# Quantum entanglement

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All our former experience with application of quantum theory seems to say: what is predicted by quantum formalism must occur in laboratory. But the essence of quantum formalism — entanglement, recognized by Einstein, Podolsky, Rosen and Schrödinger --- waited over 70 years to enter to laboratories as a new resource as real as energy.

This holistic property of compound quantum systems, which involves nonclassical correlations between subsystems, is a potential for many quantum processes, including "canonical" ones: quantum cryptography, quantum teleportation and dense coding. However, it appeared that this new resource is very complex and difficult to detect. Being usually fragile to environment, it is robust against conceptual and mathematical tools, the task of which is to decipher its rich structure. This article reviews basic aspects of entanglement including its characterization, detection, distillation and quantifying. In particular, the authors discuss various manifestations of entanglement via Bell inequalities, entropic inequalities, entanglement witnesses, quantum cryptography and point out some interrelations. They also discuss a basic role of entanglement in quantum communication within distant labs paradigm and stress some peculiarities such as irreversibility of entanglement manipulations including its extremal form — bound entanglement phenomenon. A basic role of entanglement witnesses in detection of entanglement is emphasized.

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# I. INTRODUCTION

Although in 1932 von Neumann had completed basic elements of nonrelativistic quantum description of the world, it were Einstein, Podolsky and Rosen (EPR) and Schrödinger who first recognized a "spooky" feature of quantum machinery which lies at center of interest of physics of XXI century (Einstein *et al.*, 1935; von Neumann, 1932). This feature implies the existence of global states of composite system which cannot be written as a product of the states of individual subsystems. This phenomenon, known as "entanglement", was originally called by Schrödinger "Verschränkung", which underlines an intrinsic order of statistical relations between subsystems of compound quantum system (Schrödinger, 1935).

Paradoxically, entanglement, which is considered to be the most nonclassical manifestations of quantum formalism, was used by Einstein Podolsky and Rosen in their attempt to ascribe values to physical quantities prior to measurement. It was Bell who showed the opposite: it is just entanglement which irrevocably rules out such a possibility.

In 1964 Bell accepted the EPR conclusion — that quantum description of physical reality is not complete as a working hypothesis and formalized the EPR deterministic world idea in terms of local hidden variable model (LHVM) (Bell, 1964). The latter assumes that (i) measurement results are determined by properties the particles carry prior to, and independent of, the measurement ("realism"), (ii) results obtained at one location are independent of any actions performed at spacelike separation ("locality") (iii) the setting of local apparatus are independent of the hidden variables which determine the local results ("free will"). Bell proved that the above assumptions impose constraints on statistical correlations in experiments involving bipartite systems in the form of the Bell inequalities. He then showed that the probabilities for the outcomes obtained when suitably measuring some entangled quantum state violate the Bell inequality. In this way entanglement is that feature of quantum formalism which makes impossible to simulate the quantum correlations within any classical formalism.

Greenberger, Horne and Zeilinger (GHZ) went beyond Bell inequalities by showing that entanglement of more than two particles leads to a contradiction with LHVM for nonstatistical predictions of quantum formalism (Greenberger *et al.*, 1989). Surprisingly, only in the beginning of 90's theoretical general results concerning violation of Bell inequalities have been obtained (Gisin, 1991; Popescu and Rohrlich, 1992).

Transition of entanglement from gedanken experiment to laboratory reality began in the mid-60s (Freedman and Clauser, 1972; Kocher and Commins, 1967). However it were Aspect *et al.*, who performed first a convincing test of violation of the Bell inequalities (Aspect *et al.*, 1982, 1981). Since then many kinds of beautiful and precise experimental tests of quantum formalism against the LHVM have been performed in laboratories (Bovino *et al.*, 2006a; Hasegawa *et al.*, 2003; Kwiat *et al.*, 1995; Ou and Mandel, 1988; Rowe and et al., 2001) and outsides (Tittel *et al.*, 1998, 1999; Ursin *et al.*, 2006; Weihs *et al.*, 1998). All

In fact, a fundamental nonclassical aspect of entanglement was recognized already in 1935. Inspired by EPR paper, Shrödinger analyzed some physical consequences of quantum formalism and he noticed that the two-particle EPR state does not admit ascribing individual states to the subsystems implying "entanglement of predictions" for the subsystems. Then he concluded: "Thus one disposes provisionally (until the entanglement is resolved by actual observation) of only a common description of the two in that space of higher dimension. This is the reason that knowledge of the individual systems can decline to the scantiest, even to zero, while that of the combined system remains continually maximal. Best possible knowledge of a whole does **not** include best possible knowledge of its parts — and this is what keeps coming back to haunt us" (Schrödinger, 1935)<sup>2</sup>.

Unfortunately this curious aspect of entanglement was long unintelligible, as it was related to the notion of "knowledge" in quantum context. Only in half of 90's it was formalized in terms of entropic inequalities based on Von Neumann entropy (Cerf and Adami, 1997: Horodecki and Horodecki, 1994: Horodecki et al.,  $(1996c)^3$ . The violation of these inequalities by entangled states is a signature of entanglement of quantum states, however physical meaning of this was unclear. An interesting attempt to solve this puzzle is due to Cerf and Adami (1997) in terms of conditional entropy. Soon afterwards it turned out that the latter with minus sign, called *coherent information* is a fundamental quantity responsible for capabilities of transmission of quantum information (Lloyd, 1997; Schumacher and Nielsen, 1996). The transmission is possible exactly in those situations in which "Schrödinger's demon" is "coming to haunt us" — i.e. when entropy of output system exceeds the entropy of the total system. Let us mention, that in 2005 this story has given a new twist in terms of quantum counterpart of the Slepian-Wolf theorem in classical communication (Horodecki et al., 2005h, 2006e). In this approach the violation of entropic inequalities implies the existence of negative quantum information, which is "extra" resource for quantum communication. Interestingly, only recently a direct violation of the entropic inequalities was experimentally demonstrated confirming the breaking of classical statistical order in compound quantum systems (Bovino *et al.*, 2005).

