\hat{m}' \tag{8.72}$$

of A' and S. The solution of the Liouville–von Neumann equation for S + A+A' should determine how the indications of A and A' can inform us about the initial state $\hat{r}(0)$ of S, and how the final state of S is correlated with these indications.

We readily note that such a dynamical process can not behave as an ideal measurement, since we expect that, whatever the initial state $\hat{r}(0)$ of S, its final state will be perturbed.

The equations of motion are worked out as in section 4. After elimination of the baths B and B' at lowest order in γ and γ' , the density operator \hat{D} of S + M + M' can be parametrized as in § 3.3.1 and § 4.4.1 by four functions $P_{ij}(m, m', t)$, where $i, j = \uparrow, \downarrow$ refer to S, and where the magnetizations *m* and *m'* behave as random variables. However, since the functions P_{ij} are now coupled, it is more suitable to express the dynamics in terms of $P_{MM'}(m, m', t) =$ $P_{\uparrow\uparrow} + P_{\downarrow\downarrow}$, which describes the joint probability distribution of *m* and *m'*, and of the set $C_a(m, m', t)$ defined for a = x, y

⁶⁹You should not put too much hay on your fork

⁷⁰He who embraces too much fails to catch

and z by (3.29), which describe the correlations between \hat{s}_a and the two magnets M and M'. The density operator $\hat{D}(t)$ of S + M + M' generalizing (3.18), with (3.25), (3.28) and (3.29), is

$$\hat{D}(t) = \frac{2}{NN'G(\hat{m})G(\hat{m}')} \left[P_{\rm MM'}(\hat{m}, \hat{m}', t) + \mathbf{C}(\hat{m}, \hat{m}', t) \cdot \hat{\mathbf{s}} \right].$$
(8.73)

(There is no ambiguity in this definition, since \hat{m} and \hat{m}' commute.) The full dynamics are thus governed by coupled equations for the functions $P_{MM'}(m, m', t)$ and $\mathbb{C}(m, m', t)$ which parametrize $\hat{D}(t)$. The initial state $\hat{D}(0)$ is factorized as $\hat{r}(0) \otimes \hat{R}_{M}(0) \otimes \hat{R}_{M'}(0)$, where $\hat{R}_{M}(0)$ and $\hat{R}_{M'}(0)$ describe the metastable paramagnetic states (3.45) of M and M', so that the initial conditions are

$$P_{MM'}(m, m', 0) = P_M(m, 0)P_{M'}(m', 0), \qquad \mathbf{C}(m, m', 0) = P_{MM'}(m, m', 0)\langle\hat{\mathbf{s}}(0)\rangle, \tag{8.74}$$

where $P_{\rm M}(m,0)$ and $P_{\rm M'}(m',0)$ have the Gaussian form (3.48) and where $\langle \hat{\mathbf{s}}(0) \rangle$ is the initial polarization of S.

Two types of contributions enter $\partial P_{MM'}/\partial t$ and $\partial C/\partial t$, the first one active on the time scale τ_{trunc} , and the second one on the time scale τ_{reg} , but these time scales need not be very different here. On the one hand, for given *m* and *m'*, the coupling $\hat{H}_{SA} + \hat{H}_{SA'}$ of S with the magnets M and M' behaves as a magnetic field **b** applied to S. This effective field is equal to

$$\mathbf{b}(m,m') = \frac{2Ngm}{\hbar}\hat{\mathbf{z}} + \frac{2N'g'm'}{\hbar}\hat{\mathbf{x}} = b\hat{\mathbf{u}}, \qquad b(m,m') \equiv |\mathbf{b}(m,m')| = \frac{2}{\hbar}\sqrt{N^2g^2m^2 + N'^2g'^2m'^2}, \tag{8.75}$$

where $\hat{\mathbf{z}}$ and $\hat{\mathbf{x}}$ are the unit vectors in the z- and x-direction, respectively. This yields to $\partial \mathbf{C}/\partial t$ the contribution

$$\left[\frac{\partial \mathbf{C}(m,m',t)}{\partial t}\right]_{\mathrm{MM}'} = -\mathbf{b}(m,m') \times \mathbf{C}(m,m',t).$$
(8.76)

Both the Larmor frequency b and the precession axis, characterized by the unit vector $\hat{\mathbf{u}} = \mathbf{b}/b$ in the x-z plane, depend 3516 on m and m' (whereas the precession axis was fixed along \hat{z} for a single apparatus). The distribution $P_{MM'}(m, m', t)$ is 3517 insensitive to the part $\hat{H}_{SA} + \hat{H}_{SA'}$ of the Hamiltonian, and therefore evolves slowly, only under the effect of the baths. 3518 On the other hand, $\partial P_{MM'}/\partial t$ and $\partial C/\partial t$ involve contributions from the baths B and B', which can be derived from 3519 the right-hand sides of (4.30) and (4.29). They couple all four functions $P_{MM'}$ and C, they are characterized by the time 3520 scale τ_{reg} , and they depend on all parameters of the model. In contrast with what happened for a single apparatus, the 3521 effects of the precession (8.76) and of the baths can no longer be separated. Indeed, the precession tends to eliminate 3522 the components of C(m, m', t) that are perpendicular to **b**, but the baths tend to continuously activate the creation 3523 of such components. The truncation, which for a single apparatus involved only the off-diagonal sectors and was 3524 achieved after a brief delay, is now replaced by an overall damping process taking place along with the registration, 3525 under the simultaneous contradictory effects of the couplings of M and M' with S and with the baths. 3526

Such an interplay, together with the coupling of four functions $P_{MM'}$, **C** of three variables *m*, *m'*, *t*, make the equations of motion difficult to solve, whether analytically or numerically. A qualitative analysis will, however, suffice to provide us with some interesting conclusions.

3530 8.3.2. Structure of the outcome

Note first that the positivity of the density operator (8.73), maintained by the dynamics, is expressed by the condition

$$P_{\text{MM}'}(m, m', t) \ge |\mathbf{C}(m, m', t)|,$$
(8.77)

³⁵³³ which holds at any time.

The outcome of the process is characterized by the limit, for *t* larger than the registration time τ_{reg} , of the distributions $P_{\text{MM}'}$ and **C**. In this last stage of the evolution, the interaction of M with the bath B is expected to drive it towards either one of the two equilibrium states at temperature *T*, for which the normalized distribution $P_{\text{M}\parallel}(m)$ (or $P_{\text{M}\parallel}(m)$) expressed by (3.56) is concentrated near $m = +m_{\text{F}}$ (or $m = -m_{\text{F}}$). In order to avoid the possibility of a final relaxation of M towards its metastable paramagnetic state, which may produce failures as in § 7.3.4, we consider here only a quadratic coupling J_2 . Likewise, M' is stabilized into either one of the ferromagnetic states $P'_{\text{M}\uparrow\uparrow}(m')$ (or $P_{\text{M}\downarrow}(m')$) with $m' \simeq +m'_{\rm F}$ (or $m' = -m'_{\rm F}$). Hence, $P_{\rm MM'}(m, m', t)$, which describes the statistics of the indications of the pointers, ends up as a sum of four narrow peaks which settle at $m = \varepsilon m_{\rm F}$, $m' = \varepsilon' m'_{\rm F}$, with $\varepsilon = \pm 1$, $\varepsilon' = \pm 1$, to wit,

$$P_{\mathrm{MM}'}(m,m',t) \mapsto \sum_{\varepsilon=\pm 1} \sum_{\varepsilon'=\pm 1} \mathcal{P}_{\varepsilon\varepsilon'} P_{\mathrm{M}\varepsilon}(m) P_{\mathrm{M}'\varepsilon'}(m').$$
(8.78)

The weights $\mathcal{P}_{\varepsilon\varepsilon'}$ of these peaks characterize the proportions of counts detected on M and M' in repeated experiments; they are the only observed quantities.

The precession (8.76) together with smoothing over *m* and *m'* eliminates the component C_y of **C**, so that the subsequent evolution keeps no memory of $C_y(m, m', 0)$. Thus, among the initial data (8.74) pertaining to S, only $\langle \hat{s}_x(0) \rangle$ and $\langle \hat{s}_z(0) \rangle$ are relevant to the determination of the final state: the frequencies $\mathcal{P}_{\varepsilon\varepsilon'}$ of the outcomes depend only on $\langle \hat{s}_x(0) \rangle$ and $\langle \hat{s}_z(0) \rangle$ (and on the parameters of the apparatuses).

If $\langle \hat{s}_x(0) \rangle = \langle \hat{s}_z(0) \rangle = 0$ we have $\mathcal{P}_{\varepsilon\varepsilon'} = \frac{1}{4}$ due to the symmetry $m \leftrightarrow -m, m' \leftrightarrow -m'$. Likewise, if $\langle \hat{s}_x(0) \rangle = 0$, the symmetry $m' \leftrightarrow -m'$ implies that $\mathcal{P}_{++} = \mathcal{P}_{+-}$ and $\mathcal{P}_{-+} = \mathcal{P}_{--}$. Since the equations of motion are linear, $\mathcal{P}_{++} - \mathcal{P}_{-+}$ is in this situation proportional to $\langle \hat{s}_z(0) \rangle$; we define the proportionality coefficient λ by $\mathcal{P}_{\varepsilon+} = \frac{1}{4}(1 + \varepsilon\lambda\langle \hat{s}_z(0)\rangle)$. In the situation $\langle \hat{s}_z(0) \rangle = 0$ we have similarly $\mathcal{P}_{+\varepsilon'} = \mathcal{P}_{-\varepsilon'} = \frac{1}{4}(1 + \varepsilon'\lambda'\langle \hat{s}_x(0)\rangle)$. Relying on the linearity of the equations of motion, we find altogether for an arbitrary initial state of S the general form for the probabilities $\mathcal{P}_{\varepsilon\varepsilon'}$:

$$\mathcal{P}_{\varepsilon\varepsilon'} = \frac{1}{4} \left(1 + \varepsilon \lambda \langle \hat{s}_z(0) \rangle + \varepsilon' \lambda' \langle \hat{s}_x(0) \rangle \right), \tag{8.79}$$

where $\langle \hat{s}_z(0) \rangle = r_{\uparrow\uparrow}(0) - r_{\downarrow\downarrow}(0), \langle \hat{s}_x(0) \rangle = r_{\uparrow\downarrow}(0) + r_{\downarrow\uparrow}(0)$. We term λ and λ' the *efficiency factors*.

In the long time limit, the functions $\mathbf{C}(m.m', t)$ also tend to sums of four peaks located at $m = \pm m_F$, $m' = \pm m'_F$, as implied by (8.77). With each peak is associated a direction $\mathbf{u}_{\varepsilon\varepsilon'}$, given by (8.75) where $m = \varepsilon m_F$, $m' = \varepsilon' m'_F$, around which the precession (8.76) takes place. The truncation process eliminates the component of \mathbf{C} perpendicular to $\mathbf{u}_{\varepsilon\varepsilon'}$, for each peak. Thus, if in their final state the apparatuses M and M' indicate εm_F , $\varepsilon' m'_F$, the spin S is lead into a state partly polarized in the direction $\mathbf{u}_{\varepsilon\varepsilon'}$ of the effective field **b** generated by the two ferromagnets.

3559 8.3.3. A fully informative statistical process

A well-defined indication for both pointers M and M' can be obtained here in each individual run, because the argument of § 11.2.3 holds separately for the apparatuses A and A' at the end of the process. A mere counting of the pair of outcomes ε , ε' then provides experimentally the probability (8.79).

However, the present process cannot be regarded as an ideal measurement. On the one hand, the above-mentioned correlations between the final state of S and the indications of the apparatus are not complete; they are limited by the inequality (8.77). In an ideal measurement the correlation must be *complete*: if the apparatuses are such that they provide well-defined outcomes at each run (section 11), and if for a given run we read $+m_F$ on the apparatus M measuring \hat{s}_z , the spin S must have been led by the ideal process into the pure state $|\uparrow\rangle$. Here we cannot make such assertions about an individual system, and we cannot use the process as a preparation.

On the other hand, in an ideal measurement, the outcome of the process is *unique* for both S and M in case S is initially in an eigenstate of the tested quantity. Suppose the spin S is initially oriented up in the *z*-direction, that is, $\hat{r}(0) = |\uparrow\rangle\langle\uparrow|$. The response of the apparatuses M and M' is given by (8.79) as

$$\mathcal{P}_{++} = \mathcal{P}_{+-} = \frac{1}{4}(1+\lambda), \qquad \mathcal{P}_{-+} = \mathcal{P}_{--} = \frac{1}{4}(1-\lambda),$$
(8.80)

so that there exists a probability $\frac{1}{2}(1 - \lambda)$ to *read the wrong result* $-m_{\rm F}$ on M. Indeed, without even solving the equations of motion to express the efficiency factors λ and λ' in terms of the various parameters of the model, we can assert that λ is smaller than 1: Because all $\mathcal{P}_{\varepsilon\varepsilon'}$ must be non-negative for any initial state of S, and because (8.79) has the form $\frac{1}{4}(1 + \mathbf{a} \cdot \langle \hat{\mathbf{s}}(0) \rangle)$, we must have $|\mathbf{a}| < 1$, so that λ and λ' should satisfy

$$\lambda^2 + {\lambda'}^2 \le 1,\tag{8.81}$$

and because not only \hat{s}_z but also \hat{s}_x are tested, λ' should be non zero so that the probability of failure $\frac{1}{2}(1 - \lambda)$ is finite. It is therefore clear why the attempt to perform a simultaneous *ideal* measurement of \hat{s}_x and \hat{s}_z fails. Both Born's rule

³⁵⁷⁸ and von Neumann's truncation are violated.

Nevertheless, consider a set of repeated experiments in which we read simultaneously the indications of the two 3579 apparatuses M and M'. If the runs are sufficiently numerous, we can determine the probabilities $\mathcal{P}_{\mathcal{E}}$ from the fre-3580 quencies of occurrence of the four possible outcomes $\pm m_{\rm F}$, $\pm m_{\rm F'}$. Let us assume that the coefficients λ , λ' , which 358 depend on the parameters of the model, take significant values. This requires an adequate choice of these parameters. 3582 In particular, the couplings g and g', needed to trigger the beginning of the registration, should however be small 3583 and should soon be switched off so as to reduce the blurring effect of the precession around b. This smallness is 3584 consistent with the choice of a second order transition for M, already noted. Finally, the couplings γ , γ' should ensure 3585 registration before disorder is settled. Under such conditions, inversion of eq. (8.79) yields 3586

$$\langle \hat{s}_{z}(0) \rangle = r_{\uparrow\uparrow}(0) - r_{\downarrow\downarrow}(0) = \frac{1}{\lambda} (\mathcal{P}_{++} + \mathcal{P}_{+-} - \mathcal{P}_{-+} - \mathcal{P}_{--}), \langle \hat{s}_{x}(0) \rangle = r_{\uparrow\downarrow}(0) + r_{\downarrow\uparrow}(0) = \frac{1}{\lambda'} (\mathcal{P}_{++} - \mathcal{P}_{+-} + \mathcal{P}_{-+} - \mathcal{P}_{--}).$$

$$(8.82)$$

Thus, a sequence of repeated experiments *reveals the initial expectation values of both* \hat{s}_z *and* \hat{s}_x , although these observables do not commute.

Paradoxically, as regards the determination of an unknown initial density matrix, the present process is *more informative than an ideal measurement* with a single apparatus [254]. Repeated measurements of \hat{s}_z yield $r_{\uparrow\uparrow}(0)$ (and $r_{\downarrow\downarrow}(0)$) through counting of the outcomes $\pm m_F$ of M. Here we moreover find through repeated experiments the real part of $r_{\uparrow\downarrow}(0)$. However, more numerous runs are needed to reach a given precision if λ and λ' are small. (If the parameters of the model are such that λ and λ' nearly vanish, the relaxation of M and M' is not controlled by S, all $\mathcal{P}_{\varepsilon\varepsilon'}$ lie close to $\frac{1}{4}$, and the observation of the outcomes is not informative since they are fully random.)

More generally, for a repeated process using three apparatuses M, M' and M'' coupled to \hat{s}_z , \hat{s}_x and \hat{s}_v , respectively, 3595 the statistics of readings allows us to determine simultaneously all matrix elements of the initial density operator $\hat{r}(0)$. 3596 The considered single compound apparatus thus provides full statistical information about the state $\hat{r}(0)$ of S. Our 3597 knowledge is gained indirectly, through an expression of the type (8.82) which involves both statistics and calibration 3598 so as to determine the parameters λ , λ' and λ'' . A process of the present type, although it violates the standard rules of 3599 the ideal measurement, can be regarded as a *complete statistical measurement* of the initial state of S. The knowledge 3600 of the efficiency factors allows us to determine simultaneously the statistics of the observables currently regarded as 3601 incompatible. The price to pay is the loss of precision due to the fact that the efficiency factors are less than 1, which 3602 requires a large number of runs. 3603

The dynamics thus establish a one-to-one *correspondence between the initial density matrix of* S, which embeds the whole quantum probablistic information on S, and the *classical probabilities of the various indications* that may be registered by the apparatuses at the final time. The possibility of such a mapping was considered in [68]. The size of the domain in which the counting rates may lie is limited; for instance, if S is initially polarized along z in (8.79), no $\mathcal{P}_{\varepsilon\varepsilon'}$ can lie beyond the interval $[\frac{1}{4}(1-\lambda), \frac{1}{4}(1+\lambda)]$. The limited size of the domain for the probabilities of the apparatus indications is needed to reconcile the classical nature of these probabilities with the peculiarities of the quantum probabilities of S that arise from non commutation. It also sets limitations on the precision of the measurement.

Motivated by the physics of spin-orbit interaction in solids, Sokolovski and Sherman recently studied a model related to (8.72) [261]. Two components of the spin $\frac{1}{2}$ couple not with collective magnetizations as in (8.72), but with the components of the momentum (the proper kinetic energy is neglected so that these are the only two terms in the Hamiltonian). The motivation for studing this model is the same as above: to understand the physics of simultaneous measurement for two non-commuting observables [261]. The authors show that, as a result of interaction, the average components of the momentum get correlated with the time-averaged values of the spin [instead of the initial values of the spin as in (8.78), (8.79)]. This difference relates to the fact that the model by Sokolovski and Sherman does not have macroscopic measuring apparatuses that would enforce relaxation in time.

3620 8.3.4. Testing Bell's inequality

Bell's inequality for an EPR [51] pair of spins is expressed in the CHSH form as [262]

$$|\langle \hat{s}_{a}^{(1)} \hat{s}_{a'}^{(2)} \rangle + \langle \hat{s}_{b}^{(1)} \hat{s}_{a'}^{(2)} \rangle + \langle \hat{s}_{a}^{(1)} \hat{s}_{b'}^{(2)} \rangle - \langle \hat{s}_{b}^{(1)} \hat{s}_{b'}^{(2)} \rangle| \le 2,$$
(8.83)

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which holds for classical random variables $s = \pm 1$. If $\hat{s}_a^{(1)}$ and $\hat{s}_b^{(1)}$ are the components of a quantum spin $\hat{s}^{(1)}$ in the two fixed directions *a* and *b*, $\hat{s}_{a'}^{(2)}$ and $\hat{s}_{b'}^{(2)}$ the components the other spin $\hat{s}^{(2)}$ in directions *a'* and *b'*, the left-hand side of (8.83) can rise up to $2\sqrt{2}^{71}$.

Standard measurement devices allow us to test simultaneously a pair of commuting observables, for instance $\hat{s}_a^{(1)}$ 3625 and $\hat{s}_{a'}^{(2)}$. At least theoretically, the counting rates in repeated runs *directly* provide their correlation, namely $\langle \hat{s}_{a}^{(1)} \hat{s}_{a'}^{(2)} \rangle$. However, since $\hat{s}_{a}^{(1)}$ and $\hat{s}_{b}^{(1)}$, as well as $\hat{s}_{a'}^{(2)}$ and $\hat{s}_{b'}^{(2)}$ do not commute, we need four different settings to determine the four terms of (8.83). Checking the violation of Bells inequalities thus requires combining the outcomes of *four* 3626 362 3628 incompatible experimental contexts [264, 265, 266], in each of which the spin pair is being tested through repeated 3629 runs. This necessity may be regarded as a "contextuality loophole" [267, 268]. Either hidden variables exist, and 3630 they cannot be governed by ordinary probabilities and ordinary logic, since there is no global distribution function 3631 that would yield as marginals the partial results tested in the four different contexts. Or we must admit that quantum 3632 mechanics forbids us to put together the results of these different measurements. The latter alternative is favoured by the solution of models, in which the values of physical quantities do not pre-exist but are produced during a 363 measurement process owing to the interaction between the system and the apparatus. Since these values reflect the 3635 reality of the system only within its context, it appears inconsistent to put them together [264, 265, 266, 267, 268]. 3636

In the present situation it is tempting to imagine using a combination of apparatuses of the previous type so as to simultaneously test all four non-commuting observables $\hat{s}_{a}^{(1)}$, $\hat{s}_{b}^{(1)}$, $\hat{s}_{a'}^{(2)}$, and $\hat{s}_{b'}^{(2)}$ through repeated runs. Such a unique experimental setting would bypass the contextuality loophole. However, as shown in § 8.3.3, the counting rates of the two apparatuses associated with the components $\hat{s}_{a}^{(1)}$ and $\hat{s}_{b'}^{(1)}$ of the first spin are not directly related to the statistics of these components, but only reflect them through an efficiency factor λ at most equal to $1/\sqrt{2}$. For the pair of spins, one can *deduce* a correlation such as $\langle \hat{s}_{a}^{(1)} \hat{s}_{a'}^{(2)} \rangle$ from the statistical indications of the corresponding apparatuses, but this *quantum correlation* is at least equal to *twice* the associated *observed correlation* (since $1/\lambda^2 > 2$).

Thus, with this experimental setting which circumvents the contextuallity loophole, the correlations directly ex-3644 hibited by the counting rates satisfy Bell's inequality; this is natural since the outcomes of the macroscopic apparatus 3645 are measured simultaneously and therefore have a classical nature [269]. However, from these very observations, 364 we can use standard quantum mechanics to analyse the results. We thus infer indirectly from the observations, by 364 using a mapping of the type (8.82), the tested quantum correlations (8.83) between spins components. Within a *single* 3648 set of repeated experiments where the various data are simultaneously registered, we thus acknowledge the viola-3649 tion of Bell's inequality. Here this violation no longer appears as a consequence of merging incompatible sets of 3650 measurements, but as a consequence of a theoretical analysis of the ordinary correlations produced in the apparatus. 3651

3652 9. Analysis of the results

3653 3654

And the rain from heaven was restrained Genesis 8.2

In section 3 we have introduced the Curie–Weiss model for the quantum measurement of a spin $\frac{1}{2}$ and in sections 4–8 we have discussed the dynamics of the density operator characterizing a large set of runs. For the readers who have not desired to go through all the details, and for those who did, we resume here the main points as a separate reading guide, and add pedagogical hints for making students familiar with the matter and techniques. We will discuss the solution of the quantum measurement problem for this model in section 11 by considering properties of individual runs.

3661 9.1. Requirements for models of quantum measurements

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J'ai perdu mon Eurydice⁷²

⁷¹For the establishment of Bell-type equalities for SQUIDs, see Jaeger et al. [263]
⁷²I lost my Euridice

Christoph Willibald Gluck, Orphée et Eurydice; Orfeo ed Euridice A model accounting for the various properties of ideal quantum measurements should in principle satisfy the following requirements ("R"): **R1**: simulate as much as possible real experiments; **R2**: ensure unbiased, robust and permanent registration by the pointer of A, which should therefore be macroscopic; the pointer should give at each run a well-defined indication, which requires sufficiently complex interactions within

the apparatus (dynamical stability and hierarchic structure of subensembles, see § 11.2.1); **R3**: involve an apparatus initially in a metastable state and evolving towards one or another stable state under the

influence of S, so as to amplify this signal; the transition of A, instead of occurring spontaneously, is triggered by S;

R4: include a bath where the free energy released because of the irreversibility of the process may be dumped;

R5: be solvable so as to provide a complete scenario of the joint evolution of S + A and to exhibit the characteristic times;

³⁶⁷⁶ **R6**: conserve the tested observable;

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R7: lead to a final state devoid of "Schrödinger cats"; for the whole set of runs (truncation, § 1.3.3), and to a von Neumann reduced state for each individual run;

R8: satisfy Born's rule for the registered results;

R9: produce, for ideal measurements or preparations, the required diagonal correlations between the tested system S and the indication of the pointer, as coded in the expression (9.1) for the final state of S + A;

R10: be sufficiently flexible to allow discussing processes that are not perfect measurements.

These features need not be fulfilled with mathematical rigor. A physical scope is sufficient, where violations may occur over unreachable time scales or with a negligible probability.

3685 9.2. Features of the Curie–Weiss model

The above Curie–Weiss model is satisfactory in this respect (except for the requirement **R2** which will be discussed 3686 in § 11.2.1). Its choice (section 3) has relied on a compromise between two conflicting requirements. On the one hand, 3687 the apparatus A simulates a *real object*, a magnetic dot which behaves as a magnetic memory. On the other hand, the Hamiltonian of S + A is sufficiently simple so as to afford an explicit and detailed *dynamical solution*. The registration 3689 device is schematized as a set M of N Ising spins (the magnet). The size of the dot is supposed to be much smaller than 3690 the range of the interactions, both among the N spins and between them and the tested spin S. We further simplify by 3691 taking into account only interactions between the z-components of the spins of M and S. Finally, as in a real magnetic 3692 dot, phonons (with a quasi-ohmic behavior [120, 121, 151, 152, 174]) behave as a thermal bath B which ensures 3693 equilibrium in the final state (Fig. 3.1). In spite of the schematic nature of the model, its solution turns out to exhibit 369/ a rich structure and to display the various features listed in subsection 9.1.

In particular, the choice for A = M + B of a system which can undergo a phase transition implies many properties 369 desirable for a measuring apparatus. The weakness of the interaction γ between each spin of the magnet M and 3697 the phonon bath B, maintained at a temperature T lower than T_c , ensures a long lifetime for the initial *metastable* 3698 paramagnetic state. By itself, the system M+B would ultimately relax spontaneously towards a stable state, but here 3699 3700 its transition is triggered by S. The symmetry breaking in the dynamics of the measurement produces either one of the two possible final stable ferromagnetic states, in one-to-one correspondence with the eigenvalues of the tested 3701 observable \hat{s}_z of the system S, so that the sign of the final magnetization can behave as a pointer. It is this breaking 3702 of symmetry which underlies registration, entailing the *irreversibility* of the transition from the paramagnetic to either 370.3

Che farò senza Euridice?⁷³

⁷³What shall I do without Euridice?

one of the ferromagnetic macroscopic states. Moreover, the built-in symmetry between the two possible outcomes of A prevents the appearance of *bias*.

An essential property of a measurement, often overlooked, is the ability of the apparatus A to register the indication 3706 of the pointer. Here this is ensured by the large value of the number N of spins of M, which entails a neat separation 3707 between the two ferromagnetic states of M and their extremely long lifetime. This stability warrants a permanent 3708 and *robust* registration. The large value of N is also an essential ingredient in the proof of the uniqueness of the 3709 indication fo the pointer in each run (§ 11.2.3). In both the paramagnetic state and the ferromagnetic states, the 3710 pointer variable m presents statistical fluctuations negligible as $1/\sqrt{N}$. Moreover, breaking of invariance makes 3711 quantum coherences ineffective (§ 11.2.3). The nature of the order parameter, a macroscopic magnetization, also 3712 makes the result accessible to reading, processing or printing. These properties cannot be implemented in models for 3713 which the pointer is a microscopic object. 3714

The coupling between the tested spin S and the apparatus A has been chosen in such a way that the observable \hat{s}_z is *conserved*, $[\hat{s}_z, \hat{H}] = 0$, so as to remain unperturbed during its measurement. This coupling *triggers* the beginning of the registration process, which thereby ends up in a situation which *informs us* about the the physical state of S at a certain moment, so that the process might be used as a measurement. This requires a sufficiently large value of the coupling constant *g* which characterizes the interaction of S and M.

Once the probability distribution of the magnetization *m* has left the vicinity of m = 0 to move towards either + $m_{\rm F}$ or $-m_{\rm F}$, the motion of this pointer is *driven by the bath* through the coupling γ between M and B. Somewhat later the interaction *g* between S and A becomes ineffective and can be switched off. It is the interplay between the metastability of the initial state of A, the initial triggering of M by S, and the ensuing action of B on M which ensures an *amplification* of the initial perturbation. This amplification is necessary since the indication of the pointer M, which is macroscopic, should reflect an effect caused by the tested system S, which is microscopic — the very essence of a measurement.

Such a number of adequate properties makes this model attractive, but technical developments were needed to elaborate in sections 4 to 7 a rigorous proof that the final state of S + A has the form (1.7), viz.

$$\hat{\mathcal{D}}(t_{\rm f}) = \sum_{i} \left(\hat{\Pi}_{i} \hat{r}(0) \,\hat{\Pi}_{i} \right) \otimes \hat{\mathcal{R}}_{i} = \sum_{i} p_{i} \hat{r}_{i} \otimes \hat{\mathcal{R}}_{i}, \tag{9.1}$$

where \hat{r} describes S and $\hat{\mathcal{R}}$ describes A. This form encompasses most among the required specific features of ideal quantum measurements, in particular the absence of off-diagonal terms. These developments have allowed us to *discuss the conditions* under which the process might be used as a measurement, and also to explore what happens if one or another condition is violated.

Note, however, that the final form (9.1) of the density operator of S + A concerns the statistics of a large set of runs of the measurement. This form is *necessary, but not sufficient*, to ensure that the interaction process can be regarded as an ideal measurement. It remains to elaborate the physical interpretation of this result by turning to individual measurements. We postpone his task to section 11.

9.3. Scenario of the Curie–Weiss ideal measurement: the characteristic time scales

The above study (sections 4–7) of the dynamical process undergone by S + A has revealed several successive steps involving different time scales.

3740 9.3.1. Preparation

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3742

*Co se doma uvaří, to se doma sní*⁷⁴ Czech proverb

Before S and A are coupled, A should be prepared in a metastable state. Indeed, in the old days of photography the unexposed film was metastable and could not be prevented from evolving in the dark on a time scale of months. In our magnetic case, for quartic interactions within M, the *lifetime of the paramagnetic initial state* is extremely large,

⁷⁴What is cooked home is eaten home

exponentially large in *N*. For quadratic interactions with coupling constant *J*, it was evaluated in section § 7.3.2 (eq. (7.66)) as

$$\tau_{\text{para}} = \frac{\hbar}{\gamma(J-T)} \ln \alpha \sqrt{N}, \tag{9.2}$$

where α is typically of order 1/10, and it is larger than all other characteristic times for $\alpha \sqrt{N} \gg 1$. We can thus engage the measurement process by switching on the interaction between S and M during the delay τ_{para} after preparation of A, before the paramagnetic state is spontaneously spoiled.

3751 9.3.2. Truncation

Let us recall (§ 3.3.2 and Fig. 3.2) our decomposition of the density matrix \hat{D} of the total system S + A into blocks with definite value $s_z = \uparrow, \downarrow$ of the tested spin component \hat{s}_z :

$$\hat{\mathcal{D}} = \begin{pmatrix} \hat{\mathcal{R}}_{\uparrow\uparrow} & \hat{\mathcal{R}}_{\uparrow\downarrow} \\ \hat{\mathcal{R}}_{\downarrow\uparrow} & \hat{\mathcal{R}}_{\downarrow\downarrow} \end{pmatrix}.$$
(9.3)

The first stage of the measurement process is the truncation, defined as the disappearance of the off-diagonal blocks $\hat{\mathcal{R}}_{\uparrow\downarrow}$ and $\hat{\mathcal{R}}_{\downarrow\uparrow}$ of the full density matrix (section 5). It takes place during the *truncation time*

$$\tau_{\rm red} = \frac{\hbar}{\sqrt{2N}\delta_0 g},\tag{9.4}$$

which is governed by the coupling constant *g* between S and M and the size *N* of the pointer (the fluctuation of M in the paramagnetic state is δ_0/\sqrt{N}). This characteristic time is the shortest of all; its briefness reflects an effect produced by a *macroscopic object*, the pointer M, on a *microscopic one*, the tested system S. During the delay τ_{red} , the off-diagonal components a = x, y of the spin S decay on average as $\langle \hat{s}_a(t) \rangle = \langle \hat{s}_a(0) \rangle \exp[-(t/\tau_{red})^2]$.

Over the time scale τ_{red} , only the off-diagonal blocks $\hat{\mathcal{R}}_{\uparrow\downarrow} = \hat{\mathcal{R}}_{\downarrow\uparrow}^{\dagger}$ of the overall density matrix $\hat{\mathcal{D}}$ of S + A are affected by the evolution. *Correlations* between S and M, involving larger and larger numbers $k = 1, 2, \cdots$ of spins 3760 3761 of M, such as $\langle \hat{s}_a \hat{m}^k(t) \rangle_c \propto t^k \exp[-(t/\tau_{red})^2]$ (a = x, y) are successively created in a *cascade*: They develop later 3762 and later, each one reaches a small maximum for $t = \tau_{red} \sqrt{k/2}$ and then tends to zero (§ 5.1.3 and Fig. 5.1). The 3763 information originally carried by the off-diagonal elements of the initial density matrix of S are thus transferred 376 towards correlations which couple the system S with more and more spins of M and eventually decline (§ 5.1.4). 3765 When t increases far beyond τ_{red} , all the matrix elements of $\hat{\mathcal{R}}_{\uparrow\downarrow}$ that contribute to correlations of rank $k \ll N$ tend to 3766 zero. Correlations of higher rank k, for large but finite N, are the residue of reversibility of the microscopic evolution 3767 generated by \hat{H}_{SA} (§ 5.3.2). 3768

If the total Hamiltonian of S + A did reduce to the coupling $\hat{H}_{SA} = -Ng\hat{s}_z\hat{m}$ which produces the above behavior, the truncation would be provisional, since S + A would periodically return to its initial state with the *recurrence time*

$$\tau_{\rm recur} = \frac{\pi\hbar}{2g},\tag{9.5}$$

much larger than τ_{red} (§ 5.3.1). As in spin-echo experiments, the extremely small but extremely numerous correlations created by the interaction between S and the many spins of M would conspire to progressively reconstruct the offdiagonal blocks of the initial uncorrelated state of S + A: The reversibility and simplicity of the dynamics would ruin the initial truncation.

Two possible mechanisms can prevent such recurrences to occur. In subsection 6.1 we slightly modify the model, taking into account the (realistic) possibility of a spread δg in the coupling constants g_n between S and each spin of the magnet M. The Hamiltonian (6.1) with the conditions (6.2) then produces the same initial truncation as with constant g, over the same characteristic time τ_{red} , but recurrences are now ruled out owing to the dispersion of the g_n , which produces an extra damping as $\exp[-(t/\tau_{irrev}^{M})^2]$. The *irreversibility time induced by the spreading* δg in the spin-magnet couplings,

$$\tau_{\rm irrev}^{\rm M} = \frac{\hbar}{\sqrt{2N}\delta g},\tag{9.6}$$

is intermediate between τ_{red} and τ_{recur} provided δg is sufficiently large, viz. $g/\sqrt{N} \ll \delta g \ll g$. As usual for a reversible linear evolution, a recurrence phenomenon still occurs here, but the recurrence time is inaccessibly large as shown in $\delta 6.1.2$ (see eq. (6.20)). The numerous but weak correlations between S and M, issued from the off-diagonal blocks of the initial density matrix of S, are therefore completely ineffective over any reasonable time lapse.

An alternative mechanism can also rule out any recurrence, even if the couplings between S and the spins are all equal (subsection 6.2). In this case, the required irreversibility is induced by the bath, which produces an extra decay, as exp[-NB(t)], of the off-diagonal blocks (the shape of B(t) is shown in Fig 6.1). The initial truncation of section 5, for $t \ll 1/\Gamma$, is not affected by the interaction with the bath if $NB(\tau_{red}) \ll 1$, that is, if

$$\frac{\gamma\hbar^2\Gamma^2}{8\pi N\delta_0^4 g^2} \ll 1,\tag{9.7}$$

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where Γ is the Debye cutoff on the phonon frequencies. At times *t* such that $t \gg \hbar/2\pi T$, B(t) is quasi linear and the bath produces an exponential decay, as $\exp(-t/\tau_{\text{irrev}}^{\text{B}})$, where the *bath-induced irreversibility time* is defined as

$$\tau_{\rm irrev}^{\rm B} = \frac{2\hbar \tanh g/T}{N\gamma g} \simeq \frac{2\hbar}{N\gamma T}.$$
(9.8)

This expression is a typical decoherence time, inversely proportional to the temperature *T* of B, to the bath-magnet coupling γ and to the number *N* of degrees of freedom of the system S + M. The *p*-th recurrence is then damped by a factor $\exp(-p\tau_{\text{recur}}/\tau_{\text{irrev}}^{\text{B}})$, so that the phonon bath eliminates all recurrences if $\tau_{\text{irrev}}^{\text{B}} \ll \tau_{\text{recur}}$.

At this stage, *the truncation is achieved* in the sense that the off-diagonal blocks $\hat{R}_{\uparrow\downarrow}(t)$ and $\hat{R}_{\downarrow\uparrow}(t)$ of the density operator (9.3) of S + A have practically disappeared in a definitive way. The off-diagonal correlations created during the truncation process have been irremediably destroyed at the end of this process, whereas the diagonal correlations needed to register in A the tested properties of S are not yet created. See also § 11.2.3 below.

3798 9.3.3. Registration

Just after the above processes are achieved, the diagonal blocks $\hat{\mathcal{R}}_{\uparrow\uparrow}(t)$ and $\hat{\mathcal{R}}_{\downarrow\downarrow}(t)$ as well as the marginal density operator $\hat{\mathcal{R}}(t) = \text{tr}_{S}\hat{\mathcal{D}}(t) = \hat{\mathcal{R}}_{\uparrow\uparrow}(t) + \hat{\mathcal{R}}_{\downarrow\downarrow}(t)$ of A *remain nearly unaffected*. The process cannot yet be regarded as a measurement: The pointer gives no indication, *m* is still small, and no correlation exists between A and the initial state of S.

The registration then starts and proceeds on time scales much larger than the above ones. It is a slower process 3803 because it leads to a change of a macroscopic object, the apparatus, triggered by the microscopic S. After a brief 3804 transient regime, the process becomes Markovian (§ 7.1.1). The evolution of each of the two diagonal blocks $\hat{\mathcal{R}}_{\uparrow\uparrow}(t)$ 3805 or $\hat{\mathcal{R}}_{\downarrow\downarrow}(t)$ can be expressed in terms of that of the corresponding probability distribution $P_{\uparrow\uparrow}(m,t)$ or $P_{\downarrow\downarrow}(m,t)$ for the 3806 magnetization of M, which obeys an equation of the Fokker-Planck type [230]. This equation, presenting classical 3807 features (§ 7.1.2), is governed for $P_{\uparrow\uparrow}(m, t)$ by a *drift velocity* v(m) given by (7.6) and illustrated by Figs. 7.1 and 7.2, 3808 and by a *diffusion* coefficient given by (7.7). The irreversibility of the process is exhibited by an *H*-theorem (§ 7.1.3) 3809 which implies the decrease of the free energy of M. Thus, the total entropy of M + B increases, and some energy 3810 is dumped from M to B, while the transition leads from the paramagnetic to either one of the ferromagnetic states. 381 The existence of two possible final states is associated with breaking of ergodicity, discussed for finite but large N in 3812 § 7.1.4 and subsection 7.3. 3813

For purely quadratic interactions within M (the coupling (3.7) having the form $J\hat{m}^2$), the registration proceeds in three stages (§ 7.2.3), illustrated by Figs. 7.3 and 7.5. Firstly the distribution $P_{\uparrow\uparrow}(m, t)$, initially a paramagnetic symmetric peak around m = 0, is shifted faster and faster towards the positive direction of m and it widens, under the conjugate effects of both S and B. For suitably chosen parameters, after a delay given by Eq. (7.44),

$$\tau_{\rm reg} = \frac{\hbar}{\gamma (J-T)},\tag{9.9}$$

that we term the *first registration time*, $P_{\uparrow\uparrow}(m, t)$ is entirely located in the positive region of *m*, its tail in the region m < 0 has then become negligible. Symmetrically, $P_{\downarrow\downarrow}(m, t)$ lies entirely in the m < 0 region for $t > \tau_{reg}$. Thereafter

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the coupling between M and S becomes ineffective and may be *switched off*, so that the registration is virtually, but not yet fully, achieved at this time τ_{reg} .

The last two stages describe a standard relaxation process for which the tested system S is no longer relevant. The stochastic motion of *m* is first governed mainly by the contribution of B to the drift of the magnetization *m*. The distribution $P_{\uparrow\uparrow}(m, t)$ moves rapidly towards $+m_{\rm F}$, first widening, then narrowing. We term as *second registration time* $\tau'_{\rm reg}$ the delay needed for the average magnetization to go from 0 to the vicinity of $m_{\rm F}$. It is expressed by Eq. (7.48), together with (7.47) and (7.36). During the third stage of the registration, both the drift and the diffusion generated by B establish thermal equilibrium of the pointer in an exponential process, and stabilize the distribution $P_{\uparrow\uparrow}(m, t)$ around $+m_{\rm F}$. Thus, $\hat{\mathcal{R}}_{\uparrow\uparrow}(t)$ ends up as $r_{\uparrow\uparrow}(0)\hat{\mathcal{R}}_{\uparrow}$, where $\hat{\mathcal{R}}_{\uparrow}$ denotes the ferromagnetic equilibrium state with positive magnetization, and, likewise, $\hat{\mathcal{R}}_{\downarrow\downarrow}(t)$ ends up as $r_{\downarrow\downarrow}(0)\hat{\mathcal{R}}_{\downarrow}$.

For purely quartic interactions within M (coupling as $J\hat{m}^4$), or for $3J_4 > J_2$, the transition is of first order. We can again distinguish in the registration the above three stages (§ 7.2.4), illustrated by Figs 7.4 and 7.6. Here the first stage is slowed down by the need to pass through the bottleneck $m \simeq m_c$ given by (7.34). The widening of the distribution $P_{\uparrow\uparrow}(m, t)$ is much larger than for quadratic interactions, because diffusion is effective during the large duration of the bottleneck stage. Both the first and the second registration times defined above are nearly equal here, and given by (7.51), that is,

$$\tau_{\rm reg} = \frac{\pi\hbar}{\gamma T} \sqrt{\frac{m_{\rm c}T}{g - h_{\rm c}}}, \qquad m_{\rm c} \simeq \sqrt{\frac{T}{3J}}, \qquad h_{\rm c} \simeq \frac{2}{3}Tm_{\rm c}.$$
(9.10)

The last stage is again an exponential relaxation towards the ferromagnetic state $+m_{\rm F}$ for $P_{\uparrow\uparrow}(m, t)$.

The ratio τ_{reg}/τ_{red} between the registration and truncation times, proportional to \sqrt{N}/γ , is large for two reasons, the weakness of γ and the large value of N. As usual in statistical mechanics, the coexistence of very *different time scales* is associated here with exact and approximate *conservation laws*, expressed by $[\hat{s}_z, \hat{H}] = 0$ and $[\hat{m}, \hat{H}] = [\hat{m}, \hat{H}_{MB}] \propto \sqrt{\gamma}$, which is small because $\gamma \ll 1$.

If *N* is finite, the registration is not permanent. However, the characteristic time of erasure τ_{eras} is much larger than the registration time τ_{reg} by a factor behaving as an exponential of *N* (§ 7.3.5).

The time scales involved in this Curie-Weiss measurement process present some analogy with the relaxation 3843 times in nuclear magnetic resonance [182, 183]. The truncation, i. e., the disappearance of the transverse components 3844 $\langle \hat{s}_x \rangle$ and $\langle \hat{s}_y \rangle$ and of their correlations with A, can be compared to the transverse relaxation in nuclear magnetic 3845 resonance (NMR). The truncation time τ_{red} , as well as is the relaxation time \mathcal{T}_2^* associated in NMR with a dispersion in the precession frequencies of the spins of a sample due to a non-uniformity of the field along z, are durations of 384 dephasing processes in which complex exponentials interfere destructively. By themselves, these phenomena give rise 3848 to recurrences (in our model of measurement) or to spin echoes (in NMR). The bath-induced irreversibility time τ_{irrev}^{B} 3849 is comparable to the relaxation time \mathcal{T}_2 : both characterize decoherence effects, namely the damping of recurrences 3850 in the measurement, and the complete transverse relaxation which damps the echoes in NMR. Finally the registration 3851 time characterizes the equilibration of the diagonal blocks of the density matrix \hat{D} , in the same way as the relaxation 3852 time \mathcal{T}_1 characterizes the equilibration of the longitudinal polarization of the spins submitted to the field along z. 3853

The above summary exhibits the different roles played by the two coupling constants. On the one hand, truncation is ensured entirely by the coupling *g* between S and M. Moreover, the beginning of the registration is also governed by *g*, which selects one of the alternative ferromagnetic states and which should therefore be sufficiently large. On the other hand, the coupling γ between M and B governs the registration, since the relaxation of M towards ferromagnetic equilibrium requires a dumping of energy in the bath.

3859 9.4. Conditions for ideality of the measurement

3860	What you do not wish for yourself, do not do to others
3861	Confucius

Strictly speaking, for finite values of the parameters of the model, the process that we have studied cannot be an ideal measurement in a mathematical sense. However, in a physical sense, the situation is comparable to the solution of the irreversibility paradox, which is found by disregarding correlations between inaccessibly large numbers of particles and by focusing on time scales short compared to the inaccessible Poincaré recurrence time. Here (after
 having achieved the solution in section 11) we will likewise identify physically the process with an ideal measurement,
 within negligible deviations, provided the parameters of the model satisfy some conditions.

³⁸⁶⁸ The definition of the apparatus includes a *macroscopic pointer*, so that

$$N \gg 1. \tag{9.11}$$

The temperature *T* of the bath B should lie below the transition temperature of the magnet M, which equals *J* for quadratic interactions (q = 2) and 0.363 *J* for purely quartic interactions (q = 4).

Our solution was found by retaining only the *lowest order* in the coupling between B and M. Neglecting the higher order terms is justified provided

$$\gamma \ll \frac{T}{J}.\tag{9.12}$$

This condition ensures that the autocorrelation time of the bath, \hbar/T , is short compared to the registration time (9.9) or (9.10). We have also assumed a large value for the *Debye cutoff*, a natural physical constraint expressed by

$$\hbar\Gamma \gg J. \tag{9.13}$$

The *irreversibility of the truncation*, if it is ensured by a dispersion δg of the couplings between tested spin and apparatus spins, requires a neat separation of the time scales $\tau_{red} \ll \tau_{irrev}^{M} \ll \tau_{recur}$, that is

$$\delta_0 \gg \frac{\delta g}{g} \gg \frac{1}{\pi} \sqrt{\frac{2}{N}}.$$
(9.14)

The coefficient δ_0 , the width of the initial paramagnetic distribution of $m\sqrt{N}$, is somewhat larger than 1 for q = 2(quadratic Ising interactions, Eq. (3.51) and equal to 1 for q = 4 (quartic interactions) or when using a strong RF field to initialize the magnet, so that the condition (9.14) is readily satisfied.

If the irreversibility of the truncation is ensured by the bath, we should have $NB(\tau_{\text{recur}}) = \tau_{\text{recur}}/\tau_{\text{irrev}}^{\text{B}} \gg 1$, that is

$$\gamma \gg \frac{4}{\pi N} \tanh \frac{g}{T}.$$
 (9.15)

This condition provides a lower bound on the bath-magnet coupling. An upper bound is also provided by (9.7) if we wish the initial truncation to be controlled by M only. Both bounds are easily satisfied for $N \gg 1$.

³⁸⁸³ The *coupling* g between S and M has been assumed to be rather weak,

$$g < T. \tag{9.16}$$

However, this coupling should be sufficiently strong to initiate the registration, and to ensure that the final indication of the pointer after decoupling will be $+m_F$ if S lies initially in the state $|\uparrow\rangle$, $-m_F$ if it lies initially in the state $|\downarrow\rangle$. For q = 2, this condition is not very stringent. We have seen in § 7.2.2 that it is expressed by (7.41), namely

$$g \gg \frac{(J-T)\delta_1}{\sqrt{N}}, \qquad \delta_1^2 = \delta_0^2 + \frac{T}{J-T} = \frac{T_0}{T_0 - J} + \frac{T}{J-T}.$$
 (9.17)

For purely quartic interactions $-\frac{1}{4}J\hat{m}^4$ (or for $3J_4 > J_2$) the paramagnetic state is locally stable in the absence of interaction with S. The coupling *g* should therefore be larger than some threshold, finite for large *N*,

$$g > h_{\rm c} \simeq \sqrt{\frac{4T^3}{27J}},$$
 (9.18)

so as to trigger the phase transition from m = 0 to $m = \pm m_F$ during the delay (9.10). Moreover, if we wish the decoupling between S and A to take place before the magnet has reached ferromagnetic equilibrium, g must lie sufficiently above h_c (see Eq. (7.57)). If all the above conditions are satisfied, the final state reached by S + A is physically indistinguishable from the surmise (9.1), which encompasses necessary properties of ideal measurements, to wit, truncation and unbiased registration, that is, full correlation between the indication of the apparatus and the final state of the tested system.

3895 9.5. Processes differing from ideal measurements

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In de beperking toont zich de meester⁷⁵ Le mieux est l'ennemi du bien⁷⁶ Dutch and French sayings

Violations of some among the conditions of subsection 9.4 or modifications of the model allow us to get a better insight on quantum measurements, by evaluating deviations from ideality and exploring processes which fail to be measurements, but are still respectable evolutions of coupled quantum mechanical systems.

In subsection 5.2, we modify the initial state of the apparatus, assuming that it is not prepared in an equilibrium paramagnetic state. This discussion leads us to understand truncation as a consequence of the *disordered nature of the initial state* of M, whether or not this state is pure (§ 5.2.2). For "squeezed" initial states, the rapid truncation mechanism can even fail (§ 5.2.3).

Imperfect preparation may also produce another kind of failure. In § 7.3.3 we consider a *bias in the initial state* due to the presence during the preparation stage of a parasite magnetic field which produces a paramagnetic state with non-zero average magnetization. Wrong registrations, for which M reaches for instance a negative magnetization $-m_{\rm F}$ in the final state although it is coupled to a tested spin in the state $s_z = +1$, may then occur with a probability expressed by (7.79).

Section 6 shows that *recurrences* are not washed out if the conditions Eq. (9.14) or (9.15) are not fulfilled. The probability for the *p*-th recurrence to occur is $\exp[-(p\tau_{recur}/\tau_{irrev}^{M})^{2}]$ in the first case, $\exp(-p\tau_{recur}/\tau_{irrev}^{B})$ in the second case. The process is not an ideal measurement if recurrences are still present when the outcome is read.

The violation of the condition (9.17) for q = 2 or (9.18) for q = 4 prevents the registration from taking place properly. For q = 2, if the coupling g is too weak to satisfy (9.17), the apparatus does relax towards either one of the ferromagnetic states $\pm m_F$, but it may provide a *false indication*. The probability for getting wrongly $-m_F$ for an initial state $|\uparrow\rangle$ of S, evaluated in § 7.3.3, is given by (7.79). For q = 4, the registration is aborted if (9.18) is violated: the magnet M does not leave the paramagnetic region, and its magnetization returns to 0 when the coupling is switched off.

The large number N of elements of the pointer M is essential to ensure a faithful and long-lasting registration for 3920 each individual run. It also warrants a brief truncation time, and an efficient suppression of recurrences by the bath. 3921 We study in subsection 8.1 the extreme situation with N = 2, for which \hat{m} has only two "paramagnetic" eigenstates 3922 with m = 0 and two "ferromagnetic" eigenstates with $m = \pm 1$. Although correlations can be established at the time 392 (8.20) between the initial state of S and the magnet M in agreement with Born's rule, there is no true registration. 3924 The indication of M reached at that time is lost after a delay τ_{obs} expressed by (8.15); moreover, a macroscopic extra 3925 apparatus is needed to observe M itself during this delay. On the other hand, the truncation process, governed here by 3926 the bath, is more akin to equilibration than to decoherence; it has an anomalously long characteristic time, longer than 3927 the registration time. These non-idealities of the model with N = 2 are discussed in § 8.1.5. However, such a device 3928 might be used (§ 8.1.6) to implement the idea of determining all four elements of the density matrix of S by means of 3929 repeated experiments using a single apparatus [255, 256, 257]. 3930

In subsection 8.2 we tackle the situation in which the measured observable \hat{s}_z is *not conserved* during the evolution. An ideal measurement is still feasible under the condition (8.71), but it fails if S and A are not decoupled after some delay (§ 8.2.5).

The model can also be extended (subsection 8.3) by simultaneously coupling S with *two apparatuses* A and A' which, taken separately, would measure \hat{s}_z and \hat{s}_x , respectively. The simultaneous measurement of such noncommuting observables is of course impossible. However, here again, repeated runs can provide full information on the statistics of both \hat{s}_z and \hat{s}_x in the initial state $\hat{r}(0)$ (§ 8.3.3). More generally, all the elements of the density matrix

⁷⁶Best is the enemy of good

⁷⁵Conciseness exposes the master

 $\hat{r}(0)$ characterizing an ensemble of identically prepared spins S can be determined by repeated experiments involving a compound apparatus A+A'+A'', where A, A' and A'' are simultaneously coupled to the observables \hat{s}_x , \hat{s}_y and \hat{s}_z , respectively. Indirect tests of Bell's inequalities may rely on this idea (§ 8.3.4).

3941 9.6. *Pedagogical hints*

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The path is made by walking Le mouvement se prouve en marchant African and French proverbs

Models of quantum measurements give rise to many exercises of tutorial interest, which help the students to better grasp quantum (statistical) mechanics. We have encountered above several questions which may inspire teachers. The exercises that they suggest require the use of density operators. As quantum mechanics is often taught only in the language of pure states, we present in appendix G an introduction for students on this topic.

For instance, the *treatment of a thermal bath* at lowest order in its coupling with the rest of the system (subsection 4.2 and Appendix A), although standard, deserves to be worked out by advanced students.

For a general class of models of measurement involving a pointer with many degrees of freedom, the *truncation mechanism* exhibited in § 5.1.2 shows how dephasing can eliminate the off-diagonal blocks of the density matrix of S + A over a short time through interferences.

The evaluation of the *recurrence time* for the pointer coupled with the tested system, or more generally for an arbitrary quantum system (or for a linear dynamical system) having a random spectrum (§ 6.1.2 and Appendix C) is also of general interest.

We now give two further examples of exercises for students which highlight the central steps of the quantum measurement.

3959 9.6.1. End of "Schrödinger cats"

Focusing on the Curie-Weiss model, we present here a simpler derivation of the processes which first lead to truncation and which prevent recurrences from occurring. We showed in section 6 and Appendix D that the interactions J_2 and J_4 between the spins $\hat{\sigma}^{(n)}$ of M play little role here, so that we neglect them. We further assume that M lies initially in the most disordered state (3.46), that we write out, using the notation (3.1), as

$$\hat{R}_{\mathrm{M}}(0) = \frac{1}{2^{N}} \hat{\sigma}_{0}^{(1)} \otimes \hat{\sigma}_{0}^{(2)} \otimes \cdots \otimes \hat{\sigma}_{0}^{(N)}.$$
(9.19)

This occurs for q = 4 and in the general case of $J_2 > 0$ provided the temperature of preparation T_0 in (3.51) is much higher than J_2 , so that $\delta_0 = 1$. Then, since the Hamiltonian $\hat{H}_{SA} + \hat{H}_B + \hat{H}_{MB}$ is a sum of independent contributions associated with each spin $\hat{\sigma}^{(n)}$, the spins of M behave independently at all times, and the off-diagonal block $\hat{R}_{\uparrow\downarrow}(t)$ of $\hat{D}(t)$ has the form

$$\hat{R}_{\uparrow\downarrow}(t) = r_{\uparrow\downarrow}(0)\,\hat{\rho}^{(1)}(t) \otimes \hat{\rho}^{(2)}(t) \otimes \dots \otimes \hat{\rho}^{(N)}(t), \tag{9.20}$$

where $\hat{\rho}^{(n)}(t)$ is a 2 × 2 matrix in the Hilbert space of the spin $\hat{\sigma}^{(n)}$. This matrix will depend on $\hat{\sigma}_z^{(n)}$ but not on $\hat{\sigma}_x^{(n)}$ and $\hat{\sigma}_y^{(n)}$, and it will neither be hermitean nor normalized.

The task starts with keeping the effect of the bath as in subsection 6.2, but leaves open the possibility for the coupling g_n to be random as in subsection 6.1, whence the coupling between S and A reads $\hat{H}_{SA} = -\hat{s}_z \sum_{n=1}^N g_n \hat{\sigma}_z^{(n)}$ instead of (3.5). (As simpler preliminary exercises, one may keep the $g_n = g$ as constant, and/or disregard the bath.) Each factor $\hat{\rho}^{(n)}(t)$, initially equal to $\frac{1}{2}\hat{\sigma}_0^{(n)}$, evolves according to the same equation as (4.8) for $\hat{R}_{\uparrow\downarrow}(t)$, rewritten with N = 1. (To convince oneself of the product structure (9.20), it is instructive to work out the cases N = 1 and N = 2in Eq. (4.8) or (4.18).) Admit, as was proven in subsection 6.2 and appendix D, that the effect of the bath is relevant only at times $t \gg \hbar/2\pi T$, and that in this range $\hat{\rho}^{(n)}$ evolves according to

$$\frac{\mathrm{d}\hat{\rho}^{(n)}(t)}{\mathrm{d}t} - \frac{2ig_n}{\hbar}\hat{\rho}^{(n)}\hat{\sigma}_z^{(n)} = -\frac{2\gamma}{\hbar^2} \left[\tilde{K}_-\left(\frac{2g_n}{\hbar}\right) + \tilde{K}_+\left(-\frac{2g_n}{\hbar}\right)\right] \left[\hat{\rho}^{(n)} - \frac{1}{2}\hat{\sigma}_0^{(n)}\mathrm{tr}\,\hat{\rho}^{(n)}\right].$$
(9.21)

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(Advanced students may derive this equation by noting that for N = 1, $\hat{\rho}^{(n)}$ can be identified with $P_{\uparrow\downarrow}(\hat{m} = \hat{\sigma}_z)$; starting then from Eq. (4.17) for N = 1, keeping in mind that $P_{\uparrow\downarrow}(\pm 3) = 0$ and verifying that, in the non-vanishing terms, Eq. (4.13) implies that $\Omega_i^{\pm} = \pm 2g_n s_i/\hbar$, they should show that the factors $\tilde{K}_{t>}(\Omega_{\uparrow}^-) + \tilde{K}_{t<}(\Omega_{\downarrow}^+) + \tilde{K}_{t<}(\Omega_{\downarrow}^+)$ of (4.17) reduce for $t \gg \hbar/2\pi T$ and for $J_2 = 0$ to the symmetric part of $\tilde{K}(2g_n/\hbar)$ according to (4.18) and (D.21).) Next parameterize $\hat{\rho}^{(n)}$ as

$$\hat{\rho}^{(n)}(t) = \frac{1}{2} \exp\left[-B_n(t) + i\Theta_n(t)\hat{\sigma}_z^{(n)}\right],$$
(9.22)

and derive from (9.21) the equations of motion

$$\frac{d\Theta_n}{dt} = \frac{2g_n}{\hbar} - \frac{\gamma}{\hbar^2} \left[\tilde{K} \left(\frac{2g_n}{\hbar} \right) + \tilde{K} \left(-\frac{2g_n}{\hbar} \right) \right] \sin 2\Theta_n,$$

$$\frac{dB_n}{dt} = \frac{2\gamma}{\hbar^2} \left[\tilde{K} \left(\frac{2g_n}{\hbar} \right) + \tilde{K} \left(-\frac{2g_n}{\hbar} \right) \right] \sin^2 \Theta_n,$$
(9.23)

with initial conditions $\Theta_n(0) = 0$, $B_n(0) = 0$. Keeping only the dominant contributions for $\gamma \ll 1$, use the expression (3.37) for \tilde{K} , find the solution

$$\Theta_n(t) \simeq \frac{2g_n t}{\hbar}, \qquad B_n(t) \simeq \frac{\gamma g_n}{2\hbar} \coth \frac{g_n}{T} \left(t - \frac{\hbar}{4g_n} \sin \frac{4g_n t}{\hbar} \right), \tag{9.24}$$

and compare B_n with (6.28) for B.

Eqs. (9.22), (9.24) provide the evolution of the density matrix of the spin *n* from the paramagnetic initial state $\hat{\rho}^{(n)}(0) = \frac{1}{2}\text{diag}(1, 1)$ to

$$\hat{\rho}^{(n)}(t) = \frac{1}{2} \operatorname{diag}\left(e^{2ig_n t/\hbar}, e^{-2ig_n t/\hbar}\right) \exp\left[-\frac{\gamma g_n}{2\hbar} \coth\frac{g_n}{T}\left(t - \frac{\hbar}{4g_n} \sin\frac{4g_n t}{\hbar}\right)\right].$$
(9.25)

By inserting (9.25) into (9.20) and tracing out the pointer variables, one finds the transverse polarization of S as

$$\frac{1}{2}\langle \hat{s}_x(t) - i\hat{s}_y(t) \rangle \equiv \operatorname{tr}_{S,A} \hat{\mathcal{D}}(t) \frac{1}{2} (\hat{s}_x - i\hat{s}_y) = r_{\uparrow\downarrow}(t) \equiv r_{\uparrow\downarrow}(0) \operatorname{Evol}(t),$$
(9.26)

³⁹⁸⁹ where the temporal evolution is coded in the function

$$\operatorname{Evol}(t) \equiv \left(\prod_{n=1}^{N} \cos \frac{2g_n t}{\hbar}\right) \exp\left[-\sum_{n=1}^{N} \frac{\gamma g_n}{2\hbar} \coth \frac{g_n}{T} \left(t - \frac{\hbar}{4g_n} \sin \frac{4g_n t}{\hbar}\right)\right].$$
(9.27)

To see what this describes, the student can first take $g_n = g$, $\gamma = 0$ and plot the factor |Evol(t)| from t = 0 to $5\tau_{\text{recur}}$, where $\tau_{\text{recur}} = \pi\hbar/2g$ is the time after which $|r_{\uparrow\downarrow}(t)|$ has recurred to its initial value $|r_{\uparrow\downarrow}(0)|$. By increasing N, e.g., N = 1, 2, 10, 100, he/she can convince him/herself that the decay near t = 0 becomes close to a Gaussian decay, over the characteristic time τ_{red} of Eq. (9.4). The student may demonstrate this analytically by setting $\cos 2g_n t/\hbar \approx \exp(-2g_n^2 t^2/\hbar^2)$ for small t. This time characterizes decoherence, that is, disappearance of the off-diagonal blocks of the density matrix; we called it "truncation time" rather than "decoherence time" to distinguish it from usual decoherence, which is induced by a thermal environment and coded in the second factor of Evol(t).

The exercise continues with the aim to show that $|\text{Evol}| \ll 1$ at $t = \tau_{\text{recur}}$ in order that the model describes a faithful quantum measurement. To this aim, keeping $\gamma = 0$, the student can in the first factor of Evol decompose $g_n = g + \delta g_n$, where δg_n is a small Gaussian random variable with $\langle \delta g_n \rangle = 0$ and $\langle \delta g_n^2 \rangle \equiv \delta g^2 \ll g^2$, and average over the δg_n . The Gaussian decay (6.10) will thereby be recovered, which already prevents recurrences. The student may also take e.g. N = 10 or 100, and plot the function to show this decay and to estimate the size of Evol at later times.

Next by taking $\gamma > 0$ the effect of the bath in (9.27) can be analyzed. For values γ such that $\gamma N \gg 1$ the bath will lead to a suppression. Several further tasks can be given now: Take all g_n equal and plot the function Evol(*t*); take a small spread in them and compare the results; make the small- g_n approximation $g_n \coth g_n/T \approx T$, and compare again.

At least one of the two effects (spread in the couplings or suppression by the bath) should be strong enough to prevent recurrences, that is, to make $|r_{\uparrow\downarrow}(t)| \ll |r_{\uparrow\downarrow}(0)|$ at any time $t \gg \tau_{red}$, including the recurrence times. The student can recover the conditions (9.14) or (9.15) under which the two mechanisms achieve to do so. The above study will show him/her that, in the dynamical process for which each spin $\hat{\sigma}^{(n)}$ of M independently rotates and is damped by the bath, the truncation, which destroys the expectation values $\langle \hat{s}_a \rangle$ and all correlations $\langle \hat{s}_a \hat{m}^k(t) \rangle$ (a = x or $y, k \ge 1$), arises from the precession of the tested spin \hat{s} around the *z*-axis; this is caused by the conjugate effect of the many spins $\hat{\sigma}^{(n)}$ of M, while the suppression of recurrences is either due to dephasing if the g_n are non-identical, or due to damping by the bath.

A less heavy exercise is to derive (5.27) from (5.26); hereto the student first calculates $\langle m \rangle$ and then $\langle m^2 \rangle$. Many other exercises may be inspired by sections 5 and 6, including the establishment and disappearance of the off-diagonal spin-magnet correlations (§ 5.1.3); the numerical or analytical derivation of the damping function B(t) (Appendix D); its short-time behavior obtained either as for (D.9) or from the first two terms of the short-time expansion of K(t); the analytical study of the autocorrelation functions K(t), $K_{>t}$ and $K_{<t}$ of the bath for different time scales using the complex plane technique of Appendix D.

4020 9.6.2. Simplified description of the registration process

We have seen in § 7.1.2 that the registration process looks, for the diagonal block $R_{\uparrow\uparrow}(t)$, as a classical relaxation of the magnet M towards the stable state with magnetization $+m_{\rm F}$ under the effect of the coupling *g* which behaves in this sector as a positive field. This idea can be used to describe the registration by means of the classical Fokker-Planck equation (7.1) which governs the evolution of the probability distribution $P(m, t) = P_{\uparrow\uparrow}(m, t)/r_{\uparrow\uparrow}(0)$.

⁴⁰²⁵ By assuming explicit expressions for the drift and the diffusion coefficient which enter this equation of motion, ⁴⁰²⁶ one can recover some of the results of section 7 in a form adapted to teaching.

In particular, if we keep aside the shape and the width of the probability distribution, which has a narrow peak for large N (§ 7.2.1), the center $\mu(t)$ of this peak moves according to the mean-field equation

$$\frac{\mathrm{d}\mu(t)}{\mathrm{d}t} = v[\mu(t)],\tag{9.28}$$

where v(m) is the local drift velocity of the flow of *m*., This equation can be solved once v(m) is given, and its general properties do not depend on the precise form of v(m). The first choice is phenomenological: we take v(m) proportional to -dF/dm, where *F* is the free energy (3.54), resulting in

$$v(m) = \frac{C(m)}{\hbar} \left(Jm^{q-1} + g - \frac{T}{2} \ln \frac{1+m}{1-m} \right), \tag{9.29}$$

with a dimensionless, positive function C(m) which may depend smoothly on *m* in various ways (§ 7.1.2), or even be approximated as a constant. An alternative phenomenological choice consists in deriving from detailed balance, as in $\{7.1.2\}$, the expression (7.14) for v(m), that is, within a multiplicative factor $\theta(m)$,

$$v(m) = \frac{1}{\theta(m)} \left(\tanh \frac{g + Jm^{q-1}}{T} - m \right). \tag{9.30}$$

possibly approximating θ as a constant. A more precise way is to derive v(m) from the autocorrelation function of the bath (Eq. (7.6)) as

$$v(m) = \frac{\gamma}{\hbar} (g + Jm^{q-1}) \left(1 - m \coth \frac{g + Jm^{q-1}}{T} \right).$$
(9.31)

An introductory exercise is to show that the C(m) (or the $\theta(m)$) obtained from equating (9.29) (or (9.30)) to (9.31) is

a smooth positive function, finite at the stable or unstable fixed points of Eq. (9.28), given by the condition v(m) = 0, which can in all three cases be written as $m = \tanh[(g + Jm^{q-1})/T]$.

If the coupling g is large enough, the resulting dynamics will correctly describe the transition of the magnetization 4040 from the initial paramagnetic value m = 0 to the final ferromagnetic value $m = m_{\rm F}$. Comparison between quadratic 404 interactions (q = 2) and quartic interactions (q = 4) is instructive. The student can determine in the latter case the 4042 minimum value of the coupling g below which the registration cannot take place, and convince him/herself that it 4043 does not depend on the form of C(m). Approaching this threshold from above, one observes the slowing down of the 4044 process around the crossing of the bottleneck. This feature is made obvious by comparing the Figs 7.3 and 7.4 which 4045 illustrate the two situations q = 2 and q = 4, respectively, and which were evaluated by using the form (9.30)) of v(m). 4046 The above exercise overlooks the broadening and subsequent narrowing of the profile at intermediate times, which 4047 is relevant for finite values of N. More advanced students may be proposed to numerically solve the time evolution of P(m, t), i. e., the whole registration process, at finite N, taking in the rate equations Eq. (4.16) e.g. N = 10, 100 and 1000. For the times of interest, $t \gg \hbar/\Gamma$, one is allowed to employ the simplified form of the rates from (4.33) and 4050 (4.14), and to set $\Gamma = \infty$. The relevant rate coefficients are listed at the end of Appendix B. 4051

10. Statistical interpretation of quantum mechanics

4053	A man should first direct himself in the way he should go.
4054	Only then should he instruct others
4055	Buddha

Measurements constitute privileged tools for relating experimental reality and quantum theory. The solution of 4056 models of quantum measurements is therefore expected to enlighten the foundations of quantum mechanics, in the 4057 same way as the elucidation of the paradoxes of classical statistical mechanics has provided a deeper understanding of 4058 the Second Law of thermodynamics, either through an interpretation of entropy as missing information at the micro-4059 scopic scale [57, 58, 74, 73, 80, 71, 270, 271], or through a microscopic interpretation of the work and heat concepts 4060 [72, 272, 273, 274, 275, 276, 277, 278, 279]. In fact, the whole literature devoted to the quantum measurement problem has as a background the interpretation of quantum mechanics. Conversely, some specific interpretation is needed to understand the meaning of calculations about models. The use of quantum statistical mechanics (sections 2 and 9) 4063 provides us with a density operator of the form (9.1) at the final time; before drawing physical conclusions (section 4064 11) we have to make clear what such a technical tool really means. We prefer, among the various interpretations of 4065 quantum mechanics [31, 34, 36, 280], the statistical one which we estimate the most adequate. We review below the 4066 main features of this statistical interpretation, as underlined by Park [28] and supported by other authors. It is akin to 4067 the one advocated by Ballentine [9, 48], but it does not coincide with the latter in all aspects. For a related historic 4068 perspective, see Plotnitsky [281]. 4069

4070 10.1. Principles

In its statistical interpretation, quantum mechanics presents some conceptual analogy with statistical mechanics. It has a dualistic nature, involving two types of mathematical objects, associated with a system and with its observers, respectively. On the one hand, the "observables", non-commutative random operators, describe the physical quantities related to the studied system. On the other hand, a "state" of this system, represented by a density operator, should be regarded as the probabilistic information available about it under given circumstances.

4078 4079 Hello, Dolly! It's so nice to have you back where you belong Written by Jerry Herman, sung by Louis Armstrong

In classical physics, the physical quantities are represented by *c*-numbers, that is, scalar commuting variables, possibly random in stochastic dynamics or in classical statistical mechanics. In quantum physics, we have to give up such an idea. The physical quantities cannot be directly observed or manipulated; and we are no longer allowed to imagine that they might take well-defined scalar values. The microscopic description of a system requires counterintuitive concepts, which nevertheless have a precise mathematical representation, and which will eventually turn out to perfectly fit experiments.

^{4076 10.1.1.} Physical quantities: observables

The physical quantities that we are considering are, for instance, the position, the momentum, or the components 4086 of the spin of each particle constituting the considered system, or a field at each point. The mathematical tools ac-4087 counting for such quantities in unspecified circumstances have a random nature. Termed as "observables", they are 408 elements of some specific algebra. Along the lines of Heisenberg's matrix theory [10, 11, 31, 34, 36, 48, 84, 280], 4089 they can be represented as linear operators acting in a complex Hilbert space \mathcal{H} , or as matrices once a basis is chosen 4090 in this space, which exhibits the algebraic structure. The present more abstract approach is also more general, as it 409 encompasses other representations, termed as Liouville representations [75, 282, 283] in which the product is imple-4092 mented differently; an example of these, the Wigner representation, is useful in the semi-classical limit. The structure 4093 of the set of observables, a C*-algebra [134], involves addition, multiplication by complex c-numbers, hermitean conjugation, and non-commutative product. The physical observables \hat{O} are hermitean. They play in quantum mechanics the same rôle as random variables in classical statistical mechanics, except for the essential fact that they belong 4096 to a non-commutative algebra, the structure of which fully characterizes the system [134]. Ordinary reasoning and 4097 macroscopic experience do not help us to develop intuition about such non-commuting physical quantities, and this is 4098 the main incentive for proposals of alternative interpretations of quantum mechanics [17, 19, 189, 191, 192, 284]. 4099

In some circumstances, when the observables of interest constitute a commutative subset, the peculiar aspects of 4100 quantum mechanics that raise difficulties of interpretation do not appear [134, 114, 115]. For instance, the classical 4101 probability theory is sufficient for working out the statistical mechanics of non-interacting Fermi or Bose gases at ther-4102 mal equilibrium. This simplification occurs because we deal there only with commuting observables, the occupation 4103 number operators \hat{n}_k for the single particle states $|k\rangle$, which can be treated as random c-numbers taking the discrete 4104 values $n_k = 0$ or 1 for fermions, $n_k = 0, 1, 2, \cdots$ for bosons. However, even in this simple case, it is the underlying 4105 non-commutative algebra of the creation and annihilation operators \hat{a}_k^{\dagger} and \hat{a}_k which explains why the eigenvalues of 4106 $\hat{n}_k = \hat{a}_{\iota}^{\dagger} \hat{a}_k$ are those integers. A similar situation occurs for macroscopic systems, for which classical behaviors emerge 4107 from the hidden microscopic fundamental quantum theory. The variables controlled in practice then commute, at least 4108 approximately, so that classical concepts are sufficient. Macroscopic properties such as electronic conduction versus 4109 insulation, magnetism, heat capacities, superfluidity, or the very existence of crystals all have a quantum origin but 4110 obey equations of a "classical" type, in the sense that they involve only commutative variables. Non commutation, the 4111 essence of quantum mechanics, may manifest itself only exceptionally in systems that are not microscopic, see [285] 4112 and references therein. 4113

What one calls "quantum" and "classical" depends, though, on one's definition of these terms. We have identified 4114 above a "truly quantum" behavior with non-commutativity, a deep but restrictive definition. Other viewpoints are 4115 currently expressed, such as dependence on \hbar . Quantum electrodynamics have two classical limits, wave-like when 4116 the non-commutation of the electric and magnetic fields is not effective, and particle-like when the number of photons 4117 is well defined. Moreover, the quantal or classical nature of a given concept may depend on the specific situation. The 4118 center of mass of a small metallic grain can be described by its "classical" value, while the shape of its heat capacity 4119 requires a quantum description, such as the Debye model although the concept of specific heat, its measurement, its 4120 thermodynamic aspects, are all "pre-quantal". On the other hand, in atomic clocks one needs to control the quantum 4121 fluctuations of the position of the center of mass, which is therefore not so classical. An extreme case of quantal center 4122 of mass is a mechanical resonator in its ground state or excited by one phonon [286]. 4123

4124 10.1.2. Dynamics

⁴¹²⁵ Dynamics is currently implemented in the quantum theory through the Schrödinger picture, where the observables ⁴¹²⁶ remain constant while the states (pure or mixed) evolve according to the Schrödinger or the Liouville-von Neumann ⁴¹²⁷ equation. Following the tradition, we have relied on this procedure in sections 4 to 8. The evolution then bears ⁴¹²⁸ on the wave function or the density operator, objects which characterize our information on the system. However, ⁴¹²⁹ dynamics should be regarded as a property of the system itself, regardless of its observers. It is therefore conceptually ⁴¹³⁰ enlightening to account for the evolution of an *isolated system* in the Heisenberg picture, as a change in time of its ⁴¹³¹ observables which pertain to this system.

We should then implement the dynamics as a transformation of the set of observables, represented by a linear mapping that leaves invariant the algebraic relations between the whole set of observables [10, 11, 31, 34, 36, 48, 84]. In the Hilbert space representation, this implies that the *transformation is unitary*. (In Liouville representations, where observables behave as vectors, their evolution is generated by the Liouvillian superoperator.) Denoting by t_0

the reference time at which the observables \hat{O} are defined, we can thus write the observables $\hat{O}(t, t_0)$ at the running 4136 time t as $\hat{O}(t, t_0) = \hat{U}^{\dagger}(t, t_0)\hat{O}\hat{U}(t, t_0)$, where the unitary transformation $\hat{U}(t, t_0)$ carries the set of observables from t_0 4137 to t. (In the Schrödinger picture, it is the density operator which depends on time, according to $\hat{U}(t, t_0)\hat{D}\hat{U}^{\dagger}(t, t_0)$.) 4138 The infinitesimal generator of this transformation being the Hamiltonian \hat{H} , the time-dependent observable $\hat{O}(t, t_0)$ 4139 is characterized either by the usual Heisenberg equation $i\hbar\partial\hat{O}(t,t_0)/\partial t = [\hat{O}(t,t_0),\hat{H}]$ with the boundary condition 4140 $\hat{O}(t_0, t_0) = \hat{O}$ or by the backward equation $i\hbar\partial\hat{O}(t, t_0)/\partial t_0 = [\hat{H}, \hat{O}(t, t_0)]$ with the boundary condition $\hat{O}(t, t) = \hat{O}$. 414 The backward equation, more general as it holds if \hat{H} or \hat{O} depend explicitly on time, is efficient for producing 4142 dynamical approximations, in particular for correlation functions [287]. The interest of the backward viewpoint for 4143 the registration in a measurement is exhibited in § 7.3.1, Appendix F and § 13.1.3. 4144

Automatical states in the intervalue of the i

The Heisenberg motion of observables describes a perfect and reversible transfer along time. The Heisenberg 4150 picture thus exhibits a *deterministic structure* within quantum mechanics [134]. Whereas the Schrödinger picture 4151 tangles the deterministic and probabilistic aspects of quantum mechanics within the time-dependent states $|\psi(t)\rangle$ or 4152 $\hat{D}(t)$, these two aspects are well separated in the Heisenberg picture, deterministic dynamics of the observables, 4153 probabilistic nature of the time-independent states. We will rely on this remark in subsection 13.1. The Heisenberg 4154 picture also allows to define correlations of observables taken at different times and pertaining to the same system 4155 [280, 287]. Such autocorrelations, as the Green's functions in field theory, contain detailed information about the 4156 dynamical probabilistic behavior of the systems of the considered ensemble, but cannot be directly observed through 4157 ideal measurements. 4158

4159 10.1.3. Statistical ensembles of systems

While the observables and their evolution appear as properties of the objects under study, our knowledge about 4160 them is probabilistic. The statistical interpretation highlights the fact that quantum mechanics provides us only with 4161 probabilities [9, 10, 11, 29, 31, 52, 58, 28]. Although a probabilistic theory may produce some predictions with cer-4162 tainty, most quantities that we deal with at the microscopic scale are subject to statistical fluctuations: expectation 4163 values, correlations at a given time, or autocorrelations at different times when we observe for instance the succes-4164 sive transitions of a trapped ion [288, 289]. Exact properties of individual systems can be found only in special 4165 circumstances, such as in measurements (section 11). Thus, explicitly or implicitly, our descriptions refer to statistical 4166 ensembles of systems and to repeated experiments [9, 31, 28]. Even when we describe a single object we should imag-4167 ine that it belongs to a thought ensemble \mathcal{E} [280], all elements of which are considered to be prepared under similar 4168 conditions characterized by the same set of data⁷⁷. Notice the similarity with ensemble theory in classical statistical 4169 physics, which also allows probabilistic predictions on single systems [55, 56]. However, there is no quantum system 4170 devoid of any statistical fluctuations [9, 31]. Individual events resulting from the same preparation are in general not 4171 identical but obey some probability law, even when the preparation is as complete as possible. 4172

Note that we adhere to the so-called subjective interpretation of probabilities, initiated by Bayes and Laplace, 4173 and later on advocated by Cox [290], de Finetti [291] and Jaynes [270]. According to this conception, probabilities 4174 are defined as the mathematical measure of likelihood of events. They are not inherent to the considered object 4175 alone, but are tools for making reasonable predictions about this object through consistent inference. They do not 4176 pertain to a system in itself, but characterize our information on it or on the ensemble to which it belongs. In fact, 4177 information has turned out to be a central concept in statistical physics [57, 58, 74, 73, 80, 71, 270, 271]. This 4178 idea is exemplified by spin-echo experiments [60, 61, 62, 63, 64, 65, 182, 183]. After the initial relaxation, an 4179 observer not aware of the history of the system cannot describe its spins better than by means of a completely random 4180 probability distribution. However, the experimentalist, who is able to manipulate the sample so as to let the original 4181

⁷⁷When accounting probabilistically for the cosmic microwave spectrum, one imagines the Universe to belong to an ensemble of possible universes. With a single Universe at hand, this leads to the unsolvable *cosmic variance* problem. The same ideas hold whenever probabilities are applied to a single system or event [290], and this is the subject of standard and thorough developments in books of probabilities, including already Laplace's

⁴¹⁸² magnetization revive, includes in his probablistic description the hidden correlations that keep track of the ordering ⁴¹⁸³ of the initial state. Likewise, we can assign different probabilities to the content of a coded message that we have ⁴¹⁸⁴ intercepted, depending on our knowledge about the coding [74]. Since quantum theory is irreducibly probabilistic, it ⁴¹⁸⁵ has thus a partly subjective nature — or rather "inter-subjective" since under similar conditions all observers, using the ⁴¹⁸⁶ same knowledge, will describe a quantum system in the same way and will make the same probabilistic predictions ⁴¹⁸⁷ about it. The recent developments about the use of quantum systems as information processors [42] enforce this ⁴¹⁸⁸ information-based interpretation [87, 292] (see the end of § 12.4.2).

4189 10.1.4. States

4190 4191 L'Etat c'est moi⁷⁸ Louis XIV

In the present scope, the definition of a quantum state is conceptually the same as in statistical mechanics [122, 73, 4192 28]: A state of the considered system (or more precisely a state of the real or virtual statistical ensemble \mathcal{E} of systems to 4193 which it belongs) is characterized by specifying the *collection of expectation values* $\langle \hat{O} \rangle$ of all possible observables \hat{O} 4194 of this system, that is, by the *correspondence* $\hat{O} \mapsto \langle \hat{O} \rangle$. This correspondence has the following properties [52, 58]: it 4195 is linear, it associates a real number to hermitean operators, a non-negative number to the square of an observable, and 4196 the number 1 to the unit operator. Such properties have a natural interpretation, entailing in particular that variances 4197 cannot be negative [52, 58]. The probabilistic nature of this correspondence is exemplified by the fact that it includes 4198 statistical fluctuations $\langle \hat{O}^2 \rangle - \langle \hat{O} \rangle^2$. The probability of finding for \hat{O} some eigenvalue O_i is also expressed as the 4199 expectation value $\langle \hat{\Pi}_i \rangle$ of the projection operator $\hat{\Pi}_i$ on the corresponding eigenspace of \hat{O} .

For infinite systems or fields, this definition of a state as a mapping of the algebra of observables onto commuting *c*-numbers has given rise to mathematical developments in the theory of C^* -algebras [134]. Focusing on the vector space structure of the set of observables, one then considers the states as elements of the dual vector space. For finite systems the above properties are implemented in an elementary way once the observables are represented as operators in a Hilbert space. The mapping is represented by a *density operator* \hat{D} in this Hilbert space, which is hermitean, non-negative and normalized, and which generates all the expectation values through [52, 58]

$$\hat{O} \mapsto \langle \hat{O} \rangle = \text{tr}\hat{\mathcal{D}}\hat{O}. \tag{10.1}$$

In fact, according to Gleason's theorem [50], the linearity of this correspondence for any pair of commuting observables is sufficient to ensure the existence of \hat{D} . (We use the notation \hat{D} for the generic system considered here; no confusion should arise with the state of S + A in the above sections.)

⁴²¹⁰ A tutorial introduction to density operators is presented in Appendix G.

A density operator which characterizes a state plays the rôle of a probability distribution for the non-commuting physical quantities \hat{O} since it gathers through (10.1) our whole information about an ensemble of quantum systems [52, 58, 270, 28]. As in probability theory, the amount of *missing information* associated with the state \hat{D} is measured by its von Neumann entropy [52, 58, 270].

$$S(\hat{\mathcal{D}}) = -\mathrm{tr}\hat{\mathcal{D}}\ln\hat{\mathcal{D}}.\tag{10.2}$$

For time-dependent predictions on an isolated system, Eq. (10.1) holds both in the Schrödinger picture, with 4215 fixed observables and the Liouville–von Neumann evolution for $\hat{D}(t)$, and in the Heisenberg picture, with fixed \hat{D} 4216 and observables evolving unitarily. However, two-time (and multi-time) autocorrelation functions cannot be defined 4217 within the Schrödinger picture. They are obtained as tr $\hat{D}\hat{O}_1(t_1, t_0)\hat{O}_2(t_2, t_0)$, where the observables in the Heisenberg 4218 picture refer to the physical quantities of interest and their dynamics, and where the state accounts for our knowledge 4219 about the system [134]. In particular, when defining in § 3.3.2 the autocorrelation function K(t-t') of the bath, it was 4220 necessary to express the time-dependent bath operators in the Heisenberg picture (although we eventually inserted 4221 K(t-t') into the Liouville–von Neumann equations of motion of S + M in section 4 and appendix A). 4222

⁷⁸The State, that's me

In this interpretation, what we call the state "*of a system*", whether it is pure or not, is not a property of *the considered system in itself*, but it characterizes the statistical properties of the real or virtual *ensemble* to which this system belongs [52, 58, 270, 28]. The word "*state*" itself is also misleading, since we mean by it the summary of *our knowledge* about the ensemble, from which we wish to make probabilistic predictions. The conventional expression "the state of the system" is therefore doubly improper in quantum physics, especially within the statistical interpretation [52, 58, 28], and we should not be misled by this wording — although we cannot help to use it when teaching.