The present-day entanglement theory has its roots in the key discoveries: quantum cryptography with

<sup>&</sup>lt;sup>1</sup> However so far all the above experiments suffer from loophole, see (Brunner *et al.*, 2007; Gill, 2003).

<sup>&</sup>lt;sup>2</sup> an English translation appears in Quantum Theory and Measurement, edited by J. A. Wheeler and W.H. Zurek Princeton University Press, Princeton, 1983, p.167.

<sup>&</sup>lt;sup>3</sup> The other formalization was proposed in terms of majorization relations (Nielsen and Kempe, 2001).

Bell theorem (Ekert, 1991), quantum dense coding (Bennett and Wiesner, 1992) and quantum teleportation (Bennett *et al.*, 1993)<sup>4</sup> including teleportation of entanglement of EPR pairs (so called entanglement swapping) (Bose et al., 1998; Yurke and Stoler, 1992a,b; Żukowski et al., 1993). All such effects are based on entanglement and all of them have been demonstrated in pioneering experiments (see (Boschi et al., 1998; Bouwmeester et al., 1997; Furusawa et al., 1998; Jennewein et al., 2000; Mattle et al., 1996b; Naik et al., 2000; Pan et al., 1998; Tittel et al., 2000). In fact. the above results including pioneering paper on quantum cryptography (Bennett and Brassard, 1984) and idea of quantum computation (Deutsch, 1985; Feynman, 1982; Shor, 1995; Steane, 1996a) were a basis for a new interdisciplinary domain called quantum information (Alber et al., 2001a; Bouwmeester et al., 2000; Braunstein and Pati, 2003; Bruß and Leuchs, 2007; Lo et al., 1999; Nielsen and Chuang, 2000) which incorporated entanglement as a central notion.

It has become clear that entanglement is not only subject of philosophical debates, but it is a new quantum resource for tasks which can not be performed by means of classical resources (Bennett, 1998). It can be manipulated (Bennett *et al.*, 1996c,d; Gisin, 1996b; Popescu, 1995; Raimond *et al.*, 2001), broadcasted (Buzek *et al.*, 1997), controlled and distributed (Beige *et al.*, 2000; Cirac and Zoller, 2004; Mandilara *et al.*, 2007).

Remarkably, entanglement is a resource which, though does not carry information itself, can help in such tasks as the reduction of classical communication complexity (Brukner *et al.*, 2004; Buhrman *et al.*, 2001a; Cleve and Buhrman, 1997), entanglement assisted orientation in space (Bovino *et al.*, 2006b; Brukner *et al.*, 2005), quantum estimation of a damping constant (Venzl and Freyberger, 2007), frequency standards improvement (Giovannetti *et al.*, 2004; Huelga *et al.*, 1997; Wineland *et al.*, 1992) (see in this context (Boto *et al.*, 2005)) and clock synchronization (Jozsa *et al.*, 2000). The entanglement plays fundamental role in quantum communication between parties separated by macroscopic distances (Bennett *et al.*, 1996d).

Though the role of entanglement in quantum computational speed-up is not clear (Kendon and Munro, 2006a), it has played important role in development of quantum computing, including measurement based schemes, one-way quantum computing (Raussendorf and Briegel, 2001)<sup>5</sup>, linear optics quantum computing (Knill *et al.*, 2001)<sup>6</sup>. Entanglement has also given us new insights

<sup>4</sup> Quantum teleportation with continuous variables in an infinite dimensional Hilbert space was first proposed by Vaidman (Vaidman, 1994) and further investigated theoretically by Braunstein and Kimble (Braunstein and Kimble, 1998). for understanding many physical phenomena including super-radiance (Lambert et al., 2004), superconductivity (Vedral, 2004), disordered systems (Dür et al., 2005) and emerging of classicality (Zurek, 2003a). In particular, understanding the role of entanglement in the existing methods of simulations of quantum spin systems allowed for significant improvement of the methods, as well as understanding their limitations (Anders et al., 2006: Verstraete et al., 2004b: Vidal, 2003, 2004). The role of entanglement in quantum phase transitions (Larsson and Johannesson, 2006; Latorre et al., 2004; Osborne and Nielsen, 2002; Osterloh et al., 2002; Verstraete et al., 2004a; Vidal et al., 2003) was intensively studied. Divergence of correlations at critical points is always accompanied by divergence of a suitably defined entanglement length (Verstraete et al., 2004a). The concept of entanglement length originates form earlier paper by D. Aharonov, where a critical phenomenon has been studied in context of fault-tolerant quantum computing (Aharonov, 1999).

Unfortunately quantum entanglement has three disagreeable but interesting features: It has in general very complex structure, it is very fragile to environment and it can not be increased on average when systems are not in direct contact but distributed in spatially separated regions. The theory of entanglement tries to give answers to fundamental questions as: i) How to detect optimally entanglement theoretically and in laboratory; ii) How to reverse an inevitable process of degradation of entanglement; iii) How to characterize, control and quantify entanglement.

History of the questions i) and ii) has its origin in seminal papers by Werner and Popescu (Popescu, 1995; Werner, 1989a). Werner not only gave accurate definition of separable states (those mixed states that are not entangled), but also noted that there exist entangled states that similarly to separable states, admit LHV model, hence do not violate Bell inequalities. Popescu showed (Popescu, 1995) that having system in such state, by means of local operations one can get a new state whose entanglement can be detected by Bell inequalities. This idea was developed by Gisin who used so-called filters to enhance violating of Bell inequalities (Gisin, 1996b). In fact, this idea turned out to be a trigger for a theory of entanglement manipulations (Bennett *et al.*, 1996c).

Soon afterwards, Peres showed that if the state was separable<sup>7</sup> then after *partial transpose*<sup>8</sup> of density matrix on one of the subsystems of a compound bipartite system it is still a legitimate state (Peres, 1996a). Surprisingly Peres condition appeared to be strong test for

<sup>&</sup>lt;sup>5</sup> For comprehensive review see (Browne and Briegel, 2006).