Density operators differ from distributions of the probability theory taught in mathematical courses and from densi-4230 ties in phase space of classical statistical mechanics, because the quantum physical quantities have a non-commutative 4231 nature [10, 11, 31, 34, 36, 48, 52, 58, 84, 280]. This algebraic feature, compelled by experiments in microphysics, lies at the origin of the odd properties which make quantum mechanics counterintuitive. It implies quantization. It also 4233 implies the superposition principle, which is embedded in the matrix nature of \hat{D} . It entails Heisenberg's inequality 4234 $\Delta \hat{O} \Delta \hat{O}' \geq \frac{1}{2} |\langle [\hat{O}, \hat{O}'] \rangle|$ and hence Bohr's complementarity: since the product of the variances of two non-commuting 4235 observables has a lower bound, it is only in a fuzzy way that we can think simultaneously of quantities such as the 4236 position and the momentum (or the wavelength) of a particle, contrary to what would happen in classical statistical 4237 mechanics. Thus the non-commutation of observables implies the existence of intrinsic fluctuations, and the quantum 4238 theory is irreducibly probabilistic [10, 11, 31, 34, 36, 48, 84, 280, 28].

One should note, however, that the non-commutation of two observables does not necessarily imply that they present quantum fluctuations. For instance, if two operators do not commute, there may exist states (their common eigenstates) in which both have well-defined values. As an example, in states with orbital momentum zero, the components \hat{L}_x and \hat{L}_y vanish without any statistical fluctuation. (This does not contradict the Heisenberg inequality $\Delta \hat{L}_x \Delta \hat{L}_y \geq \frac{1}{2} \hbar |\langle \hat{L}_z \rangle|$, because both sides vanish in this case; more general uncertainty relations for orbital momentum are given in [293].) Conversely, two commuting observables may fluctuate in some states, even pure ones.

In the statistical interpretation, we should refrain from imagining that the observables might take well-defined 1210 but undetectable values in a given state, and that the uncertainties about them might be a mere result of incomplete knowledge. The very concept of physical quantities has to be dramatically changed. We should accept the idea 4248 that quantum probabilities, as represented by a density operator, do not simply reflect as usual our ignorance about 4249 supposedly preexisting values of physical quantities (such as the position and the momentum of a particle), but arise 4250 because our very conception of physical quantites as scalar numbers, inherited from macroscopic experience, is not in 4251 adequacy with microscopic reality [10, 11, 31, 34, 36, 48, 84, 280]. Macroscopic physical quantities take scalar values 4252 that we can observe, in particular for a pointer, but the scalar values that we are led to attribute to microscopic (non-4253 commuting) observables are the outcome of inferences which are indirectly afforded by our measurement processes. 4254

From an epistemological viewpoint, the statistical interpretation of quantum theory has a dualistic nature, both objective and subjective. On the one hand, observables are associated with the physical properties of a real system. On the other hand, in a given circumstance, the reality of this system is "veiled" [303], and what we call "state" refers to the information available to observers.

4259 10.2. Resulting properties

4260 10.2.1. Contextuality

Non-commutativity gives rise to odd phenomena that force us to overturn some of our ways of thinking. According 426 the above viewpoint, the violation of Bell's inequalities [27, 29, 31, 34, 262] should be attributed to the nonto 4262 commutative nature of the distribution \hat{D} rather than to non-locality; quantum mechanics does not involve ordinary 4263 probabilities nor ordinary correlations. The violation of the classical inequality, observed experimentally [294, 295], 4264 arises when one puts together outcomes of measurements performed in different experimental contexts, and this may 4265 itself be a probem [264, 265, 266, 267, 268]. The discussion of § 8.3.4 shows how quantum and ordinary correlations 4266 may be reconciled in the context of a thought experiment where one attempts to measure simultaneously, with a unique 4267 setting, all spin components.

⁴²⁶⁹ Other quantum phenomena, involving properties satisfied exactly rather than statistically, may be regarded as ⁴²⁷⁰ failures of ordinary logic. This is exemplified by the GHZ paradox [34, 36, 296], that we now recall.

The GHZ setup is as follows: Consider six observables \hat{B}_i and \hat{C}_i (i = 1, 2, 3) such that $\hat{B}_i^2 = \hat{C}_i^2 \equiv \hat{I}, \hat{C}_1 \hat{C}_2 \hat{C}_3 \equiv \hat{I},$ and with commutators $[\hat{B}_i, \hat{B}_j] = [\hat{C}_i, \hat{C}_j] = 0, [\hat{B}_i, \hat{C}_i] = 0$ and $\hat{B}_i \hat{C}_j = -\hat{C}_j \hat{B}_i$ for $i \neq j$. (A physical realization is

provided with 3 spins, employing their $\hat{\sigma}_x$ and $\hat{\sigma}_z$ operators by taking $\hat{B}_1 = \hat{\sigma}_x^{(1)}$, $\hat{C}_1 = \hat{\sigma}_z^{(2)} \hat{\sigma}_z^{(3)}$, and likewise⁷⁹. In the 4273 pure state $|\varphi\rangle$ characterized by $\hat{B}_i \hat{C}_i |\varphi\rangle = |\varphi\rangle$, each one of the three statements " B_i takes the same value as C_i ", where 427 $B_i = \pm 1$ and $C_i = \pm 1$ are the values taken by the observables \hat{B}_i and \hat{C}_i , is separately true, and can be experimentally 4275 checked. However, these three statements cannot be true *together*, since the identity $\hat{C}_1 \hat{C}_2 \hat{C}_3 \equiv \hat{I}$ seems to entail that 4276 $B_1B_2B_3 = +1$ in the considered state, whereas the algebra implies $B_1B_2B_3 = -1$, which is confirmed experimentally 4277 [38]. Indeed we are not even allowed to think simultaneously about the values of B_1 and C_2 , for instance, since these 427 observables do not commute. It is not only impossible to measure them simultaneously but it is even "forbidden" (i. 4279 e., devoid of any physical meaning) to imagine in a given system the simultaneous existence of numerical values for 4280 them⁸⁰. 4281

Such paradoxical effects imply that one cannot attribute to the system alone the numerical values taken by its observables. Indeed, these values are joint properties of observables and states as expressed by Eq. (10.1). Since the "state" simply encompasses information gathered through instruments of observation, a value of an observable characterises some ensemble tested by a given instrument. This value has no existence before measurement, it emerges indirectly after interaction with the apparatus, and can be defined only with reference to the experimental setting which might measure them. Different ensembles are tested by different instruments. The statements of quantum mechanics are meaningful and can be logically combined only if one can imagine an *unique experimental context* in which the quantities involved might be simultaneously measured.

4290 10.2.2. Preparations of states

4291 4292 Que sera, sera⁸¹ Jay Livingston and Ray Evans; sung by Doris Day in The man who knew too much

In order to analyze theoretically quantum phenomena, we need to associate with the considered situation the state that describes adequately the system (or rather the set of systems of the considered ensemble). In particular, to study a dynamical process in the Schrödinger picture, we must specify the initial state. Such an assignment can be performed in various ways, depending on the type of preparation of the system [114, 115].

Textbooks often stress *complete preparations*, in which a complete set of commuting observables is controlled; see Refs. [114, 115] for a recent conceptual discussion that goes beyond the average text-book level. The state \hat{D} is then the projection on the common eigenvector of these observables determined by their given eigenvalues. (This unambiguous determination of \hat{D} should not hide its probabilistic nature.) The control of a single observable may in fact be sufficient to allow a complete preparation of a pure state, in case one is able to select a non-degenerate eigenvalue that characterizes this state. Atoms or molecules are currently prepared thereby in their non-degenerate ground state [289].

As indicated in § 1.1.4, the ideal measurement of an observable \hat{s} (like the spin component \hat{s}_z in the Curie–Weiss model considered in the bulk of the present work) of a system S, followed by the selection of the outcome A_i of the pointer constitutes a *preparation through measurement*. If the density operator of S before the process is $\hat{r}(0)$, this selection produces the filtered state $\Pi_i \hat{r}(0) \Pi_i$, where Π_i denotes the projection operator onto the eigenspace associated with the eigenvalue s_i of \hat{s} (see § 11.3.1). This theoretical scheme of preparing states via measurements was realized experimentally [289, 339].

There are however other, *macroscopic* methods of preparing quantum states [55, 56]. They are much more incomplete [55, 56]. Usually they provide on the quantum system of interest a number of data much too small to characterize a single density operator. As in ordinary probability theory, for describing a macropscopic preparation, one can rely on some criterion to select among the allowed \hat{D} 's the least biased one [270]. For instance, if the only known data are the expectation values of some observables, Laplace's "principle of insufficient reason" yields the least biased density operator, among all those compatible with the available data, as the one that maximizes the entropy (10.2) [297, 298, 299]. In particular, the energy of a small object can be controlled by macroscopic means, exchange of heat or of work; depending on the type of control, the maximum entropy criterion leads us to assign a different distribution

⁸¹What will be, will be

⁷⁹More precisely, $\hat{B}_1 = \hat{\sigma}_x^{(1)} \hat{\sigma}_0^{(2)} \hat{\sigma}_0^{(3)}$, $\hat{C}_1 = \hat{\sigma}_0^{(1)} \hat{\sigma}_z^{(2)} \hat{\sigma}_z^{(3)}$, $\hat{B}_2 = \hat{\sigma}_0^{(1)} \hat{\sigma}_x^{(2)} \hat{\sigma}_0^{(3)}$, $\hat{C}_2 = \hat{\sigma}_z^{(1)} \hat{\sigma}_0^{(2)} \hat{\sigma}_z^{(3)}$, etc., so that $\hat{I} = \hat{\sigma}_0^{(1)} \hat{\sigma}_0^{(2)} \hat{\sigma}_0^{(3)}$

⁸⁰In popular accounts the attempts to do so are explained to fail, after which the situation is often called "mind boggling" [14]. But the only mind boggling point is that one should *not* make these attempts

to this object [58, 73, 71]. This distribution should be verified experimentally. For instance, if one controls only the expectation value of its energy, which is free to fluctuate owing to exchanges with a large bath, the least biased state is the canonical one. Alternatively, for a non-extensive system such that the logarithm of its level density is not concave, another type of thermal equilibrium (locally more stable) can be established [300] through a different preparation involving the confinement of the energy in a narrow range. Within this range, the maximum entropy criterion leads us to attribute the same probability to all allowed levels and to adopt a microcanonical distribution.

The fact that macroscopic states cannot be characterized completely entails that in measurement models the apparatus should be supposed to have initially been prepared in a mixed state. Thus, the discussion of the quantum measurement problem within the statistical interpretation does need the existence of macroscopic preparations that are different from preparations via quantum measurements.

4328 10.2.3. Mixed states and pure states

4329	Something is rotten in the state of Denmark
4330	Shakespeare, Hamle

Most textbooks introduce the principles of quantum mechanics by relying on pure states $|\psi\rangle$, which evolve according to the Schrödinger equation and from which the expectation value of any observable \hat{O} can be evaluated as $\langle \psi | \hat{O} | \psi \rangle$ [4, 84]. Mixed states, represented by density operators, are then constructed from pure states [4, 84]. This form of the principles entail the above-mentioned laws, namely, the Liouville–von Neumann (or the Heisenberg) equation of motion and the properties of the mapping (10.1) (linearity, reality, positivity and normalization).

Within the statistical interpretation, there is at first sight little conceptual difference between pure states and mixed states, since in both cases the density operator behaves as a non-Abelian probability distribution that realizes the correspondence (10.1) [9, 10, 11, 31, 48, 52, 58, 73]. As a mathematical specificity, pure states are those for which all eigenvalues but one of the density operator \hat{D} vanish, or equivalently those for which the von Neumann entropy $S(\hat{D})$ vanishes. They appear thus as extremal among the set of Hermitean positive normalized operators, in the form $|\psi\rangle\langle\psi|$. However, a major physical difference⁸², stressed by Park [28], exists, *the ambiguity in the decomposition of a mixed state into pure states*. This question will play an important rôle in section 11, and we discuss it below.

Let us first note that a mixed state \hat{D} can always be decomposed into a weighted sum of projections over pure states, according to

$$\hat{\mathcal{D}} = \sum_{k} |\phi_k\rangle v_k \langle \phi_k|.$$
(10.3)

It is then tempting to interpret this decomposition as follows. Each of the pure states $|\phi_k\rangle$ would describe systems 4345 belonging to an ensemble \mathcal{E}_k , and the ensemble \mathcal{E} described by $\hat{\mathcal{D}}$ would be built by extracting a proportion v_k 4346 of systems from each ensemble \mathcal{E}_k . Such an interpretation is consistent with the definition of quantum states as 4347 mappings (10.1) of the set of observables onto their expectation values, since (10.3) implies $\langle \hat{O} \rangle = \sum_k v_k \langle \phi_k | \hat{O} | \phi_k \rangle =$ 4348 tr $\hat{D}\hat{O}$. It is inspired by classical statistical mechanics, where a mixed state, represented by a density in phase space, 4349 can be regarded in a unique fashion as a weighted sum over pure states localized at given points in phase space. 4350 However, in quantum mechanics, the state \hat{D} (unless it is itself pure) can be decomposed as (10.3) in an infinity of 4351 *different ways.* For instance, the 2 × 2 density operator $\hat{D} = \frac{1}{2}\hat{\sigma}_0$ which represents an unpolarized spin $\frac{1}{2}$ might be 4352 interpreted as describing a spin polarized either along +z with probability $\frac{1}{2}$ or along -z with probability $\frac{1}{2}$; but these 4353 two possible directions of polarization may also be taken as +x and -x, or as +y and -y; the same isotropic state 4354 $\hat{D} = \frac{1}{2}\hat{\sigma}_0$ can also be interpreted by assuming that the direction of polarization is fully random [31, 48]. Within the 4355 statistical interpretation of quantum mechanics, this ambiguity of the decompositions of \hat{D} prevents us from selecting 4356 a "fundamental" decomposition and to give a sense to the pure states $|\phi_k\rangle$ and the weights v_k entering (10.3). 4357

⁸²Another essential difference between pure and mixed states is especially appealing to intuition [78, 252]. Consider a system in a state represented by a density operator \hat{D} whose eigenvalues are non-degenerate and differ from zero. Consider then a set of observables that have non-degenerate spectra. Then none of such observables can produce definite results when measured in the state \hat{D} [252]. In other words, all such observables have non-zero dispersion in \hat{D} . This statement has been suitably generalized when either \hat{D} or the observables have degeneracies in their spectra; see Appendix C of Ref. [252]. In contrast, for a pure density operator $|\psi\rangle\langle\psi|$ all observables that have $|\psi\rangle$ as eigenvector are dispersionless. Pure and mixed states also differ as regards their preparation and as regards their determination via measurements (e.g., the number of observables to be measured for a complete state determination) [301]

More generally, we may decompose the given state $\hat{\mathcal{D}}$ into a weighted sum

$$\hat{\mathcal{D}} = \sum_{k} \nu_k \hat{\mathcal{D}}_k \tag{10.4}$$

of density operators \hat{D}_k associated with subensembles \mathcal{E}_k . But here again, such a decomposition can always be performed *in an infinity of different ways, which appear as contradictory*. Due to this ambiguity, splitting the ensemble \mathcal{E} described by \hat{D} into subensembles \mathcal{E}_k , described either by pure states as in (10.3) or by mixed states as in (10.4), is physically meaningless (though mathematically correct) if no other information than \hat{D} is available.

The above indetermination leads us to acknowledge an important difference between pure and mixed quantum 4363 states [9, 31, 48, 84, 28, 302]. If a statistical ensemble \mathcal{E} of systems is described by a pure state, any one of its 4364 subensembles is also described by the same pure state, since in this case (10.4) can include only a single term. If for instance a set of spins have been prepared in the polarized state $|\uparrow\rangle$, the statistical prediction about any subset are 4366 embedded in $|\uparrow\rangle$ as for the whole set. In contrast, the existence of many decompositions (10.3) or (10.4) of a mixed 4367 state \hat{D} describing an ensemble \mathcal{E} implies that there exists many ways of splitting this ensemble into subensembles \mathcal{E}_k 4368 that would be described by different states \hat{D}_k . In particular, pure states $|\phi_k\rangle$ that would underlie as in (10.3) a mixed \hat{D} 4369 cannot a posteriori be identified unambiguously by means of experiments performed on the ensemble of systems. In 4370 the statistical interpretation, such underlying pure states have no physical meaning. More generally, decompositions 4371 of the type (10.4) can be given a meaning only if the knowledge of \hat{D} is completed with extra information, allowing 4372 one to identify, within the considered ensemble \mathcal{E} described by $\hat{\mathcal{D}}$, subensembles \mathcal{E}_k that do have a physical meaning [31, 48, 302]. 4374

According to this remark, since the outcome of a large set of measurements is represented by a mixed state $\hat{D}(t_f)$, this state can be decomposed in many different ways into a sum of the type (10.4). The decomposition (9.1), each term of which is associated with an indication A_i of the pointer, is not the only one. This ambiguity of $\hat{D}(t_f)$, as regards the splitting of the ensemble \mathcal{E} that it describes into subensembles, will be discussed in § 11.1.3, and we will show in §§ 11.2 and 13.1 that *the dynamics of the process removes this ambiguity* by privileging the decomposition (9.1) and yields a physical meaning to each of its separate terms, thus allowing us to make statements about individual systems.

4381 10.2.4. Ensembles versus aggregates

We have assumed above that the density operator \hat{D} and the corresponding ensemble \mathcal{E} were given *a priori*. In practice, the occurrence of a mixed state \hat{D} can have various origins. An incomplete preparation (§ 10.2.2) always yields a mixed state, for instance, the initial state $\hat{\mathcal{R}}(0)$ of the apparatus in a measurement model. The mixed nature of a state may be enhanced by dynamics, when some randomness occurs in the couplings or when approximations, justified for a large system, are introduced; this is illustrated by the final state $\hat{\mathcal{D}}(t_f)$ of a measurement process.

Density operators have also been introduced by Landau in a different context [31, 48, 84]. Consider a compound 4387 system $S_1 + S_2$. Its observables are the operators that act in the Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, and its states $\hat{\mathcal{D}}$ are 4388 characterized by the correspondence (10.1) in the space \mathcal{H} . If we are interested only in the subsystem S₁, disregarding 4389 the properties of S_2 and the correlations between S_1 and S_2 , the relevant observables constitute the subalgebra of 4390 operators acting in \mathcal{H}_1 , and the correspondence (10.1) is implemented in the subspace \mathcal{H}_1 by means of the mixed 4391 density operator $\hat{D}_1 = \text{tr}_2 \hat{D}$. Suppose for instance that in an ensemble \mathcal{E} of pairs S_1 , S_2 of spins $\frac{1}{2}$ prepared in the singlet pure state $\frac{1}{\sqrt{2}}(|\uparrow\rangle_1|\uparrow\rangle_2 - |\downarrow\rangle_1|\downarrow\rangle_2)$, we wish to describe only the spin S_1 . Its marginal state in the considered 4392 4393 ensemble \mathcal{E} is again the unpolarized state, represented by $\hat{\mathcal{D}}_1 = \frac{1}{2}\hat{\sigma}_0$. Isotropy is here built in, from this definition of 4394 the state of the spin S_1 .

In all such cases, the state \hat{D} describes a statistical ensemble \mathcal{E} , and the argument of § 10.2.3 entails the impossibility of splitting unambiguous this ensemble into subensembles described by well defined pure states.

Another approach to density operators, initiated by von Neumann [4], consists in constructing them from pure states, by following a path converse to that of § 10.2.3. We start from a collection of statistical ensembles \mathcal{E}_k of systems prepared in pure states $|\phi_k\rangle$. We build a new set \mathcal{E} by *extracting randomly* each individual system of \mathcal{E} from one among the ensembles \mathcal{E}_k , the probability of this extraction being v_k . If we have lost track of the original ensemble \mathcal{E}_k from which each drawing was performed, we have no means to acknowledge in which pure state $|\phi_k\rangle$ a given system of \mathcal{E} was originally lying. The expectation value, for this system, of any quantity is then given by $\langle \hat{O} \rangle = \sum_k v_k \langle \phi_k | \hat{O} | \phi_k \rangle = \text{tr} \hat{D} \hat{O}$, and we are led to assign to it the mixed state defined by (10.3). Here again,

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the ambiguity of § 10.2.3 is present: If two different constructions have lead to the same state \hat{D} , they cannot be distinguished from each other in any measurement.

A further important point should be stressed. The above procedure of *randomly selecting* the elements extracted from the ensembles \mathcal{E}_k produces a set \mathcal{E} of systems which is a *bona fide ensemble*. Indeed, a statistical ensemble must have an essential property, the *statistical independence* of its elements, and this property is here ensured by the randomness of the drawings. Thus, our full information about the ensemble, and not only about each of its individual systems, is embedded in the density operator $\hat{\mathcal{D}}$. In the ensemble \mathcal{E} obtained *after mixing*, the pure states $|\phi_k\rangle$ have been *completely lost*, although they were originally meaningful. In other words, no observation of an ensemble \mathcal{E} obtained by merging subensembles \mathcal{E}_k can reveal the history of its elaboration.

Another, slightly different construction, also inspired by von Neumann's idea, is preferred by some authors, see, e.g., [303]. In this alternative procedure, a (non random) number n_k of systems is extracted from each ensemble \mathcal{E}_k so as to constitute a set \mathcal{A} having $n = \sum_k n_k$ elements, which we term as an *aggregate*. Losing again track of the origin of each system of \mathcal{A} , we have to assign to any individual system of \mathcal{A} the density operator (10.3), with $v_k = n_k/n$. However, in spite of this analogy with the ensemble \mathcal{E} constructed above, we will acknowledge an important difference between the two situation, due to the nature of the numbers v_k , which are *probabilities* for \mathcal{E} , *proportions* for \mathcal{A} .

As an illustration, let us consider the aggregate \mathcal{R}_z built by gathering $n_1 = \frac{1}{2}n$ spins prepared in the pure state 4420 $|\uparrow\rangle$ ($s_z = +1$) and $n_2 = \frac{1}{2}n$ spins prepared in the pure state $|\downarrow\rangle$ ($s_z = -1$), and by forgetting the original state of 4421 each spin. Each individual spin of the aggregate \mathcal{A}_z in then described by the unpolarized density operator $\hat{\mathcal{D}} = \frac{1}{2}\hat{\sigma}_0$, 4422 exactly as each spin of the ensemble \mathcal{E}_{z} , obtained by picking up states $|\uparrow\rangle$ or $|\downarrow\rangle$ randomly with equal probabilities. 4423 Nevertheless, the joint statistics of two systems belonging to the aggregate \mathcal{A}_r differs from that of two spins belonging 4424 to the ensemble \mathcal{E}_z (which are statistically independent). Indeed, the systems of an aggregate are correlated, due to the construction procedure. In our spin example, this is flagrant for $n_1 = n_2 = 1$. More generally, if $\hat{\sigma}_z$ is simultaneously measured on all n spins of the aggregate \mathcal{A}_z , the correlations will be expressed by the equality of the number of 4427 outcomes \uparrow and \downarrow . If the ideal measurement bears on n-1 spins, we can predict for the last spin the sign of σ_z 4428 with 100% confidence. For an ensemble \mathcal{E}_z containing *n* spins, we cannot infer anything about the *n*'th spin from the 4429 outcomes of previous measurements on the n - 1 other ones. 4430

Altogether, *an aggregate is not a statistical ensemble*, because its elements are *correlated with one another*. A random selection is needed in von Neumann's procedure of defining mixed states, so as to ensure the statistical independence required for ensembles.

The above point was purely classical (since we dealt with the z-component only), but it can have quantum im-4434 plications. Prepare another aggregate \mathcal{A}_x with n_1 spins oriented in the +x-direction and n_2 spins oriented in the 4435 -x-direction. Consider likewise the ensemble \mathcal{E}_x built by randomly selecting spins in the -x- and +x-directions, with 4436 equal probabilities. Any single system belonging to either \mathcal{A}_z or \mathcal{E}_z or \mathcal{A}_x or \mathcal{E}_x is described by the same unpolarized 4437 density operator $\frac{1}{2}\hat{\sigma}_0$. However, differences occur when correlations between systems are accounted for. We first re-4438 member that the ensembles \mathcal{E}_z and \mathcal{E}_x are undistinguishable. In contrast, the two aggregates \mathcal{A}_z and \mathcal{A}_x have different 4439 properties. Measuring for \mathcal{A}_x , as above for \mathcal{A}_z , the components $\hat{\sigma}_z$ of all the *n* spins of \mathcal{A}_x does not show up the 4440 correlations that were exhibited for \mathcal{R}_z : Instead of finding exactly $\frac{1}{2}n$ spins up and $\frac{1}{2}n$ spins down, we find outcomes 4441 that are statistically independent, and characterized by a same binomial law as in the case of the ensembles \mathcal{E}_z or \mathcal{E}_x . 4442 The correlations within \mathcal{A}_x occur between x-components. 4443

Hence, failing to distinguish aggregates from ensembles leads to the inevitable conclusion that "two ensembles having the same density matrix can be distinguished from each other" [303]. This statement has influenced similar conclusions by other authors, see e.g. [304]. The persistent occurrence of such an idea in the literature (see [305]) demonstrates that the difference between ensembles and aggregates is indeed far from being trivial. In the light of the above discussion, and in agreement with [306] and [307], we consider such statements as incorrect. Indeed, *two aggregates having for a single system the same density matrix can be distinguished from each other* via two-system (or many-system) measurements, but *two statistical ensembles cannot*.

4451 11. The quantum measurement problem within the statistical interpretation

All's well that ends well Shakespeare

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- 4453

In section 9 we have resumed the detailed solution of the Curie-Weiss model. As other models of measurement 4454 treated in the framework of quantum statistical dynamics (section 2), it yields, for the compound system S + A at the 4455 end of the process, a density operator $\hat{D}(t_f)$ which satisfies the properties required for ideal measurements. However, 4456 we have already stressed that such a result, although necessary, is not sufficient to afford a complete understanding 4457 of quantum measurements. Indeed, the statistical interpretation of quantum mechanics emphasizes the idea that this 4458 theory, whether it deals with pure or mixed states, does not govern individual systems but statistical ensembles of 4459 systems (§ 10.1.3). Within this statistical interpretation, the density operator $\mathcal{D}(t_f)$ accounts in a probabilistic way for 4460 a large set \mathcal{E} of similar runs of the experiment, whereas a measurement involves properties of individual runs. What 4461 can then we say about the individual runs? This question is the core of the present section. 4462

The remaining challenge is to elucidate the *quantum measurement problem*, that is, to explain why *each individual run provides a definite outcome*, for both the apparatus and the tested system. As we will discuss, this property is not granted by the knowledge of $\hat{D}(t_f)$, an object associated with the full set \mathcal{E} . Within the statistical interpretation, we may deal only with ensembles; the individual systems that we wish to consider should be embedded in some subensembles of \mathcal{E} , to be eventually characterised by a specific outcome.

4468 11.1. Formulating the problem

4469 We discuss below how the quantum measurement problem arises in the framework of the statistical interpretation.

4470 11.1.1. A physical, but simplistic and circular argument

⁴⁴⁷¹ Une idée simple mais fausse s'impose toujours face à une idée juste mais compliquée⁸³ Alexis de Tocqueville

As shown by the review of section 2 and by the Curie–Weiss example of section 3, many models of ideal quantum 4473 measurements rely on the following ideas. The apparatus A is a macroscopic system which has several possible stable 4474 states $\hat{\mathcal{R}}_i$ characterized by the value A_i of the (macroscopic) pointer variable. If A is initially set into a metastable 4475 state $\hat{\mathcal{R}}(0)$, it may spontaneously switch towards one or another state $\hat{\mathcal{R}}_i$ after a long time. In a measurement, this 4476 transition is triggered by the coupling with the tested object S, it happens faster, and it creates correlations such that, if 4477 the apparatus reaches the state $\hat{\mathcal{R}}_i$, the tested observable \hat{s} takes the value s_i . The neat separation between the states $\hat{\mathcal{R}}_i$ 1170 and their long lifetime, together with the lack of survival of "Schrödinger cats", suggest that each individual process has a unique outcome, characterized by the indication A_i of the pointer and by the value s_i for the observable \hat{s} of the 4480 system S. 4481

This intuitive argument, based on current macroscopic experimental observation and on standard classical theories of phase transitions, is nevertheless delusive. Although its outcome will eventually turn out to be basically correct, it postulates the very conclusion we wish to justify, namely that the apparatus reaches in each run one or another among the states $\hat{\mathcal{R}}_i$. This idea is based on a classical type of reasoning applied blindly to subtle properties of quantum ensembles, which is known to produce severe mistakes (prescribed ensemble fallacy) [308, 309, 310, 311]. We ought to analyze quantum measurements by means of rigorous quantum theoretical arguments, in order to explain why the indication of the apparatus is unique in a single experiment.

4489 11.1.2. Where does the difficulty lie?

4490 4491 Δεν βρέτηεκα τίτλο με του όρο⁸⁴ Aesop

Let us remind that the most detailed results established for models of ideal measurements of ideal measurements treated within quantum theory (section 2), and in particular for the Curie–Weiss model (sections 3-9), refer to a *large statistical ensemble* \mathcal{E} *of runs* involving a coupled evolution of S and A. We wish, however, to focus on *individual runs* so as to explain in particular why, at the end of each run, the pointer gives a well-defined indication A_i . This property agrees with our macroscopic experience and seems trivial, but it is not granted in the quantum framework.

⁸³A simple but wrong idea always prevails over a right but complex idea

⁸⁴After all is said and done, more is said than done

The outcomes of statistical mechanical treatments are represented by the evolution of the density operator $\hat{D}(t)$ of the compound system S + A, from the initial state

$$\hat{\mathcal{D}}(0) = \hat{r}(0) \otimes \hat{\mathcal{R}}(0), \tag{11.1}$$

4499 to the final state

$$\hat{\mathcal{D}}(t_{\rm f}) = \sum_{i} p_i \hat{\mathcal{D}}_i, \qquad \hat{\mathcal{D}}_i = \hat{r}_i \otimes \hat{\mathcal{R}}_i, \qquad p_i = \operatorname{tr}_S \hat{r}(0) \hat{\Pi}_i, \qquad p_i \hat{r}_i = \hat{\Pi}_i \hat{r}(0) \hat{\Pi}_i. \tag{11.2}$$

⁴⁵⁰⁰ In the Curie–Weiss model its explicit form is the expression (3.20), that is,

$$\hat{\mathcal{D}}(t_{\rm f}) = p_{\uparrow} |\uparrow\rangle\langle\uparrow|\otimes\hat{\mathcal{R}}_{\uparrow} + p_{\perp}|\downarrow\rangle\langle\downarrow|\otimes\hat{\mathcal{R}}_{\parallel}.$$
(11.3)

As we wish to interpret this result physically, we recall its nature. The state $\hat{D}(t)$ provides a faithful probabilistic account for the dynamics of *a large set* \mathcal{E} *of runs* of similarly prepared experiments, but *nothing more*. Quantum mechanics is our most fundamental theory, but even a complete solution of its dynamical equations refers only to the statistics of an ensemble \mathcal{E} . The description of *individual processes* is *excluded* (§ 10.1.3): As any quantum state, (11.2) is irreducibly probabilistic. In fact, probabilities occur for many other reasons (§ 12.1.2), which have not necessarily a quantum origin.

The specific form of the expression (11.2) for the final state of S + A properly accounts for all the features of ideal 4507 measurements that are related to a large set of runs. On the one hand, its off-diagonal blocks have been truncated. This 4508 is a necessary property: the presence of elements in off-diagonal blocks would imply, as will be shown in § 11.2.1, that 4509 some runs would not provide a well-defined outcome, rendering the solution of the measurement problem impossible. 4510 In fact, von Neumann's reduction implies that each individual run should end up in a state represented by a diagonal 4511 block $\hat{\mathcal{D}}_i$, so that the ensemble \mathcal{E} obtained by putting together these runs should be represented by a weighted sum of blocks $\hat{\mathcal{D}}_i$. On the other hand, the values of the weights p_i occurring in (11.2) agree with Born's rule. Finally, 4513 (11.2) exhibits the expected complete correlation between the final state \hat{r}_i of S and that $\hat{\mathcal{R}}_i$ of A characterized by the 4514 indication A_i of the pointer. 4515

Nevertheless, the properties of ideal measurements require the consideration of individual runs, or at least of subensembles of \mathcal{E} . The correlation existing in (11.2) means that, *if* A_i *is observed*, S will be described by \hat{r}_i . However, nothing in $\hat{\mathcal{D}}(t_f)$ warrants that one can observe some well-defined value of the pointer in an individual run [155, 156, 157, 158, 159, 160, 161], so that the standard classical interpretation cannot be given to this quantum correlation. Likewise, Born's rule means that a proportion p_i of individual runs end up in the state $\hat{\mathcal{D}}_i$. The validity of this rule requires $\hat{\mathcal{D}}(t_f)$ to have the form (11.2); but conversely, as will be discussed in § 11.1.3, the sole result (11.2) is not sufficient to explain Born's rule which requires the counting of the individual runs tagged by the outcome A_i . And of course von Neumann's reduction requires a selection of the runs having produced a given outcome.

If quantum mechanics were based on the same kind of probabilities as classical physics, it would be obvious 4524 infer statistically the properties of individual systems from the probability distribution governing the statistical to 4525 ensemble to which they belong. At first sight, the description of a quantum ensemble by a density operator seems 4526 analogous to the description of an ensemble of classical statistical mechanics by a probability density in phase space 4527 or to the description of some ensemble of events by ordinary probabilities. We must acknowledge, however, a major difference. In ordinary probability theory, one can distinguish exclusive properties, one of which unambiguously 4529 occurs for each individual event. When we toss a coin, we get either heads or tails. In contrast, a quantum state is 4530 plagued by the impossibility of analysing it in terms of an exclusive alternative, as demonstrated by the example of an 4531 unpolarized spin $\frac{1}{2}$ (§ 10.2.3). We are not allowed to think, in this case, that the spin may lie either in the +z (or the 4532 -z) direction, since we might as well have thought that it lay either in the +x (or the -x) direction. 4533

This ambiguity of a mixed quantum state may also be illustrated, in the Curie–Weiss model, by considering the final state of the magnet M alone. For the ensemble \mathcal{E} , it is described by the density operator $\hat{R}_{\rm M}(t_{\rm f}) = P_{\rm M}^{\rm dis}(\hat{m}, t_{\rm f})/G(\hat{m})$, where the probability distribution $P_{\rm M}^{\rm dis}(m, t_{\rm f})$ is strongly peaked around the two values $m = m_{\rm F}$ and $m = -m_{\rm F}$ of the pointer variable m, with the weights p_{\uparrow} and p_{\downarrow} . In standard probability theory this would imply that for a single system m takes either the value $m_{\rm F}$ or the value $-m_{\rm F}$. However, in quantum mechanics, an individual system should be regarded as belonging to some subensemble \mathcal{E}'_k of \mathcal{E} . We may imagine, for instance, that this subensemble is described by a pure state $|\psi\rangle$ such that $|\langle m, \eta |\psi\rangle|^2$ presents the same two peaks as $P_M(m, t_f)$, where we noted as $|m, \eta\rangle$ the eigenstates of \hat{m} (the other quantum number η takes a number G(m) of values for each m). This state lies astride the two ferromagnetic configurations, with coherences, so that the magnetization of the considered individual system cannot have a definite sign. From the sole knowledge of $\hat{R}_M(t_f)$, we cannot infer the uniqueness of the macroscopic magnetization.

Thus, albeit both quantum mechanics and classical statistical mechanics can be formulated as theories dealing with statistical ensembles, going to individual systems is automatic in the latter case, but problematic in the former case since it is impossible to characterize unambiguously the ensembles of \mathcal{E} .

4548 11.1.3. The real task: determine proper subensembles

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Horresco referens⁸⁵ Virgil, Aeneid

Remember first that, when quantum mechanics is used to describe an individual system, the density operator that characterizes its state refers either to a real or to a thought ensemble (§ 10.1.3). If we consider a real set \mathcal{E} of measurement processes, each individual outcome should be embedded in a *real subset* of \mathcal{E} . We are thus led to study the various possible splittings of \mathcal{E} into subensembles.

A superficial examination of the final state (11.2) suggests the following argument. In the same way as we may 4555 obtain an unpolarized spin state by merging two populations of spins separately prepared in the states $|\uparrow\rangle$ and $|\downarrow\rangle$, let 4556 us imagine that we have prepared many compound systems S + A in the equilibrium states \hat{D}_i . We build ensembles \mathcal{E}_i , 4557 each of which contains a proportion p_i of systems in the state \hat{D}_i , merge them into a single one \mathcal{E} and lose track of this construction. The resulting state for the ensemble \mathcal{E} is identified with (11.2) and all predictions made thereafter about 4559 \mathcal{E} will be the same as for the state $\hat{\mathcal{D}}(t_f)$ issued from the dynamics of the measurement process. It is tempting to admit 4560 conversely that the set \mathcal{E} of runs of the measurement may be split into subsets \mathcal{E}_i , each of which being characterized 4561 by the state \hat{D}_i . This would be true in ordinary probability theory. If the reasoning were also correct in quantum 4562 mechanics, we would have proven that each run belongs to one of the subsets \mathcal{E}_i , so that it leads S + A to one or 4563 another among the states \hat{D}_i at the time t_f , and that its outcome is well-defined. 4564

Here as in § 11.1.1 the above argument is fallacious. Indeed, as illustrated by the example of unpolarized spins, as stressed in § 10.2.3, and contrarily to a state in classical statistical mechanics, a mixed state \hat{D} can be split in *many different incompatible ways* into a weighted sum of density operators which are more informative than \hat{D} . Here, knowing the sole final state $\hat{D}(t_f)$ for the set \mathcal{E} of runs, we can decompose it not only according to (11.2), but alternatively into one out of many different forms

$$\hat{\mathcal{D}}(t_{\rm f}) = \sum_{k} \nu_k \hat{\mathcal{D}}'_k(t_{\rm f}), \qquad (11.4)$$

where the set of states $\hat{D}'_k(t_f)$, possibly pure, differ from the set \hat{D}_i : The very concept of decomposition is *ambiguous*. If we surmise, as we did above above when we regarded \mathcal{E} as the union of (thought) subensembles \mathcal{E}_i described by \hat{D}_i , that the density operator $\hat{D}'_k(t_f)$ is associated with a (thought) subset \mathcal{E}'_k of \mathcal{E} containing a fraction v_k of runs of the measurement, we stumble upon a physical contradiction: The full set \mathcal{E} of runs could be partitioned in different ways, so that a given run would belong both to a subset \mathcal{E}_i and to a subset \mathcal{E}'_k , but then we could not decide whether its final state is \hat{D}_i or $\hat{D}'_k(t_f)$, which provide different expectation values.

The various decompositions (11.2) and (11.4) are purely *mathematical properties*. Unless we succeed to identify some physical process that selects one of them, the very fact that they formally exist precludes the task considered here, namely to explain the uniqueness of individual measurements, the quantum measurement problem. In the present context, *only one decomposition may be physically meaningful*, i. e., may correspond to the splitting of the real set \mathcal{E} of runs described by $\hat{\mathcal{D}}(t_f)$ into actually existing subsets. If we wish to remain within standard quantum mechanics we can only identify this physical decomposition by extending to subensembles the dynamical analysis which has

⁸⁵I shiver while I am telling it

produced $\hat{D}(t_f)$ from $\hat{D}(0)$ for the full ensemble. Nothing a priori warrants that the set of states \hat{D}_i will then play a privileged role.

The above ambiguity is well known in the literature [22, 310, 308, 311]. Not paying attention to its existence, and

then imposing by hand the desired separation into subensembles, was called the "*prescribed ensemble fallacy*" [311]. The question does not seem to have yet been resolved in the context of proper measurement models, but we attempt to answer it below.

To illustrate the harmfulness of this ambiguity, consider the simple case of a Curie–Weiss model with N = 2(subsection 8.1). Although it cannot be regarded as an ideal measurement, it will clearly exhibit the present difficulty. After elimination of the bath, after reduction and under the conditions (8.7), the state of S + M at a time t_f such that $\tau_{reg} \ll t_f \ll \tau_{obs}$ has the form

$$\hat{D}(t_{\rm f}) = p_{\uparrow} \hat{D}_{\uparrow} + p_{\downarrow} \hat{D}_{\downarrow}, \qquad (11.5)$$

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where \hat{D}_{\uparrow} is the projection onto the pure state $|\uparrow, \uparrow\rangle$ characterized by the quantum numbers $s_z = 1, m = 1$, and likewise \hat{D}_{\downarrow} the projection onto $|\downarrow, \downarrow\rangle$ with $s_z = m = -1$. This form suggests that individual runs of the measurement should lead as expected either to the state $|\uparrow, \uparrow\rangle$ or to the state $|\downarrow, \downarrow\rangle$ with probabilities given by the Born rule. However, this conclusion is not granted since we can also decompose $\hat{D}(t_f)$ according, for instance, to

$$\hat{D}(t_{\rm f}) = \nu_1 \hat{D}'_1 + \nu_2 \hat{D}'_2, \tag{11.6}$$

where \hat{D}'_1 is the projection onto $\sqrt{p_{\uparrow}/v_1} \cos \alpha |\uparrow, \uparrow\rangle + \sqrt{p_{\downarrow}/v_1} \sin \alpha |\downarrow, \downarrow\rangle$ and \hat{D}'_2 the projection onto $\sqrt{p_{\uparrow}/v_2} \sin \alpha \times \frac{1}{2}$, $\uparrow\uparrow, \uparrow\rangle - \sqrt{p_{\downarrow}/v_2} \cos \alpha |\downarrow, \downarrow\rangle$, with α arbitrary and $v_1 = p_{\uparrow} \cos^2 \alpha + p_{\downarrow} \sin^2 \alpha = 1 - v_2$. Nothing then would prevent a single real run to belong to the subensemble described by the first term of (11.6) and thus end up in the state \hat{D}'_1 where neither *m* nor s_z are well-defined. In spite of the suggestive form of (11.5), we cannot give any physical interpretation to its separate terms, on account of the existence of alternative formally similar decompositions (11.6).

In order to interpret the results drawn from the solution of models, it is therefore essential to determine not only the state $\hat{D}(t)$ for the full ensemble \mathcal{E} of runs of the measurement, but also the final state of S + A for any subensemble of runs. Only then may one be able to assign to an individual system, after the end of the process, a density operator more informative than $\hat{D}(t_f)$ and to derive from it the required properties of an ideal measurement.