<sup>&</sup>lt;sup>6</sup> It has been shown that linear optics quantum computing can viewed as a measurement based one-way computing (Popescu,

<sup>2006).</sup> 

<sup>&</sup>lt;sup>7</sup> More formal definitions of entangled and separable states are given in the next section.

<sup>&</sup>lt;sup>8</sup> The positive partial transpose condition is called the Peres criterion or the PPT criterion of separability.

entanglement<sup>9</sup>. As the partial transpose is a *positive* map it was realized that the positive maps can serve as the strong detectors of entanglement. However they cannot be implemented directly in laboratory, because they are unphysical<sup>10</sup>. Fortunately there is "footbridge"-Jamiołkowski isomorphism (Jamiołkowski, 1972a) which allowed to go to physical measurable quantities — Hermitian's operators. This constitutes a necessary and sufficient condition separability on the both, physical level of observables and nonphysical one engaging positive maps (Horodecki et al., 1996a). This characterization of entanglement although nonoperational, provides a basis for general theory of detection of entanglement. A historical note is here in order. Namely it turned out that both the general link between separability and positive maps as well as the Peres-Horodecki theorem were first known and expressed in slightly different language as early as in 70s (Choi, 1972) (see also (Osaka, 1991; Størmer, 1963; Woronowicz, 1976)). The rediscovery by Peres and Horodecki's brought therefore powerful methods to entanglement theory as well as caused revival of research on positive maps, especially the so-called nondecomposable ones.

Terhal was the first to construct a family of indecomposable positive linear maps based on entangled quantum states (Terhal, 2000b). She also pointed out, that a violation of a Bell inequality can formally be expressed as a witness for entanglement  $(Terhal, 2000a)^{11}.$ Since then theory of entanglement witnesses was intensively developed (Brandao, 2005; Brandao and Vianna, 2006; Bruß et al., 2002; Gühne et al., 2003; Kiesel et al., 2005; Lewenstein et al., 2000; Toth and Gühne, 2005a), including a nonlinear generalization (Gühne and Lewenstein, 2004b; Gühne and Lütkenhaus, 2006a) and the study of indistinguishable systems (Eckert et al., 2002; Schliemann et al., 2001).

The concept of entanglement witness was applied to different problems in statistical systems (Brukner *et al.*, 2006; Cavalcanti *et al.*, 2006; Wieśniak *et al.*, 2005; Wu *et al.*, 2005), quantum cryptography (Curty *et al.*, 2005) quantum optics (Stobińska and Wódkiewicz, 2005; Stobińska and Wódkiewicz, 2006), condensed-matter nanophysics (Blaauboer and DiVincenzo, 2005), bound entanglement (Hyllus *et al.*, 2004), experimental realization of cluster states (Vallone *et al.*, 2007), hidden nonlocality (Masanes *et al.*, 2007). About this time in precision experiments a detection of multipartite entanglement using entanglement witness operators was performed (Altepeter, 2005; Barbieri *et al.*, 2003; Bourennane *et al.*, 2004; Haffner *et al.*, 2005; Leibfried *et al.*, 2005; Lu *et al.*, 2007; Mikami *et al.*, 2005; Resch *et al.*, 2005; Roos *et al.*, 2004).

As one knows, the main virtue of entanglement witnesses is that they provide an economic way of detection of entanglement, that does not need full (tomographic) information about the state. It arises a natural question: how to estimate optimally the amount of entanglement of compound system in an unknown state if only incomplete data in the form of averages values of some operators detecting entanglement are accessible? This question posed in 1998 lead to the new inference scheme for all the processes where entanglement is measured. It involves a principle of minimization of entanglement under a chosen measure of entanglement with constrains in the form of incomplete set of data from experiment (Horodecki et al., 1999c). In particular, minimization of entanglement measures (entanglement of formation and relative entropy of entanglement) under fixed Bell-type witness constraint was obtained. Subsequently the inference scheme based of the minimization of entanglement was successfully applied (Audenaert and Plenio, 2006; Eisert et al., 2007; Gühne et al., 2006b) to estimate entanglement measures from the results of recent experiments measuring entanglement witnesses. This results show clearly that the entanglement witnesses are not only economic indicators of entanglement but they are also helpful in estimating the entanglement content.

In parallel to entanglement witnesses the theory of positive maps was developed which provides, in particular strong tools for the detection of entanglement (Benatti *et al.*, 2004; Breuer, 2006a; Cerf *et al.*, 1999; Chruściński and Kossakowski, 2006; Datta *et al.*, 2006b; Hall, 2006; Horodecki and Horodecki, 1999; Kossakowski, 2003; Majewski, 2004; Piani, 2004; Piani and Mora, 2006; Terhal, 2000b). Strong inseparability criteria beyond the positive maps approach were also found (Chen and Wu, 2003; Clarisse and Wocjan, 2006; Devi *et al.*, 2007; Gühne, 2004; Gühne *et al.*, 2007; Hofmann and Takeuchi, 2003; Horodecki *et al.*, 2006d; Mintert *et al.*, 2005b; Rudolph, 2000).

Separability criteria for continuous variables were also proposed (see Sec. XVII.D). The necessary and sufficient condition for separability of Gaussian states of a bipartite system of two harmonic oscillators has been found independently by Simon and Duan *et al.* (Duan *et al.*, 2000; Simon, 2000). Both approaches are equivalent. Simon's criterion is direct generalization of partial transpose to CV systems, while Duan *et al.* started with local uncertainty principles.

Soon afterwards (Werner and Wolf, 2001a) found bound entangled Gaussian states. Since then theory of entanglement continuous variables have been developed in many directions, especially for Gaussian states which are accessible at the current stage of technology (see (Braunstein and Pati, 2003) and references therein). For the latter states the problem of entanglement versus separability was solved completely: operational necessary

<sup>&</sup>lt;sup>9</sup> For low dimensional systems it turned out to be necessary and sufficient condition of separability what is called Peres-Horodecki criterion (Horodecki *et al.*, 1996a).

 $<sup>^{10}</sup>$  See however (Horodecki and Ekert, 2002).

 $<sup>^{11}</sup>$  The term "entanglement witness" for operators detecting entanglement was introduced in (Terhal, 2000a).

and sufficient condition was provided in (Giedke *et al.*, 2001b). This criterion therefore detects all bipartite bound entangled Gaussian states. Interestingly McHugh *et al.* constructed a large class of non-Gaussian two-mode continuous variable states for which separability criterion for Gaussian states can be employed (McHugh *et al.*, 2007).

Various criteria for continuous variables were obtained (Agarwal and Biswas, 2005; Hillery and Zubairy, 2006; Mancini *et al.*, 2002; Raymer *et al.*, 2003). A powerful separability criterion of bipartite harmonic quantum states based on partial transpose was derived which includes all the above criteria as a special cases (Miranowicz and Piani, 2006; Shchukin and Vogel, 2005b).

The present day entanglement theory owes its form in great measure to the discovery of the concept of entanglement manipulation (Bennett et al., 1996c; Popescu, 1995). It was realized (Bennett et al., 1996c) that a natural class of operations suitable for manipulating entanglement is that of local operations and classical communication (LOCC), both of which cannot bring in entanglement for free. So established distant lab (or LOCC) paradigm plays a fundamental role in entanglement theory. Within the paradigm many important results have been obtained. In particular, the framework for pure state manipulations have been established, including reversibility in asymptotic transitions of pure bipartite states (Bennett et al., 1996b), connection between LOCC pure state transitions and majorization theory (Schumacher and Nielsen, 1996) as well as a surprising effect of catalysis (Jonathan and Plenio, 1999; Vidal and Cirac, 2001, 2002). Moreover, inequivalent types of multipartite entanglement have been identified (Bennett et al., 2001; Dür et al., 2000b).

Since in laboratory one usually meets mixed states representing *noisy entanglement*, not much useful for quantum information processing, there was a big challenge: to reverse the process of degradation of entanglement by means of some active manipulations. Remarkably in (Bennett *et al.*, 1996c) it was shown that it is possible to distill pure entanglement from noisy one in asymptotic regime. It should be noted that in parallel, there was intense research aiming at protection of quantum information in quantum computers against decoherence. As a result error correcting codes have been discovered (Shor, 1995; Steane, 1996a). Very soon it was realized that quantum error correction and distillation of entanglement are in fact inherently interrelated (Bennett *et al.*, 1996d).

A question fundamental for quantum information processing has then immediately arisen: "Can noisy entanglement be always purified?". A promising result was obtained in (Horodecki *et al.*, 1997), where all twoqubit noisy entanglement were showed to be distillable. However, soon afterwards a no-go type result (Horodecki *et al.*, 1998a) has revealed dramatic difference between pure and noisy entanglement: namely, there exists *bound entanglement*. It destroyed the hope that noisy entanglement can have more or less uniform structure: instead we encounter peculiarity in the structure of entanglement. Namely there is *free* entanglement that can be distilled and the bound one - very weak form of entanglement. Passivity of the latter provoked intensive research towards identifying any tasks that would reveal its quantum features. More precisely any entangled state that has so called *positive partial transpose* cannot be distilled. First explicit examples of such states were provided in (Horodecki, 1997). Further examples (called bound entangled) were found in (Bennett *et al.*, 1999b; Bruß and Peres, 2000; Werner and Wolf, 2001a); see also (Bruß and Leuchs, 2007; Clarisse, 2006b) and references therein.

Existence of bound entanglement has provoked many questions including relations to local variable model and role as well as long standing and still open question of existence of bound entangled states violating PPT criterion, which would have severe consequences for communication theory.

Due to bound entanglement, the question can every entanglement be used for some useful quantum task? had stayed open for a long time. Only quite recently positive answer both for bipartite as well as multipartite states was given by Masanes (Masanes, 2005a,b) in terms of so-called activation (Horodecki et al., 1999a). This remarkable result allows to define entanglement not only in negative terms of Werner's definition (a state is entangled if it is not mixture of product states) but also in positive terms: a state is entangled, if it is a resource for a nonclassical task.

One of the most difficult and at the same time fundamental questions in entanglement theory, is quan-Remarkably, two fundamentifying entanglement. tal measures. entanglement distillation (Bennett et al., 1996b,c,d), and what is now called *entanglement cost* (Bennett et al., 1996d; Hayden et al., 2001), appeared in the context of manipulating entanglement, and have an operational meaning. Their definitions brings in mind thermodynamical analogy (Horodecki *et al.*, 1998b; Popescu and Rohrlich, 1997; Vedral and Plenio, 1998), since they describe two opposite processes — creation and distillation, which ideally should be reverse of each Indeed, reversibility holds for pure bipartite other. states, but fails for noisy as well as multipartite entanglement. The generic gap between these two measures shows a fundamental irreversibility (Horodecki et al., 1998a; Vidal and Cirac, 2001; Yang et al., 2005a). This phenomenon has its origin deeply in the nature of noisy entanglement, and its immediate consequence is the nonexistence of a unique measure of entanglement.

Vedral *et al.* (Vedral and Plenio, 1998; Vedral *et al.*, 1997b) have proposed an axiomatic approach to quantifying entanglement, in which a "good" measure of entanglement is any function that satisfy some postulates. The leading idea (Bennett *et al.*, 1996d) is that entanglement should not increase under local operations and

classical communication so called monotonicity condition. The authors proposed a scheme of obtaining based on the concept on distance from separable states, and introduced one of the most important measures of entanglement, the so called *relative entropy of entanglement* (Vedral and Plenio, 1998; Vedral *et al.*, 1997b) (for more comprehensive review see (Vedral, 2002)). Subsequently, a general mathematical framework for entanglement measures was worked out by Vidal (Vidal, 2000), who concentrated on the axiom of monotonicity (hence the often used term "entanglement monotone").