To this end, one might *postulate that a measuring apparatus is a macroscopic device which produces at each run a well-defined value for the pointer variable*, a specific property which allows registration. (This idea is somewhat reminiscent of Bohr's view that the apparatus is classical.) Thus, the apparatus would first be treated as a quantum object so as to determine the solution $\hat{D}(t)$ of the Liouville–von Neumann equation for the full ensemble \mathcal{E} , and would then be postulated to behave classically so as to determine the states of the subensembles to which the individual runs belong. No contradiction would arise, owing to the reduced form found for $\hat{D}(t_f)$. (This viewpoint differs from that of the quantum–classical models of section 2.2.)

Although expedient, such an approach is unsatisfactory. It is obviously unjustified in the above N = 2 case. To really solve the measurement problem, we need to *explain the behaviour of the apparatus in individual runs* by relying on the sole principles of quantum mechanics, instead of supplementing them with a doubtful postulate. We now show that the task of understanding from quantum dynamics the uniqueness of measurement outcomes is feasible, at least for sufficiently elaborate models of quantum measurements. In fact, we will prove in the forthcoming subsections that the quantum Curie–Weiss model for a magnetic dot M + B can be modified so as to explain the *classical behaviour of its ferromagnetic phases*, and hence the full properties of the measurement.

4619 11.2. The states describing subensembles

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De hond bijt de kat niet⁸⁶ Les chiens ne font pas des chats⁸⁶ Dutch and French sayings

⁸⁶Dogs do not beget cats

Several steps will be required to show that the density matrix for any subensemble has the same truncated structure as the one which describes the full ensemble \mathcal{E} . We rely for illustration on the Curie–Weiss model, and first include for orientation a tentative solution, seemingly natural but unsuccessful (§ 11.2.2), and then another one, effective but not yet fully satisfactory (§ 11.2.3), before we present a final, more general treatment (§ 11.2.4).

4627 11.2.1. Hierarchic structure of physical subensembles

We first note that the question of splitting the state $\hat{D}(t)$ of S + A into a weighted sum of states $\hat{D}'_k(t)$, each one 4628 associated with a subensemble \mathcal{E}'_k , can be raised only after the truncation has taken place, at a time τ_{split} such that 4629 $\tau_{\text{split}} \gg \tau_{\text{trunc}}$ and under conditions that exclude recurrences. This is not a strong constraint, since we anyhow intend 4630 to wait until t_f , the end of the registration. Among the various possible states \hat{D}'_k that one can imagine to associate with subensembles of \mathcal{E} , one should never find at least those which possess at the final time non-zero elements in off-4632 diagonal blocks. (These elements only have to cancel out in the sum over k of (11.4).) Such a situation is exemplified 4633 by (11.6) for $\cos 2\alpha \neq 0$, in which case \hat{D}'_1 possesses off-diagonal terms $|\uparrow, \uparrow\rangle\langle\downarrow, \downarrow|$ and $|\downarrow, \downarrow\rangle\langle\uparrow, \uparrow|$, and hence contains 4634 (due to the positivity of \hat{D}'_1) both the diagonal terms $|\uparrow,\uparrow\rangle\langle\uparrow,\uparrow|$ and $|\downarrow,\downarrow\rangle\langle\downarrow,\downarrow|$. An individual run of this type could 4635 not yield a well-defined macroscopic value for the pointer.

⁴⁶³⁷ A model suitable to fully explain an ideal measurement must yield for S + A, at the end of each run, one or another ⁴⁶³⁸ among the states \hat{D}_i defined by (11.2). We do not have direct access to individual runs, but should regard them as ⁴⁶³⁹ embedded in subensembles. Consider then an arbitrary subensemble of real runs drawn from the full ensemble \mathcal{E} and ⁴⁶⁴⁰ containing a proportion q_i of individual runs of the type *i*. We expect this subensemble to be described at the end of ⁴⁶⁴¹ the measurement process by a density operator of the form

$$\hat{\mathcal{D}}_{\text{sub}}(t_{\text{f}}) = \sum_{i} q_{i} \hat{\mathcal{D}}_{i}.$$
(11.7)

(For the full set \mathcal{E} , the weights have been denoted by p_i^{87} .)

We will refer to the essential property (11.7) as the *hierarchic structure of subensembles*. The existence of this common form for all subensembles is a *consistency property*. It is trivially satisfied in ordinary probability theory where all subensembles are constructed from the same building blocks, but it is not granted in quantum mechanics due to the infinity of different ways of splitting the state of \mathcal{E} into elementary components as in (10.3), or into subensembles as in (10.4). The existence of *the hierarchic structure removes this ambiguity* stressed in § 11.1.3. We wish therefore to prove that the final state of *any subensemble of* \mathcal{E} *has the form* (11.7).

Our strategy will again rely on a dynamic analysis, now not for the whole ensemble as before, but for an arbitrary 4649 subensemble. Consider, at the time t_{split} , some splitting (11.4) of \mathcal{E} into subensembles \mathcal{E}'_{k} . We select one of these, 4650 denoted as \mathcal{E}_{sub} and described for $t > t_{split}$ by the state $\hat{\mathcal{D}}_{sub}(t)$. Since $\hat{\mathcal{D}}_{sub}(t_{split})$ is issued from the decomposition 4651 (11.4) of $\hat{D}(t_{\text{split}})$, it presents an arbitrariness limited by the positivity of $\hat{D}(t_{\text{split}}) - \nu_k \hat{D}_{\text{sub}}(t_{\text{split}})$ for a sizeable value of 4652 v_k . We will then study, at least in the Curie–Weiss model, the Liouville–von Neumann evolution of the state $\hat{D}_{sub}(t)$ 4653 from $t = t_{split}$ to $t = t_f$, and will prove that it relaxes towards the form (11.7). We could not decide beforehand whether 4654 \mathcal{E}_{sub} was real or virtual, but all real subensembles will anyhow be accounted for by this treatment, which will therefore 4655 yield the desired conclusion. 4656

4657 11.2.2. Attempt of early truncation

In order to try to establish the hierarchical structure of subensembles, which requires at least the elimination from $\hat{D}_{sub}(t)$ of its off-diagonal blocks, it is natural to try to approach the problem as in section 5. We take a short splitting time t_{split} , satisfying $\tau_{trunc} \ll t_{split} \ll \tau_{reg}$, so that $\hat{D}(t_{split})$ has the form $\sum_i \hat{\Pi}_i \hat{r}(0) \hat{\Pi}_i \otimes \hat{\mathcal{R}}(0)$ issued from $\hat{D}(0)$ by projecting out its off-diagonal blocks; the state $\hat{\mathcal{R}}(0)$ of the apparatus has not yet been significantly affected. The initial condition $\hat{D}_{sub}(t_{split})$ is found from some decomposition of the simple truncated state $\hat{D}(t_{split})$. To follow the fate of the subensemble \mathcal{E}_{sub} , we have to solve the equations of motion of section 4. The situation is the same as in section 5, except for the replacement of the initial condition $\hat{D}(0)$ by $\hat{D}_{sub}(t_{split})$. If the truncation mechanisms of sections 5

⁸⁷In case \mathcal{E} is split into some set of disjoint subensembles, each p_i of \mathcal{E} is a weighted sum of the corresponding coefficients q_i for these subensembles

and 6 are effective, the elements present in the off-diagonal blocks of $\hat{D}_{sub}(t)$ disappear over the short time scale τ_{trunc} , as they did for $\hat{D}(t)$. The state $\hat{D}_{sub}(t)$ is thus dynamically unstable against truncation. Later on, the diagonal blocks that remain after this relaxation will evolve as in section 7, and give rise to ferromagnetic states for M, so that $\hat{D}_{sub}(t_f)$ will eventually reach the form (11.7).

⁴⁶⁶⁹ Unfortunately, the truncation mechanism based on the coupling between S and M is not efficient for all possible ⁴⁶⁷⁰ initial states $\hat{D}_{sub}(t_{split})$ arising from a decomposition of $\hat{D}(t_{split})$. We have seen in section 5 that it relies on the ⁴⁶⁷¹ presence of a width in the initial paramagnetic probability distribution $P_M(m, 0)$ of the pointer variable, and that it ⁴⁶⁷² may fail for "squeezed" initial states of M (§ 5.2.3). While the full state $\hat{D}(t_{split})$ involves a width of order $1/\sqrt{N}$ ⁴⁶⁷³ for $P_M(m, 0)$, this property is not necessarily satisfied for its components $\hat{D}_{sub}(t_{split})$, so that we cannot prove for an ⁴⁶⁷⁴ arbitrary subensemble \mathcal{E}_{sub} that $\hat{D}_{sub}(t)$ finally reaches the form (11.7).

In fact, the Curie–Weiss model as it stands was too crude for our present purpose. Indeed, only a single variable, the combination $\hat{m} = (1/N) \sum_n \hat{\sigma}_z^{(n)}$ of the pointer observables, enters its dynamics. The irreversible process that ensures the hierarchic structure of the subensembles is more elaborate than the truncation process of $\hat{D}(t)$ and requires dynamics involving many variables.

4679 11.2.3. Dynamical instability for the pointer alone

The relaxation of arbitrary states of subensembles towards (11.7) thus cannot be achieved by the interaction \hat{H}_{SM} , so that we have to rely on the Hamiltonian of the apparatus itself. The dynamical mechanism responsible for this 4681 relaxation cannot work at an early stage, when the invariance of the state of M has not yet been broken. We must 4682 therefore consider the instability of subensembles at a late time t_{split} such that M, after having been triggered by S, 4683 has reached for the full ensemble \mathcal{E} a mixture of the two ferromagnetic states. The time t_{split} at which we imagine 468 splitting \mathcal{E} into subensembles \mathcal{E}'_t is now taken at the end of the registration, just before the time t_f , so that the initial 4685 state $\hat{D}_{sub}(t_{split})$ of the considered dynamical process is one element of some decomposition (11.4) of $\hat{D}(t_f)$. As the registration is achieved at that time t_{split} , the interaction \hat{H}_{SA} is then supposed to have been *switched off*, so that the 4687 apparatus will relax by itself without any effect on the system. 4688

Such decompositions (11.4) of $\hat{D}(t_f)$ are made simpler if we replace, in the expression (11.2), each *canonical* ferromagnetic equilibrium state \hat{R}_i by a *microcanonical* state; this is justified for large *N*. Tracing out the bath, which reduces \hat{D} to \hat{D} , we will therefore consider arbitrary decompositions of the analogue for S+M of the state (11.3), that is,

$$\hat{D}(t_{\rm f}) = p_{\uparrow} \hat{\Pi}_{\uparrow} \otimes \hat{R}^{\mu}_{\uparrow} + p_{\downarrow} \hat{\Pi}_{\downarrow} \otimes \hat{R}^{\mu}_{\downarrow}, \qquad (11.8)$$

where $\hat{\Pi}_{\uparrow} = |\uparrow\rangle\langle\uparrow|$ and $\hat{\Pi}_{\downarrow} = |\downarrow\rangle\langle\downarrow|$. The two occurring microcanonical states of M are expressed as (with the index μ for microcanonical)

$$\hat{R}^{\mu}_{\uparrow\uparrow} = \frac{1}{G} \sum_{\eta} |m_{\rm F}, \eta\rangle \langle m_{\rm F}, \eta|, \qquad \hat{R}^{\mu}_{\downarrow\downarrow} = \frac{1}{G} \sum_{\eta} |-m_{\rm F}, \eta\rangle \langle -m_{\rm F}, \eta|, \qquad (11.9)$$

where $|m, \eta\rangle$ denote the eigenstates $|\sigma_z^{(1)}, \dots, \sigma_z^{(n)}, \dots, \sigma_z^{(N)}\rangle$ of \hat{H}_M , with $m = (1/N) \sum_n \sigma_z^{(n)}$; the index η takes a number G(m) of values, and the degeneracy G(m) of the levels, expressed by (3.23), is large as an exponential of N; for shorthand we have denoted $G(m_F) = G(-m_F)$ as G.

As (11.8) is an operator in the 2*G*-dimensional space spanned by the basis $|\uparrow\rangle \otimes |m_{\rm F}, \eta\rangle$, $|\downarrow\rangle \otimes |-m_{\rm F}, \eta\rangle$, any density operator $\hat{D}_{\rm sub}(t_{\rm split})$ issued from the decomposition of $\hat{D}(t_{\rm f})$ is a linear combination of projections over pure states $|\Psi(0)\rangle$ of the form [71, 309, 312]

$$|\Psi(0)\rangle = \sum_{\eta} U_{\eta} |\uparrow\rangle \otimes |m_{\rm F}, \eta\rangle + \sum_{\eta} V_{\eta} |\downarrow\rangle \otimes |-m_{\rm F}, \eta\rangle, \qquad (11.10)$$

with some coefficients U_{η} , V_{η} . For shorthand we chose in (11.10) t_{split} as a new origin of time. It will therefore be sufficient for our purpose to take $\hat{D}_{\text{sub}}(t_{\text{split}} + t) = |\Psi(t)\rangle\langle\Psi(t)|$, and to prove that $|\Psi(t_{\text{f}})\rangle\langle\Psi(t_{\text{f}})|$ has for arbitrary values of U_{η} , V_{η} the same form as (11.8) within replacement of p_{\uparrow} and p_{\downarrow} by other coefficients. (Owing to our choice of a micro-canonical model, the "cat terms" $|\uparrow\rangle\langle\downarrow|\otimes|m_{\rm F},\eta\rangle\langle-m_{\rm F},\eta'|$ in $|\Psi(0)\rangle\langle\Psi(0)|$, and their hermitean conjugates, fully correlate S and M, in spite of splitting the ensemble \mathcal{E} into subensembles.) Our problem amounts to show that any such $|\Psi(0)\rangle$ is dynamically unstable and that $|\Psi(t)\rangle\langle\Psi(t)|$ decays on a time scale short compared to $t_{\rm f}$ towards an *incoherent sum of microcanonical distributions*, as

$$|\Psi(t)\rangle\langle\Psi(t)| \to q_{\uparrow}\hat{\Pi}_{\uparrow} \otimes \hat{R}^{\mu}_{\uparrow} + q_{\downarrow}\hat{\Pi}_{\downarrow} \otimes \hat{R}^{\mu}_{\downarrow}, \qquad q_{\uparrow} = \sum_{\eta} |U_{\eta}|^{2}, \quad q_{\downarrow} = \sum_{\eta} |V_{\eta}|^{2}.$$
(11.11)

This is the microcanonic relaxation process which was discussed in literature several times and under various assumptions; see Ref. [313] for an early review and Refs. [314, 315, 316, 317, 314] for further recent results.

Obviously, our simple Curie–Weiss model as defined in section 3 is inappropriate for this task. Indeed, all the 4710 states $|\uparrow\rangle \otimes |m_{\rm F}, \eta\rangle$ and $|\downarrow\rangle \otimes |-m_{\rm F}, \eta\rangle$ are eigenstates with the same eigenvalue of both the coupling $\hat{H}_{\rm SM}$ and the Ising 4711 Hamiltonian \hat{H}_M , so that $\hat{H}_{SM} + \hat{H}_M$ has no effect on $|\Psi(t)\rangle\langle\Psi(t)|$. Whether S is still coupled to M or not at the time 4712 $t_{\rm split}$ thus makes no difference. Moreover, the coupling $\hat{H}_{\rm MB}$ with the bath was adequate to allow dumping of energy 4713 from M to B during the registration, whereas we need here transitions between states $|m_{\rm F}, \eta\rangle$ and $|m_{\rm F}, \eta'\rangle$ with equal 4714 energies. We must therefore improve the model, by supplementing the original Hamiltonian of subsection 3.2 with 4715 weak interactions \hat{V}_{M} which may induce the required transitions among the spins of M. As these transitions should not 4716 affect *m*, the perturbation $\hat{V}_{\rm M}$ has the form $\hat{V}_{\rm M} = \hat{V}_{\uparrow\uparrow} + \hat{V}_{\downarrow\downarrow}$, where $\hat{V}_{\uparrow\uparrow}$ and $\hat{V}_{\downarrow\downarrow}$ act in the subspaces $|m_{\rm F}, \eta\rangle$ and $|-m_{\rm F}, \eta\rangle$, 471 respectively, so that $\hat{V}_{\uparrow}| - m_{\rm F}, \eta \rangle = \hat{V}_{\downarrow}|m_{\rm F}, \eta \rangle = 0.$ 4718

In order to find explicitly the time dependence of $|\Psi(t)\rangle\langle\Psi(t)|$, we should specialize the model. A simple possibility consists in taking \hat{V}_{\uparrow} and \hat{V}_{\downarrow} as *random matrices* [233]. We shall thus average $|\Psi(t)\rangle\langle\Psi(t)|$ over a Gaussian unitary ensemble of statistically independent random Hermitean matrices \hat{V}_{\uparrow} and \hat{V}_{\downarrow} of size *G*, with a weight proportional to

$$\exp\left[-\frac{2G}{\Delta^2}\left(\operatorname{tr}\hat{V}_{\uparrow\uparrow}^2 + \operatorname{tr}\hat{V}_{\downarrow\downarrow}^2\right)\right].$$
(11.12)

The matrix elements of $\hat{V}_{\rm M}$ have a very small typical size Δ/\sqrt{G} , where we remind that *G* is large as an exponential of *N*. The *G* energy levels of $\hat{H}_{\rm M} + \hat{V}_{\rm M}$ in the subspace $|m_{\rm F}, \eta\rangle$ are no longer degenerate, and, subtracting the fixed value from $\hat{H}_{\rm M}$, their density obeys Wigner's semi-circle law $(2/\pi\Delta^2)\sqrt{\Delta^2 - E^2}$ since $G \gg 1$. We do not wish the perturbation $\hat{V}_{\rm M}$ to spoil the above analysis of the original Curie–Weiss model; its effect, measured for large *G* by the parameter Δ , should therefore be sufficiently weak so as to produce a widening Δ small compared to the fluctuation of the energy in the canonical distribution. Since the fluctuation of \hat{m} in the latter distribution is of order $1/\sqrt{N}$, we should take, according to (3.7),

$$\Delta \ll \sqrt{N(J_2 + J_4)}.\tag{11.13}$$

Returning to $|\Psi(t)\rangle\langle\Psi(t)|$, we notice that the system S behaves as a spectator, so that we need only to study, in the space of M, the time dependence of the operators

$$\hat{X}_{\uparrow}^{\eta\eta'}(t) \equiv \exp(-i\hat{V}_{\uparrow}t/\hbar)|m_{\rm F},\eta\rangle\langle m_{\rm F},\eta'|\exp(i\hat{V}_{\uparrow}t/\hbar), \qquad (11.14)$$

$$\hat{Y}^{\eta\eta'}(t) \equiv \exp(-i\hat{V}_{\uparrow}t/\hbar)|m_{\rm F},\eta\rangle\langle -m_{\rm F},\eta'|\exp(i\hat{V}_{\downarrow}t/\hbar), \qquad (11.15)$$

and of the operators $\hat{X}_{\downarrow}^{\eta\eta'}(t)$ and $\left[\hat{Y}^{\eta'\eta}(t)\right]^{\dagger}$ obtained by interchanging \uparrow and \downarrow . (Recall that we have chosen t_{split} as a new origin of time.) Because \hat{V}_{\uparrow} and \hat{V}_{\downarrow} are statistically independent, the evaluation of $\hat{Y}^{\eta\eta'}(t)$ simply involves the separate averages of $\exp(-i\hat{V}_{\uparrow}t/\hbar)$ and of $\exp(i\hat{V}_{\downarrow}t/\hbar)$, which for symmetry reasons are proportional to the unit operator. We can therefore evaluate the time dependence of $\hat{Y}^{\eta\eta'}(t)$ through the trace

$$\phi(t) \equiv \frac{1}{G} \operatorname{tr} \overline{\exp(-i\hat{V}t/\hbar)} = \frac{2}{\pi\Delta^2} \int_{-\Delta}^{\Delta} dE \ \sqrt{\Delta^2 - E^2} \exp(-iEt/\hbar) = \frac{\tau_{\rm sub}}{t} J_1\left(\frac{2t}{\tau_{\rm sub}}\right),\tag{11.16}$$

where we made use the semi-circle law for the density of eigenvalues mentioned above. This expression exhibits the characteristic time τ_{sub} associated with the relaxation of the subensembles:

$$\tau_{\rm sub} = \frac{2\hbar}{\Delta}.\tag{11.17}$$

⁴⁷³⁷ Notice that τ_{sub} does not depend on the huge size *G* of our Hilbert space. We wish τ_{sub} to be short compared to the ⁴⁷³⁸ registration time τ_{reg} given by (9.9) or (9.10). As $N \gg 1$ and $\gamma \ll 1$, the condition (11.13) permits easily a value of Δ ⁴⁷³⁹ such that $\tau_{sub} \ll \tau_{reg}$. From (11.15) and (11.16), we find that $\hat{Y}^{\eta\eta'}(t)$ behaves as $\hat{Y}^{\eta\eta'}(t) = f_Y(t)\hat{Y}^{\eta\eta'}(0)$, where

$$f_Y(t) = \phi^2(t) = \left(\frac{\tau_{\rm sub}}{t}\right)^2 J_1^2 \left(\frac{2t}{\tau_{\rm sub}}\right) \approx \left[1 - \left(\frac{t}{\tau_{\rm sub}}\right)^2\right], \qquad (t \ll \tau_{\rm sub}),$$
$$\sim \frac{1}{\pi} \left(\frac{\tau_{\rm sub}}{t}\right)^3 \sin^2 \left(\frac{2t}{\tau_{\rm sub}} - \frac{\pi}{4}\right), \qquad (t \gg \tau_{\rm sub}). \tag{11.18}$$

Accordingly, the off-diagonal blocks of $|\Psi(t)\rangle\langle\Psi(t)|$, which involve both ferromagnetic states $m_{\rm F}$ and $-m_{\rm F}$, decay for $t \gg \tau_{\rm sub}$ as Eq. (11.18). It is thus seen that the coherent contributions $\hat{Y}^{\eta\eta'}$ fade out over the short time $\tau_{\rm sub}$.

The time dependence of $f_Y(t)$ includes a slow decrease in t^{-3} and oscillations, unusual features for a physical decay. These peculiarities result from the sharp behavior of the level density at $E = \pm \Delta$. We will show in § 11.2.4 how such a pathology can be cured by relying on more realistic models which give rise to a usual exponential decay. To evaluate $\hat{X}_{\uparrow\uparrow}^{\eta\eta'}(t)$, we imagine that the two exponentials of (11.14) are expanded in powers of \hat{V}_{\uparrow} and that Wick's theorem is used to express the Gaussian average over (11.12) in terms of the averages $\overline{\hat{V}_{\eta\eta'}\hat{V}_{\eta'\eta}} = \Delta^2/4G$. We thus

theorem is used to express the Gaussian average over (11.12) in terms of the averages $v_{\eta\eta'}v_{\eta'\eta} = \Delta^2/4G$. We thus find a diagrammatic expansion [318, 314] for the matrix elements of $\hat{X}^{\eta\eta'}_{\uparrow}(t)$ in the basis $|m_{\rm F}, \eta\rangle$. Apart from the factor ($-i)^n i^{n'}/n!n'!$ arising from the expansion of the exponentials, each line of a diagram carries a contraction

$$\frac{t^2}{\hbar^2} \,\overline{\hat{V}_{\eta\eta'}} \hat{V}_{\eta'\eta} = \frac{\Delta^2 t^2}{4\hbar^2 G} = \frac{1}{G} \left(\frac{t}{\tau_{\rm sub}} \right)^2,\tag{11.19}$$

and each summation over an internal index η brings in a factor *G*. The structure of the contractions (11.19) implies that each index must come in a right-left pair. Hence, for $\eta \neq \eta'$, the sole non-vanishing matrix element of $\hat{X}_{\uparrow\uparrow}^{\eta\eta'}(t)$ is $\langle m_{\rm F}, \eta | \hat{X}_{\uparrow\uparrow}^{\eta\eta'}(t) | m_{\rm F}, \eta' \rangle$. Among the contributions to this matrix element, the only diagrams that survive in the large-*G* limit are those which involve as many summations over indices η as contractions. This excludes in particular all diagrams containing contractions astride the left and right exponentials of (11.14). The evaluation of $\langle m_{\rm F}, \eta | \hat{X}_{\uparrow\uparrow}^{\eta\eta'}(t) | m_{\rm F}, \eta' \rangle$ thus involves the same factorization as in $\langle m_{\rm F}, \eta | \hat{Y}^{\eta\eta'}(t) | -m_{\rm F}, \eta' \rangle$, and this simply produces the factor $[\phi(t)]^2$. We therefore find, for $\eta \neq \eta'$, that $\hat{X}_{\uparrow\uparrow}^{\eta\eta'}(t) = \phi^2(t) \hat{X}_{\uparrow\uparrow}^{\eta\eta'}(0)$ tends to 0 just as (11.18). For $\hat{X}_{\uparrow\uparrow}^{\eta\eta}(t)$, the pairing of indices shows that the sole non-vanishing elements are $\langle m_{\rm F}, \eta | \hat{X}_{\uparrow\uparrow}^{\eta\eta}(t) | m_{\rm F}, \eta \rangle$, the outcome

For $\hat{X}_{\uparrow\uparrow}^{\eta\eta}(t)$, the pairing of indices shows that the sole non-vanishing elements are $\langle m_{\rm F}, \eta | \hat{X}_{\uparrow\uparrow}^{\eta\eta}(t) | m_{\rm F}, \eta \rangle$, the outcome of which does not depend on η , and, for $\eta \neq \eta'$, $\langle m_{\rm F}, \eta' | \hat{X}_{\uparrow\uparrow}^{\eta\eta}(t) | m_{\rm F}, \eta' \rangle$, which depends neither on η nor on η' . Moreover, according to the definitions (11.9) and (11.14), we note that tr $\hat{X}_{\uparrow\uparrow}^{\eta\eta}(t) = 1$, so that $\hat{X}_{\uparrow\uparrow}^{\eta\eta}(t)$ must have the general form

$$\hat{X}^{\eta\eta}_{\uparrow}(t) = f_X(t)\,\hat{X}^{\eta\eta}_{\uparrow}(0) + \left[1 - f_X(t)\right]\,\hat{R}^{\mu}_{\uparrow}.$$
(11.20)

In the large-*G* limit, the same analysis as for $\langle m_{\rm F}, \eta | \hat{X}^{\eta\eta'}_{\uparrow}(t) | m_{\rm F}, \eta' \rangle$ holds for $\langle m_{\rm F}, \eta | \hat{X}^{\eta\eta}_{\uparrow}(t) | m_{\rm F}, \eta \rangle$, and we find likewise $f_X(t) = \phi^2(t)$, so that the first term of (11.20) again decays as (11.18). (A direct evaluation of $\langle m_{\rm F}, \eta' | \hat{X}^{\eta\eta}_{\uparrow\uparrow}(t) | m_{\rm F}, \eta' \rangle$ for $\eta \neq \eta'$, which contributes to the second term of (11.20), would be tedious since this quantity, small as 1/G, involves correlations between the two exponentials of (11.14).) Thus, on the time scale $\tau_{\rm sub}$, the operators $\hat{X}^{\eta\eta'}_{\uparrow\uparrow}(t)$ fade out for $\eta \neq \eta'$ and *tend to the microcanonical distribution* for $\eta = \eta'$. Let us gather the above results. Starting from the ensemble \mathcal{E} described by the state (11.8), we consider any (real or virtual) subensemble \mathcal{E}_{sub} described by a state $\hat{D}_{sub}(t_{split})$ issued from a decomposition of $\hat{D}(t_{split})$. In the present model \hat{D}_{sub} evolves according to

$$\hat{D}_{sub}(t_{split} + t) = \phi^2(t)\hat{D}_{sub}(t_{split}) + [1 - \phi^2(t)]\hat{D}_{split}(t_f),$$
(11.21)

4767 where

$$\hat{D}_{\text{split}}(t_{\text{f}}) = q_{\uparrow}\hat{\Pi}_{\uparrow} \otimes \hat{R}^{\mu}_{\uparrow} + q_{\downarrow}\hat{\Pi}_{\downarrow} \otimes \hat{R}^{\mu}_{\downarrow}, \qquad q_{\uparrow,\downarrow} = \text{tr } \hat{D}_{\text{split}}(t_{\text{f}})\hat{\Pi}_{\uparrow,\downarrow}, \qquad (11.22)$$

and where $\phi^2(t)$, expressed by (11.18), (11.17), decreases over the very short time scale τ_{sub} . The state of any subensemble therefore relaxes rapidly to the expected asymptotic form (11.22), fulfilling the hierarchic structure at the final time t_f . *Decoherence and equilibration take place simultaneously*.

Note that the above relaxation is a property *of the magnet alone*, if we deal with *broken invariance in the quantum framework*. So we momentarily disregard the system S and measurements on it. Consider the perfectly symmetric process of § 7.3.2 (fig 7.7) which brings a statistical ensemble \mathcal{E} of magnets M from the paramagnetic state to the quantum mixture $\hat{R}_{M}(t_{f}) = P_{M}^{dis}(\hat{m}, t_{f})/G(\hat{m})$ of both ferromagnetic states. To simplify the discussion we replace the canonical distribution $P_{M}^{dis}(m, t_{f})$ by a microcanonical distribution located at m_{F} and $-m_{F}$. The mixed state $\hat{R}_{M}(t_{f})$ can be decomposed, as indicated at the end of § 11.1.2, into a weighted sum of projections $|\psi\rangle\langle\psi|$ onto pure states (notice that Ψ in (11.10) refers not only to M but also to S, that we disregard here)

$$|\psi\rangle = \sum_{\eta} U'_{\eta} |m_{\rm F}, \eta\rangle + \sum_{\eta} V'_{\eta} |-m_{\rm F}, \eta\rangle, \qquad (11.23)$$

each of which describes a subensemble of \mathcal{E} and contains coherent contributions astride $m = m_{\rm F}$ and $m = -m_{\rm F}$. Let 4778 us imagine that such a pure state has been prepared at some initial time. Then, in the present model including random 4779 interactions, it is *dynamically unstable* and decays into $\sum_{\eta} |U'_{\eta}|^2 \hat{R}^{\mu}_{\uparrow} + \sum_{\eta} |V'_{\eta}|^2 \hat{R}^{\mu}_{\downarrow}$ on the time scale τ_{sub} . Starting from $|\psi\rangle\langle\psi|$ we are left after a while with an incoherent superposition of microcanonical equilibrium states of M. 4780 478 Contrary to the initial state, this final situation can be interpreted classically as describing individual events, in each 4782 of which *m* takes a well-defined value, either $m_{\rm F}$ or $-m_{\rm F}$. Quantum dynamics thus allows us, at least in the present 4783 model, to by-pass the postulate about the apparatus suggested the end of § 11.1.3. Quantum magnets (and, more 4784 generally, macroscopic quantum systems having several equilibrium states) can just relax rapidly into well-defined 4785 unique macroscopic states, and in that sense behave as classical magnets (systems), as one would expect. 4786

Let us return to the measurement setting, where S and M are correlated. We have proved in the present model 4787 that the surmise (11.7) is justified for any subensemble, and that the set of subensembles possess at the final time 4788 $t_{\rm f}$ the hierarchic structure which removes the quantum ambiguity associated with the splitting of the full ensemble 4789 of runs. The solution of the measurement problem thus relies on specific properties of the apparatus, especially of 4790 its pointer M. We had already dwelt on the large number of degrees of freedom of M, needed to let it reach several 4791 possible equilibrium states. Now, we wish coherent states astride these equilibrium states to be dynamically unstable 4792 so that the pointer can yield well-separated indications; the present model shows that this is achieved owing to the 4793 macroscopic size of M and to a sufficient complexity of the internal interactions \hat{V}_{M} . Moreover these interactions 4794 equalize the populations of all levels of each microcanonical equilibrium state. 4795

4796 11.2.4. Dynamical instability in more realistic models

As usual in the applications of the random matrix theory, the use of a random interaction $\hat{V}_{\rm M}$ may be justified by the idea that, among the possible interactions, most will have similar dynamic effects, and that these effects can be exhibited by averaging $|\Psi(t)\rangle\langle\Psi(t)|$ over $\hat{V}_{\rm M}$. Nevertheless, although our choice of a Gaussian randomness (11.12) was mathematically sensible and provided the desired result, this choice was artificial. We have noted above that it yields a non-exponential decay (11.18) of $f_Y(t) = f_X(t)$, which is not satisfactory. In fact, by assuming that all the matrix elements of \hat{V}_{\uparrow} have comparable sizes, we have put all the states $|m_{\rm F}, \eta\rangle$ on an equal footing and disregarded their structure in terms of the spins $\sigma_z^{(n)}$. Such a \hat{V}_{\uparrow} is rather unphysical, as it produces transitions from $|m_{\rm F}, \eta\rangle$ to $|m_{\rm F}, \eta'\rangle$ that involve flip-flops of many spin pairs with the same amplitude as transitions that involve a single flip-flop. (The total spin remains unchanged in these dynamics.)

A more realistic model should involve, for instance, as interaction \hat{V}_{\uparrow} a sum of terms $\hat{\sigma}_{-}^{(n)}\hat{\sigma}_{+}^{(n')}$, which keep *m* fixed and produce single flip-flops within the set $|m_{\rm F}, \eta\rangle = |\sigma_z^{(1)}, \dots, \sigma_z^{(n)}, \dots, \sigma_z^{(N)}\rangle$. The number of significant elements of the $G \times G$ matrix \hat{V}_{\uparrow} is then of order *G* rather than G^2 as for Gaussian ensembles. This idea can be implemented in a workable model by taking for \hat{V}_{-} and \hat{V}_{-} effect on the set \hat{V}_{-} and \hat{V}_{-} and \hat{V}_{-} and \hat{V}_{-} and \hat{V}_{-} and \hat{V}_{-} as for Gaussian ensembles. 4806 4807 4808 in a workable model by taking for \hat{V}_{\uparrow} and \hat{V}_{\downarrow} other types of random matrices. If, for instance, the level density 1800 associated with \hat{V}_{\uparrow} is Gaussian instead of satisfying the semi-circle law, the relaxation will be exponential. One possible realization of this exponential relaxation scenario is achieved via a class of random matrices, where the 481 distribution of eigenvalues is factorized from that of the eigenvectors. This case corresponds to the homogeneous 4812 (Haar's) distribution. The above Gaussian ensemble, with the distribution of the eigenvalues satisfying the semi-circle 4813 law, also belongs to this class [230]. Appendix H justifies that if the distribution of the eigenvalues is taken to be 4814 Gaussian (independent from the Haar distribution of the eigenvectors), the relaxation is indeed exponential. 4815

One can justify the use of random matrices from a different, open-system perspective. We have assumed till now 4816 that the decay (11.21) was due to interactions within the spins of M. Alternatively, a concrete physical mechanism 4817 involving the bath B can efficiently produce the same decay. Instead of being governed by \hat{V}_{M} as above, the evolution of $|\Psi(t)\rangle$ is now governed by an interaction \hat{V}_{MB} with the bath. In contrast to the spin-boson interaction \hat{H}_{MB} defined by 4819 (3.10) which affects the spins of M one by one and which produces the registration, this interaction \hat{V}_{MB} does not affect 4820 the energy of M, and thus consists of flip-flops of spin pairs. It gives rise to transitions within the subspaces $|m_{\rm F},\eta\rangle$ or 4821 $|-m_F,\eta\rangle$, which can be described as a *quantum collisional process*. Successive brief processes take place within M + 4822 B. Each such "collision" may be produced by one among the various elements k of the bath, which act independently. 4823 Its effect on M is thus described in the subspaces $|m_F, \eta\rangle$ and $|-m_F, \eta\rangle$ by either one of the unitary transformations $\hat{U}^k_{\uparrow\uparrow}$ 4824 and \hat{U}_{\parallel}^k associated with the element k of B. It is then fully legitimate to treat the effective Hamiltonians for M entering 4825 each \hat{U}^k_{\uparrow} and \hat{U}^k_{\downarrow} as random matrices. Their randomness arises here from tracing out the bath. 4826

⁴⁸²⁷ This collisional approach is worked out in Appendix I. It is shown to produce the required decay (11.11) of ⁴⁸²⁸ $|\Psi(t)\rangle\langle\Psi(t)|$ through the two effects already described in § 11.2.3: the *disappearance of the coherent contributions* ⁴⁸²⁹ of the marginal density matrix of M, and the *microcanonical relaxation*. The process is rapid, because the colli-⁴⁸³⁰ sions produce transitions between kets having the same energy, and the decay is *exponential* as expected on physical ⁴⁸³¹ grounds.

Altogether, simple models such as the Curie–Weiss model of section 3 can provide, for the full set \mathcal{E} of runs 4832 of a measurement issued from the initial state (11.1), the final state (11.2) issued from two relaxation processes, 4833 the truncation, and the registration which fully correlates the system and the pointer. However, ideal measurements 4834 require a property, less easy to satisfy, the *hierarchic structure* of the subensembles of \mathcal{E} , expressed by the special form 4835 (11.7) of their states, which are constructed from the same building blocks $\hat{\mathcal{D}}_i$ as the state (11.2) for the full ensemble. 4836 We have just seen that more elaborate dynamical models involving suitable interactions within the apparatus must be 4837 introduced to establish this property. Then, if a state having a form different from (11.7) occurs for some subensemble at the end of the process, it is dynamically unstable and undergoes a new type of rapid relaxation towards a form 4839 (11.7). This mechanism removes the ambiguity in the possible decompositions (11.4) of the state $\hat{D}(t_f)$ describing the 4840 full ensemble \mathcal{E} : all subensembles will at the final time t_f be described by states of the form (11.7), the only physical 4841 ones at the end of the process. 4842

4843 11.3. Emergence of uniqueness and of classical features in measurements

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4845 4846 *Luctor et emergo*⁸⁸ *Fluctuat nec mergitur*⁸⁹ Devices of the often flooded Dutch province Zeeland and of the city of Paris

We are now in position to tackle the quantum measurement problem within the statistical interpretation of quantum mechanics. The dynamical establishment of the hierarchic structure (11.7) for any state $\hat{D}_{sub}(t_f)$ that may arise from

⁸⁸ I fight and emerge

⁸⁹She is agitated by the stream, but does not sink

any decomposition of $\hat{D}(t_{\rm f})$, whether virtual as in (11.4) or real, allows us to focus upon the *real subsets of runs* of the measurement. As regards these subsets, the large set \mathcal{E} has exactly the same structure as an ordinary, classical statistical ensemble. The quantum ambiguity of § 11.1.3 has disappeared; each term of the decomposition (11.2) has become meaningful since a state $\hat{D}_{\rm sub}(t_{\rm split})$ issued from an arbitrary decomposition (11.4) would anyhow relax to a state of the form (12.7). The explanation of all properties of ideal quantum measurements will rely on this *hierarchic structure* which results from the *elimination of the quantum ambiguity* and which gives an answer to the quantum measurement problem.

4856 11.3.1. Individual processes, reduction and preparations through measurement

4857 4858 *Cogito, ergo sum*⁹⁰ René Descartes

The common final form (11.7) of all density operators describing subsets \mathcal{E}_{sub} displays a classical feature: the 4859 properties i now appear as *exclusive*, and the weights q_i have exactly the same additivity property as *ordinary prob*-4860 *abilities* when disjoint subsets are amalgamated. (The diagonal blocks \hat{D}_i themselves retain a quantum probabilistic 4861 nature.) In the same way as a probability law associated with a classical statistical ensemble can be interpreted in 4862 terms of individual events (§ 11.1.2), we are now allowed to consider individual runs. Relying on the unambiguous 4863 splitting of any real subset \mathcal{E}_{sub} into blocks $\hat{\mathcal{D}}_i$ at the time t_f , we then identify the least mixed subset \mathcal{E}_i to which some 4864 single run belongs; its density matrix contains the sole block \hat{D}_i . We also interpret q_i , for any subset \mathcal{E}_{sub} , as the 4865 relative frequency of occurrence of the runs having the property i. Thus, the considered individual run ends up in one 4866 among the states $\hat{D}_i = \hat{r}_i \otimes \hat{\mathcal{R}}_i$ for S + A, and it has a *well-defined outcome i*.

We have therefore explained, at least in the present model, the phenomenon of *reduction*, that is, the production, in each individual run, of *one* among the diagonal blocks of the truncated final density matrix (11.2) of S+A, which describes the whole set \mathcal{E} and arises from (11.1). The possibility of making such a statement about individual processes in spite of the irreducibly probabilistic nature of quantum mechanics (in its statistical interpretation) is founded on the special dynamics of the apparatus, as shown in § 11.2.3.

We stressed in § 10.1.3 that the concept of "state" of an individual system refers, in the statistical interpretation, 4873 to a thought statistical (sub-)ensemble to which this system belongs. In a measurement, two different states may thus be assigned to S + A at the end of a single run. Before acknowledging the outcome of the process, we have to regard 4875 it as an element of the full set \mathcal{E} of runs issued from the initial state $\hat{\mathcal{D}}(0)$ of Eq (11.1), and we thus assign to S + A 4876 the state $\hat{D}(t_f)$ of Eq. (11.2) (which involves correlations between S and A). After having read the specific outcome 4877 A_i , we have learnt that the considered single run belongs to the subset \mathcal{E}_i which has emerged from the dynamics, so 4878 that we thus assign to S + A the more informative state \hat{D}_i (which has the uncorrelated form $\hat{r}_i \otimes \hat{\mathcal{R}}_i$). Predictions 4879 about experiments performed on S after the considered run should therefore be made from the weakly truncated state 4880 $\sum_i \hat{\Pi}_i \hat{r}(0) \hat{\Pi}_i$ if the result is not read, and from the reduced state \hat{r}_i if the result A_i has been selected. 4881

Whereas the transformation of the state $\hat{D}(0)$ into $\hat{D}(t_f)$ is a *real physical process, reduction* from the state $\hat{D}(t_f)$ to the state \hat{D}_i has *no dynamical meaning*. It is simply an *updating of our probabilistic description*, allowed by the acquisition of the information A_i which characterizes the new narrower ensemble \mathcal{E}_i . The state \hat{D}_i retains through \hat{r}_i some features inherited from the initial state $\hat{D}(0)$, but not all due to irreversibility of truncation and registration, and it accounts in addition for the knowledge of A_i . Measurement can thus indeed be regarded as *information processing*; the amounts of information acquired and lost are characterized by the entropy balance of § 1.2.4.

The information needed to partition \mathcal{E} into its subsets \mathcal{E}_i characterized by their density operator $\hat{\mathcal{D}}_i$ is embedded in the indication A_i of the pointer. The uniqueness of this indication is explained by the dynamics of the reduction⁹¹. The macroscopic size of the pointer then allows observing, storing or processing the outcome. The complete correlations established between S and A by the registration, exhibited in (11.2), entail *uniqueness of the outcome* s_i for the *tested observable* \hat{s} of S in each run⁹². A *filtering of the runs of an ideal measurement* that are tagged by the indication A_i

⁹⁰I think, therefore I exist

 $^{^{91}}$ The physical argument of § 11.1.1 turns out to be "not even wrong". It also turns out that we do not need the additional postulate alluded to at of the end of § 11.1.3, owing to realistic interactions which act within the apparatus at the end of the proces, and which need not play a major role in the truncation and registration

 $^{^{92}}$ But of course there are no well-defined results for observables that do not commute with \hat{s}
of the pointer therefore constitutes a *preparation* of the system S (as was anticipated in § 1.1.4) [31]. Lying formerly (for the ensemble \mathcal{E}) in the state $\hat{r}(0)$, this system lies after measurement and selection of the subset \mathcal{E}_i in the *final prepared* state \hat{r}_i . In the Curie–Weiss model, this final filtered state is pure, $\hat{r}_{\uparrow} = |\uparrow\rangle\langle\uparrow|$ or $\hat{r}_{\downarrow} = |\downarrow\rangle\langle\downarrow|$ or, shortly, $|\uparrow\rangle$ or $|\downarrow\rangle$.