At first sight it could seem that measures of entanglement do not exhibit any ordered behavior. Eisert and Plenio have showed numerically that entanglement measures do not necessarily imply the same ordering of states (Eisert and Plenio, 1999). It was further shown analytically in (Miranowicz and Grudka, 2004) However, it turns out that there are constraints for measures that satisfy suitable postulates relevant in asymptotic regime. Namely, any such measure must lie between two extreme measures that are just two basic operational measures distillable entanglement  $E_D$  and entanglement cost  $E_C$ (Horodecki et al., 2000b). This can be seen as reflection of a more general fact, that *abstractly* defined measures, provide bounds for operational measures which are of interest as they quantify how well some specific tasks can be performed.

The world of entanglement measures, even for bipartite states, still exhibits puzzles. As an example may serve an amazing phenomenon of locking of entanglement (Horodecki *et al.*, 2005d). For most of known bipartite measures we observe a kind of collapse: after removing a single qubit, for some states, entanglement dramatically goes down.

Concerning multipartite states, some bipartite entanglement measures such as relative entropy of entanglement or robustness of entanglement (Vidal and Tarrach, 1999) easily generalize to multipartite states. See (Barnum and Linden, 2001; Eisert and Briegel, 2001) for other early candidates for multipartite entanglement measures. In multipartite case a new ingredient comes in: namely, one tries to single out and quantify "truly multipartite" entanglement. First measure that reports genuinely multipartite properties is the celebrated "residual tangle" of (Coffman et al., 2000). It is clear, that in multipartite case even quantifying entanglement of pure states is a great challenge. Interesting new schemes to construct measures for pure states have been proposed (Miyake, 2003; Verstraete *et al.*, 2003).

Entanglement measures allowed for analysis of dynamical aspects of entanglement<sup>12</sup>, including entanglement decay under interaction with environment (Ban, 2006; Carvalho *et al.*, 2005; Dodd and Halliwell, 2004; Ficek and Tanaś, 2006; Jakóbczyk and Jamróz, 2004; Kim *et al.*, 2002; Maloyer and Kendon, 2007; Mintert *et al.*, 2005a; Miranowicz, 2004a; Montangero, 2004; Shresta *et al.*, 2005; Wang *et al.*, 2006; Yi and Sun, 1999; Yu and Eberly, 2004; Życzkowski *et al.*, 2002) entanglement production, in the course of quantum computation (Kendon and Munro, 2006b; Parker and Plenio, 2002) or due to interaction between subsystems. The latter problem gave rise to a notion of "entangling power" (Linden *et al.*, 2005; Zanardi *et al.*, 2000) of a unitary transformation, which can be seen as a higher-level entanglement theory dealing with entanglement of operations, rather than entanglement of states (Collins *et al.*, 2001; Eisert *et al.*, 2000b; Harrow and Shor, 2005; Linden *et al.*, 2005).

Interestingly even two-qubit systems can reveal not trivial features of entanglement decay. Namely, the state of the two entangled qubits, treated with unitary dynamics and subjected to weak noise can reach the set of separable states (see Sec. II) in finite time while coherence vanishes asymptotically (Życzkowski et al., 2002). Such an effect was investigated in detail in more realistic scenarios (Dodd and Halliwell, 2004; Ficek and Tanaś, 2006:Jakóbczyk and Jamróz, 2004; Lastra et al., 2007; Tolkunov et al., 2005; Vaglica and Vetri, 2007; Wang et al., 2006; Yu and Eberly, 2004, 2007) (see also (Jordan et al., 2007)). It was called "sudden death of entanglement" (Yu and Eberly, 2006) and demonstrated in the ingenious experiment (Almeida et al., 2007) (see in this context (Santos et al., 2006)).

Note that apart from entanglement decay a positive effects of environment was investigated: entanglement generated by interference in the measurement process (Bose et al., 1999; Cabrillo et al., 1999), cavity-lossinduced generation of entangled atoms (Plenio et al., 1999), atom-photon entanglement conditioned by photon detection (Horodecki, 2001c), generation of entanglement from white noise (Plenio and Huelga, 2002), entanglement generated by interaction with a common heat bath (Braun, 2002), noise-assisted preparation of entangled atoms inside a cavity (Yi et al., 2003), environment induced entanglement in markovian dissipative dynamics (Benatti et al., 2003), entanglement induced by the spin chain (Yi *et al.*, 2006). It has been demonstrated (Lamata et al., 2007), that it is possible to achieve an arbitrary amount of entanglement between two atoms by using spontaneous emitted photons, linear optics and projective measurements.

One of the pillars of the theory of entanglement is its connection with quantum cryptography, in particular, with its subdomain — theory of privacy, as well as a closely related classical privacy theory (Collins and Popescu, 2002; Gisin and Wolf, 1999, 2000). It seems that the most successful application of quantum entanglement is that it provides basic framework for quantum cryptography (despite the fact that the basic key distribution protocol BB84 does not use entanglement directly.). This is not just a coincidence. It

 $<sup>^{12}</sup>$  See (Amico  $et~al.,~2007;~{\rm Yu}$  and Eberly, 2005) and references therein.

appears, that entanglement is a quantum equivalent of what is meant by privacy. Indeed the main resource for privacy is a secret cryptographic key: correlations shared by interested persons but not known by any other person. Now, in the *single* notion of entanglement two fundamental features of privacy are encompassed in an ingenuous way. If systems are in pure entangled state then at the same time (i) systems are correlated and (ii) no other system is correlated with them. The interrelations between entanglement and privacy theory are so strong, that in many cases cryptographic terminology seems to be the most accurate language to describe entanglement (see e.g. (Devetak and Winter, 2005)). An example of a back-reaction — from entanglement to privacy — is the question of existence of bound information as a counterpart of bound entanglement (Gisin and Wolf, 2000). In fact, the existing of such phenomenon conjectured by Gisin and Wolf was founded by (Acin et al., 2004b) for multipartite systems.