In this circumstance, quantum mechanics, although irreducibly probabilistic and dealing with ensembles, can pro-4897 vide certainty about s for an individual system after measurement and selection of the indication of the pointer. While 4898 answering, within the statistical interpretation, Bohr's modest query "What can we say about...?" [320], we have given 4899 a partial answer to Einstein's query "What is...?" [23]. The solution of models involving not only the interaction of 4900 the microscopic object with a macroscopic apparatus but also appropriate interactions within this apparatus thus explains the emergence of a well-defined answer in a single measurement, a property interpreted as the emergence of a "physical reality". However, although the outcome of each individual process is unique, it could not have been 4903 predicted. The current statement "the measurement is responsible for the appearance of the uniqueness of physical 4904 reality" holds only for the considered single system and for the tested observable, and only after measurement with 4905 selection of the result. 4906

4907 11.3.2. Repeatability of ideal measurements
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Korean proverb Another property that allows us to approach physical reality within the statistical interpretation is the repeatability

It is a bad plowman that quarrels with his ox

Another property that allows us to approach physical reality within the statistical interpretation is the repeatability of ideal measurements⁹³. Suppose two successive ideal measurements are performed on the same system S, first with an apparatus A, then, independently, with a similar apparatus A'. The second process does not affect A, and generates for S and A' the same effect as the first one, as exhibited by Eq (11.2). Hence, the initial state

$$\hat{\mathcal{D}}(0) = \hat{r}(0) \otimes \hat{\mathcal{R}}(0) \otimes \hat{\mathcal{R}}'(0) \tag{11.24}$$

4914 of S + A + A' becomes at the time t_f between the two measurements

$$\hat{\mathcal{D}}(t_{\rm f}) = \sum_{i} p_i \hat{r}_i \otimes \hat{\mathcal{R}}_i \otimes \hat{\mathcal{R}}'(0), \qquad (11.25)$$

4915 and

$$\hat{\mathcal{D}}(t_{\rm f}') = \sum_{i} p_i \hat{r}_i \otimes \hat{\mathcal{R}}_i \otimes \hat{\mathcal{R}}_i' \tag{11.26}$$

at the final time $t'_{\rm f}$ following the second process. For the whole statistical ensemble \mathcal{E} , a complete correlation is therefore exhibited *between the two pointers*. In an individual process, the second measurement does not affect S. We can even retrodict, from the observation of the value A'_i for the pointer of the apparatus A', that S lies in the state \hat{r}_i not only at the final time $t'_{\rm f}$, but already at the time $t_{\rm f}$, the end of the first measurement.

4920 11.3.3. Classical probabilities

4921 4922 *De oudjes doen het nog goed* ⁹⁴ Dutch expression

We have inferred from the hierarchic structure that, for any subensemble \mathcal{E}_{sub} , the coefficients q_i entering (11.7) can be interpreted as the relative numbers of runs with outcome i (§ 11.3.1). Let us return to the full ensemble \mathcal{E} of runs, described at the final time by $\hat{\mathcal{D}}(t_f)$. The weight $p_i = \text{tr}_S \hat{r}(0)\hat{\Pi}_i$ therefore appears as the proportion of individual

 $^{^{93}}$ It can be shown that the sole property of repeatability implies reduction in the weak sense, for the marginal state of S [52] 94 The oldies are doing well still

runs described by the reduced state \hat{D}_i . This *fully justifies Born's rule*. The *census of the indications of the pointer* in a large number of runs affords *a partial determination of the initial density matrix* $\hat{r}(0)$ of S (its diagonal elements for the Curie–Weiss model).

The quantity p_i can also be interpreted as the ordinary probability of finding the outcome A_i , s_i in each run. From the density operator $\hat{r}(0)$ which has no classical interpretation, the measurement process has thus extracted *classical probabilities*. Because of the one-by-one selection procedure of the runs, these probabilities are defined within the *frequency interpretation* [28].

4933 11.4. The ingredients of the solution of the measurement problem

4934	Bring vor, was wahr ist;
4935	schreib' so, daß es klar ist
4936	und verficht's, bis es mit dir gar ist ⁹⁵
4937	Ludwig Boltzmann

Altogether, as in statistical mechanics [55, 56, 73], qualitatively new features emerge in an ideal measurement 4938 process, with a *near certainty*. The explanation of the appearance, within the quantum theory, of properties seemingly 4939 in contradiction with this very theory relies on several ingredients, exhibited by the detailed solution of the Curie-4940 Weiss model. (i) The macroscopic size of the apparatus allows the pointer to relax towards one or another among some possible values, within weak statistical or quantum fluctuations; these outcomes remain unchanged for a long time 4942 and can be *read* or *processed*; the choice of the a priori equivalent aternatives is triggered by the tested system. (ii) 4943 Statistical considerations help us to disregard unlikely events. (iii) The special dynamics of the process must produce 4944 several effects. The truncation, initiated by the interaction between the tested system and the pointer, eliminates the 4945 off-diagonal blocks of \hat{D} which would prevent any classical interpretation. The *registration*, too often overlooked 4946 in theoretical considerations, which requires a triggering by the system and a dumping of energy towards the bath, 1917 creates the needed correlations between the system and the pointer. The registration also lets the apparatus reach, in the state $\hat{D}(t)$, at large t, a mixture of the possible final states; this paves the way to the process of § 11.2.3, where more elaborate but possibly very small interactions within the apparatus ensure that all subensembles reach at the final 4950 time the hierarchic structure required for reduction. This last step explains how statements about individual systems 4951 and how classical features may emerge from measurement processes in spite of the quantum oddities (§§ 10.2.1 and 4952 10.2.3) associated with the irreducibly probabilistic nature of the theory [10, 11, 31, 48, 52, 58]. 4953

As the symmetry breaking for phase transitions, a breaking of unitarity takes place, entailing an apparent violation of the superposition principle for $S + A^{96}$. Here also, there cannot exist any breaking in the strict mathematical sense for a finite apparatus and for finite parameters. Nevertheless, this acknowledgement has no physical relevance: the approximations that underlie the effective breaking of unitarity are justified for the evaluation of physically sensible quantities.

However, the type of emergence that we acknowledge here is more subtle than in statistical mechanics, although 4959 both arise from a change of scale. In the latter case, emergence bore on phenomena that have no microscopic equiva-4960 lent, such as irreversibility, phase transitions or viscosity. In quantum measurements, it bears on concepts. Quantum 4961 theory, which is fundamentally probabilistic, deals with ensembles, but measurements reveal properties of *individ*-4962 ual systems, a fact that we understand within this very theory. The tested physical quantity, random at the mi-4963 croscopic level, comes out with a well-defined value. Ordinary probabilities, ordinary correlations, emerge from a non-commutative physics, and thus afford a *classical interpretation* for the outcome of the measurement. Thus, ideal 4965 measurements establish a bridge between the macroscopic scale, with its every day's life features, and the micro-4966 scopic scale, giving us access to microscopic quantities presenting unusual quantum features and impossible to grasp 4967 directly⁹⁷. In the measurement device we lose track of the non-commutative nature of observables, which constitutes 4968

⁹⁵Put forward what is true, write it such that it is clear, and fight for it till it is finished with you

⁹⁶ As the tested system interacts with the apparatus, it is not an isolated system, so that the breaking of unitarity in its evolution is trivial

 $^{^{97}}$ A more artificial link between microscopic and macroscopic scales was established by Bohr [320] – see also [84, 321, 322] – by postulating the classical behavior of the measuring apparatus. Though we consider that the apparatus must be treated as a quantum object, we have noticed (§ 11.2.3) that quantum dynamics lets the pointer variable reach some classical features

the deep originality of quantum mechanics and which gives rise to its peculiar types of correlations and of probabilities, and we thus recover familiar macroscopic concepts. (The disappearance of non-commuting observables will be seen to arise naturally from the Heisenberg dynamics in § 13.1.4.)

4972 12. Lessons from measurement models

4973	Cette leçon vaut bien un fromage, sans doute ⁹⁸
4974	Jean de La Fontaine, Le Corbeau et le Renard

A microscopic interpretation of the entropy concept has been provided through the elucidation of the irreversibility paradox [54, 55, 56, 72]. Likewise, most authors who solve models of quantum measurements (section 2) aim at elucidating the measurement problem so as to get insight on the interpretation of quantum mechanics. We gather below several ideas put forward in this search, using as an illustration the detailed solution of the Curie–Weiss model presented above, and we try to draw consequences on the interpretation of quantum physics. These ideas deserve to be taken into account in future works on measurement models.

4981 12.1. About the nature of the solutions

4982	La Nature est un temple où de vivants piliers
4983	Laissent parfois sortir de confuses paroles;
4984	L'homme y passe à travers des forêts de symboles
4985	<i>Qui l'observent avec des regards familiers</i> ⁹⁹
4986	Charles Baudelaire, Les fleurs du mal, Correspondances

The most important conclusion that can be drawn from the solution of models is that one can reach a *full understanding of ideal measurements through standard quantum statistical mechanics*. Within a *minimalist* interpretation of quantum mechanics, the sole use of *Hamiltonian dynamics is sufficient to explain all the features of ideal measurements*. In particular, *uniqueness* of the outcome of each run and *reduction* can be derived only from the Hamiltonian dynamics of the macroscopic pointer. *Unconventional interpretations are not needed*.

4992 12.1.1. Approximations are needed

4993 4994 Fire could leave ashes behind Arab proverb

As stressed in § 1.2.1, a measurement is an *irreversible* process, though governed by the *reversible* von Neumann equation of motion for the coupled system S + A. This apparent contradiction cannot be solved with mathematical rigor if the compound system S + A is finite and all its observables are under explicit control. As in the solution of the irreversibility paradox (§ 1.2.2), some approximations, justified on physical grounds, should be introduced [54, 55, 56, 174, 120]. We must accept the idea that theoretical analyses of quantum measurements are approximate [323].

For instance, when solving the Curie–Weiss model, we were led to neglect some contributions, which strictly 5001 speaking do not vanish for a finite apparatus A = M + B, but which are very small under the conditions of subsection 5002 9.4. For the diagonal blocks $\hat{\mathcal{R}}_{\uparrow\uparrow}$ and $\hat{\mathcal{R}}_{\downarrow\downarrow}$, the situation is the same as for ordinary thermal relaxation processes 5003 [174, 120, 121]: the invariance under time reversal is broken through the *elimination of the bath* B, performed by 5004 keeping only the lowest order terms in γ and by treating the spectrum of B as continuous (section 4). Correlations 5005 within B and between B and M+S are thus disregarded, and an irreversible nearly exact Fokker–Planck equation [230] 5006 for the marginal operators $\hat{R}_{\uparrow\uparrow}$ and $\hat{R}_{\downarrow\downarrow}$ thus arises from the exact reversible dynamics. For the off-diagonal blocks $\hat{R}_{\uparrow\downarrow}$ 500 and $\hat{R}_{\uparrow\downarrow}$, correlations between S and a large number, of order N, of spins of M are also discarded (section 5). Such 5008 correlations are ineffective, except for recurrences; but these recurrences are damped either by a randomness in the 5009

⁹⁸Surely, this lesson is worth a cheese

⁹⁹Nature is a temple where living pillars / Let sometimes emerge confused words;

Man passes there through forests of symbols / Which watch him with familiar glances

coupling between S and M (subsection 6.1) or by the bath (subsection 6.2). We will return to this point in § 12.2.3. 5010 We also showed that, strictly speaking, *false or aborted registrations* may occur but that they are very rare (§ 7.3.4 501 and § 7.3.5). 5012 Mathematically rigorous theorems can be proved in statistical mechanics by going to the thermodynamic limit of 5013 infinite systems [134]. In the Curie–Weiss model, the disappearance of $\hat{R}_{\uparrow\downarrow}$ and $\hat{R}_{\downarrow\uparrow}$ would become exact in the limit 5014 where $N \to \infty$ first, and then $t \to \infty$. However, in this limit, we lose track of the time scale τ_{trunc} , which tends to 0. 5015 Likewise, the weak coupling condition $\gamma \to 0$, needed to justify the elimination of the bath, implies that τ_{reg} tends to

 ∞ . Physically sensible time scales are obtained only *without limiting process* and at the price of approximations. 5017

12.1.2. Probabilities are omnipresent 501

5019 5020

5016

O Fortuna, imperatrix mundi¹⁰⁰ Carmina Burana

Although the dynamics of S + A is deterministic, randomness occurs in the solution of measurement models 502 for several reasons. On the one hand, quantum physical quantities are blurred due to the non-commutation of the 5022 observables which represent them, so that quantum mechanics is irreducibly probabilistic (section 10 and [10, 11, 31, 5023 48, 52, 58]). On the other hand, the large size of the apparatus, needed to ensure registration, does not allow us to 5024 describe it at the microscopic scale; for instance it lies after registration in a thermal equilibrium (or quasi-equilibrium) 5025 state. Thus, both conceptually and technically, we are compelled to analyse a quantum measurement by relying on 5026 the formalism of quantum statistical mechanics. 5027

Moreover, as shown in subsection 5.2, some randomness is needed in the *initial state* of the apparatus. Indeed, for some specific initial pure states, the truncation process may fail, in the same way, for instance, as some exceptional 502 initial configurations of a classical Boltzmann gas with uniform density may produce after some time a configuration 5030 with non uniform density. For realistic models of quantum measurements, which are of rising interest for q-bit 5031 processing in quantum information theory, experimental noise and random errors should also be accounted for [319]. 5032 Recognizing thus that a quantum measurement is a process of quantum statistical mechanics has led us to privilege 5033 the statistical interpretation of quantum mechanics, in which an assertion is "certain" if its probability is close to one. 5034 For instance, the probability of a false registration does not vanish but is small for large N (§ 7.3.3). Still, the statistical 5025 solution of the quantum measurement problem does not exclude the existence of a hidden variable theory that would describe individual measurements, the statistics of which would be given by the probabilistic theory, that is, the 503 standard quantum mechanics; see [284] for a recent review of hidden variable theories. 5038

12.1.3. Time scales 5039

5040 504

De tijd zal het leren¹⁰¹ Dutch proverb

Understanding a quantum measurement requires mastering the dynamics of the process during its entire duration 5042 [91]. This is also important for experimental purposes, especially in the control of quantum information. Even when 5043 the number of parameters is small, a measurement is a complex process which takes place over several time scales, as 5044 exhibited by the solution of the Curie–Weiss model (subsection 9.3). There, the truncation time turns out to be much 504 shorter than the registration time. This feature arises from the large number of degrees of freedom of the pointer M 50 (directly coupled to S) and from the weakness of the interaction between M and B. The large ratio that we find for 504 $\tau_{\rm reg}/\tau_{\rm trunc}$ allows us to distinguish in the process a rapid disappearance of the off-diagonal blocks of the density matrix 5048 of S + A. After that, the registration takes place as if the density matrix of S were diagonal. The registration times are 5049 also not the same for quartic or quadratic interactions within M. The final process of dynamical instability of M, that 5050 allows reduction (§ 11.2.3), is also rapid owing to the large size of the pointer. 5051

In the variant of the Curie–Weiss model with N = 2 (subsection 8.1), the orders of magnitude of the truncation 5052 and registration times are reversed. A large variety of results have been found in other models for which the dynamics 5053 was studied (section 2). This should encourage one to explore the dynamics of other, more and more realistic models. 505

¹⁰¹Time will tell

¹⁰⁰Oh Fortune, empress of the world

However, it is essential that such models ensure a crucial property, the dynamical establishment of the hierarchic structure of subensembles (§ 11.2.1).

⁵⁰⁵⁷ 12.1.4. May one think in terms of underlying pure states?

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5059 5060 Als de geest uit de fles is, krijg je hem er niet makkelijk weer in¹⁰² Dutch proverb

The solutions of the Curie–Weiss model and of many other models have relied on the use of density operators. 5061 We have argued (§ 10.2.3) that, at least in the statistical interpretation, the non uniqueness of the representations 5062 (10.3) of mixed states as superpositions of pure states makes the existence of such underlying pure states unlikely. 5063 Here again, Ockam's razor works against such a representation, which is not unique and more complicated than the 5064 framework of quantum statistical mechanics, and which in general would not permit explicit calculations. Moreover, 50 it is experimentally completely unrealistic to assume that the apparatus has been initially prepared in a pure state. 506 Nevertheless, although pure states are probabilistic entities, it is not a priori wrong to rely on other interpretations 5067 in which they are regarded as more fundamental than density operators [14], and to afford the latter a mere status of 5068 technical tools, used to describe both the initial state and the evolution. 5069

We can compare this situation with that of the irreversibility paradox for a gas (§ 1.2.2). In that case, although it 5070 is technically simpler to tackle the problem in the formalism of statistical mechanics, one may equivalently explain 5071 the emergence of irreversibility by regarding the time-dependent density in phase space as a mathematical object 5072 that synthetizes the trajectories and the random initial conditions [55, 56]. The dynamics is then accounted for by 507 Hamilton's equations instead of the Liouville equation, whereas the statistics bears on the initial conditions. (We 5074 stressed, however, in § 10.2.3, that although density operators and densities in phase space have a similar status, 5075 quantum pure states differ conceptually from points in phase space due to their probabilistic nature; see also [55, 56] 5076 in this context.) 507

Likewise, in the Curie–Weiss measurement model, one may theoretically imagine to take as initial state of A a 5078 pure state, S being also in a pure state. Then at all subsequent times S + A lies in pure states unitarily related to one 5079 another. However it is impossible in any experiment to prepare A = M + B in a pure state. What can be done is to prepare M and B in thermal equilibrium states, at a temperature higher than the Curie temperature for M, lower for 508 B. Even if one wishes to stick to pure states, one has to explain *generic experiments*. As in the classical irreversibility 5082 problem, this can be done by weighing the possible initial pure states of A = M + B as in $\hat{\mathcal{R}}(0)$, assuming that M is a 5083 typical paramagnetic sample and B a typical sample of the phonons at temperature T. This statistical description in 5084 terms of weighted pure states governed by the Schrödinger equation is technically the same as the above one based on 5085 the density operator $\hat{D}(t)$, governed by the Liouville–von Neumann equation, so that the results obtained above or the 5086 full ensemble \mathcal{E} of runs are recovered in a statistical sense for most relevant pure states. As regards the expectation 5087 values in the ensemble \mathcal{E} of physical quantities (excluding correlations between too many particles), the typical final 508 pure states are equivalent to $\hat{D}(t_f)$. Very unlikely events will never be observed over reasonable times for most of 508 these pure states (contrary to what happens for the squeezed initial states of M considered in § 5.2.3). 5090

However, it does not seem feasible to transpose to the mere framework of pure states the explanation of reduction given in section 11, based on the unambiguous splitting of the mixed state $\hat{D}(t_f)$. This splitting is needed to identify the real subsets of runs of the measurement, and it has no equivalent in the context of pure states.

5094 12.2. About truncation and reduction

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De duivel steekt in het detail¹⁰³ Le diable est dans les détails¹⁰³ Dutch and French proverb

 $^{^{102}}$ When the genie is out of the bottle, it is not easy to get it in again 103 The devil is in the details

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Het klopt als een bus¹⁰⁴ Dutch expression

In many approaches, starting from von Neumann [4, 17, 21] the word "collapse" or "reduction" is taken in a weak acception, *referring to* S *alone*. Such theoretical analyses involve only a proof that, in a basis that diagonalizes the tested observable, the off-diagonal blocks of the marginal density matrix $\hat{r}(t)$ of S fade out, but not necessarily those of the full density matrix $\hat{D}(t)$ of S + A. In the Curie–Weiss model, this would mean that $\hat{r}_{\uparrow\downarrow}(t)$ and $\hat{r}_{\downarrow\uparrow}(t)$, or the expectation values of the *x*- and *y*-components of the spin S, fade out, but that $\hat{R}_{\uparrow\downarrow}(t)$ and $\hat{R}_{\downarrow\uparrow}(t)$, which characterize the correlations between the pointer M and these components, do not necessarily disappear.

Let us show that the presence of non negligible elements in the off-diagonal blocks of the final state $\hat{D}(t_f)$ of S 5107 + A is prohibited for ideal measurements. Remember first the distinction between truncation and reduction (§ 1.1.2 5108 and \S 1.3.3). Both terms refer to the compound system S + A, but while the truncation is the disappearance of the 510 off-diagonal blocks in the matrix $\hat{D}(t_f)$ that describes the full ensemble \mathcal{E} of runs of the measurement, the reduction 5110 is the assignment of the final state \hat{D}_i to a subset \mathcal{E}_i of \mathcal{E} . More precisely, once the uniqueness of the outcome of each 5111 run is ensured (subsections 11.2 and 11.3), one can sort out the runs that have produced the specific indication A_i of 5112 the pointer. In each such run, the system S lies in the state \hat{r}_i and the apparatus A in the state \hat{R}_i , hence S + A lies 5113 in the state $\hat{\mathcal{D}}_i$. Born's rule implies that the proportion of runs in \mathcal{E}_i is p_i . Collecting back the subsets \mathcal{E}_i into \mathcal{E} , we 5114 find that this full set must be described by the state $\sum_i p_i \hat{D}_i$, which is a truncated one. It is therefore essential, when 5115 solving a model of ideal measurement, to prove the strong truncation property, for S + A, as we did in sections 5 and 5116 6, a prerequisite to the proof of reduction. A much more stringent result must thereafter be proven (§ 11.2.1), the 511 "hierarchic property" (11.7), according to which the state \hat{D}_{sub} of S+A must have the form $\sum_i q_i \hat{D}_i$ for arbitrary, real 5118 or virtual, subensembles \mathcal{E}_{sub} of \mathcal{E} . 5119

The weak type of truncation is the mere result of disregarding the off-diagonal correlations that exist between S and 5120 A. This procedure of tracing out the apparatus has often been considered as a means of circumventing the existence 512 of "Schrödinger cats" issued from the superposition principle [32, 33, 176, 177, 178, 179]. However, this tracing 5122 procedure as such does not have a direct physical meaning [14, 68]. While satisfactory for the statistical predictions 5123 about the final marginal state of S, which has the required form $\sum_i p_i \hat{r}_i$, the lack of a complete truncation for S + 512 A keeps the quantum measurement problem open since the apparatus is left aside. Indeed, the proof of uniqueness 5125 of § 11.2.3 takes as a starting point the state $\hat{D}(t_f)$ for \mathcal{E} where truncation and registration have already taken place, 5126 and moreover this proof involves only the apparatus. Anyhow, tracing out the apparatus eliminates the correlations 5127 between the system and the indications of the pointer, which are the very essence of a measurement (subsection 11.3). 5128 Without them we could not get any information about S. This is why the elimination of the apparatus in a model is 5129 generally considered as a severe weakness of such a model [17], that even led to the commandment "Thou shalt not 5130 trace" [33]. 5131

5132 So indeed, theory and practice are fundamentally related. The elimination of the apparatus in the theory of mea-5133 surements is no less serious than its elimination in the experiment!

5134 12.2.2. The truncation is a material phenomenon; the reduction involves both dynamics and "observers"

*Weh! Ich ertrag' dich nicht*¹⁰⁵ Johann Wolfgang von Goethe, Faust, part one

The truncation of the density matrix of S + A appears in measurement models as an irreversible change, occurring with a nearly unit probability during the dynamical process. It has a material effect on this compound system, modifying its properties as can be checked by subsequent measurements. In the Curie–Weiss model, this effect is the disappearance of correlations between the pointer and the components \hat{s}_x and \hat{s}_y of S. Though described statistically for an ensemble, the joint truncation of S + A thus appears as a purely dynamical, real phenomenon.

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¹⁰⁴It really fits

¹⁰⁵Beware, I can't stand you

The reduction has a more subtle status. It also relies on a dynamical instability process governed by the Liouville-5142 von Neumann equation (§ 11.2.3), which takes place by the end of the measurement for any subensemble \mathcal{E}_{sub} of \mathcal{E} 5143 (whereas the truncation took place earlier and for the full ensemble \mathcal{E}). Moreover, reduction requires the *selection* 514 of the subset \mathcal{E}_i of runs characterized by the value A_i of the pointer variable. This selection, based on a gain of 5145 *information about* A, allows the *updating of the state* $\hat{D}(t_f)$, which plays the role of a probability distribution for 5146 the compound system S + A embedded in the ensemble \mathcal{E} , into the state $\hat{\mathcal{D}}_i$ which refers to the subensemble about 5147 which we have collected information. Subsequent experiments performed on this subensemble will be described by 5148 $\hat{\mathcal{D}}_i$ (whereas we should keep $\hat{\mathcal{D}}(t_f)$ for experiments performed on the full set \mathcal{E} without sorting). 5149

The idea of an "observer", who selects the subset \mathcal{E}_i of systems so as to assign to them the density operator $\hat{\mathcal{D}}_i$, therefore underlies the reduction, as it underlies any assignment of probability. However, "observation" is meant here as "identification and sorting of runs" through discrimination of the outcomes of the pointer. Such a "reading" does not require any "conscious observer". The "observer" who selects the runs A_i will in fact, in many experiments, be a macroscopic automatic device triggered by the pointer. An outstanding example is the sophisticated automatic treatment of the information gathered by detectors in particle physics, achieved in order to select the extremely rare events of interest.

5157 12.2.3. Physical extinction versus mathematical survival of the off-diagonal sectors

5158I have not failed.5159I've just found 10,000 ways that won't work5160Thomas A. Edison

Many works on quantum measurement theory stumble over the following paradox. The evolution of the density 5161 matrix $\hat{D}(t)$ of the isolated system S + A is unitary. Hence, if \hat{D} is written in a representation where the full Hamil-5162 tonian \hat{H} is diagonal, each of its matrix elements is proportional to a complex exponential $\exp(i\omega t)$ (where $\hbar\omega$ is 5163 a difference of eigenvalues of \hat{H}), so that its modulus remains constant in time. In the ideal case where the tested 516 observable \hat{s} commutes with \hat{H} , we can imagine writing $\hat{D}(t)$ in a common eigenbasis of \hat{s} and \hat{H} ; the moduli of the 5165 matrix elements of its off-diagonal block $\hat{\mathcal{R}}_{\uparrow\downarrow}(t)$ are therefore independent of time. Such a basis was used in sections 5166 5 and 6.1 where the bath played no rôle; in section 6.2, the term \hat{H}_{MB} does not commute with \hat{s} , and likewise in most 516 other models the full Hamiltonian is not diagonalizable in practice. In such a general case, the moduli of the matrix 516 elements of $\mathcal{R}_{\uparrow\downarrow}(t)$, in a basis where only \hat{s} is diagonal, may vary, but we can ascertain that the norm Tr $\mathcal{R}_{\uparrow\downarrow}(t)\mathcal{R}_{\uparrow\uparrow}^{\dagger}(t)$ re-5169 mains invariant. This mathematically rigorous property seems in glaring contradiction with the physical phenomenon 5170 of truncation, but both are valid statements, the former being undetectable, the latter being important in practice for 5171 measurements. 5172

In which sense are we then allowed to say that the off-diagonal block $\hat{\mathcal{R}}_{\uparrow\downarrow}(t)$ decays? The clue was discussed in § 6.1.2: The physical quantities of interest are weighted sums of matrix elements of $\hat{\mathcal{D}}$, or here of its block $\hat{\mathcal{R}}_{\uparrow\downarrow}$. For instance, the off-diagonal correlations between \hat{s}_x or \hat{s}_y and the pointer variable \hat{m} are embedded in the characteristic function (5.14), which reads

$$\Psi_{\uparrow\downarrow}(\lambda,t) \equiv \langle \hat{s}_{-}e^{i\lambda\hat{m}} \rangle = \mathrm{Tr}_{A}\hat{\mathcal{R}}_{\uparrow\downarrow}(t)e^{i\lambda\hat{m}},\tag{12.1}$$

where the trace is taken over A = B + M. Likewise, the elimination of the bath B, which is sensible since we cannot 5177 control B and have no access to its corelations with M and S, produces $\hat{R}_{\uparrow\downarrow} = \text{Tr}_B \hat{\mathcal{R}}_{\uparrow\downarrow}$, which contains our whole off-5178 diagonal information, and which is a sum of matrix elements of the full density matrix \hat{D} . We are therefore interested 5179 only in weighted sums of complex exponentials, that is, in *almost periodic functions* (in the sense of Harald Bohr¹⁰⁶). 5180 5181 For a large apparatus, these sums involve a large number of terms, which will usually have incommensurable frequencies. Depending on the model, their large number reflects the large size of the pointer or that of some environment. 5182 The situation is the same as for a large set of coupled harmonic oscillators [151, 152, 153, 174, 120, 121], which 5183 in practice present damping although some exceptional quantities involving a single mode or a few modes oscillate. 5184

¹⁰⁶The mathematician and olympic champion Harald Bohr, younger brother of Niels Bohr, founded the field of almost periodic functions. For a recent discussion of his contributions, see the expository talk "The football player and the infinite series" of H. Boas [180]

In § 6.1.2 we have studied a generic situation where the frequencies of the modes are random. The random almost periodic function F(t) defined by (6.14) then exhibits a decay over a time scale proportional to $1/\sqrt{N}$; Poincaré *recurrences* are not excluded, but occur only after *enormous times* — not so enormous as for chaotic evolutions but still large as exp(exp N).

The above contradiction is therefore apparent. The off-diagonal blocks cannot vanish in a mathematical sense since their norm is constant. However, all quantities of physical interest in the measurement process combine many complex exponentials which interfere destructively, so that everything takes place as if $\hat{\mathcal{R}}_{\uparrow\downarrow}$ did vanish at the end of the process. The exact final state of S + A and its reduced final state are thus equivalent with respect to all physically reachable quantities in the sense of Jauch [93]. Admittedly, one may imagine some artificial quantities involving few exponentials; or one may imagine processes with huge durations. But such irrealistic circumstances are not likely to be encountered by experimentalists in a near future.

Note that the matrix elements of the marginal state $\hat{r}(t)$ of S, obtained by tracing out the apparatus, are again obtained by summing a very large number of matrix elements of $\hat{D}(t)$. We can thus understand that the decay of the off-diagonal elements of \hat{r} is easier to prove than the truncation of the full state of S + A.

5199 12.2.4. The preferred basis issue

5200 5201 Lieverkoekjes worden niet gebakken¹⁰⁷ Dutch saying

Realistic models must explain why the truncation does not take place in an arbitrary basis but in the specific basis 5202 in which the tested observable of S is diagonal. This leads to the question of determining which mechanism selects 5203 this basis; intuitively, it is the very apparatus that the experimentalist has chosen and, in the Hamiltonian, the form 5204 of the interaction between S and A. One has, however, to understand precisely for each specific apparatus how the 520 dynamics achieve this property. For the Curie–Weiss model and for similar ones, the tested observable is directly 5206 coupled through (3.5) with the pointer observable \hat{m} , and the preferred basis problem is readily solved because the 5207 initial truncation is a mere result of the form of this coupling and of the large number of degrees of freedom of the 5208 pointer M. The finite expectation values $\langle \hat{s}_x \rangle$ and $\langle \hat{s}_y \rangle$ in the initial state of S are thereby transformed into correlations 5209 with many spins of the pointer, which eventually vanish (sections 5 and 6). Pointer-induced reduction thus takes place, 5210 as it should, in the eigenbasis of the tested observable. 521

We have also shown (§ 6.2.4) that in this model the suppression of the recurrences by the bath, although a *decoherence* phenomenon, is *piloted by the spin-magnet interaction* which selects the decoherence basis. When it is extended to a microscopic pointer, the Curie–Weiss model itself exhibits the preferred basis difficulty (§ 8.1.5). In the large *N* model, the final dynamical instability process (subsection 11.2) ensures that the reduction takes place in the same basis as the truncation. This basis should therefore have been determined by the dynamics at an earlier stage.

In other models, a decoherence generated by a random environment would have no reason to select this basis [32, 33, 40, 176, 177, 178, 179]. It is therefore essential, in models where truncation and registration are caused by some bath or some environment, to show how the interaction \hat{H}_{SA} determines the basis where these phenomena take place.

5221 12.2.5. Dephasing or bath-induced decoherence?

We reserve here the word "decoherence" to a process generated by a random environment, such as a thermal 5222 bath. We have just recalled that, in the Curie-Weiss model with large N, the initial truncation is ensured mainly by a 5223 dephasing effect, produced by the interaction between the system and the pointer; the bath only provides one of the 5224 two mechanisms that prevent recurrences from occurring after reduction (subsection 6.2). We have contrasted this 5225 direct mechanism with bath-induced decoherence (§ 5.1.2). In particular, our truncation time τ_{trunc} does not depend 5226 on the temperature as does usually a decoherence time, and it is so short that the bath B is not yet effective. Later on, 522 the prohibition of recurrences by the bath in this model is a subtle decoherence process, which involves resonance and 5228 which implies all three objects, the tested spin, the magnet and the bath (\S 6.2.4) 5229

⁵²³⁰ We have shown (§ 5.1.2 and § 6.1.2) that more general models with macroscopic pointers can also give rise to ⁵²³¹ direct truncation by the pointer. However, in models involving a microscopic pointer (see subsections 2.1, 2.4.1, 2.5

¹⁰⁷"I-prefer-this" cookies are not baked, i.e., you won't get what you want

and 8.1), the truncation mechanism can only be a bath-induced decoherence [32, 33, 40, 176, 177, 178, 179], and the occurrence of a preferred truncation basis is less easy to control.

As regards the dynamical instability mechanism, which ensures the hierarchical structure and thus allows reduction (section 11), it may either arise from interactions within the pointer itself (§ 11.2.3), or be induced by the bath (§ 11.2.4 and appendix I). Although this process includes a kind of decoherence or self-decoherence, it presents very specific features associated with the breaking of invariance of the pointer. It involves two sets of levels associated with the two possible indications of the pointer, all at the same energy. The coherences astride the two sets of levels rapidly disappear, but during the same time lapse, each set also reaches microcanonical ferromagnetic equilibrium.

5240 12.3. About registration

In order to regard a dynamical process as an ideal measurement, we need it to account for registration, a point too often overlooked. Indeed, we have seen (section 11) that not only truncation but also registration are prerequisites for the establishment of the uniqueness of the outcome in each individual run. The mechanism that ensures this property relies on the dynamics of the *sole macroscopic apparatus* and on its *bistability*; it may therefore be effective only after registration. Of course, registration is also our sole access to the microscopic tested system.

5246 12.3.1. The pointer must be macroscopic

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*Iedere keer dat hij het verhaal vertelde, werd de vis groter*¹⁰⁸ Dutch expression

Like the truncation, the registration is a material process, which affects the apparatus and creates correlations 525 between it and the tested system. This change of A must be detectable: We should be able to read, print or process 5251 the results registered by the pointer, so that they can be analysed by "automatic observers". In the Curie-Weiss model, 5252 the apparatus simulates a magnetic memory, and, under the conditions of subsection 9.4, it satisfies these properties 5253 required for registrations (section 7). The apparatus is *faithful*, since the probability of a wrong registration, in which 5254 the distribution $P_{\uparrow\uparrow}(m, \tau_{\rm reg})$ would be sizeable for negative values of m, is negligible, though it does not vanish in 5255 a mathematical sense. The registration is robust since both ferromagnetic states represented by density operators 5256 yielding magnetizations located around $+m_{\rm F}$ and $-m_{\rm F}$ are stable against weak perturbations, such as the ones needed 525 to read or to process the result. 5258

The registration is also *permanent*. This is an essential feature, not only for experimental purposes but also because 5259 the solution of the quantum measurement problem (section 11) requires the state $\hat{D}(t)$ to have reached the form (11.2). 5260 However, this permanence, or rather quasi-permanence, may again be achieved only in a physical sense (§ 11.1.1), just 5261 as the broken invariance associated with phase transitions is only displayed at physical times and not at "truly infinite 5262 times" for finite materials. Indeed, in the Curie-Weiss model, thermal fluctuations have some probability to induce in 5263 the magnetic dot transitions from one ferromagnetic state to the other. More generally, information may spontaneously 5264 be erased after some delay in any finite registration device, but this delay can be extremely long, sufficiently long for 526 our purposes. For our magnetic dot, it behaves as an exponential of N owing to invariance breaking, see Eq. (7.84). 5266

⁵²⁶⁷ The enhancement of the effect of S on A is ensured by the metastability of the initial state of A, and by the ⁵²⁶⁸ irreversibility of the process, which leads to a stable final state.

All these properties require a *macroscopic* pointer (§ 1.2.1), and not only a macroscopic apparatus. In principle, the models involving a large bath but a small pointer are therefore unsatisfactory for the aim of describing ideal measurements. In many models of quantum measurement (section 2), including the Curie-Weiss model for N = 2(subsection 8.1), the number of degrees of freedom of the pointer is not large. We have discussed this situation, in which an ideal measurement can be achieved, but only if the small pointer is coupled at the end of the process to a *further, macroscopic apparatus* ensuring amplification and true registration of the signal.

Altogether the macroscopic pointer behaves in its final state as a *classical object* which may lie in either one or the other of the states characterized within negligible fluctuation by the value A_i of the pointer observable \hat{A} . (In the Curie-Weiss model, $A_i \simeq \pm m_F$ is semiclassical, while $s_i = \pm 1$ is quantal). This crucial point has been established in section 11. Theoretically, nothing prevents us from imagining that the pointer M lies in a quantum state including coherences

¹⁰⁸Every time he told the story, the caught fish became bigger

across $m = m_{\rm F}$ and $-m_{\rm F}$. For the full ensemble \mathcal{E} (section 7), such a situation does not occur during the slow registration process due to the spin-apparatus interaction which creates complete correlations. For any subensemble $\mathcal{E}_{\rm sub}$, coherences might exist near the end of the process, but according to section 11, they would rapidly disappear, owing to the large size of M and to suitable interactions within the apparatus (§ 11.2.3 and § 11.2.4). The correlations between the signs of s_z and of *m* produced during the registration, and the uniqueness property (§ 11.3.1), then separate the two sectors. The large size of the pointer is therefore essential for a complete solution of the ideal measurement problem.

5286 12.3.2. Does the registration involve observers?

5287 5288 *Hij stond erbij en keek er naar*¹⁰⁹ Dutch saying

We have seen that truncation does not involve observers. Likewise, conscious observers are irrelevant for the registration, which is a physical process, governed by a Hamiltonian. Once the registration of the outcome has taken place, the correlated values of $A_i \simeq \pm m_F$ and $s_i = \pm 1$ take an *objective* character, since any observer will read the same well-defined indication A_i at each run. "Forgetting" to read off the registered result will not modify it in any way. Anyhow, nothing prevents the *automatic processing* of the registered data, in view of further experiments on the tested system (§ 12.2.2).

We thus cannot agree with the idealist statement that "the state is a construct of the observer". Although we interpret the concept of probabilities as a means for making predictions from available data (§ 10.1.4), a state reflects real properties of the physical system acquired through its preparation, within some undetermined effects due to the non-commutative nature of the observables.

5299 12.3.3. What does "measuring an eigenvalue" mean?

A measurement process is an experiment which creates in the apparatus an image of some property of the tested 5300 system. From a merely experimental viewpoint alone, one cannot know the observable of S that is actually tested, but 5301 experience as well as theoretical arguments based on the form of the interaction Hamiltonian may help to determine 5302 which one. From the observed value A_i of the pointer variable, one can then infer the corresponding eigenvalue s_i of the 530 measured operator (that appears in the interaction Hamiltonian), provided the correlation between A_i and s_i is complete 530 (an example of failure is given in § 7.3.3). In the Curie-Weiss model the observed quantity is the magnetization of 5305 M; we infer from it the eigenvalue of \hat{s}_z . The statement of some textbooks "only eigenvalues of an operator can be 5306 measured" refers actually to the pointer values, which are in one-to-one correspondence with the eigenvalues of the 5307 tested observable provided the process is an ideal measurement. The eigenvalues of an observable as well as the 5308 quantum state of S are abstract mathematical objects associated with a microscopic probabilistic description, whereas 5309 the physical measurement that reveals them indirectly relates to the macroscopic pointer variable. 5310

⁵³¹¹ 12.3.4. Did the registered results preexist in the system?

Å gå over bekken etter vann¹¹⁰ Norwegian expression

After the measurement process has taken place and after the outcome of the apparatus has been read, we can assert that the apparatus lies in the state $\hat{\mathcal{R}}_i$ characterized by the value A_i of the pointer while the system lies in the final projected state \hat{r}_i (Eq. (11.2)). We can also determine the weights p_i from the statistics of the various outcomes A_i . However a quantum measurement involves not only a change in A that reflects a property of S, but also a change in S (§ 1.1.2). In an ideal measurement the latter change is minimal, but we have to know precisely which parts of the initial state $\hat{r}(0)$ are conserved during the process so as to extract information about it from the registered data.

⁵³²⁰ Consider first the whole ensemble of runs of the experiment. Together with the theoretical analysis it provides the ⁵³²¹ set of final states \hat{r}_i and their weights p_i . The corresponding marginal density operator $\sum_i p_i \hat{r}_i$ of S is obtained from ⁵³²² $\hat{r}(0)$ by keeping only the diagonal blocks, the off-diagonal ones being replaced by 0. We thus find a partial statistical

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¹⁰⁹He stood there and watched, i.e., he did not assist

¹¹⁰To cross the stream to get water

information about the initial state: all probabilistic properties of the tested observable \hat{s} remain unaffected, as well as those pertaining to observables that commute with \hat{s} . (The amount of information retained is minimal, see § 1.2.4.) Some retrodiction is thus possible, but it is merely statistical and partial.

Consider now a single run of the measurement, which has provided the result A_i . The fact that S is thereafter in 5326 the state \hat{r}_i with certainty does not mean that it was initially in the same state. In fact no information about the initial 5327 state $\hat{D}(0)$ is provided by reading the result A_i , except for the fact that the expectation value in $\hat{D}(0)$ of the projection 532 on the corresponding eigenspace of \hat{A} does not vanish. For a spin $\frac{1}{2}$, if we have selected at the end of a single run 5329 the value $s_z = 1$, we can only ascertain that the system was not in the pure state $|\downarrow\rangle$ at the initial time; otherwise 5330 its polarization could have been arbitrary. In contrast, a classical measurement may leave the system invariant, in 533 which case we can retrodict from the observation of A_i that the measured quantity took initially the value s_i . For an 5332 individual quantum measurement, retrodiction is impossible, and devoid of physical meaning, due to the probabilistic 5333 nature of observables and to the irreversibility of the process. The property " \hat{s} takes the value s_i " did not preexist the 5334 process. It is only in case all runs provide the outcome A_i that we can tell that S was originally in the state \hat{r}_i . One 5335 should therefore beware of some realist interpretations in which the value s_i is supposed to preexist the individual 5336 measurement¹¹¹: they do not take properly into account the perturbation brought in by the measurement [31]. 5337

5338 12.4. Ideal measurements and interpretation of quantum mechanics

5339	An expert is a man who has made all the mistakes which can be made,
5340	in a narrow field
5341	Niels Bohr

Quantum measurements throw bridges between the microscopic reality, that we grasp through quantum theory, and the macroscopic reality, easier to apprehend directly. The images of the microscopic world that we thus get appear more "natural" (i.e., more customary) than the counter-intuitive quantum laws, although they emerge from the underlying quantum concepts (subsection 11.3). However, the interpretation of the latter concepts is subject to ongoing debate. In particular, as a measurement is a means for gaining information about a physical quantity pertaining to some state of a system, the meaning of "physical quantity" and of "state" should be made clear.

⁵³⁴⁸ 12.4.1. The statistical interpretation is sufficient to fully explain measurements

Հավ է մրջնի գլուխ լինես քան առյուծի պոչ¹¹² Armenian proverb

Many authors treat quantum measurements as irreversible processes of quantum statistical mechanics involving interaction between the tested system and a macroscopic apparatus or a macroscopic environment (section 2). The natural tool in such approaches is the density operator of the system S + A, which can be regarded as representing a state in the statistical interpretation of quantum mechanics (§ 10.1.4). Implicitly or explicitly, we have relied throughout the present work on this interpretation, resumed in section 10.

A *classical* measurement can be regarded as a means to exhibit, through an apparatus A, some *pre-existing* property of an *individual* system S. In the statistical interpretation of a quantum measurement, we deal with the joint evolution of an *ensemble* of systems S + A, the outcome of which indirectly reveals only *some probabilistic properties* of the initial state of S [10, 11, 31, 48, 52, 58]. The value s_i of the tested observable \hat{s} inferred from the observation of the indication A_i of the pointer *did not preexist* the process, even though we can assert that it is taken by S after an ideal measurement where A_i has been registered and selected.

A preliminary step in a measurement model is the assignment to the apparatus at the initial time of a density operator $\hat{\mathcal{R}}(0)$, namely, in the Curie–Weiss model (§ 3.3.2 and § 3.3.3), a paramagnetic state for M and a thermal equilibrium state for B. The preparation of this initial state is of the macroscopic type, involving a control of only few variables such as energy. The *assignment of a density operator* is based, according to the statistical interpretation,

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¹¹¹ In a hidden variable description that enters discussions of Bell inequalities in the BCHSH setup, one should thus describe the measured variable not as a "predetermined" value set only by the pair of particles (Bell's original setup) but as depending on the hidden variables of both the pair and the detector (Bell's extended setup). See Ref. [268] for a discussion of an assumption needed in that setup

¹¹²Better to be an ant's head than a lion's tail

on probabilistic arguments (§ 10.2.2), in particular on the maximum entropy criterion which underlies the choice of canonical distributions. (A preparation through measurement is excluded, not only because the apparatus is macroscopic, but also logically, since the measurement that we wish to explain by a model would depend on a preceding measurement.)

The next stages of the solution, truncation and registration (sections 4 to 7), are mere relaxation processes of 5370 quantum statistical mechanics, governed by the Liouville-von Neumann equation, which lead the state of S + A from 537 $\hat{D}(0)$ to $\hat{D}_{exact}(t_f)$ for the large ensemble \mathcal{E} of runs. Approximations justified under the conditions of subsection 9.4 5372 let us replace $\hat{\mathcal{D}}_{exact}(t_f)$ by $\hat{\mathcal{D}}(t_f)$. The breaking of unitarity entailed by this replacement can be understood, in the 5373 interpretation of a state as a mapping (10.1) of the observables onto their expectation values, as a restriction of this 537 mapping to the "relevant observables" [58]. Indeed, if we disregard the "irrelevant" observables associated with cor-53 relations between an inaccessibly large number of particles, which are completely ineffective if no recurrences occur, 5376 both states $\hat{D}(t_f)$ and $\hat{D}_{exact}(t_f)$ realize the same correspondence (10.1) for all other, accessible observables \hat{O} . The 5377 entropy $S[\hat{D}(t_f)]$, larger than $S[\hat{D}_{exact}(t_f)] = S[\hat{D}(0)]$, (§ 1.2.4 and [72]), enters the framework of the general concept 5378 of *relevant entropies* associated with a reduced description from which irrelevant variables have been eliminated [58]. 5379

Within the informational definition (10.1) of states in the statistical interpretation, we may acknowledge a restriction of information to relevant observables when eliminating either the environment in models for which this environment induces a decoherence, or the bath B in the Curie–Weiss model (subsection 4.1). In the latter case, the states $\hat{D}(t)$ and $\hat{D}(t) \otimes \hat{R}_{\rm B}(t)$ (Fig. 3.2) should be regarded as equivalent if we disregard the inaccessible observables that correlate B with M and S.

Still another equivalence of "states" in the sense of (10.1) will be encountered in § 13.1.5, where the Curie– Weiss model is reconsidered in the Heisenberg picture. There the evolution of most off-diagonal observables lets them vanish at the end of the process, so that they become irrelevant. The (time-independent) density matrix and the resulting truncated one are therefore equivalent after the time t_f , since they carry the same information about the only remaining diagonal observables. Note also that, in the statistical interpretation, it is natural to attribute the quantum specificities (§ 10.2.1) to the non commutation of the observables; in the Heisenberg picture, the effective commutation at the time t_f of those which govern the measurement sheds another light on the emergence of classicality (§ 13.1.4).

We have stressed that, in the statistical interpretation, a quantum state does not describe an individual system, but 5392 an ensemble (§ 10.1.3). The solution $\hat{\mathcal{D}}(t)$ of the Liouville von-Neumann equation for S + A describes fully, but in a 5393 probabilistic way, a large set \mathcal{E} of runs originated from the initial state $\mathcal{D}(0)$: quantum mechanics treats statistics of 5394 processes, not single processes. However, the solution of the quantum measurement problem requires to distinguish, 5395 at the end of the process, single runs or at least subensembles \mathcal{E}_i of \mathcal{E} having yielded the outcome A_i for the pointer. A 5396 measurement is achieved only after reading, collecting, processing or selecting the result of each individual process, 5397 so as to interpret its results in every day's language [320]. It is essential to understand how ordinary logic, ordinary 53 probabilities, ordinary correlations, as well as exact statements about individual systems may emerge at our scale 5399 from quantum mechanics in measurement processes, even within the statistical interpretation which is foreign to such 5400 concepts. Although $\hat{D}(t)$ appears as an adequate tool to account for truncation and registration, it refers to the full set 5401 \mathcal{E} , and its mere determination is not sufficient to provide information about subsets. The difficulty lies in the *quantum* 5402 *ambiguity* of the decomposition of the mixed state $\hat{D}(t_f)$ into states describing subensembles (§ 10.2.3 and § 11.1.3). 5403 We have achieved the task of understanding ideal measurements in section 11 by relying on a dynamical mechanism 5404 according to which the macroscopic apparatus retains quantum features only over a brief delay. This provides the 5405 unambiguous splitting of \mathcal{E} into the required subsets \mathcal{E}_i . 540

5407 12.4.2. Measurement models in other interpretations

As shown above, standard quantum mechanics within the statistical interpretation provides a satisfactory explanation of all the properties, including odd ones, of quantum measurement processes. Any other interpretation is of course admissible insofar as it yields the same probabilistic predictions. However, the *statistical interpretation*, in the present form or in other forms, as well as alternative equivalent interpretations, *is minimalistic*. Since it has been sufficient to explain the crucial problem of measurement, we are led to leave aside at least those interpretations which require additional postulates, while keeping the same probabilistic status.

In particular, we can eliminate the variants of the "orthodox" *Copenhagen interpretation* in which it is postulated that two different types of evolution may exist, depending on the circumstances, a Hamiltonian evolution if the system is isolated, and a sudden change producing von Neumann's reduction and Born's rule if the system S undergoes an ⁵⁴¹⁷ ideal measurement [4, 172]. We can rule out the second type of evolution, since we have seen in detail (section 11) that ⁵⁴¹⁸ the standard Liouville–von Neumann evolution alone, when applied to arbitrary subensembles, is sufficient to explain ⁵⁴¹⁹ the reduction. The apparent violation of the superposition principle is understood as the result of suitable interactions ⁵⁴²⁰ within the macroscopic apparatus, together with standard treatments of quantum statistical mechanics. It is therefore ⁵⁴²¹ legitimate to abandon the "postulate of reduction", in the same way as the old "quantum jumps" have been replaced ⁵⁴²² by transitions governed by quantum electrodynamics. It is also useless to postulate the uniqueness of the outcome of ⁵⁴²³ individual runs (§ 11.1.3).

Interpretations based on *decoherence* by some environment underlie many models (subsection 2.7). The detailed study of section 11 shows, however, that a proper explanation of reduction requires a special type of decoherence, which accounts for the bistability (or multistability) of the apparatus (§ 12.2.5). Decoherence models in which a special mode of the environment is considered as "pointer mode" [33, 32] are unrealistic, since, by definition, the environment cannot be manipulated or read off. See also the discussion of this issue in [69].

Many interpretations are motivated by a wish to describe individual systems, and to get rid of statistical ensembles. The consideration of *conscious observers* was introduced in this prospect. However, the numerous models based on the S + A dynamics show that a measurement is a *real dynamical process*, in which the system undergoes a physical interaction with the apparatus that modifies both, as can be shown by performing subsequent experiments.

Reduction in an individual measurement process has often been regarded as a kind of bifurcation which may lead 543 the single initial state $\hat{\mathcal{D}}(0)$ towards several possible outcomes $\hat{\mathcal{D}}_i$, a property seemingly at variance with the linearity 5434 of quantum mechanics. In the interpretation of Bohm and de Broglie [18, 24], such a bifurcation occurs naturally. 5435 Owing to the introduction of trajectories piloted by the wave function, a one-to-one correspondence exists between 5436 the initial and the final point of each possible trajectory; the initial point is governed by a classical probability law 5437 determined by the initial quantum wave function, while the set of trajectories end up as separate bunches, each of 5438 which is associated with an outcome *i*. Thus, the final subsets \mathcal{E}_i reflect pre-existing subsets of \mathcal{E} that already existed 5439 at the initial time. In spite of this qualitative explanation of reduction, the trajectories, which refer to the coupled 5440 system S + A, are so complicated that models relying on them seem out of reach.

At the other extreme, the reality of collapse is denied in Everett's many-worlds interpretation [25, 26]. A measurement is supposed to create several branches in the "relative state", one of which only being observed, but no dynamical mechanism has been proposed to explain this branching.

The same concern, describing individual quantum processes, has led to a search for sub-quantum mechanics [20, 31, 325]. Although new viewpoints on measurements might thus emerge, such drastic changes do not seem needed in this context. Justifications should probably be looked for at scales where quantum mechanics would fail, hopefully at length scales larger than the Planck scale so as to allow experimental tests.

Of particular interest in the context of measurements are the information-based interpretations [52, 58, 74, 79, 80, 544 292], which are related to the statistical interpretation (§ 10.1.3 and § 10.1.4). Indeed, an apparatus can be regarded 5450 as a device which *processes information* about the system S, or rather about the ensemble \mathcal{E} to which S belongs. The 5451 initial density operator $\hat{r}(0)$, if given, gathers our information about some preliminary preparation of S. During the 5452 process, which leads \mathcal{E} to the final truncated state $\hat{r}(t_f) = \sum_i p_i \hat{r}_i$ (Eq. (1.10)), all the off-diagonal information are lost. 5453 However, the correlations created between S and A then allow us to gain indirectly information on S by reading the 5454 outcome of the pointer, to select the corresponding subensemble \mathcal{E}_i , and to update our information about \mathcal{E}_i as \hat{r}_i . The 5455 amounts of information involved in each step are measured by the entropy balance of § 1.2.4. 5456

5457 12.4.3. Empiricism versus ontology

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5459

Einstein, stop telling God what to do Niels Bohr

There is no general agreement about the purpose of science [303]¹¹³. Is our task only to explain and predict phenomena? Does theoretical physics provide only an imperfect mathematical image of reality? Or is it possible to uncover the very nature of things? This old debate, more epistemologic than purely scientific, cannot be skipped since it may inflect our research. The question has become more acute with the advent of quantum physics, which deals

¹¹³The present authors do not regard science as having a unique purpose

with a "veiled reality" [303]. Physicists, including the authors of the present article, balance between two extreme 5464 attitudes, illustrated by Bohr's pragmatic question [320]: "What can we say about...?" facing Einstein's ontological 5465 question [23]: "What is ...?" The latter position leads one to ask questions about individual systems and not only about 5466 general properties, to regard quantum mechanics as an *incomplete* theory and to look for hidden "elements of reality". 5467 This opposition may be illustrated by current discussions about the status of pure states. In the statistical in-5468 terpretation, there is no conceptual difference between pure and mixed states (§ 10.1.4); both behave as probability 5469 distributions and involve the observer. In order to reject the latter, many authors with ontological aspirations afford 5470 pure states a more fundamental status, even though they acknowledge their probabilistic character, a point also crit-5471 icized by van Kampen [14]. Following von Neumann's construction of density operators (in analogy to densities in 5472 phase space of classical statistical mechanics), they regard pure states as building blocks rather than special cases of mixed states. In a decomposition (10.3) of a density operator \hat{D} associated with an ensemble \mathcal{E} , they consider that 547 each individual system of \mathcal{E} has its own ket. In this *realist interpretation* [31], two types of probabilities are distin-5475 guished [14]: "merely quantal" probabilities are interpreted as properties of the individual objects through $|\phi_k\rangle$, while 5476 the weights v_k are interpreted as ordinary probabilities associated with our ignorance of the structure of the statistical 547 ensemble \mathcal{E} . Such an interpretation might be sensible if the decomposition (10.3) were unique. We have stressed, 5478 however, its ambiguity (§ 10.2.3 and § 11.1.3); as a consequence, the very collection of pure states $|\phi_k\rangle$ among which 5479 each individual system is supposed to lie cannot even be imagined. It seems therefore difficult to imagine the existence 548 of "underlying pure states" which would carry more "physical reality" than \hat{D} [302, 308]. The distinction between 548 the two types of probabilities on which decompositions (10.3) rely is artificial and meaningless [10, 11]. 5482

Landau's approach to mixed states may inspire another attempt to regard a pure state as an intrinsic description of an individual system [14, 84]. When two systems initially in pure states interact, correlations are in general established between them and the marginal state of each one becomes mixed. To identify a pure state, one is led to embed any system, that has interacted in the past with other ones, within larger and larger systems. Thus, conceptually, the only individual system lying in a pure state would be the whole Universe [186, 188], a hazardous extrapolation [10, 11]. Not to mention the introduction in quantum mechanics of a hypothetic multiverse [25, 324].

Such considerations illustrate the kind of difficulties to be faced in a search for realist interpretations, a search 5489 which, however, is legitimate since purely operational interpretations present only a blurred image of the microscopic 5490 reality and since one may long for a description that would uncover hidden faces of Nature [303]. Among the proposed 5491 realist interpretations, one should distinguish those which provide exactly the same outcomes as the conventional 5492 quantum mechanics, and that can therefore neither be verified nor falsified. They have been extensively reviewed 5493 [17, 19, 31, 36, 189, 191, 192, 284] (see also references in § 1.1.1), and we discussed above some of them in connection 5494 with models of measurements. Many involve hidden variables of various kinds (such as Bohm and de Broglie's 5495 bunches of trajectories or such as stochastic backgrounds) or hidden structures (such as consistent histories, see 549 subsection 2.9). 5497

Other approaches attempt to go "beyond the quantum". They resort, for instance, to stochastic electrodynamics [325, 326, 327, 328], to quantum Langevin equations [31], to nonlinear corrections to quantum mechanics such as in the GRW approach [17, 89, 190], or to speculations about quantum gravitation [329]. The sole issue issue to close the Einstein–Bohr debate in such fields is a search for testable specific predictions [23, 320].

For the time being, empirical approaches appear satisfactory "for all practical purposes" [330]. The statistical 5502 interpretation, either in the form put forward by Blokhintsev [10, 11] and Ballentine [9, 48] or in the form presented 5503 above, is empirical and minimalist: It regards quantum mechanics only as a means for deriving predictions from 550 available data. It is related to partly subjective interpretations that focus on information [292], since information is 5505 akin to probability. We have seen (section 11) that, although the statistical interpretation is irreducibly probabilistic, 5506 involving both the system (as regards the observables and their evolution) and the observers (as regards the state), 5507 although it only deals with statistical ensembles, it suffices in conjunction with dynamics to account for individual 5508 behaviours in ideal measurements. The same epistemological attitude is shared by phenomenological-minded people, 5509 and is advocated, for instance, by Park [28], van Kampen [14] and de Muynck [31]. It can be viewed as a common 5510 ground for all physicists, as stressed by Laloë [34], whose "correlation interpretation" emphasizes predictions as cor-551 relations between successive experiments. A more extreme philosophical position, the rejection of any interpretation, 5512 is even defended by Fuchs and Peres in [331]. According to such positions, quantum theory has the modest task of 5513 accounting for the statistics of results of experiments or of predicting them. It deals with what we know about reality, 5514

and does not claim to unveil an underlying reality per se¹¹⁴. Bohr himself shared [320] this conception when he said (see [321, 322] for a list of Bohr's quotations): "*There is no quantum world*. *There is only an abstract quantum physical description*. *It is wrong to think that the task of physics is to find out how nature is*. *Physics concerns what we can say about nature*."

5519 13. What next?

Այս ապուրը դեռ շատ ջուր կքաշի ^{!15}
<i>Il va couler encore beaucoup d'eau sous les ponts</i> ¹¹⁶
Er zal nog heel wat water door de Rijn moeten ¹¹⁷
Armenian, French and Dutch proverbs

Much can still be learnt from models, even about the ideal quantum measurements on which we have focused. 5524 Various features of measurements and their incidence on interpretations of quantum mechanics have been explained 5525 by the many models reviewed in section 2. However, the treatments based on quantum statistical mechanics provide, 5526 as final state describing the outcome of a large set of runs of the measurement, a mixed state. Such a state cannot be 5527 decomposed unambiguously into components that would describe subsets of runs (§ 11.1.3), so that a further study 5528 was required to explain the uniqueness of the outcome of each run. A dynamical mechanism that achieves this task 5529 has been proposed (§§ 11.2.3 and 11.2.4 and appendices H and I). Adapting it to further models should demonstrate 5530 the generality of such a solution of the measurement problem. 553

Alternative approaches should also be enlightening. We suggest some paths below.

5533 13.1. Understanding ideal measurements in the Heisenberg picture

Some insight can be gained by implementing the dynamics of the measurement process in the Heisenberg picture (§ 10.1.2) rather than in the more familiar Schrödinger picture. Both pictures are technically equivalent but the Heisenberg picture will provide additional understanding. It is then the observables $\hat{O}(t, t_0)$ which evolve, in terms of either the running time *t* or of the reference time t_0 . By taking t_0 as the initial time $t_0 = 0$, an observable $\hat{O}(t, t_0)$ is governed for an isolated system by the Heisenberg equation

$$i\hbar \frac{d\hat{O}(t,0)}{dt} = [\hat{O}(t,0),\hat{H}]$$
 (13.1)

with the initial condition $\hat{O}(0,0) = \hat{O}$, while the states assigned at the reference time $t_0 = 0$ remain constant. This formulation presents a conceptual advantage; it clearly dissociates *two features* of quantum mechanics, which in the Schrödinger picture are merged within the time-dependent density operator. Here, the *deterministic evolution* is carried by the observables, which represent random physical quantities; on the other hand, our whole *probabilistic information* about these quantities is embedded in the time-independent density operator¹¹⁸.

We can thus account for the dynamics of a system in a general way, without having to specify its probabilistic description in the particular situation we wish to describe. The use of the Heisenberg picture has therefore an incidence on the interpretation of quantum mechanics. Whereas the Schrödinger picture only allows us to describe dynamics of the statistical ensemble represented by the density operator, we can regard the equation of motion (13.1) as pertaining to an *individual system*¹¹⁹. It is only when evaluating expectation values as tr[$\hat{D}\hat{O}(t, 0)$] that we have to embed the studied system in a statistical ensemble.

¹¹⁴This point may be illustrated on the double slit experiment. While the particle-wave duality allows to imagine that electrons or photons "go through both slits simultaneously", some authors find it hard to accept this for large objects such as bucky balls [332] or viruses [333]

¹¹⁵Preparing this porridge still requires much water

¹¹⁶Much water will still flow under the bridges

¹¹⁷Quite some water will still have to flow through the Rhine river

¹¹⁸We use the term "observables" in the sense of "operator-valued random physical quantities" (§ 10.1.1), not of "outcomes of observations". The latter quantities (frequencies of occurrence, expectation values, variances) are joint properties (10.1) of "states" (i. e., density operators playing the role of quantum probabilities) and observables

¹¹⁹As understood, in the statistical interpretation, to belong to an ensemble of identically prepared members

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Moreover, when a measurement is described in the Schrödinger picture, the density operator of S + A undergoes two types of changes, the time dependence from $\hat{D}(0)$ to $\hat{D}(t_f)$, and the restriction to \hat{D}_i if the outcome A_i is selected. The temptation of attributing the latter change to some kind of dynamics will be eluded in the Heisenberg picture, where only the observables vary in time.

Let us sketch how the Curie–Weiss model might be tackled in the Heisenberg picture.

5555 13.1.1. Dynamical equations

The equations of motion (13.1) which couple the observables to one another have the same form as the Liouville– 5556 von Neumann equation apart from a sign change and from the boundary conditions. Thus, their analysis follows the 5557 same steps as in section 4. Elimination of the bath takes place by solving at order γ the equations (13.1) for the bath 5558 observables $\hat{B}_{a}^{(n)}(t,0)$, inserting the result into the equations for the observables of S + M and averaging over the state 5559 $\hat{R}_{\rm B}$ of B; this provides integro-differential equations that couple the observables $\hat{s}_a(t,0)$ of S and those $\hat{\sigma}_a(t,0)$ of M 5560 (a = x, y or z). The conservation of \hat{s}_z implies, instead of the decoupling between the four blocks $\uparrow\uparrow, \downarrow\downarrow, \uparrow\downarrow, \downarrow\uparrow$ of the 5561 Schrödinger density matrix, the decoupling between four sets of observables, the diagonal observables proportional 5562 to $\hat{\Pi}_{\uparrow} \equiv \frac{1}{2}(1+\hat{s}_z)$ and $\hat{\Pi}_{\downarrow} \equiv \frac{1}{2}(1-\hat{s}_z)$, and the *the off-diagonal observables proportional to* \hat{s}_{-} and \hat{s}_{+} , respectively. 5563 Finally the symmetry between the various spins of M allows us again to deal only with \hat{m} , so that the dynamics bears 556 on the observables $\hat{\Pi}_{\uparrow} f(\hat{m})$, $\hat{\Pi}_{\downarrow} f(\hat{m})$, $\hat{s}_{-} f(\hat{m})$ and $\hat{s}_{+} f(\hat{m})$, coupled within each sector. 5565

5566 13.1.2. Dynamics of the off-diagonal observables

The evolution (13.1) of the off-diagonal observables generated over very short times $t \ll \tau_{\text{recur}} = \pi \hbar/2g$ by $\hat{H}_{\text{SA}} = -Ng\hat{s}_{z}\hat{m}$ (section 5) is expressed by

$$\hat{s}_{-}(t) = \hat{s}_{-} \exp \frac{2iNg\hat{m}t}{\hbar}, \qquad \hat{m}(t) = \hat{m}.$$
 (13.2)

Instead of the initial truncation exhibited in the Schrödinger picture, we find here a rapid oscillation, which will entail a damping after averaging over the canonical paramagnetic state of M.

The suppression of recurrences through the non-identical couplings of subsection 6.1 replaces $Ng\hat{m}$ by $\sum_n (g + \delta g_m)\hat{\sigma}_z^{(n)}$ in (13.2), a replacement which after averaging over most states will produce damping. The bath-induced mechanism of subsection 6.2 introduces, both in $\hat{m}(t)$ and in the right side of (13.2), observables pertaining to the bath which are regarded as unreachable. Tracing out B then produces the damping of recurrences for the off-diagonal observables.

⁵⁵⁷⁶ We have shown (§§ 11.2.3 and 11.2.4) that reduction can result from a decoherence produced by a random inter-⁵⁵⁷⁷ action within M or by a collisional process. In the Heisenberg picture, the result is again the decay towards 0 of the ⁵⁵⁷⁸ off-diagonal observables $|\uparrow\rangle\langle\downarrow|\otimes|m_{\rm F},\eta\rangle\langle-m_{\rm F},\eta'|$ and $|\downarrow\rangle\langle\uparrow|\otimes|-m_{\rm F},\eta\rangle\langle m_{\rm F},\eta'|$.

5579 13.1.3. Establishment of system–apparatus correlations

The evolution of the *diagonal observables* in the Heisenberg picture is analogous to the registration of section 7, but it is represented by more general equations than in the Schrödinger picture. Indeed, denoting by $\delta_{\hat{m},m}$ the projection operator on the eigenspace associated with the eigenvalue *m* of \hat{m} , we now have in the sector $\uparrow\uparrow$ to look at the dynamics of the time-dependent observables $\hat{\Pi}_{\uparrow}\delta_{\hat{m},m}(t,0)$, instead of the dynamics of their expectation values $P_{\uparrow\uparrow}^{\text{dis}}(m,t)$ in the specific state $\hat{D}(0)$ of S + M as in section 7. The solution of the equations of motion has the form

$$\hat{\Pi}_{\uparrow}\delta_{\hat{m},m}(t_{\rm f},0) = \sum_{m'} K_{\uparrow}(m,m')\hat{\Pi}_{\uparrow}\delta_{\hat{m},m'}.$$
(13.3)

The kernel $K_{\uparrow}(m, m')$ represents the transition probability of the random order parameter \hat{m} from its eigenvalue m' at the time 0 to its eigenvalue m at the time $t_{\rm f}$, under the effect of the bath and of a field +g. It is obtained by taking the long-time limit of the Green's function defined by Eq. (7.58), and we infer its properties from the outcomes of section 7. As m' is arbitary, we must deal here with a *bifurcation* (as in subsection 7.3). For m' larger than some negative threshold, $K_{\uparrow}(m, m')$ is concentrated near $m \simeq +m_{\rm F}$; this will occur in particular if m' is small, of order

 $1/\sqrt{N}$. However, if m' is negative with sufficiently large |m'|, it will be sent towards $m = -m_{\rm F}$. Likewise, $K_{\downarrow}(m, m')$ is 5590 concentrated around $m \simeq -m_{\rm F}$ if |m'| is sufficiently small (or if m' is negative), but around $m \simeq +m_{\rm F}$ if m' is positive 559 and sufficiently large. The complete correlations required for the process to be a measurement will be created only 5592 after averaging over a state of the pointer concentrated around m' = 0. 5593

At later times, around t_f , the process of § 11.2.3 produces the irreversible decay of the diagonal observables 5594 $|\uparrow\rangle\langle\uparrow|\otimes|m_{\rm F},\eta\rangle\langle m_{\rm F},\eta'|$ and $|\downarrow\rangle\langle\downarrow|\otimes|-m_{\rm F},\eta\rangle\langle-m_{\rm F},\eta'|$ towards $\delta_{\eta\eta'}\hat{R}^{\mu}_{\uparrow\uparrow}$ and $\delta_{\eta\eta'}\hat{R}^{\mu}_{\downarrow\downarrow}$, respectively. Notice that while the 5595 initial observables involve here the full set $\hat{\sigma}_z^{(n)}$, their evolution narrows this set, leading it only towards the projection 5596 operators on $\hat{m} = m_{\rm F}$ and $\hat{m} = -m_{\rm F}$. 5597

13.1.4. Fate of observables at the final time 5598

Physical data come out in the form tr $\hat{D}^{\text{Heis}}\hat{O}(t,0)$ where $\hat{D}^{\text{Heis}} = \hat{D}^{\text{Schr}}(0) = \hat{r} \otimes \hat{\mathcal{R}}$ is time-independent, namely 5599 just the initial state in the Schrödinger picture. The success of an ideal measurement process now appears as the joint 5600 result of the algebraic properties that result in the expressions of the time-dependent observables, and of some specific 5601 properties of the initial preparation of the apparatus embedded in $\hat{\mathcal{R}}(0)$. On the one hand, the width in $1/\sqrt{N}$ of the 5602 initial paramagnetic distribution $P_{\rm M}^{\rm dis}(m,0)$ is sufficiently large so that the oscillations (13.2) of $\hat{s}_{-}(t)$ are numerous 5603 and *interfere destructively* on the time scale τ_{red} . On the other hand, it is sufficiently narrow so as to *avoid wrong* registrations: The final probability distribution $P_{\uparrow\uparrow}^{dis}(m, t_f)$ for the pointer is the expectation value of (13.3) over \hat{D}^{Heis} , 560 5605 and the concentration near the origin of $\hat{m} = m'$ in $\hat{\mathcal{R}}(0)$ entails the concentration near $+m_{\rm F}$ of $P_{\uparrow\uparrow}^{\rm dis}(m, t_{\rm f})$. 5606

Some intuition about ideal measurements may be gained by acknowledging the decay of the off-diagonal ob-5607 servables during the process and their effective disappearance¹²⁰ after the time $t_{\rm f}$. The evolution of the diagonal 5608 observables also implies that, under the considered circumstances, only the eigenspaces of \hat{m} associated with eigen-5609 values close to $m_{\rm F}$ and $-m_{\rm F}$ survive at $t_{\rm f}$. The only observables remaining at the end of the process, $\Pi_{\rm f} \delta_{\hat{m},m}(t_{\rm f},0)$ 5610 with *m* close to m_F expressed by (Eq b) and $\hat{\Pi}_{\downarrow} \delta_{\hat{m},m}(t_f, 0)$ with *m* close to $-m_F$, belong to an *abelian algebra*. It is 5611 therefore natural to regard them as ordinary random variables governed by standard probabilities, and to use daily 5612 reasoning which allows statements about individual events. The singular features of quantum mechanics which arose 5613 from non-commutativity (§ 10.2.1) can be disregarded. The *emergence of classicality* in measurement processes now 5614 appears as a *property of the Heisenberg dynamics* of the observables. 5615

13.1.5. Truncation 5616

We now turn to the states describing the ensemble \mathcal{E} of runs and its subensembles. Remember that in the statistical 5617 interpretation and in the Heisenberg picture, a "state" is a time-independent mathematical object that accounts for 5618 our information about the evolving observables (§ 10.1.4). Equivalently, the density operator gathers the expectation 5619 values of all observables at any time. The assignment of a density matrix \hat{D}^{Heis} to the whole set \mathcal{E} of runs of the 5620 measurement relies on information acquired before the interaction process (i.e., the measurement) and embedded 5621 in the states \hat{r} , \hat{R}_{M} and \hat{R}_{B} of S, M and B. These information allow us to describe the statistics of the whole process between the times 0 and t_{f} through the equations of motion (13.1) and the density operator $\hat{D}^{\text{Heis}} = \hat{r}(0)\otimes\hat{R}_{M}(0)\otimes\hat{R}_{B}(0)$ 5622 5623 describing the set \mathcal{E} . 5624

However, the vanishing at t_f of the off-diagonal observables (at least of all accessible ones) entails that their *expec*-5625 tation values vanish, not only for the full set \mathcal{E} of runs of the measurement but also for any subset. The information 5626 about them, that was embedded at the beginning of the process in the off-diagonal blocks of $\mathcal{D}^{\text{Heis}}$, have been *irreme*-5627 diably lost at the end, so that these off-diagonal blocks become irrelevant after measurement. For the whole ensemble 5628 \mathcal{E} , and for any probabilistic prediction at times $t > t_f$, it makes no difference to replace the state $\hat{\mathcal{D}}^{\text{Heis}}$ by the sum of 5629 its diagonal blocks according to 5630

$$\hat{\mathcal{D}}^{\text{Heis}} \mapsto \hat{\mathcal{D}}_{\text{red}}^{\text{Heis}} = \sum_{i} p_i \hat{\mathcal{D}}_i^{\text{Heis}}, \qquad \hat{\mathcal{D}}_i^{\text{Heis}} = \hat{\Pi}_i \hat{r}(0) \hat{\Pi}_i \otimes \hat{R}_{\text{M}}(0) \otimes \hat{R}_{\text{B}}(0), \qquad (i = \uparrow, \downarrow).$$
(13.4)

 $^{^{120}}$ In fact, the disappearance of the off-diagonal observables is approximate for finite N and is not complete: We disregard the inaccessible observables, whether they belong to the bath or they are associated with correlations of a macroscopic number of particles. The suppression of all the accessible off-diagonal observables relies on the mechanism of § 11.2.3, itself based on the concentration of \hat{m} around $\pm m_{\rm F}$

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This reasoning sheds a new light on the interpretation of *truncation*, which in the Schrödinger picture appeared as the result of an irreversible evolution of the state. In the present Heisenberg picture, truncation comes out as the mere replacement (13.4), which is nothing but an innocuous and convenient elimination of those *parts of the state* \hat{D}^{Heis} *which have become irrelevant*, because the corresponding *observables have disappeared* during the measurement process.

5636 13.1.6. Reduction

The argument given at the end of § 13.1.2 then allows us to assign states to the *subensembles of* \mathcal{E} which can be distinguished at the time $t_{\rm f}$, after the observables have achieved their evolution and after decoupling of S and A. Here, however, the states, which do not depend on time, can be directly constructed from $\hat{\mathcal{D}}^{\rm Heis}$ or from the equivalent expression (13.4). The vanishing of the off-diagonal observables themselves simplifies the discussion since we can always eliminate the off-diagonal blocks of a state associated with any subensemble. The diagonal observables display the same correlations between the system and the pointer as in (13.4), so that any subset of runs of the measurement can be represented by the state

$$\hat{\mathcal{D}}_{\text{sub}}^{\text{Heis}} = \sum_{i} q_i \hat{\mathcal{D}}_i^{\text{Heis}}.$$
(13.5)

which is the basis for all predictions about the considered subensemble at times later than t_f . (Note that the inclusion of elements in the off-diagonal blocks of (13.5) would not change anything since there are no surviving observables in these blocks.)

The uniqueness of the outcome of each individual run comes out from (13.5) through the same argument as in § 11.3.1. A well-defined indication *i* of the pointer is the *additional piece of information* that allows us to assign to the system S + A, which then belongs to the subensemble \mathcal{E}_i , the state $\hat{\mathcal{D}}_i^{\text{Heis}}$. Retaining only one diagonal block of $\hat{\mathcal{D}}^{\text{Heis}}$ in (13.4) amounts to *upgrade our probabilistic description* ¹²¹.

The specific features of the Heisenberg representation were already employed in literature for arguing that this 5651 representation (in contrast to that by Schrödinger) has advantages in explaining the features of quantum measure-5652 ments [334, 335, 336]. In particular, Rubin argued that obstacles preventing a successful application of the Everett 565 interpretation to quantum measurements are absent (or at least weakened) in the Heisenberg representation [335, 336]. 565 Certain aspects of the analysis by Rubin do not depend on the assumed Everett interpretation and overlap with the 5655 presentation below (that does not assume this interpretation). Blanchard, Lugiewicz and Olkiewicz employed the 5656 decoherence physics within the Heisenberg representation for showing that it accounts more naturally (as compared 565 to the Schrödinger representation) for the emergence of classical features in quantum measurements [334]. Their 5658 approach is phenomenological (and shares the criticisms we discussed in section 2.2), but the idea of an emergent 565 Abelian (classical) algebra again overlaps with the preliminary results reported above. The emergent Abelian algebra 5660 is also the main subject of the works by Sewell [137, 138, 139] and Requardt [69] that we already reviewed in section 566 2.4.3. In particular, Requardt explains that closely related ideas were already expressed by von Neumann and van 5662 Kampen (see references in [69]). 5663

As shown by this reconsideration of the Curie–Weiss model, the Heisenberg picture enlightens the truncation, reduction and registration processes, by exhibiting them as a purely dynamical phenomena and by explaining their generality. Although mathematically equivalent to the Schrödinger picture, it suggests more transparent interpretations, owing to a separate description of the dynamics of quantum systems and of our probabilistic knowledge about them. A better insight on other models of measurement should therefore be afforded by their treatment in the Heisenberg picture.

¹²¹In the Schrödinger picture, the expectation value of any (time-independent) observable for the subensemble \mathcal{E}_i was found from the state $\hat{\mathcal{D}}_i$ of Eq. (11.21). Here, it is obtained from the evolution (13.3) and the state $\hat{\mathcal{D}}_i^{\text{Heis}}$ of Eq (13.4). The state $\hat{\mathcal{D}}_i$ results from $\hat{\mathcal{D}}_i^{\text{Heis}}$ by integrating the Liouville–von Neumann equation from t = 0 to t_f

5670 13.2. Other types of measurements

We have only dealt in this article with ideal quantum measurements, in which information about the initial state 567 of the tested system S is displayed by the apparatus at some later time, and in which the final state of the system 5672 S is obtained by projection. Other realistic setups, e.g. of particle detectors or of avalanche processes, deserve 5673 to be studied through models. Measurements of a more elaborate type, in which some quantum property of S is 5674 continuously followed in time, are now being performed owing to experimental progress [288, 289, 337, 338, 339]. For instance, non-destructive (thus non-ideal) repeated observations of photons allow the study of quantum jumps 5676 [338], and quantum-limited measurements, in which a mesoscopic detector accumulates information progressively 5677 [340], are of interest to optimize the efficiency of the processing of q-bits. Quantum measurements are by now 5678 employed for designing feedback control processes [339, 341], a task that in the classical domain is routinely done 5679 via classical measurements. 5680

Such experiments seem to reveal properties of individual systems, in apparent contradiction with the statistical 5681 interpretation of quantum mechanics. However, as in ideal measurements, repeated observations of the above type on 568 identically prepared systems give different results, so that they do not give access to trajectories in the space of the 5683 tested variables, but only to *autocorrelation functions* presenting quantum fluctuations. It seems timely, not only for 5684 conceptual purposes but to help the development of realistic experiments, to work out further models, in particular 5685 for such quantum measurements in which the whole history of the process is used to gather information. In this 5686 context we should mention the so-called weak measurements [342] that (in a sense) minimize the back-action of the 5687 measurement device on the measured system, and – although they have certain counterintuitive features – can reveal 5688 the analogues of classical concepts in quantum mechanics; e.g., state determination with the minimal disturbance, 5689 classical causality [343, 344, 345, 346, 347], and even mapping out of the complete wave function [348] or of the average trajectories of single photons in a double-slit experiment [349]. 5691

Apart from such foreseeable research works, it seems desirable to make educational progress by taking into ac-5692 count the insights provided by the solution of models of quantum measurement processes. The need of quantum 5693 statistical mechanics to explain these processes, stressed all along this paper, and the central role that they play in the 5694 understanding of quantum phenomena, invite us to a reformation of teaching at the introductory level. The statistical 5695 interpretation, as sketched in subsection 10.1, is in keeping with the analysis of measurements. Why not introduce the 5696 concepts and bases of quantum mechanics within its framework. This "minimal" interpretation seems more easily as-5697 similable by students than the traditional approaches. It thus appears desirable to foster the elaboration of new courses 56 and of new textbooks, which should hopefully preserve the forthcoming generations from bewilderment when being 5699 first exposed to quantum physics ... and even later! 5700

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- 5708 Appendices
- 5709 A. Elimination of the bath
- 5710 5711

*De ballast van je afgooien*¹²² Dutch expression

¹²²To throw the ballast from you

Taking $\hat{H}_0 = \hat{H}_S + \hat{H}_{SA} + \hat{H}_M$ and \hat{H}_B as the unperturbed Hamiltonians of S + M and of B, respectively, and denoting by \hat{U}_0 and \hat{U}_B the corresponding evolution operators, we consider the full evolution operator associated with $\hat{H} = \hat{H}_0 + \hat{H}_B + \hat{H}_{MB}$ in the interaction representation. We can expand it as

$$\hat{U}_{0}^{\dagger}(t)\,\hat{U}_{B}^{\dagger}(t)\,e^{-i\hat{H}t/\hbar}\approx\hat{I}-i\hbar^{-1}\int_{0}^{t}\mathrm{d}t'\hat{H}_{\mathrm{MB}}\left(t'\right)+O\left(\gamma\right),\tag{A.1}$$

⁵⁷¹⁵ where the coupling in the interaction picture is

$$\hat{H}_{\rm MB}(t) = \sqrt{\gamma} \sum_{n,a} \hat{U}_0^{\dagger}(t) \,\hat{\sigma}_a^{(n)} \hat{U}_0(t) \,\hat{B}_a^{(n)}(t) \,, \tag{A.2}$$

5716 with $\hat{B}_{a}^{(n)}(t)$ defined by (3.34).

⁵⁷¹⁷ We wish to take the trace over B of the exact equation of motion eq. (4.1) for $\hat{D}(t)$, so as to generate an equation ⁵⁷¹⁸ of motion for the density operator $\hat{D}(t)$ of S + M. In the right-hand side the term tr_B $[\hat{H}_B, \hat{D}]$ vanishes and we are left ⁵⁷¹⁹ with

$$i\hbar \frac{\mathrm{d}D}{\mathrm{d}t} = \left[\hat{H}_0, \hat{D}\right] + \mathrm{tr}_{\mathrm{B}}\left[\hat{H}_{\mathrm{MB}}, \hat{D}\right]. \tag{A.3}$$

The last term involves the coupling \hat{H}_{MB} both directly and through the correlations between S + M and B which are created in $\mathcal{D}(t)$ from the time 0 to the time t. In order to write (A.3) more explicitly, we first exhibit these correlations. To this aim, we expand $\mathcal{D}(t)$ in powers of $\sqrt{\gamma}$ by means of the expansion (A.1) of its evolution operator. This provides, using $\hat{U}_0(t) = \exp[-i\hat{H}_0t/\hbar]$,

$$\hat{U}_{0}^{\dagger}(t)\,\hat{U}_{B}^{\dagger}(t)\,\hat{\mathcal{D}}(t)\,\hat{U}_{B}(t)\,\hat{U}_{0}(t)\approx\hat{\mathcal{D}}(0)-i\hbar^{-1}\left[\int_{0}^{t}\mathrm{d}t'\hat{H}_{\mathrm{MB}}(t'),\hat{D}(0)\,\hat{R}_{B}(0)\right]+O(\gamma)\,.\tag{A.4}$$

Insertion of the expansion (A.4) into (A.3) will allow us to work out the trace over B. Through the factor $\hat{R}_{B}(0)$, this trace has the form of an equilibrium expectation value. As usual, the elimination of the bath variables will produce memory effects as obvious from (A.4). We wish these memory effects to bear only on the bath, so as to have a short characteristic time. However the initial state which enters (A.4) involves not only $\hat{R}_{B}(0)$ but also $\hat{D}(0)$, so that a mere insertion of (A.4) into (A.3) would let $\hat{D}(t)$ keep an undesirable memory of $\hat{D}(0)$. We solve this difficulty by re-expressing perturbatively $\hat{D}(0)$ in terms of $\hat{D}(t)$. To this aim we note that the trace of (A.4) over B provides

$$U_0^{\dagger}(t)\hat{D}(t)\hat{U}_0(t) = \hat{D}(0) + O(\gamma).$$
(A.5)

⁵⁷³⁰ We have used the facts that the expectation value over $\hat{R}_{\rm B}(0)$ of an odd number of operators $\hat{B}_a^{(n)}$ vanishes, and that ⁵⁷³¹ each $\hat{B}_a^{(n)}$ is accompanied in $\hat{H}_{\rm MA}$ by a factor $\sqrt{\gamma}$. Hence the right-hand side of (A.5) as well as that of (A.3) are power ⁵⁷³² series in γ rather than in $\sqrt{\gamma}$.