The fact that entanglement represents correlations that cannot be shared by third parties, is deeply connected with monogamy - the basic feature of entanglement. In 1999 Coffman, Kundu and Wootters first formalized monogamy in quantitative terms, observing that there is inevitable trade-off between the amount of entanglement that a qubit A can share with a qubit  $B_1$  and entanglement which same qubit A shares with some other qubit  $B_2$  (Coffman *et al.*, 2000). In fact, already in 1996 the issue of monogamy was touched in (Bennett et al., 1996d), where it was pointed out that no system can be EPR correlated with two systems at the same time, which had direct consequences for entanglement distillation. Monogamy expresses unshareability of entanglement (Terhal, 2004), being not only central to cryptographic applications, but allows to shed new light to physical phenomena in many body systems such as frustration effect leading to highly correlated ground state (see e.g. (Dawson and Nielsen, 2004)).

The entanglement was also investigated within the framework of special relativity and quantum field theory (Alsing and Milburn, 2002; Caban and Rembieliński, 2005; Czachor, 1997; Jordan et al., 2006; Peres et al., 2005:Summers and Werner, 1985: Terno. 2004)(see comprehensive review and references there in (Peres and Terno, 2004)). In particular, entanglement for indistinguishable many body systems was investigated in two complementary directions. The first (canonical) approach bases on the tensor product structure and the canonical decomposition for the states (Eckert et al., 2002; Li et al., 2001; Paskauskas and You, 2001; Schliemann et al., 2001), while the second one is based on the occupation-number representation (Zanardi, 2002; Zanardi and Wang, 2002). Moreover Cirac and Verstraete (Verstraete and Cirac, 2003) considered notion of entanglement in the context of superselection rules more generally. However it seems that there is still controversy concerning what is entanglement for indistinguishable particles (Wiseman and Vaccaro,

2003). Recently, the group-theory approach to entanglement has been developed in (Korbicz and Lewenstein, 2006) and its connection with non-commutativity was found.

In general, the structure of quantum entanglement appears to be complex (see in (Gurvits and Barnum, 2002)) and many various parameters, measures, inequalities are needed to characterize its different aspects (see (Alber *et al.*, 2001a; Bengtsson and Życzkowski, 2006; Bruß, 2002; Bruß *et al.*, 2002; Bruß and Leuchs, 2007; Eckert *et al.*, 2003; Terhal, 2002)).

Finally, the list of experiments dealing with entanglement<sup>13</sup> quickly grows: "Entanglement over long distances (Marcikic et al., 2004; Peng et al., 2005; Tittel et al., 1998, 1999; Weihs et al., 1998), entanglement between many photons (Zhao et al., 2004) and many ions (Haeffner et al., 2005), entanglement of an ion and a photon (Blinov et al., 2004; Volz et al., 2006), entanglement of mesoscopic systems (more precisely entanglement between a few collective modes carried by many particles) ((Altewischer et al., 2002; Fasel et al., 2005; Julsgaard et al., 2004)), entanglement swapping (Jennewein et al., 2002; Pan et al., 1998), the transfer of entanglement between different carriers ((Tanzilli et al., 2005)), etc." (Gisin, 2005). Let us add few recent experiments: multiphoton path entanglement (Eisenberg et al., 2005), photon entanglement from semiconductor quantum dots (Akopian *et al.*, 2006)<sup>14</sup>. teleportation between a photonic pulse and an atomic ensemble (Sherson *et al.*, 2006), violation of the CHSH inequality measured by two observers separated by 144 km (Ursin et al., 2006), purification of two-atom entanglement (Reichle et al., 2006), increasing entanglement between Gaussian states (Ourjoumtsev et al., 2007), creation of entangled six photon graph states (Lu *et al.*, 2007).

# II. ENTANGLEMENT AS A QUANTUM PROPERTY OF COMPOUND SYSTEMS

We accustomed to the statement that on the fundamental level Nature required quantum description rather than classical one. However the full meaning of this and its all possible experimental and theoretical implications are far from triviality (Jozsa, 1999) In particular the "effect" of replacement of classical concept of phase space by abstract Hilbert space makes a gap in the description of composite systems. To see this consider multipartite system consisting of n subsystems. According to classical description the total state space of the system is the *Cartesian* product of the n subsystem spaces implying,

<sup>&</sup>lt;sup>13</sup> Historically the first experimental realization of a two-qubit entangling quantum gate (CNOT) is due to (Monroe *et al.*, 1995).

<sup>&</sup>lt;sup>14</sup> There was a controversy whether in previous experiment of (Stevenson *et al.*, 2006) entanglement was actually detected (Lindner *et al.*, 2006)

that the total state is always a product state of the n separate systems. In contrast according to the quantum formalism the total Hilbert space H is a *tensor* product of the subsystem spaces  $H = \bigotimes_{l=1}^{n} H_l$ . Then the superposition principle allows us to write the total state of the system in the form:

$$|\psi\rangle = \sum_{\mathbf{i}_n} c_{\mathbf{i}_n} |\mathbf{i}_n\rangle,\tag{1}$$

where  $\mathbf{i}_n = i_1, i_2...i_n$  is the multiindex and  $|\mathbf{i}_n\rangle = |i_1\rangle \otimes |i_2\rangle ... \otimes |i_n\rangle$ , which cannot be, in general, described as a product of states of individual subsystems<sup>15</sup>  $|\psi\rangle \neq |\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_n\rangle$ .

This means, that it is in general not possible, to assign a single state vector to any of n subsystems. It express formally a phenomenon of entanglement, which in contrast to classical superposition, allows to construct exponentially large superposition with only linear amount of physical resources. It is just what allows to perform nonclassical tasks. The states on left hand side (LHS) of (1) appear usually as a result of direct physical interactions. However, the entanglement can be also generated indirectly by application of the projection postulate (entanglement swapping).