⁵⁷³³ We can now rewrite the right-hand side of (A.4) in terms of $\hat{D}(t)$ instead of $\hat{D}(0)$ by means of (A.5), then insert ⁵⁷³⁴ the resulting expansion of $\hat{D}(t)$ in powers of $\sqrt{\gamma}$ into (A.3). Noting that the first term in (A.4) does not contribute to ⁵⁷³⁵ the trace over B, we find

$$\frac{d\hat{D}}{dt} - \frac{1}{i\hbar} \left[\hat{H}_{0}, \hat{D} \right] = -\frac{1}{\hbar^{2}} \operatorname{tr}_{B} \int_{0}^{t} dt' \left[\hat{H}_{\mathrm{MB}}, \hat{U}_{B} \hat{U}_{0} \left[\hat{H}_{\mathrm{MB}} \left(t' \right), \hat{U}_{0}^{\dagger} \hat{D} \hat{U}_{0} \hat{R}_{B} \left(0 \right) \right] \hat{U}_{0}^{\dagger} \hat{U}_{B}^{\dagger} \right] + O\left(\gamma^{2}\right), \tag{A.6}$$

where \hat{D} , \hat{U}_{B} and \hat{U}_{0} stand for $\hat{D}(t)$, $\hat{U}_{B}(t)$ and $\hat{U}_{0}(t)$. Although the effect of the bath is of order γ , the derivation has required only the first-order term, in $\sqrt{\gamma}$, of the expansion (A.4) of $\mathcal{D}(t)$.

The bath operators $\hat{B}_{a}^{(n)}$ appear through \hat{H}_{MB} and $\hat{H}_{MB}(t')$, and the evaluation of the trace thus involves only the equilibrium autocorrelation function (3.33). Using the expressions (3.10) and (A.2) for \hat{H}_{MB} and $\hat{H}_{MB}(t')$, denoting the memory time t - t' as u, and introducing the operators $\hat{\sigma}_{a}^{(n)}(u)$ defined by (4.4), we finally find the differential equation (4.5) for $\hat{D}(t)$.

5742 B. Representation of the density operator of S + M by scalar functions

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*Je moet je niet beter voordoen dan je bent*¹²³ Dutch proverb

⁵⁷⁴⁵ We first prove that, if the operators $\hat{R}_{ij}(t)$ in the Hilbert space of M depend only on \hat{m} , the right hand side of (4.8) ⁵⁷⁴⁶ has the same property.

The operators $\hat{\sigma}_{+}^{(n)} = \frac{1}{2} \left(\hat{\sigma}_{x}^{(n)} + i \hat{\sigma}_{y}^{(n)} \right)$ and $\hat{\sigma}_{-}^{(n)} = \left(\hat{\sigma}_{+}^{(n)} \right)^{\dagger}$ raise or lower the value of *m* by $\delta m = 2/N$, a property expressed by

$$[\hat{\sigma}_{+}^{(n)}, \hat{\sigma}_{z}^{(n)}] = -2\hat{\sigma}_{+}^{(n)}, \qquad \hat{\sigma}_{+}^{(n)}\hat{m} = (\hat{m} - \delta m)\,\hat{\sigma}_{+}^{(n)}. \tag{B.1}$$

5749 The last identity can be iterated to yield

$$\hat{\sigma}_{+}^{(n)}\hat{m}^{k} = (\hat{m} - \delta m)\,\hat{\sigma}_{+}^{(n)}\hat{m}^{k-1} = \dots = (\hat{m} - \delta m)^{k}\,\hat{\sigma}_{+}^{(n)},\tag{B.2}$$

so that for every function that can be expanded in powers of \hat{m} , but does not otherwise depend on the $\hat{\sigma}_a^{(k)}$, it holds that

$$\hat{\sigma}_{+}^{(n)} f(\hat{m}) = f(\hat{m} \mp \delta m) \hat{\sigma}_{+}^{(n)}.$$
 (B.3)

In order to write explicitly the time-dependent operators $\hat{\sigma}_{a}^{(n)}(u,i)$ defined by (4.7) with the definition (4.6), it is convenient to introduce the notations

$$m_{\pm} = m \pm \delta m = m \pm \frac{2}{N}, \tag{B.4}$$

$$\Delta_{\pm} f(m) = f(m_{\pm}) - f(m).$$
(B.5)

The time-dependent operators (4.7) are then given by (u = t - t') is the memory time; $i = \uparrow, \downarrow$

$$\hat{\sigma}_{z}^{(n)}(u,i) = \hat{\sigma}_{z}^{(n)},$$
(B.6)

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$$\hat{\sigma}_{+}^{(n)}(u,i) = \frac{1}{2} \left[\hat{\sigma}_{x}^{(n)}(u,i) + i \hat{\sigma}_{y}^{(n)}(u,i) \right] = e^{-i\hat{H}_{i}u/\hbar} \hat{\sigma}_{+}^{(n)} e^{i\hat{H}_{i}u/\hbar} = \hat{\sigma}_{+}^{(n)} e^{-i\hat{\Omega}_{i}^{+}u} = e^{i\hat{\Omega}_{i}^{-}u} \hat{\sigma}_{+}^{(n)} = [\hat{\sigma}_{-}^{(n)}(u,i)]^{\dagger}, \quad (B.7)$$

where we used (B.3) and where the operators $\hat{\Omega}^+_{\uparrow}$, $\hat{\Omega}^-_{\downarrow}$, $\hat{\Omega}^-_{\downarrow}$, $\hat{\Omega}^-_{\downarrow}$ are functions of \hat{m} defined by $\hat{\Omega}^{\pm}_i = \Omega^{\pm}_i(\hat{m})$ and by

$$\hbar\Omega_i^{\pm}(m) = \Delta_{\pm}H_i(m) = H_i(m \pm \delta m) - H_i(m).$$
(B.8)

If in the right-hand side of (4.8) the operator \hat{R}_{ij} depends only on \hat{m} at the considered time, the terms with a = zcancel out on account of (B.6). The terms with a = x and a = y, when expressed by means of (B.7), generate only products of $\hat{\sigma}_{+}^{(n)} \hat{\sigma}_{-}^{(n)}$ or $\hat{\sigma}_{-}^{(n)} \hat{\sigma}_{+}^{(n)}$ by functions of \hat{m} . This can be seen by using (B.3) to bring $\hat{\sigma}_{+}^{(n)}$ and $\hat{\sigma}_{-}^{(n)}$ next to each other through commutation with \hat{R}_{ij} . Since $\hat{\sigma}_{+}^{(n)} \hat{\sigma}_{-}^{(n)} = 1 - \hat{\sigma}_{-}^{(n)} \hat{\sigma}_{+}^{(n)} = \frac{1}{2} \left(1 + \hat{\sigma}_{z}^{(n)}\right)$, we can then perform the summation over *n*, which yields products of some functions of \hat{m} by the factor

$$\sum_{n} \hat{\sigma}_{+}^{(n)} \hat{\sigma}_{-}^{(n)} = N - \sum_{n} \hat{\sigma}_{-}^{(n)} \hat{\sigma}_{+}^{(n)} = \frac{N}{2} \left(1 + \hat{m} \right), \tag{B.9}$$

itself depending only on \hat{m} . Hence, if \hat{R}_{ij} is a function of the operator \hat{m} only, this property also holds for $d\hat{R}_{ij}(t)/dt$ given by (4.8). Since, except in section 5.2, it holds at the initial time, it holds at any time.

The equations of motion (4.8) for $\hat{R}_{ij}(t)$ are therefore equivalent to the corresponding equations for $P_{ij}(m, t)$ which we derive below. The matrices $\hat{R}_{ij}(t)$ which characterize the density operator of S + M are parametrized as $\hat{R}_{ij}(t) =$

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¹²³Don't pretend to be more than you are

 $R_{ij}(\hat{m}) = P_{ij}^{\text{dis}}(\hat{m}, t)/G(\hat{m})$; in the continum limit, we introduced $P_{ij}(m, t) = (N/2)P_{ij}^{\text{dis}}(m, t)$. We first note that the autocorrelation function K(t) enters (4.8) through integrals of the form

$$\tilde{K}_{t>}(\omega) = \int_{0}^{t} du e^{-i\omega u} K(u) = (1/2\pi i) \int_{-\infty}^{+\infty} d\omega' \{ [\exp[i(\omega' - \omega)t] - 1]/(\omega' - \omega) \} \tilde{K}(\omega'),$$

$$\tilde{K}_{t<}(\omega) = \int_{-t}^{0} du e^{-i\omega u} K(u) = \int_{0}^{t} du e^{i\omega u} K(-u) = \left[\tilde{K}_{t>}(\omega) \right]^{*}.$$
(B.10)

As shown above, only the contributions to (4.8) with a = x or a = y survive owing to (B.6). The first term is transformed, by relying successively on (B.7), (B.10), (B.3) and (B.9), into

$$\int_{0}^{t} du \sum_{n} \sum_{a=x,y} K(u) \hat{\sigma}_{a}^{(n)}(u,i) \hat{R}_{ij} \hat{\sigma}_{a}^{(n)} = 2 \int_{0}^{t} du \sum_{n} K(u) \left[e^{i\hat{\Omega}_{i}^{-}u} \hat{\sigma}_{+}^{(n)} R_{ij}(\hat{m}) \hat{\sigma}_{-}^{(n)} + e^{i\hat{\Omega}_{i}^{+}u} \hat{\sigma}_{-}^{(n)} R_{ij}(\hat{m}) \hat{\sigma}_{+}^{(n)} \right]$$

$$= N \tilde{K}_{t>} \left(-\hat{\Omega}_{i}^{-} \right) R_{ij}(\hat{m} - \delta m) \left(1 + \hat{m} \right) + N \tilde{K}_{t>} \left(-\hat{\Omega}_{i}^{+} \right) R_{ij}(\hat{m} + \delta m) \left(1 - \hat{m} \right).$$
(B.11)

From the relation $R_{ij}(m) = P_{ij}^{\text{dis}}(m) / G(m)$ (see Eq. (3.28)), we get

$$(1 \mp m)R_{ij}(m_{\pm}) = (1 \mp m)\frac{P_{ij}^{\text{dis}}(m_{\pm})}{G(m_{\pm})} = \frac{1 \pm m_{\pm}}{G(m)}P_{ij}^{\text{dis}}(m_{\pm}),$$
(B.12)

so that we can readily rewrite (B.11) in terms of $P_{ij}(\hat{m}) = \frac{1}{2}NP_{ij}^{\text{dis}}(\hat{m})$ instead of \hat{R}_{ij} . The same steps allow us to express the other three terms of (4.8) in a similar form. Using also $\Delta_+\Omega_i^- = \Delta_+[H_i(m - \delta m) - H_i(m)] = -\Omega_i^+$ and $\Delta_-\Omega_i^+ = -\Omega_i^-$, where Δ_+ and Δ_- were defined by (B.4) and (B.5), we find altogether, after multiplying by G(m),

$$\frac{\mathrm{d}}{\mathrm{d}t}P_{ij}(m,t) - \frac{1}{i\hbar} \left[H_i(m) - H_j(m) \right] P_{ij}(m,t) = \frac{\gamma N}{\hbar^2} \Delta_+ \left\{ (1+m) \left[\tilde{K}_{t>}(\Omega_i^-) + \tilde{K}_{t<}(\Omega_j^-) \right] P_{ij}(m,t) \right\} + \frac{\gamma N}{\hbar^2} \Delta_- \left\{ (1-m) \left[\tilde{K}_{t>}(\Omega_i^+) + \tilde{K}_{t<}(\Omega_j^+) \right] P_{ij}(m,t) \right\}, \quad (B.13)$$

For i = j this equation simplifies into Eq. (4.16), due to both the cancellation in the left-hand side and the appearance of the combination (4.17) in the right-hand side.

Since it is an instructive exercise for students to numerically solve the full quantum dynamics of the registration process at finite *N*, we write out here the ingredients of the dynamical equation (B.13) for $P_{\uparrow\uparrow}$ and $P_{\downarrow\downarrow}$. As we just indicated above, this equation simplifies for i = j into (4.16). Moreover, in the registration regime, we can replace $\tilde{K}_{t>}(\omega) + \tilde{K}_{t<}(\omega) = \tilde{K}_t(\omega)$ by $\tilde{K}(\omega)$, defined in (3.37). The rates entering Eq. (4.16) or Eq. (B.13) for i = j have therefore the form

$$\frac{\gamma N}{\hbar^2} \tilde{K}(\omega) = \frac{N\hbar\omega}{8J\tau_J} \left[\coth\left(\frac{1}{2}\beta\hbar\omega\right) - 1 \right] \exp\left(-\frac{|\omega|}{\Gamma}\right), \tag{B.14}$$

where the timescale $\tau_J = \hbar/\gamma J$ can be taken as a unit of time. The variable ω in $\tilde{K}(\omega)$ takes the values Ω_i^{\pm} , with $i = j = \uparrow$ or \downarrow , which are explicitly given by (4.14) in terms of the discrete variable *m*. It can be verified that, for $\Gamma \gg J/\hbar$, the omission of the Debye cut-off in (B.14) does not significantly affect the dynamics.

5783 C. Evaluation of the recurrence time for a general pointer

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For what cannot be cured, patience is best Irish proverb

⁵⁷⁸⁶ We consider here general models for which the tested observable \hat{s} is coupled to a pointer through the Hamiltonian ⁵⁷⁸⁷ (6.12) where the pointer observable \hat{m} has Q eigenvalues behaving as independent random variables. The probability ⁵⁷⁸⁸ distribution $p(\omega_q)$ for the corresponding eigenfrequencies $\omega_q \equiv Ng(s_i - s_j)m_q/\hbar$ which enter the function $\Re F(t) =$ $Q^{-1} \sum_{q} \cos \omega_{q} t$ is taken as (6.16). For shorthand we denote from now on in the present appendix by F(t) the real part $\mathcal{R}F$ of the function defined in § 6.1.2 by (6.14).

We wish to evaluate the probability $\mathcal{P}(f, t)$ for F(t) to be larger than some number f at a given time $t \gg \Delta \omega$. This probability is deduced from the characteristic function for F(t) through

$$\mathcal{P}(f,t) = \overline{\theta\left[F(t) - f\right]} = \int_{-\infty}^{+\infty} \frac{d\lambda}{2\pi(i\lambda+0)} e^{-iQ\lambda f} \overline{e^{iQ\lambda F(t)}} = \int_{-\infty}^{+\infty} \frac{d\lambda}{2\pi(i\lambda+0)} \left[e^{-i\lambda f} \int d\omega p(\omega) e^{i\lambda\cos\omega t}\right]^Q.$$
(C.1)

Since $t \gg 1/\Delta\omega$, the factor $p(\omega)$ in the integrand varies slowly over the period $2\pi/t$ of the exponential factor exp $i\lambda \cos \omega t$. This exponential may therefore be replaced by its average on ω over one period, which is the Bessel function $J_0(\lambda)$. The integral over ω then gives unity, and we end up with

$$\mathcal{P}(f,t) = \int_{-\infty}^{+\infty} \frac{\mathrm{d}\lambda}{2\pi \left(i\lambda + 0\right)} \exp\{Q\left[\ln J_0\left(\lambda - i0\right) - i\lambda f\right]\}.$$
(C.2)

For $Q \gg 1$, the exponent has a saddle point λ_s given as function of f by

$$\lambda_{\rm s} \equiv -iy, \qquad \frac{I_1(y)}{I_0(y)} = f, \qquad \frac{{\rm d}f}{{\rm d}y} = 1 - \frac{f}{y} - f^2,$$
 (C.3)

5797 and we find

$$\mathcal{P}(f,t) = \frac{1}{y} \left(2\pi Q \frac{df}{dy} \right)^{-1/2} \exp\left\{ -Q \left[yf - \ln I_0(y) \right] \right\}.$$
(C.4)

⁵⁷⁹⁸ We now evaluate the average duration $\overline{\delta t}$ of an excursion of F(t) above the value f. To this aim, we determine the ⁵⁷⁹⁹ average curvature of F(t) at a peak, reached for values of the set ω_q such that F(t) > f. The quantity

$$\overline{\theta[F(t) - f]} \frac{\mathrm{d}^2 F(t)}{\mathrm{d}t^2} \tag{C.5}$$

is obtained from (C.1) by introducing in the integrand a factor

$$\frac{-\int d\omega p\left(\omega\right)\omega^{2}\cos\omega t \ e^{-i\lambda\cos\omega t}}{\int d\omega p\left(\omega\right)e^{i\lambda\cos\omega t}} = \frac{-i\Delta\omega^{2}J_{1}\left(\lambda\right)}{J_{0}\left(\lambda\right)},$$
(C.6)

where we used $t\Delta\omega \gg 1$. The saddle-point method, using (C.3), then provides on average, under the constraint F(t) > f,

$$\frac{1}{F(t)}\frac{\mathrm{d}^2 F(t)}{\mathrm{d}t^2} = -\Delta\omega^2. \tag{C.7}$$

A similar calculation shows that, around any peak of F(t) emerging above f, the odd derivatives of F(t) vanish on average while the even ones are consistent with the gaussian shape (6.17), rewritten for $f^{-1}F(t')$ in terms of $t' - t < 1/\Delta\omega$. This result shows that the shape of the dominant term of (6.19) is not modified by the constraint F(t) > f. Hence, if F(t) reaches a maximum $f + \delta f$ at some time, the duration of its excursion above f is

$$\delta t = \frac{2}{\Delta \omega} \sqrt{\frac{2\delta f}{f}}.$$
(C.8)

From (C.4) we find the conditional probability density for F(t) to reach $f + \delta f$ if F(t) > f, as $Qye^{-Qy\delta f}$, and hence

$$\overline{\delta t} = \frac{1}{\Delta \omega} \sqrt{\frac{2\pi}{Qyf}}.$$
(C.9)

Since the probability $\mathcal{P}(f,t)$ for a recurrence to occur at the time *t* does not depend on this time, and since the average duration of the excursion is $\overline{\delta t}$, the average delay between recurrences is here

$$\tau_{\rm recur} = \frac{\overline{\delta t}}{\mathcal{P}(f,t)} = \frac{2\pi}{\Delta\omega} \sqrt{\frac{y}{f}} \frac{\mathrm{d}f}{\mathrm{d}y} e^{\mathcal{Q}[yf - \ln I_0(y)]},\tag{C.10}$$

⁵⁸¹⁰ where *y* is given by $I_1(y) = f I_0(y)$.

For *f* sufficiently small so that $\ln I_0(f) \simeq f^2$ (for f = 0.2 the relative error is 1%), we find from (C.3) that $y \simeq 2f$, and this expression of the recurrence time reduces to (6.20), that is exponentially large in Q.

We notice that in this derivation the shape of the eigenvalue spectrum $p(\omega)$ hardly played any role, we only used that it is smooth on the scale $2\pi/t$, where *t* is the observation time. So after times $t \gg 2\pi/\Delta\omega$, where the individual levels are no longer resolved, there will be an exponentially long timescale for the pointer to recur.

D. Effect of the bath on the off-diagonal sectors of the density matrix of S + M

⁵⁸¹⁷ Dopóty dzban wodę nosi, dopóki mu się ucho nie urwie ¹²⁴

5818

5819 D.1. Full expression of $P_{\uparrow\downarrow}$ for large N

In Eq. (6.22) we have parametrized $P_{\uparrow\downarrow}(m, t)$ in terms of the function A(m, t), which satisfies

$$\frac{\partial A}{\partial t} = \frac{2igm}{\hbar} - \frac{1}{NP_{\uparrow\downarrow}} \frac{\partial P_{\uparrow\downarrow}}{\partial t},\tag{D.1}$$

with A(m, 0) = 0. In subsection 4.4, we have derived the equation (4.29) for $P_{\uparrow\downarrow}$, from which A(m, t) can be obtained for large *N* at the two relevant orders (finite and in 1/*N*). As we need A(m, t) only at linear order in γ , we can replace in (4.29) the quantity $X_{\uparrow\downarrow}(m, t)$ by its value for $\gamma = 0$,

$$X \equiv X_{\uparrow\downarrow}(m,t) = \frac{2igt}{\hbar} - \frac{m}{\delta_0^2},$$
 (D.2)

which contains no 1/N term. We then insert (4.29) in (D.1) to obtain

$$\frac{\partial A(m,t)}{\partial t} = \frac{\gamma}{\hbar^2} \left\{ \left(1 - e^{2X}\right)(1+m)\tilde{K}_- + \left(1 - e^{-2X}\right)(1-m)\tilde{K}_+ - \frac{2}{N} \left[\frac{\partial [(1+m)\tilde{K}_- e^X]}{\partial m} e^X - \frac{\partial [(1-m)\tilde{K}_+ e^{-X}]}{\partial m} e^{-X}\right] \right\}, (D.3)$$

where the combinations $\tilde{K}_{\pm}(m, t) = \tilde{K}_{t>}(\Omega_{\uparrow}^{\pm}) + \tilde{K}_{t<}(\Omega_{\downarrow}^{\pm})$ were introduced in (4.19). The functions $\tilde{K}_{t>}(\omega)$ and $\tilde{K}_{t<}(\omega) = \tilde{K}_{t>}^{*}(\omega)$ were defined by (3.36), (3.37), (4.10) and (4.11), and the frequencies Ω_{\uparrow}^{\pm} and $\Omega_{\downarrow}^{\pm}$ by (4.14). The initial condition is A(m, 0) = 0.

5828 D.2. Expansion for small m

The above result holds for arbitrary values of *m* and *t*. However, since in $P_{\uparrow\downarrow}(m, t)$ the values of *m* remain small as $1/\sqrt{N}$, only the first three terms in the expansion

$$A(m,t) \approx B(t) - i\Theta(t)m + \frac{1}{2}D(t)m^2, \qquad (D.4)$$

are relevant. The time-dependence of these three functions, which vanish for t = 0, will be elementary so that we will work out only their time derivatives, which are simpler and which result from (D.3).

5833 We note as Ω the frequency defined by

$$\Omega \equiv \frac{2g}{\hbar} \equiv \frac{\pi}{\tau_{\text{recur}}},\tag{D.5}$$

Polish proverb

¹²⁴A jug carries water only until its handle breaks off

which is related to the period τ_{recur} of the recurrences that arise from the leading oscillatory term $\exp(2iNgmt/\hbar)$ in (6.22) with *m* taking the discrete values (3.22). We can then rewrite, up to the order m^2 and up to corrections in 1/N,

$$\Omega_{\uparrow}^{\pm} \approx \mp \Omega \mp \frac{2J_2m}{\hbar}, \qquad \Omega_{\downarrow}^{\pm} \approx \pm \Omega \mp \frac{2J_2m}{\hbar}, \qquad X = i\Omega t - \frac{m}{\delta_0^2}. \tag{D.6}$$

The expressions (4.10) and (4.11) for $\tilde{K}_{t>}(\omega)$ or $\tilde{K}_{t<}(\omega)$ then provide for their combinations (4.19) the expansion

$$\tilde{K}_{\pm}(m,t) \approx e^{\pm i\Omega t} \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{\pi} \,\tilde{K}\left(\omega \mp \frac{2J_2m}{\hbar}\right) \frac{\omega\sin\omega t - \Omega\sin\Omega t \mp i\Omega(\cos\Omega t - \cos\omega t)}{\omega^2 - \Omega^2} + O\left(\frac{1}{N}\right). \tag{D.7}$$

The required functions B(t), $\Theta(t)$ and D(t) are obtained by inserting (D.4) and (D.7) into (D.3). While the term of order 1/N in B(t) provides a finite factor in $P_{\uparrow\downarrow}(m, t)$, the terms of order 1/N in $\Theta(t)$ and D(t) provide negligible contributions. However that may be, it will be sufficient for our purpose to evaluate only the finite contribution to B(t)and the large *t* approximations for $\Theta(t)$ and D(t).

5841 D.3. The damping term B(t)

To find B(t), we simply set m = 0 in (D.3) and (D.7). Next we employ the expression (3.37) for $\tilde{K}(\omega)$ and take advantage of the symmetry of the integrand with respect to ω , which allows us to keep only the symmetric part of $\tilde{K}(\omega)$. This yields

$$\frac{\mathrm{d}B}{\mathrm{d}t} = \frac{4\gamma\Omega\sin\Omega t}{\hbar^2} \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{\pi} \tilde{K}(\omega) \frac{\cos\Omega t - \cos\omega t}{\omega^2 - \Omega^2} = \frac{\gamma\Omega\sin\Omega t}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}\omega\,\omega\,\coth\frac{\hbar\omega}{2T}\,\exp\left(-\frac{|\omega|}{\Gamma}\right) \frac{\cos\Omega t - \cos\omega t}{\omega^2 - \Omega^2}.$$
(D.8)

where we discarded corrections of order 1/*N*. This entails the result for *B*(*t*) presented in Eq. (6.25) of the main text. For $t \ll 1/\Gamma$, (D.8) reduces to $dB/dt \sim (\gamma \Gamma^2 \Omega^2 / 2\pi) t^3$ and hence

$$B(t) \sim \frac{\gamma \Gamma^2 \Omega^2}{8\pi} t^4 = \frac{\gamma \Gamma^2 g^2}{2\pi \hbar^2} t^4. \tag{D.9}$$

The ω integral in Eq. (6.25) for B(t) can be easily carried out numerically and the result is plotted in Fig 6.1 for typical values of the parameters. It is nevertheless instructive to carry out this integral explicitly. This calculation is hindered by the non-analyticity of our Debye cutoff. However, since the result is not expected to depend significantly on the shape of the cutoff (Γ is the largest frequency of the model), we may replace the exponential cutoff in (3.37) by a quasi Lorentzian cutoff,

$$\exp\left(-\frac{|\omega|}{\Gamma}\right) \mapsto \frac{4\tilde{\Gamma}^4}{4\tilde{\Gamma}^4 + \omega^4}; \qquad \tilde{K}(\omega) \mapsto \frac{\hbar^2 \omega}{4(e^{\beta\hbar\omega} - 1)} \frac{4\tilde{\Gamma}^4}{4\tilde{\Gamma}^4 + \omega^4}, \tag{D.10}$$

where the factors 4 are introduced for later convenience. This expression ensures convergence while being analytic with simple poles. The cutoff (D.10) provides for B(t) the same short time behavior as (D.9) if we make the connection

$$\tilde{\Gamma} = \sqrt{\frac{2}{\pi}} \Gamma. \tag{D.11}$$

In order to integrate the thus modified version of (D.8) over ω , we first split $\cos \omega t$ into $\frac{1}{2} \exp i\omega t + \frac{1}{2} \exp -i\omega t$ and then slightly rotate the integation contour so that ω passes below $+\Omega$ and above $-\Omega$, instead of passing through these poles. For each of the terms we can close the contour either in the upper or lower half-plane, such that it decays for $|\omega| \to \infty$, and pick up the residues at the various poles. The first set of poles, arising from the denominator of (D.8), consist of $\pm \Omega$; since they lie on the real ω -axis, they will produce a non-decaying long time behavior. The second set of poles arise from the coth, as exhibited by the expansion

$$\coth \frac{\hbar\omega}{2T} = \sum_{n=-\infty}^{\infty} \frac{2T}{\hbar(\omega - i\Omega_n)}, \qquad \Omega_n \equiv \frac{2\pi nT}{\hbar}, \tag{D.12}$$

where the sum is meant as principal part for $n \to \pm \infty$; the frequencies Ω_n are known as Matsubara frequencies. Thirdly, the cutoff (D.10) provides the four poles $\pm \tilde{\Gamma} \pm i\tilde{\Gamma}$. We can also take advantage of the symmetry $\omega \to -\omega$, which associates pairwise complex conjugate residues. Altogether, we find

$$\frac{1}{\gamma\Omega}\frac{\mathrm{d}B}{\mathrm{d}t} = \operatorname{coth}\frac{\hbar\Omega}{2T}\frac{\tilde{\Gamma}^4}{4\tilde{\Gamma}^4 + \Omega^4}(1 - \cos 2\Omega t) + \frac{T}{\hbar}\sum_{n=1}^{\infty}\frac{\Omega_n}{\Omega_n^2 + \Omega^2}\frac{4\tilde{\Gamma}^4}{4\tilde{\Gamma}^4 + \Omega_n^4}\left[\sin 2\Omega t - 2\exp(-\Omega_n t)\sin\Omega t\right] \\ + \frac{1}{2}\Im\left\{\operatorname{coth}\frac{(1+i)\hbar\tilde{\Gamma}}{2T}\frac{\tilde{\Gamma}^2}{2\tilde{\Gamma}^2 + i\Omega^2}\left[\sin 2\Omega t - 2\exp[-(1-i)\tilde{\Gamma}t]\sin\Omega t\right]\right\} + O\left(\frac{1}{N}\right).$$
(D.13)

Now *B* is easily obtained by integrating this from 0 to t,

$$B(t) = \frac{\gamma}{2} \coth \frac{g}{T} \frac{\tilde{\Gamma}^4}{4\tilde{\Gamma}^4 + \Omega^4} (2\Omega t - \sin 2\Omega t) \\ + \sum_{n=1}^{\infty} \frac{4\gamma \tilde{\Gamma}^4 \Omega_n T}{\hbar (4\tilde{\Gamma}^4 + \Omega_n^4)} \left[\frac{\sin^2 \Omega t}{\Omega^2 + \Omega_n^2} + 2\Omega \frac{(\Omega \cos \Omega t + \Omega_n \sin \Omega t) \exp(-\Omega_n t) - \Omega}{(\Omega^2 + \Omega_n^2)^2} \right]$$
(D.14)
$$- \frac{\gamma \tilde{\Gamma}^2}{2} \Re \left\{ \coth \frac{(1+i)\hbar \tilde{\Gamma}}{2T} \left[\frac{\sin^2 \Omega t}{\Omega^2 - 2i\tilde{\Gamma}^2} + 2\Omega \frac{(\Omega \cos \Omega t + (1-i)\tilde{\Gamma} \sin \Omega t) \exp[-(1-i)\tilde{\Gamma} t] - \Omega}{(\Omega^2 - 2i\tilde{\Gamma}^2)^2} \right] \right\},$$

where we made the residues at $(\pm 1 \pm i)\tilde{\Gamma}$ look as much as possible like the ones at Ω_n .

With these exact results in hand, let us discuss the relative sizes of the various terms. The above complete formula exhibits some contributions that become exponentially small for sufficiently large *t*. Such contributions are essential to ensure the behavior (D.9) of *B* for $t \ll 1/\tilde{\Gamma}$, and also its behavior for $t \ll \hbar/2\pi T$, but can be neglected otherwise. Moreover, we have $\hbar\tilde{\Gamma} \gg T$ and $\tilde{\Gamma} \gg \Omega$; hence, within exponentially small corrections, the third term of (D.13) reduces, for $t \gg 1/\tilde{\Gamma}$, to $-\Omega^2 \sin(2\Omega t)/8\tilde{\Gamma}^2$ and is therefore negligible compared to the first two terms. In the first term of (D.13), the Debye cutoff is irrelevant, but it is needed in the second term to ensure convergence of the series. Restoring our exponential cutoff, we can write this series as

$$\frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{n}{n^2 + a^2} e^{-bn}, \qquad a \equiv \frac{\hbar\Omega}{2\pi T} \ll 1, \qquad b \equiv \frac{2\pi T}{\hbar\Gamma} \ll 1, \tag{D.15}$$

⁵⁸⁷² which, within corrections of order a^2 , is equal to

$$\frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{1}{n} e^{-bn} = -\frac{1}{2\pi} \ln\left(1 - e^{-b}\right) \sim \frac{1}{2\pi} \ln\frac{h\Gamma}{2\pi T}.$$
(D.16)

Altogether, returning to our original notations through use of (D.5), we find from the first two terms of (D.13), for $t \gg \hbar/2\pi T$:

$$\frac{\tau_{\text{recur}}}{\gamma} \frac{\mathrm{d}B}{\mathrm{d}t} = \frac{\pi}{4} \coth \frac{g}{T} \left(1 - \cos \frac{2\pi t}{\tau_{\text{recur}}} \right) + \frac{1}{2} \ln \frac{\hbar\Gamma}{2\pi T} \sin \frac{2\pi t}{\tau_{\text{recur}}}.$$
(D.17)

5875 Likewise, the function B(t) itself behaves in this region as

$$B(t) = \frac{\gamma\pi}{4} \coth \frac{g}{T} \left(\frac{t}{\tau_{\text{recur}}} - \frac{1}{2\pi} \sin \frac{2\pi t}{\tau_{\text{recur}}} \right) + \frac{\gamma}{4\pi} \ln \frac{\hbar\Gamma}{2\pi T} \left(1 - \cos \frac{2\pi t}{\tau_{\text{recur}}} \right) - \frac{\gamma\zeta(3)}{\pi^3} \frac{g^2}{T^2}, \tag{D.18}$$

⁵⁸⁷⁶ where the last piece arises, in the considered approximation, from the last term of the sum in (D.15).

⁵⁸⁷⁷ D.4. Approximations for $\Theta(t)$ and D(t)

⁵⁸⁷⁸ We have just seen that the dominant contribution to B(t) in the region $t \gg \hbar/2\pi T$ originates from the poles $\omega = \pm \Omega$ ⁵⁸⁷⁹ of the integrand of (D.8). Likewise, as we need only an estimate of $\Theta(t)$ and D(t), we will evaluate approximately ⁵⁸⁸⁰ the integral in (D.7) by picking up only the contributions of these poles. As we did for B(t), we deform and close the ⁵⁸⁸¹ integration contour in the upper or in the lower half-plane, but we now disregard the singularities of $\tilde{K}(\omega \mp 2J_2m/\hbar)$. ⁵⁸⁹² This approximation amounts to make the replacements

$$\frac{\omega \sin \omega t - \Omega \sin \Omega t}{\omega^2 - \Omega^2} \mapsto \frac{\pi}{2} \cos(\Omega t) [\delta(\omega - \Omega) + \delta(\omega + \Omega)], \tag{D.19}$$

$$\frac{\omega^2 - \Omega^2}{\omega^2 - \Omega^2} \xrightarrow{2} \frac{2}{\omega^2 - \Omega^2} \mapsto \frac{\pi}{2} \sin(\Omega t) [\delta(\omega - \Omega) + \delta(\omega + \Omega)], \qquad (D.20)$$

which as we have seen are justified for $t \gg \hbar/2\pi T$. As a result, we find the time-independent expressions for \tilde{K}_{\pm} ,

$$\tilde{K}_{\pm} \approx \frac{1}{2} [\tilde{K}(\Omega \mp 2J_2 m) + \tilde{K}(-\Omega \mp 2J_2 m)].$$
(D.21)

We now return to our original notations by use of (D.5) for Ω and (D.6) for *X*, rewriting the dominant part of (D.3) as

$$\frac{\tau_{\text{recur}}}{\gamma} \frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\pi}{2\hbar g} \left[(1 - e^{2X})(1 + m)\tilde{K}_{-} + (1 - e^{-2X})(1 - m)\tilde{K}_{+} \right]. \tag{D.22}$$

In order to generate $\Theta(t)$ and D(t) through the expansion (D.4) of A(m, t) in powers of *m*, we insert into (D.22) the expansions

$$\left[1 - e^{\pm 2X}\right](1 \pm m) \approx \left[1 - e^{\pm 2i\Omega t}\right] \pm \left[1 + e^{\pm 2i\Omega t}\left(\frac{2}{\delta_0^2} - 1\right)\right]m + 2e^{\pm 2i\Omega t}\left(\frac{1}{\delta_0^2} - \frac{1}{\delta_0^4}\right)m^2,\tag{D.23}$$

$$\frac{4}{\hbar}\tilde{K}_{\pm} \approx g \coth \frac{g}{T} \pm J_2 m - \frac{J_2^2}{T^2 \sinh^2 g/T} \left(\coth \frac{g}{T} - \frac{T}{g} \right) m^2.$$
(D.24)

Gathering, in the resulting expansion of A(m, t), the terms in *m*, we find (for $g \ll T$)

$$\frac{\tau_{\text{recur}}}{\gamma} \frac{d\Theta}{dt} = -\frac{\pi}{4} \left[\left(\frac{2}{\delta_0^2} - 1 \right) \coth \frac{g}{T} + \frac{J_2}{g} \right] \sin \frac{2\pi t}{\tau_{\text{recur}}} \sim -\frac{\pi}{4} \left[\left(\frac{2}{\delta_0^2} - 1 \right) \frac{T}{g} + \frac{J_2}{g} \right] \sin \frac{2\pi t}{\tau_{\text{recur}}}, \tag{D.25}$$

5889 which is integrated as

$$\Theta(t) \sim -\frac{\gamma}{8g} \left[\left(\frac{2}{\delta_0^2} - 1 \right) T + J_2 \right] \left[1 - \cos \frac{2\pi t}{\tau_{\text{recur}}} \right].$$
(D.26)

Likewise, the terms in m^2 yield

$$\frac{\tau_{\text{recur}}}{\gamma} \frac{dD}{dt} \sim \frac{\pi}{2} \left[\frac{J_2^2}{T^2 \sinh^2 g/T} \left(\coth \frac{g}{T} - \frac{T}{g} \right) - \frac{J_2}{g} \right] \left(1 - \cos \frac{2\pi t}{\tau_{\text{recur}}} \right) + \frac{\pi}{2} \left[2 \coth \frac{g}{T} \left(\frac{1}{\delta_0^2} - \frac{1}{\delta_0^4} \right) - \frac{2J_2}{g\delta_0^2} \right] \cos \frac{2\pi t}{\tau_{\text{recur}}}.$$
(D.27)

5891 The first bracket simplifies for $g \ll T$ into

$$\frac{J_2^2}{T^2 \sinh^2 g/T} \left(\coth \frac{g}{T} - \frac{T}{g} \right) - \frac{J_2}{g} \sim \frac{J_2}{g} \left(\frac{J_2}{3T} - 1 \right).$$
(D.28)

We shall only need the values of D(t) at the recurrence times $p\tau_{\text{recur}}$. Integration of the factors $\cos 2\pi t/\tau_{\text{recur}}$ generates $\sin 2\pi t/\tau_{\text{recur}}$, which vanishes at these times. We have therefore the compact result

$$D(p\tau_{\text{recur}}) \simeq p \times D(\tau_{\text{recur}}) = p \, \frac{\pi \gamma}{2} \frac{J_2}{g} \left(\frac{J_2}{3T} - 1\right). \tag{D.29}$$

5894 E. Time dependence of the registration process

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 Time heals all wounds

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 Proverb

The location $\mu(t)$ of the peak of the distribution P(m, t) increases in time according to (7.30) where $\phi(m)$ is defined by (7.25). We wish in § 7.2.3 and § 7.2.4 to obtain an algebraic approximation for $\mu(t)$ at all times. To this aim, we will represent $1/\nu(\mu)$ by its Mittag-Leffler expansion

$$\frac{\gamma T}{\hbar v(m)} \equiv \frac{1}{\phi(m)[1 - m \coth \phi(m)]} = \sum_{i} \frac{m_i}{[(1 - m_i^2)(\mathrm{d}\phi/\mathrm{d}m_i) - 1]\phi(m_i)} \frac{1}{m - m_i},\tag{E.1}$$

which sums over all real or complex values $m = m_i$ where v(m) = 0.