In practice we encounter mixed states rather than pure. Entanglement of mixed states is not longer equivalent to being non-product, as in the case of pure states. Instead, one calls a mixed state of n systems entangled if it cannot be written as a convex combination of product states<sup>16</sup> (Werner, 1989a):

$$\rho \neq \sum_{i} p_{i} \rho_{1}^{i} \otimes \ldots \otimes \rho_{n}^{i}.$$
 (2)

The states which are *not* entangled in the light of the above definition are called *separable*. In practice it is hard to decide if a given states is separable or entangled basing on the definition itself. Therefore one of the fundamental problems concerning entanglement is the so called *separability* problem (see Secs. VI-X).

It should be noted in the above context that the active definition of entanglement states was proposed recently. Namely entangled states are the ones that cannot be simulated by classical correlations (Masanes et al., 2007). This interpretation defines entanglement in terms of the behavior of the states rather than in terms of preparation of the states.

*Example.* for bipartite systems the Hilbert space  $H = H_1 \otimes H_2$  with dim $H_1 = \dim H_2 = 2$  is spanned by the

four Bell-state entangled basis

$$\left|\psi^{\pm}\right\rangle = \frac{1}{\sqrt{2}}(\left|0\right\rangle\left|1\right\rangle\pm\left|1\right\rangle\left|0\right\rangle) \quad \left|\phi^{\pm}\right\rangle = \frac{1}{\sqrt{2}}(\left|0\right\rangle\left|0\right\rangle\pm\left|1\right\rangle\left|1\right\rangle).$$
(3)

These states (called sometimes EPR states) have remarkable properties namely if one measures only at one of the subsystems one finds it with equal probability in state  $|0\rangle$  or state  $|1\rangle$ . However, the result of the measurements for both subsystems are perfectly correlated. This is just feature which was recognized by Schrödinger: we know nothing at all about the subsystems, although we have maximal knowledge of the whole system because the state is pure (see Sec. V). There is another holistic feature, that unitary operation of only one of the two subsystems suffices to transform from any Bell states to any of the other three states. Moreover, Braunstein *et al.* showed that the Bell states are eigenstates of the Bell operator (16) and they maximally violate Bell-CHSH inequality (17) (see Sec. IV) (Braunstein *et al.*, 1992).

The Bell states are special cases of bipartite maximally entangled states on Hilbert space  $C^d \otimes C^d$  given by

$$|\psi\rangle = U_A \otimes U_B |\Phi_d^+\rangle_{AB} \tag{4}$$

where

$$|\Phi_d^+\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^d |i\rangle |i\rangle \tag{5}$$

is the "canonical" maximally entangled state. In this paper, a maximally entangled state will be also called EPR (Einstein-Podolsky-Rosen) state or singlet state, since it is equivalent to the true singlet state up to local unitary transformations (for d = 2 we call it also e-bit). We will also often drop the index d.

The question whether a mixture of Bell states is still entangled, is quite nontrivial. Actually it is the case if and only if one of eigenvalues is greater than  $\frac{1}{2}$  (see Sec. VI).

So far the most widely used source of entanglement are entangled-photon states produced by nonlinear process of parametric down-conversion of type I or of type II corresponding to weather entangled photons of the down-conversion pair are generated with the same polarization or orthogonal polarization respectively. In particular using parametric down-conversion one can produce Bell-state entangled basis (3). There are also many other sources of entangled quantum systems, for instance: entangled photon pairs from calcium atoms (Kocher and Commins, 1967), entangled ions prepared in electromagnetic Paul traps (Meekhof et al., 1996), entangled atoms in quantum electrodynamic cavities (Raimond et al., 2001), long-living entanglement between macroscopic atomic ensembles (Hald et al., 1999; Julsgaard et al., 2001), entangled microwave photons from quantum dots (Emary et al., 2005), entanglement between nuclear spins within a single molecule (Chen *et al.*, 2006a) entanglement between light and an atomic ensembles (Muschik et al., 2006).

<sup>&</sup>lt;sup>15</sup> Sometimes instead of notation  $|\psi\rangle \otimes |\phi\rangle$  we use simply  $|\psi\rangle |\phi\rangle$  and for  $|i\rangle \otimes |j\rangle$  even shorter  $|ij\rangle$ .

 $<sup>^{16}</sup>$  Note, that classical probability distributions can *always* be written as mixtures of product distributions.

In the next section, we will present briefly pioneering entanglement-based communications schemes using Bell entangled states.

# III. PIONEERING EFFECTS BASED ON ENTANGLEMENT

#### A. Quantum key distribution based on entanglement

The first invention of quantum information theory, which involves entanglement, was done by A. Ekert (Ekert, 1991). There were two well known facts: existence of highly correlated state<sup>17</sup>

$$|\psi^{-}\rangle = \frac{1}{\sqrt{2}}(|0\rangle|1\rangle - |1\rangle|0\rangle) \tag{6}$$

and Bell inequalities (violated by these states).

Ekert showed that if put together, they become useful producing private cryptographic key. In this way he discovered entanglement based quantum key distribution as opposed to the original BB84 scheme which uses directly quantum communication. The essence of the protocol is as follows: Alice and Bob can obtain from a source the EPR pairs. Measuring them in basis  $\{|0\rangle, |1\rangle\}$ , Alice and Bob obtain string of perfectly (anti)correlated bits, i.e. the key. To verify whether it is secure, they check Bell inequalities on selected portion of pairs. Roughly speaking, if Eve knew the values that Alice and Bob obtain in measurement, this would mean that the values exist before the measurement, hence Bell's inequalities would not be violated. Therefore, if Bell inequalities are violated, the values do not exist before Alice and Bob measurement, so it looks like nobody can know them.<sup>18</sup> First implementations of Ekert's cryptography protocol has been performed using polarizationentangled photons from spontaneous parametric downconversion (Naik et al., 2000) and entangled photons in energy-time (Tittel *et al.*, 2000).