5901 E.1. Registration for second-order transition of M

*Nou zeg ik het voor de tweede keer*¹²⁵ Dutch expression

For q = 2, it is sufficient for our purpose to keep in the expansion (E.1) only the real poles m_i . This truncation does not affect the vicinity of the (stable or unstable) fixed points where the motion of $\mu(t)$ is slowest, and provides elsewhere a good interpolation provided T/J is not too small. Three values m_i occur here, namely $-m_B$, $m_{\uparrow} \simeq m_F$ and $m_{\downarrow} \simeq -m_F$, with $m_B \ll m_F$, so that we find over the whole range $0 < \mu < m_F$, through explicit integration of (7.30),

$$\frac{t}{\tau_{\rm reg}} = \ln \frac{m_{\rm B} + \mu}{m_{\rm B}} + a \ln \frac{m_{\rm F}^2}{m_{\rm F}^2 - \mu^2},\tag{E.2}$$

⁵⁹⁰⁸ where the coefficient a, given by

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$$a = \frac{T(J-T)}{J[T - J(1 - m_{\rm E}^2)]},\tag{E.3}$$

decreases with temperature from a = 1 at T = 0 to $a = \frac{1}{2}$ for T = J. For short times, such that $\mu \ll m_F$, we recover from the first term of (E.2) the evolution (7.43) of $\mu(t)$. When μ approaches m_F , the second term dominates, but as long as $m_F - m$ is of order m_B the time needed for μ to reach m is of order $\tau_{reg} \ln(m_F/m_B)$. We define the cross-over by

 $r g \ll T$ into

¹²⁵Now I say it for the second time

writing that the two logarithms of (E.2) are equal, which yields $\mu = m_{\rm F} - \frac{1}{2}m_{\rm B}$. The time $\tau'_{\rm reg}$ during which $\mu(t)$ goes from 0 to $m_{\rm F} - \frac{1}{2}m_{\rm B}$, termed the second characteristic registration time, is then given by (7.48), that is,

$$\tau'_{\rm reg} = \tau_{\rm reg}(1+a)\ln\frac{m_{\rm F}}{m_{\rm B}}.$$
(E.4)

⁵⁹¹⁴ When μ approaches $m_{\rm F}$ in the regime $m_{\rm F} - \mu \ll m_{\rm B}$, we can invert (E.2) as

$$\mu(t) = m_{\rm F} \left[1 - \frac{1}{2} \left(\frac{m_{\rm F}}{m_{\rm B}} \right)^{1/a} \exp\left(-\frac{t}{a \tau_{\rm reg}} \right) \right],\tag{E.5}$$

which exhibits the final exponential relaxation. We can also invert this relation in the limiting cases $T \to J$ and $T \to 0$. If *T* lies close to the transition temperature, we have $m_{\rm F} \sim \sqrt{3(J-T)/J}$ and $a = \frac{1}{2}$. Provided the coupling is weak so that $m_{\rm B} = g/(J-T) \ll m_{\rm F}$, we find

$$\mu(t) = \frac{m_{\rm B}m_{\rm F}}{m_{\rm B}^2 + m_{\rm F}^2 e^{-2t/\tau_{\rm reg}}} \left[\sqrt{m_{\rm B}^2 + (m_{\rm F}^2 - m_{\rm B}^2)e^{-2t/\tau_{\rm reg}}} - m_{\rm B}m_{\rm F}e^{-2t/\tau_{\rm reg}} \right].$$
(E.6)

This expression encompasses all three regimes of § 7.2.3, namely, $\mu \sim m_{\rm B} t / \tau_{\rm reg}$ for $t \ll \tau_{\rm reg}$, μ running from $m_{\rm B}$ to $m_{\rm F}$ for *t* between $\tau_{\rm reg}$ and $\tau'_{\rm reg}$, and

$$\mu(t) \approx m_{\rm F} \left(1 - \frac{m_{\rm F}^2}{2m_{\rm B}^2} e^{-2t/\tau_{\rm reg}} \right)$$
(E.7)

for $t - \tau'_{reg} \gg \tau_{reg}$. In the low temperature regime ($T \ll J$, with $m_B \sim g/J$ and $a \sim 1$), we can again invert (E.2) as

$$\mu(t) = \frac{1}{2m_{\rm B}} \left[\sqrt{4m_{\rm B}^2 \left(m_{\rm F}^2 - m_{\rm B}^2\right) + \left(2m_{\rm B}^2 - m_{\rm F}^2 e^{-t/\tau_{\rm reg}}\right)^2} - m_{\rm F}^2 e^{-t/\tau_{\rm reg}} \right],\tag{E.8}$$

encompassing the same three regimes; for $t - \tau'_{reg} \gg \tau_{reg}$, we now have

$$\mu(t) \approx m_F \left(1 - \frac{m_F}{2m_B} e^{-t/\tau_{\rm reg}} \right). \tag{E.9}$$

⁵⁹²² E.2. Registration for first-order transition of M

For $J_4 \neq 0$, such as the q = 4 case with $J_2 = 0$ and $J_4 = J$, we need to account for the presence of the minimum of v(m) at $m = m_c$. To this aim, we still truncate the Mittag-Leffler expansion (E.1) of 1/v(m). However, we now retain not only the real poles but also the two complex poles near m_c which govern the minimum of v(m). These poles are located at

$$m_{\rm c} \pm i\delta m_{\rm c}, \qquad \delta m_{\rm c}^2 = \frac{m_{\rm c}(1-m_{\rm c}^2)^2}{1+2m_{\rm c}^2}\frac{g-h_{\rm c}}{T} \sim m_{\rm c}\left(\frac{g}{T}-\frac{2m_{\rm c}}{3}\right).$$
 (E.10)

The real pole associated with the repulsive fixed point lies at $-m_{\rm B} \sim -2m_{\rm c}$, and the ferromagnetic poles lie close to $\pm m_{\rm F} \sim \pm 1$. We have thus, at lowest order in $T/J \simeq 3m_{\rm c}^2$ and in $g/T \sim 2m_{\rm c}/3$, but with T/J sufficiently large so that we can drop the other complex poles,

$$\frac{\gamma T}{\hbar \nu(m)} = \frac{m_{\rm c} - \frac{1}{2}(m - m_{\rm c})}{(m - m_{\rm c})^2 + \delta m_{\rm c}^2} + \frac{1}{3(m + 2m_{\rm c})} + \frac{2Tm}{J(1 - m^2)}.$$
(E.11)

Hence the time-dependence of the peak $\mu(t)$ of $P_{\uparrow\uparrow}(m, t)$ is given through integration of (7.30) as

$$\frac{t}{\tau_{\rm reg}} = \frac{1}{\pi} \left(\frac{\pi}{2} + \arctan \frac{\mu - m_{\rm c}}{\delta m_{\rm c}} \right) + \frac{\delta m_{\rm c}}{\pi m_{\rm c}} \left[\frac{1}{4} \ln \frac{m_{\rm c}^2}{(\mu - m_{\rm c})^2 + \delta m_{\rm c}^2} + \frac{1}{3} \ln \frac{\mu + 2m_{\rm c}}{2m_{\rm c}} + \frac{T}{J} \ln \frac{1}{1 - \mu^2} \right], \tag{E.12}$$

⁵⁹³¹ where we introduced the registration time

$$\tau_{\rm reg} \equiv \frac{\pi \hbar m_{\rm c}}{\gamma T \delta m_{\rm c}} = \frac{\pi \hbar}{\gamma T} \sqrt{\frac{m_{\rm c} T}{g - h_{\rm c}}},\tag{E.13}$$

5932 with $m_c = \sqrt{T/3J} = 3h_c/2T$.

The initial evolution (7.50) is recovered from (E.12) for $\mu \ll m_c$ and $t \ll \hbar/\gamma T$. It matches the bottleneck stage in which $\mu(t)$ varies slowly around the value m_c on the time scale τ_{reg} . Then, the right-hand side of (E.12) is dominated by its first term, so that the magnetization increases from $m_c - \delta m_c$ to $m_c + \delta m_c$ between the times $t = \tau_{reg}/4$ and $t = 3\tau_{reg}/4$, according to:

$$\mu(t) = m_{\rm c} - \delta m_{\rm c} \cot an \, \frac{\pi t}{\tau_{\rm reg}}.\tag{E.14}$$

⁵⁹³⁷ After μ passed the bottleneck, for $\mu - m_c \gg \delta m_c$, (E.12) provides

$$t = \tau_{\rm reg} + \tau_1 \left(-\frac{m_{\rm c}}{\mu - m_{\rm c}} + \frac{1}{2} \ln \frac{m_{\rm c}}{\mu - m_{\rm c}} + \frac{1}{3} \ln \frac{\mu + 2m_{\rm c}}{2m_{\rm c}} + \frac{T}{J} \ln \frac{1}{1 - \mu^2} \right), \tag{E.15}$$

which is nearly equal to τ_{reg} within corrections of order $\tau_1 = \hbar/\gamma T$, as long as μ is not very close to 1. The final exponential relaxation takes place on the still shorter scale $\hbar/\gamma J$.

5940 **F. Effects of bifurcations**

In subsection 7.3 we consider situations in which Suzuki's slowing down is present, namely the preparation of the initial metastable state for q = 2 and the possibility of false registrations. We gather here some derivations.

The Green's function G(m, m', t - t') associated to the equation (7.1) for $P_{\rm M}(m, t)$ will be obtained from the backward equation

$$\frac{\partial}{\partial t'}G(m,m',t-t') + v(m')\frac{\partial}{\partial m'}G(m,m',t-t') + \frac{1}{N}[w(m')\frac{\partial^2}{\partial m'^2}G(m,m',t-t')] = -\delta(m-m')\delta(t-t'), \quad (F.1)$$

where *t'* runs down from t + 0 to 0. Introducing the time scale τ_{reg} defined by (7.44) and using the expression (7.42) for v(m') for small *m'* together with the related $w(m') \approx \gamma gt/\hbar$, we have to solve the equation

$$\left[\tau_{\rm reg}\frac{\partial}{\partial t'} + (m_{\rm B} + m')\frac{\partial}{\partial m'} + \frac{1}{N}\frac{T}{J - T}\frac{\partial^2}{\partial m'^2}\right]G(m, m', t - t') = 0, \tag{F.2}$$

with the boundary condition $G(m, m', 0) = \delta(m - m')$. Its solution in terms of m' has the Gaussian form

$$G(m,m',t) = A(m,t) \sqrt{\frac{N}{2\pi D(m,t)}} \exp\left\{-\frac{N[m'-\mu'(m,t)]^2}{2D(m,t)}\right\},$$
(F.3)

where the coefficients μ' , *D* and *A* should be found by insertion into (F.2).

As in § 7.2.3, the evolution of $P_{\rm M}(m, t)$ takes place in three stages: (i) *widening* of the initial distribution, which here takes place over the bifurcation $-m_{\rm B}$; (ii) *drift* on both sides of $-m_{\rm B}$ towards $+m_{\rm F}$ and $-m_{\rm F}$; (iii) narrowing around $+m_{\rm F}$ and $-m_{\rm F}$ of the two final peaks, which evolve *separately towards equilibrium*. We are interested here only in the first two stages. During the first stage, the relevant values of *m* lie in the region where the approximation (7.59) holds. The functions of *m* and *t*: μ' , *D* and *A*, satisfy according to (F.2) the equations

$$\tau_{\rm reg}\frac{\partial\mu'}{\partial t} = -m_{\rm B} - \mu', \qquad \frac{1}{2}\tau_{\rm reg}\frac{\partial D}{\partial t} = \frac{T}{J-T} - D, \qquad \tau_{\rm reg}\frac{\partial A}{\partial t} = -A, \tag{F.4}$$

¹²⁶Whether bitten by the dog or the cat, the result is equal

and the boundary condition $G(m, m', 0) = \delta(m - m')$ for t' = t - 0 yields

$$\mu' = -m_{\rm B} + (m + m_{\rm B})e^{-t/\tau_{\rm reg}}, \qquad D = \frac{T}{J - T}(1 - e^{-t/\tau_{\rm reg}}), \qquad A = e^{-t/\tau_{\rm reg}}.$$
 (F.5)

5957 As function of m, the probability

$$P_{\rm M}(m,t) = \int {\rm d}m' G(m,m',t) P_{\rm M}(m',0) \tag{F.6}$$

given by (F.3), (F.5) involves fluctuations which increase exponentially as $\exp(t/\tau_{reg})$.

In the second stage, the time is sufficiently large so that $P_{M}(m, t)$ extends over regions of *m* where the linear approximation (7.59) for v(m) fails; we must account for the decrease of |v(m)|, which vanishes at $m = \pm m_F$. We therefore cannot comply directly with the boundary condition for G(m, m', t - t') at t' = t since it requires *m'* to be large as *m*. However, during this second stage $P_{M}(m, t)$ is not peaked, so that diffusion is negligible compared to drift. The corresponding Green's function, with its two times *t* and *t'* taken during this stage, is given according to (7.32) by

$$G(m,m',t-t') = \frac{1}{\nu(m)} \delta\left(t-t' - \int_{m'}^{m} \frac{\mathrm{d}m''}{\nu(m'')}\right).$$
(F.7)

We can now match the final time of (F.3), (F.5) with the initial time of (F.7), using the convolution law for Green's functions. This yields an approximation for G(m, m', t) valid up to the final equilibration stage. We therefore define the function $\mu'(m, t)$ by the equation

$$t = \int_{\mu'(m,t)}^{m} \frac{\mathrm{d}m''}{v(m'')},\tag{F.8}$$

of which (F.5) is the approximation for small *m* and μ' . For $m > -m_B$, we have $m > \mu' > -m_B$ and v(m'') > 0; for $m < -m_B$ we have $m < \mu' < -m_B$ and v(m'') < 0. We also note that the convolution replaces $A = e^{-t/\tau_{reg}}$ by

$$A(m,t) = \frac{v[\mu'(m,t)]}{v(m)} = \frac{\partial \mu'(m,t)}{\partial m}.$$
(F.9)

Altogether the Green's function (F.3) reads

$$G(m,m',t) = \frac{v(\mu')}{v(m)} \sqrt{\frac{N(J-T)}{2\pi T (1-e^{-2t/\tau_{\rm reg}})}} \exp\left[-\frac{N(J-T)(m'-\mu')^2}{2T (1-e^{-2t/\tau_{\rm reg}})}\right],\tag{F.10}$$

where $\mu' = \mu'(m, t)$ is found through (F.8). The resulting distribution function $P_{\rm M}(m, t)$, obtained from (F.6), (F.10) and $P_{\rm M}(m, 0) \propto \exp[-N(m - \mu_0)^2/2\delta_0^2]$, is expressed by (F.10) or, in the main text, by (7.61) with (7.63) for $\delta_1(t)$. Notice that here we allowed for a finite value μ_0 of the average magnetization in the initial state.

⁵⁹⁷³ We have studied in § 7.3.2 the evolution of $P_{\rm M}(m, t)$ for g = 0 and for an unbiased initial state. For $m_{\rm B} = g/(J - T) \neq 0$ and a non-vanishing expectation value of μ_0 of m in the initial state, the dynamics of $P_{\rm M}(m, t)$ is ⁵⁹⁷⁵ explicitly found from (F.10) by noting that $m_{\rm B} \ll m_{\rm F}$; the expression (E.1) for v(m) thus reduces to

$$\frac{1}{\tau_{\rm reg}\nu(m)} = \frac{1}{m+m_{\rm B}} + \frac{2am}{m_{\rm F}^2 - m^2},\tag{F.11}$$

with $\tau_{\text{reg}} = \hbar/\gamma(J - T)$ and *a* defined by (E.3). Hence, the relation (F.8) between μ' , *m* and *t* reads

$$\frac{t}{\tau_{\rm reg}} = \ln \frac{m + m_{\rm B}}{\mu' + m_{\rm B}} + a \ln \frac{m_{\rm F}^2 - {\mu'}^2}{m_{\rm E}^2 - m^2}.$$
(F.12)

For large *N*, the quantities μ' , m_0 and m_B are small as $1/\sqrt{N}$, except at the very large times when $P_M(m, t)$ is concentrated near $+m_F$ and $-m_F$. We can thus write (7.60) as

$$P_{\rm M}(m,t) = \frac{1}{\sqrt{\pi}} \frac{\partial \xi}{\partial m} e^{-(\xi - \xi_0)^2},\tag{F.13}$$

⁵⁹⁷⁹ where we introduced the functions

$$\xi(m,t) = \sqrt{3a} \frac{m + m_{\rm B}}{m_{\rm F}} \left(\frac{m_{\rm F}^2}{m_{\rm F}^2 - m^2}\right)^a \frac{\delta_1}{\delta_1(t)} e^{-(t - \tau_{\rm flat})/\tau_{\rm reg}},\tag{F.14}$$

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$$\xi_0(t) \equiv \sqrt{\frac{N}{2} \frac{m_{\rm B} + \mu_0}{\delta_1(t)}}.$$
(F.15)

The characteristic time τ_{flat} is the same as (7.69), it is large as $\frac{1}{2} \ln N$. The function $\delta_1(t)$ and the parameter δ_1 are defined in (7.63)

The expression (F.13) encompasses (7.64), (7.70), (7.74) and (7.79), which were established in the special case where the distribution is symmetric ($m_{\rm B} = \mu_0 = 0$) and/or when *m* is small as $1/\sqrt{N}$. For $t \gg \tau_{\rm reg}$ we reach Suzuki's scaling regime characterized by the scaling parameter (F.14), in which $\delta_1(t)$ reduces to the constant δ_1 and in which $m_{\rm B}$ can be disregarded. The asymmetry of $P_{\rm M}(m, t)$ then arises only from the constant ξ_0 . Even in the presence of this asymetry, the time $t = \tau_{\rm flat}$ still corresponds to a flat $P_{\rm M}(m, t)$, in the sense that the curvature of $P_{\rm M}(m, \tau_{\rm flat})$ at m = 0vanishes.

5989 G. Density operators for beginners

5990 5991 Begin at the beginning and go on till you come to the end: then stop Lewis Carroll, Alice's Adventures in Wonderland

In elementary courses of quantum mechanics, a state is usually represented by a vector $|\psi\rangle$ in Hilbert space (or a 5992 ket, or a wave function). Such a definition is too restrictive. On the one hand, as was stressed by Landau [84, 353], 5993 if the considered system is not isolated and presents quantum correlations with another system, its properties cannot 599 be described by means of a state vector. On the other hand, as was stressed by von Neumann [4], an incomplete 5995 preparation does not allow us to assign a unique state vector to the system; various state vectors are possible, with some 5996 probabilities, and the formalism of quantum statistical mechanics is needed. Both of these circumstances occur in a 5997 measurement process: The tested system is correlated to the apparatus, and the apparatus is macroscopic. The opinion, 5998 too often put forward, that the (mixed) post-measurement state cannot be derived from the Schrödinger equation, 5999 originates from the will to work in the restricted context of pure states. This is why we should consider, to understand 6000 quantum measurement processes, the realistic case of a mixed initial state for the apparatus, and subsequently study 600 the time-dependent mixed state for the tested system and the apparatus. 6002

The more general formulation of quantum mechanics that is needed requires the use of density operators, and is presented in section 10 in the context of the statistical interpretation of quantum mechanics. We introduce here, for teaching purposes, an elementary introduction to § 10.1.4. In quantum (statistical) mechanics, a state is represented by a *density operator* \hat{D} or, in a basis $|i\rangle$ of the Hilbert space, by a *density matrix* $\langle i|\hat{D}|j\rangle$. The *expectation value* in this state of an observable \hat{O} (itself represented on the basis $|i\rangle$ by the matrix $\langle i|\hat{O}|j\rangle$) is equal to

$$\langle \hat{O} \rangle = \operatorname{tr} \hat{\mathcal{D}} \hat{O} = \sum_{ij} \langle i | \hat{\mathcal{D}} | j \rangle \langle j | \hat{O} | i \rangle.$$
(G.1)

This concept encompasses as a special case that of state vector, as the expectation value of \hat{O} in the state $|\psi\rangle$,

$$\langle \hat{O} \rangle = \langle \psi | \hat{O} | \psi \rangle = \sum_{ij} \langle \psi | j \rangle \langle j | \hat{O} | i \rangle \langle i | \psi \rangle, \tag{G.2}$$

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is implemented by associating with $|\psi\rangle$ the density operator $\hat{D} = |\psi\rangle\langle\psi|$ or the density matrix $\langle i|\hat{D}|j\rangle = \langle i|\psi\rangle\langle\psi|j\rangle$, referred to as a "pure state" in this context.

⁶⁰¹¹ Density operators have several characteristic properties. (i) They are *Hermitean*, $\hat{D} = \hat{D}^{\dagger}$, (i. e., $\langle j|\hat{D}|i\rangle =$ ⁶⁰¹² $\langle i|\hat{D}|j\rangle^*$), implying that the expectation value (G1) of a Hermitean observable is real. (ii) They are *normalized*, ⁶⁰¹³ tr $\hat{D} = 1$, meaning that the expectation value of the unit operator is 1. (iii) They are *non-negative*, $\langle \phi | \hat{D} | \phi \rangle \ge 0 \forall | \phi \rangle$, ⁶⁰¹⁴ meaning that the variance $\langle \hat{O}^2 \rangle - \langle \hat{O} \rangle^2$ of any Hermitean observable \hat{O} is non-negative. A density operator can be ⁶⁰¹⁵ diagonalized; its eigenvalues are real, non negative, and sum up to 1. For a pure state $\hat{D} = |\psi\rangle\langle\psi|$, all eigenvalues ⁶⁰¹⁶ vanish but one, equal to 1.

In the Schrödinger picture, the evolution of the time-dependent density operator $\hat{D}(t)$ is governed by the Hamiltonian *H* of the system if it is isolated. The *Liouville–von Neumann equation of motion*,

$$i\hbar \frac{\mathrm{d}\hat{\mathcal{D}}(t)}{\mathrm{d}t} = [\hat{H}, \hat{\mathcal{D}}(t)],\tag{G.3}$$

generalizes the Schrödinger equation $i\hbar d|\psi\rangle/dt = \hat{H}|\psi\rangle$, or, in the position basis, $i\hbar d\psi(x)/dt = \hat{H}\psi(x)$, which governs the motion of pure states. The evolution of $\hat{D}(t)$ is unitary; it conserves its eigenvalues.

⁶⁰²¹ In quantum statistical mechanics, the *von Neumann entropy*

$$S(\hat{\mathcal{D}}) = -\mathrm{tr}\hat{\mathcal{D}}\ln\hat{\mathcal{D}} \tag{G.4}$$

is associated with \hat{D} . It characterizes the amount of information about the system that is missing when it is described by \hat{D} , the origin of values of *S* being chosen as S = 0 for pure states. If $S(\hat{D}) \neq 0$, \hat{D} can be decomposed in an infinite number of ways into a sum of projections onto pure states (§ 10.2.3).

The concept of density operator allows us to define the *state of a subsystem*, which is not feasible in the context of state vectors or pure states. Consider a compound system $S_1 + S_2$, described in the Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$ by a density operator $\hat{\mathcal{D}}$. This state is represented, in the basis $|i_1, i_2\rangle$ of $\mathcal{H}_1 \otimes \mathcal{H}_2$, by the density matrix $\langle i_1, i_2 | \hat{\mathcal{D}} | j_1, j_2 \rangle$. Suppose we wish to describe the subsystem S_1 alone, that is, to evaluate the expectation values of the observables O_1 pertaining only to the Hilbert space \mathcal{H}_1 and thus represented by matrices $\langle i_1 | O_1 | j_1 \rangle$ in \mathcal{H}_1 , or $\langle i_1 | O_1 | j_1 \rangle \delta_{i_2,j_2}$ in $\mathcal{H}_1 \otimes \mathcal{H}_2$. These expectation values are given by

$$\langle \hat{O}_1 \rangle = tr_1 \hat{\mathcal{D}}_1 \hat{O}_1 = \sum_{i_1, j_1} \langle i_1 | \hat{\mathcal{D}}_1 | j_1 \rangle \langle j_1 | \hat{O}_1 | i_1 \rangle, \tag{G.5}$$

where the matrix $\langle i_1 | \hat{\mathcal{D}}_1 | j_1 \rangle$ in the Hilbert space \mathcal{H}_1 is defined by

$$\langle i_1 | \hat{\mathcal{D}}_1 | j_1 \rangle = \sum_{i_2} \langle i_1, i_2 | \hat{\mathcal{D}} | j_1, i_2 \rangle.$$
(G.6)

The partial trace $\hat{D}_1 = \text{tr}_2 \hat{D}$ on the space \mathcal{H}_2 is therefore, according to (G1), the density operator of the subsystem S₁. If the subsystems S₁ and S₂ interact, the evolution of \hat{D}_1 should in principle be determined by solving (G3) for the the density operator \hat{D} of the compound system, then by taking the partial trace at the final time. The elimination of the bath (subsection 4.1) followed this procedure. The evolution of a subsystem is in general not unitary, because it is not an isolated system.

The formalism of density operators is more flexible than that of pure states: It affords the possibility not only of changing the basis in the Hilbert space, but also of performing linear transformations in the vector space of observables, which mix the left and right indices of observables $\langle i|\hat{O}|j\rangle$ and of density matrices $\langle i|\hat{D}|j\rangle$. The resulting *Liouville representations of quantum mechanics* [75, 282, 283] are useful in many circumstances. They include for instance the *Wigner representation*, suited to study the semi-classical limit, and the *polarization representation* for a spin, currently used by experimentalists, in which any operator is represented by its coordinates on the basis (3.1) of the space of operators; in the present work, the parametrization of the state \hat{D} of S + M by $P_{\rm M}^{\rm dis}(m)$ and $C_a^{\rm dis}(m)$ enters this framework (Eqs. (3.18), (3.26), (3.28), (3.29)).

H. Evolution generated by random matrices from the factorized ensemble 6045

For they have sown the wind, and they shall reap the whirlwind Hosea 8.7

The purpose of this Appendix is to work out Eq. (11.14) of the main text, where the average is taken over an 6048 ensemble of random Hamiltonians with the eigenvector distribution factorized from the eigenvalue distribution. The 6049 eigenvectors are then distributed with the uniform (Haar) measure, while we are free to choose the eigenvalue distri-6050 bution (e.g. from some plausible physical arguments). The case where the random matrix elements are Gaussian and 6051 distributed identically belongs to this class [233]. For simplicity we shall deal here with the microcanonical relaxation 6052 of one set of states. The extension to two sets (the case discussed in the main text) is straightforward. 605

We thus need to determine the average evolution [inside this Appendix we take $\hbar = 1$] 6054

$$\overline{\hat{U}_{\uparrow\uparrow}\hat{\rho}\hat{U}_{\uparrow\uparrow}^{\dagger}} = \overline{e^{-it\hat{V}_{\uparrow\uparrow}}\hat{\rho}\,e^{it\hat{V}_{\uparrow\uparrow}}},\tag{H.1}$$

where \hat{V}_{\uparrow} is a random matrix generated according to the above ensemble, and where $\hat{\rho}$ is an initial density matrix; see 6055 Eq. (11.14) of the main text in this context. To calculate (H.1) we introduce 6056

$$\hat{U}_{\uparrow}\hat{\rho}\hat{U}_{\uparrow}^{\dagger} = \sum_{\alpha=1}^{G} \langle\psi_{\alpha}|\hat{\rho}|\psi_{\alpha}\rangle|\psi_{\alpha}\rangle\langle\psi_{\alpha}| + \sum_{\alpha\neq\beta}^{G} \langle\psi_{\alpha}|\hat{\rho}|\psi_{\beta}\rangle|\psi_{\alpha}\rangle\langle\psi_{\beta}| \ e^{it(E_{\beta}-E_{\alpha})}, \tag{H.2}$$

where 6057

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$$\hat{U}_{\uparrow}(t) = \sum_{\alpha=1}^{G} e^{-itE_{\alpha}} |\psi_{\alpha}\rangle \langle\psi_{\alpha}|$$
(H.3)

is the eigenresolution of $\hat{U}_{\uparrow}(t)$. 6058

We now average (H.2) over $|\psi_{\alpha}\rangle$ assuming that they are distributed uniformly (respecting the constraints of ortog-6059 onality and normalization). This averaging will be denoted by an overline, 6060

$$\overline{\hat{U}_{\uparrow\uparrow}\hat{\rho}\hat{U}_{\uparrow\uparrow}^{\dagger}} = G\overline{\langle\psi_1|\hat{\rho}|\psi_1\rangle|\psi_1\rangle\langle\psi_1|} + \overline{\langle\psi_1|\hat{\rho}|\psi_2\rangle|\psi_1\rangle\langle\psi_2|}\sum_{\alpha\neq\beta}^G e^{it(E_{\beta}-E_{\alpha})}.$$
(H.4)

It suffices to calculate $\langle \psi_1 | \hat{\rho} | \psi_1 \rangle | \psi_1 \rangle \langle \psi_1 |$, since $\langle \psi_1 | \hat{\rho} | \psi_2 \rangle | \psi_1 \rangle \langle \psi_2 |$ will be deduced from putting t = 0 in (H.4). The 6061 calculation is straightforward: 6062

$$\overline{\langle \psi_1 | \hat{\rho} | \psi_1 \rangle | \psi_1 \rangle \langle \psi_1 |} = (c_{40} - c_{22}) \hat{\rho} + c_{22} \hat{1}$$
(H.5)

where 6063

$$c_{40} = \frac{\int_{0}^{\infty} \prod_{\alpha=1}^{G} (x_{\alpha} dx_{\alpha}) x_{1}^{4} \delta\left[\sum_{\alpha} x_{\alpha}^{2} - 1\right]}{\int_{0}^{\infty} \prod_{\alpha=1}^{G} (x_{\alpha} dx_{\alpha}) \delta\left[\sum_{\alpha} x_{\alpha}^{2} - 1\right]}, \qquad c_{22} = \frac{\int_{0}^{\infty} \prod_{\alpha=1}^{G} (x_{\alpha} dx_{\alpha}) x_{1}^{2} x_{2}^{2} \delta\left[\sum_{\alpha} x_{\alpha}^{2} - 1\right]}{\int_{0}^{\infty} \prod_{\alpha=1}^{G} (x_{\alpha} dx_{\alpha}) \delta\left[\sum_{\alpha} x_{\alpha}^{2} - 1\right]}.$$
 (H.6)

The integration variables in (H.6) refer to random components of a normalized vector. Expectedly, (H.5) is a linear 606 combination of $\hat{\rho}$ and the unit matrix, because only this matrix is invariant with respect to all unitary operators. 6065 The calculation of (H.6) brings

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$$c_{40} = 2c_{22}, \qquad c_{22} = \frac{1}{G(G+1)}.$$
 (H.7)

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⁶⁰⁶⁷ Using (H.7, H.5) in (H.2) we obtain:

$$\overline{\hat{U}_{\uparrow\uparrow}\hat{\rho}\hat{U}_{\uparrow\uparrow}^{\dagger}} = \frac{1}{(G+1)(G-1)}(G\hat{1}-\hat{\rho}) + \frac{1}{(G+1)(G-1)}\left(\hat{\rho}-\frac{\hat{1}}{G}\right)\left|\sum_{\alpha=1}^{G}e^{itE_{\alpha}}\right|^{2}.$$
(H.8)

For sufficiently large times the sum $\sum_{\alpha=1}^{G} e^{itE_{\alpha}}$ goes to zero. Neglecting factors $O(\hat{\rho}G^{-1})$ we get from (H.8) that

$$\overline{\hat{U}_{\Uparrow}\hat{\rho}\hat{U}_{\Uparrow}^{\dagger}} \to \frac{\hat{1}}{G}.$$
(H.9)

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The considered arbitrary initial state $\hat{\rho}$ thus tends to the microcanonical distribution only provided that $G \gg 1$.

The relaxation in (H.9) will be exponential, if we assume that the eigenvalues in (H.8) are Gaussian, e.g., assuming that they are independently distributed with zero average and dispersion Δ we get in the limit $G \gg 1$: $\sum_{\alpha=1}^{G} \exp(itE_{\alpha}) \propto \exp(-t^2\Delta^2)$. Obviously, the same relaxation scenario (under the stated assumptions) will hold for the off-diagonal components; see Eq. (11.15) of the main text.

The reason of the non-exponential relaxation for the Gaussian ensemble is that all the non-diagonal elements of the random matrix are taken to be identically distributed. This makes the distribution of the eigenvalues bounded (the semi-circle law). If the elements closer to the diagonal will be weighted stronger, the distribution of the eigenvalues will be closer to the Gaussian. The above factorized ensemble models this situation.

6078 I. Collisional relaxation of subensembles and random matrices

Collisions have a relaxing effect Anonymous

The purpose of this Appendix is to show that the evolution produced by a random Hamiltonian—which is normally regarded as a description of a closed, complex quantum system—may be generated within an open-system dynamics. This enlarges the scope and applicability of the random matrix approach.

6084 I.1. General discussion

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The ideas of collisional relaxation are well-known in the context of the classical Boltzmann equation. It is possible to extend the main ideas of the linearized Boltzmann equation (independent collisions with equilibrated systems) to the quantum domain [350, 351, 352]. We shall first describe this scenario in general terms and then apply it to the specific situation described in § 11.2.4.

Each collision is an interaction between the target quantum system \mathbb{T} and a bath particle B. The interaction lasts a finite but short amount of time. Then another collision comes, etc. The bath particles are assumed to be independent of each other and thermalized. Each collision is generated by the Hamiltonian

$$\hat{H}_{\mathbb{T}+\mathbf{B}} = \hat{H}_{\mathbb{T}} + \hat{H}_{\mathbf{B}} + \hat{H}_{\mathbf{I}},\tag{I.1}$$

where $\hat{H}_{\mathbb{T}}$ and $\hat{H}_{\mathbf{B}}$ are the Hamiltonians of \mathbb{T} and B, respectively, and where $\hat{H}_{\mathbf{I}}$ is the interaction Hamiltonian. Each collision is spontaneous and obeys the strict energy conservation:

$$[\hat{H}_{\rm I}, \hat{H}_{\rm B} + \hat{H}_{\rm T}] = 0. \tag{I.2}$$

This condition guarantees that there are no energy costs for switching the collisional interaction \hat{H}_{I} on and off.

⁶⁰⁹⁵ The initial density matrix of B is assumed to be Gibbsian (this assumption can be relaxed)

$$\hat{\rho}_{\rm B} = \frac{1}{Z_{\rm B}} \exp[-\beta \hat{H}_{\rm B}] \tag{I.3}$$

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with Hamiltonian $\hat{H}_{\rm B}$ and temperature $1/\beta = T > 0$. The target system starts in an arbitrary initial state $\rho_{\rm T}$ and has Hamiltonian $\hat{H}_{\rm T}$. The initial state of ${\rm T} + {\rm B}$ is $\hat{\rho}_{{\rm T}+{\rm B}} = \hat{\rho}_{\rm T} \otimes \hat{\rho}_{\rm R}$. The interaction between them is realized via a unitary operator \hat{V} , so that the final state is

$$\hat{\rho}_{\mathbb{T}+\mathbf{B}}' = \hat{\mathcal{V}}\hat{\rho}_{\mathbb{T}+\mathbf{B}}\hat{\mathcal{V}}^{\dagger}, \quad \hat{\rho}_{\mathbb{T}}' = \operatorname{tr}_{\mathbf{B}}\hat{\rho}_{\mathbb{T}+\mathbf{B}}'.$$
(I.4)

For the second collision, the initial state of \mathbb{T} and the second bath particle \mathbf{B}_2 is $\hat{\rho}'_{\mathbb{T}} \otimes \hat{\rho}_{\mathbf{R}_2}$, etc.

Let the energy levels of \mathbb{T} involved in the interaction with B be degenerate: $\hat{H}_{\mathbb{T}} \propto \hat{1}$. Using (I.1–I.4) and going to the eigenresolution of $\rho_{\rm B}$ we see that the evolution of \mathbb{T} in this case can be described as a mixture of unitary processes (note that in \hat{U}^k below k is an index and not the power exponent)

$$\hat{\rho}_{\mathbb{T}}' = \sum_{k} \lambda_k \hat{U}^k \hat{\rho}_{\mathbb{T}} \, \hat{U}^{k\,\dagger},\tag{I.5}$$

$$\hat{U}^{k} = \exp\left(-i\delta\langle k|\hat{H}_{\mathbf{I}}|k\rangle\right), \qquad \hat{U}^{k}U^{k\dagger} = \hat{1},$$
(I.6)

where $\{\lambda_k\}$ and $\{|k\rangle\}$ are the eigenvalues and eigenvectors, respectively, of $\hat{\rho}_B$ and δ is the interaction time. Eq. (I.5) holds for all subsequent collisions; now k in (I.5) is a composite index. Within this Appendix we put $\hbar = 1$.

Note that the mixture of unitary processes increases the von Neumann entropy $S_{vN}[\hat{\rho}_T] = -\text{tr}[\hat{\rho}_T \ln \hat{\rho}_T]$ of \mathbb{T} ; this is the concavity feature of S_{vN} . Hence after sufficiently many collisions \mathbb{T} will relax to the microcanonic density matrix $\hat{\rho}_T \propto \hat{1}$ that has the largest entropy possible.

The same process (I.5) can be generated assuming the random Hamiltonian $\langle k|\hat{H}_{I}|k\rangle$, and then averaging over it. This is closely related to section 11.2.3 of the main text, where we postulated the random Hamiltonian $V_{\rm M} = \langle k|\hat{H}_{I}|k\rangle$ as a consequence of complex interactions. For the purpose of section 11.2.3, T amounts to S + M (system + magnet) and the complex interactions are supposed to take place in M. In contrast, the averaging in (I.5) arises due to tracing the bath out. If the \hat{U}^k mutually commute, (I.5) means averaging over varying phases, i.e. it basically represents a (partial) dephasing in the common eigenbasis of \hat{U}^k .

We shall apply the collisional relaxation to the target system $\mathbb{T} = S + M$ after the measurement, so without the S-M coupling. We can directly apply mixtures of unitary processes for describing the relaxation; see (I.5). Following to the discussion in section 11.2.3 of the main text [see the discussion before (11.12)], we assume that each unitary operator \hat{U}^k in the mixture (I.5) will have the following block-diagonal form:

$$\hat{U}^{k} = \hat{\Pi}_{\uparrow} \hat{U}^{k}_{\uparrow} \hat{\Pi}_{\uparrow} + \hat{\Pi}_{\Downarrow} \hat{U}^{k}_{\downarrow} \hat{\Pi}_{\Downarrow}, \qquad (I.7)$$

⁶¹¹⁸ where in view of (11.10) of the main text we defined the following projectors

$$\hat{\Pi}_{\uparrow} = \sum_{\eta} |m_F, \eta\rangle \langle m_F, \eta|, \qquad \hat{\Pi}_{\downarrow} = \sum_{\eta} |-m_F, \eta\rangle \langle -m_F, \eta|.$$
(I.8)

Eq. (I.7) is now to be applied to (11.9) of the main text, which yields

$$\hat{U}^{k}|\Psi\rangle\langle\Psi|\hat{U}^{k\dagger} = \sum_{\eta\eta'} U_{\eta}U_{\eta'}^{*}|\uparrow\rangle\langle\uparrow|\otimes\hat{U}_{\uparrow}^{k}|m_{F},\eta\rangle\langle m_{F},\eta'|\hat{U}_{\uparrow\uparrow}^{k\dagger} + \sum_{\eta\eta'} V_{\eta}V_{\eta'}^{*}|\downarrow\rangle\langle\downarrow|\otimes\hat{U}_{\downarrow\downarrow}^{k}|-m_{F},\eta\rangle\langle-m_{F},\eta'|\hat{U}_{\downarrow\downarrow}^{k\dagger} + \left[\sum_{\eta\eta'} U_{\eta}V_{\eta'}^{*}|\uparrow\rangle\langle\downarrow|\otimes\hat{U}_{\uparrow\uparrow}^{k}|m_{F},\eta\rangle\langle-m_{F},\eta'|\hat{U}_{\downarrow\downarrow}^{k\dagger} + \text{h.c.}\right],$$
(I.9)

⁶¹²⁰ where h.c. means the hermitean conjugate of the last term.

6121 I.2. Gaussian random matrix ensemble: characteristic time within the collisional relaxation scenario

As we saw in the main text [section 11.2.3], the relaxation generated by the Gaussian ensemble of random Hamiltonians (where the elements of the random matrix Hamiltonian are identically distributed Gaussian random variables) is not exponential. From the viewpoint of the collisional relaxation, the averaging over a random matrix ensemble
corresponds to a single collision. We now show that taking into account many short collisions can produce exponential relaxation.

Our technical task is to work out Eq. (11.14) of the main text for multiple collisions. We introduce a shorthand $\hat{\rho}(0) = |m_F, \eta\rangle\langle m_F, \eta|$ and recall that within this Appendix $\hbar = 1$. Following the assumptions we made in section 11.2.3 of the main text [see Eq. (11.12)] we write $\hat{U}_{\uparrow,\downarrow} = e^{-it\hat{V}_{\uparrow,\downarrow}}$, where \hat{V}_{\uparrow} and \hat{V}_{\downarrow} are independent random matrices: the elements $V_{\uparrow,\eta\eta'}$ of \hat{V}_{\uparrow} in the basis $|m_F,\eta\rangle$ (and of \hat{V}_{\downarrow} in $|-m_F,\eta\rangle$) are independent, identically distributed random quantities with zero average and variance

$$\overline{V_{\uparrow,\eta_1\eta_2}V_{\uparrow,\eta_3\eta_4}} = \overline{V_{\downarrow,\eta_1\eta_2}V_{\downarrow,\eta_3\eta_4}} = \frac{\Delta^2}{4G}\delta_{\sigma_1\sigma_4}\delta_{\sigma_2\sigma_3}.$$
(I.10)

Note that taking the distribution of non-diagonal elements different from that of the diagonal elements will not change anything essential below.

⁶¹³⁴ We shall now assume that the duration δ of each collision is small and work out the post-collision state $\hat{U}_{\uparrow}\hat{\rho}(t)U_{\uparrow}^{\dagger} = e^{-i\delta\hat{V}_{\uparrow}}\hat{\rho}(t)e^{i\delta\hat{V}_{\uparrow}}$:

$$e^{-i\delta\hat{V}_{\uparrow}}\hat{\rho}(t)e^{i\delta\hat{V}_{\uparrow}} = \hat{\rho}(t) - i\delta[\hat{V}_{\uparrow},\hat{\rho}] - \frac{\delta^2}{2} \left\{ \hat{V}_{\uparrow}\hat{\rho}(t) + \hat{\rho}(t)\hat{V}_{\uparrow} - 2\hat{V}_{\uparrow}\hat{\rho}(t)\hat{V}_{\uparrow} \right\} + O(\delta^3).$$
(I.11)

6136 Averaging with help of (I.10) produces

$$\overline{\hat{V}_{\uparrow\uparrow}\hat{\rho}} = 0, \qquad \overline{\hat{V}_{\uparrow\uparrow}^2\hat{\rho}} = \overline{\hat{\rho}\hat{V}_{\uparrow\uparrow}^2} = \frac{1}{4}\Delta^2\hat{\rho}, \qquad \overline{\hat{V}_{\uparrow\uparrow}\hat{\rho}\hat{V}_{\uparrow\uparrow}} = \frac{\Delta^2}{4G}\mathrm{tr}(\hat{\rho})\hat{1}.$$
(I.12)

6137 This brings

$$\hat{\rho}(t+\delta) = \hat{\rho}(t) - \frac{1}{4}\delta^2 \Delta^2 \left[\hat{\rho}(t) - \frac{\hat{1}}{G} \right] + O[\delta^4 \Delta^4].$$
(I.13)

6138 If the factor $O[\delta^4 \Delta^4]$ in (I.13) is neglected, *i.e.* if

$$\delta^2 \Delta^2 \ll 1, \tag{I.14}$$

6139 (I.13) can be extended to a recurrent relation for all subsequent collisions:

$$\hat{\rho}(n\delta) = \hat{\rho}((n-1)\delta) - \frac{1}{4}\delta^2 \Delta^2 \left[\hat{\rho}((n-1)\delta) - \frac{\hat{1}}{G}\right],\tag{I.15}$$

where n = 1, 2, ... is the number of collisions. Eq. (I.15) is solved as

$$\hat{\rho}(n\delta) = (1 - \frac{1}{4}\delta^2\Delta^2)^n \hat{\rho}(0) + \frac{\hat{1}}{G} \left[1 - (1 - \frac{1}{4}\delta^2\Delta^2)^n \right].$$
(I.16)

It is seen from (I.16) that the relaxation time of $\hat{\rho}(n\delta) \rightarrow \hat{1}/G$ is

$$-\frac{\delta}{\ln[1-\frac{1}{4}\delta^2\Delta^2]}.$$
(I.17)

⁶¹⁴² We now want to satisfy several conditions: (*i*) the magnitude $\sqrt{\hat{V}_{\uparrow\uparrow}^2} = \Delta/2$ of the random Hamiltonian has to be ⁶¹⁴³ much smaller than *N*, because the random Hamiltonian has to be thermodynamically negligible. (*ii*) The relaxation ⁶¹⁴⁴ time (I.17) has to be very short for a large (but finite) *N*. (*iii*) Condition (I.14) has to hold.

All these conditions can be easily satisfied simultaneously by taking, e.g., $\Delta \propto N^{\gamma}$ and $\delta \propto N^{-\chi}$, where

$$2\gamma > \chi > \gamma, \qquad \gamma < 1. \tag{I.18}$$

Now the relaxation time will be $\propto N^{\chi-2\gamma} \ll 1$, while (I.14) will hold, because $N^{2(\gamma-\chi)} \ll 1$.

The same derivation applies to non-diagonal elements $\hat{U}_{\uparrow}|m_F,\eta\rangle\langle -m_F,\eta'|U_{\downarrow}^{\dagger} = e^{-i\delta\hat{V}_{\uparrow}}|m_F,\eta\rangle\langle -m_F,\eta'|e^{i\delta\hat{V}_{\downarrow}}$ in (I.9). Instead of (I.16) we get

$$\hat{\rho}(n\delta) = (1 - \frac{1}{4}\delta^2 \Delta^2)\hat{\rho}((n-1)\delta), \qquad \hat{\rho}(0) = |m_F, \eta\rangle \langle -m_F, \eta'|,$$
(I.19)

with the same form of the characteristic time as for the exponential relaxation $\hat{\rho}(n\delta) \to 0$ for $n \to \infty$.

6150 References

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