After Ekert's idea, the research in quantum cryptography could have taken two paths. One was to treat violation of Bell inequality merely as a confirmation that Alice and Bob share good EPR states, as put forward in (Bennett *et al.*, 1992), because this is sufficient for privacy: if Alice and Bob have true EPR state, then nobody can know results of their measurements. This is what actually happened, for a long time only this approach was developed. In this case the eavesdropper Eve obeys the rules of quantum mechanics. We discuss it in Sec. XIX. The second path was to treat EPR state as the source of strange correlations that violate Bell inequality (see Sec. IV). This leads to new definition of security: against eavesdropper who do not have to obey the rules of quantum mechanics, but just the no-faster-then-light communication principle. The main task of this approach, which is an unconditionally secure protocol has been achieved only recently (Barrett *et al.*, 2005; Masanes *et al.*, 2006; Masanes and Winter, 2006).

### B. Quantum dense coding

In quantum communication there holds a reasonable bound on the possible miracles stemming from quantum formalism. This is the *Holevo bound* (Holevo, 1973). Roughly speaking it states, that one qubit can carry at most only one bit of classical information. In 1992, Bennett and Wiesner have discovered a fundamental primitive, called dense coding, which can come around the Holevo bound. Dense coding allows to communicate two classical bits by sending one a priori entangled qubit.

Suppose Alice wants to send one of four messages to Bob, and can send only one qubit. To communicate two bits sending one qubit she needs to send a qubit in one of  $2^2 = 4$  states. Moreover the states need to be mutually orthogonal, as otherwise Bob will have problems with discriminating them, and hence the optimal bound 2 will not be reached. But there are only two orthogonal states of one qubit. Can entanglement help here? Let Alice and Bob instead share a EPR state. Now the clever idea comes: it is not that qubit which is sent that should be in four orthogonal state, but the pair of entangled qubits together. Let us check how it works. Suppose Alice and Bob share a singlet state (6). If Alice wants to tell Bob one of the four events  $k \in \{0, 1, 2, 3\}$  she rotates her qubit (entangled with Bob) with a corresponding transformation  $\sigma_k$ :

$$\sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$
$$\sigma_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \ -i\sigma_3 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$
(7)

The singlet state (6) rotated by  $\sigma_k$  on Alice's qubit becomes the corresponding  $|\psi_k\rangle$  Bell state<sup>19</sup>. Hence  $|\psi_k\rangle = [\sigma_k]_A \otimes I_B |\psi_0\rangle$  is orthogonal to  $|\psi_{k'}\rangle = [\sigma_{k'}]_A \otimes I_B |\psi_0\rangle$ 

<sup>&</sup>lt;sup>17</sup> This state is also referred to as *singlet*, *EPR state* or *EPR pair*. If not explicitly stated, we will further use these names to denote any maximally entangled state, also in higher dimensions, see section VI.B.3.

<sup>&</sup>lt;sup>18</sup> In fact, the argument is more subtle. This is because in principle the values which did not preexist could come to exist in a way that is immediately available to a third party - Eve, i.e. the values that were not known to anybody could happen to be known to everybody when they come to exist. To cope with this problem Ekert has used the fact that singlet state can not be correlated with any environment. Recently it turned out that one can argue basing solely on Bell inequalities by means of so called monogamy of nonlocal correlations (Acin *et al.*, 2006a; Barrett *et al.*, 2005, 2006; Masanes and Winter, 2006).

<sup>&</sup>lt;sup>19</sup> In correspondence with Bell basis defined in Eq. (3) we have here  $|\psi_0\rangle = |\psi^-\rangle$ ,  $|\psi_1\rangle = |\phi^-\rangle$ ,  $|\psi_2\rangle = |\psi^+\rangle$ ,  $|\psi_3\rangle = |\phi^+\rangle$ .

for  $k \neq k'$  because Bell states are mutually orthogonal. Now if Bob gets Alice's half of the entangled state, after rotation he can discriminate between 4 Bell states, and infer k. In this way Alice sending one qubit has given Bob log 4 = 2 bits of information.

Why does not this contradict the Holevo bound? This is because the communicated qubit was a priori entan*aled* with Bob's qubit. This case is not covered by Holevo bound, leaving place for this strange phenomenon. Note also that as a whole, two qubits have been sent: one was needed to share the EPR state. One can also interpret this in the following way: sending first half of singlet state (say it is during the night, when the channel is cheaper) corresponds to sending one bit of *potential* communication. It is thus just as creating the possibility of communicating 1 bit in future: at this time Alice may not know what she will say to Bob in the future. During day, she knows what to say, but can send only one qubit (the channel is expensive). That is, she sent only one bit of actual communication. However sending the second half of singlet as in dense coding protocol she uses both bits: the *actual* one and *potential* one, to communicate in total 2 classical bits. Such an explanation assumes that Alice and Bob have a good quantum memory for storing EPR states, which is still out of reach of current technology. In the original dense coding protocol, Alice and Bob share pure maximally entangled state. The possibility of dense coding based on partially entangled pure and mixed states in multiparty settings were considered in (Barenco and Ekert, 1995; Bose et al., 2000; Bruß et al., 2005; Hausladen et al., 1996; Mozes et al., 2005; Ziman and Buzek, 2003) (see Sec. XIV.F. The first experimental implementation of quantum dense coding was performed in Innsbruck (Mattle et al., 1996a), using as a source of polarization-entangled photons (see further experiments with using nuclear magnetic resonance (Fang et al., 1999), two-mode squeezed vacuum state (Mizuno et al., 2005) and controlled dense coding with EPR state for continuous variable (Jietai et al., 2003)).

#### C. Quantum teleportation

Suppose Alice wants to communicate to Bob an *unknown quantum bit*. They have at disposal only a classical telephone, and one pair of entangled qubits. One way for Alice would be to measure the qubit, guess the state based on outcomes of measurement and describe it to Bob via telephone. However, in this way, the state will be transferred with very poor fidelity. In general an *unknown* qubit can not be described by classical means, as it would become cloneable, which would violate the main principle of quantum information theory: *a qubit in an unknown quantum state cannot be cloned* (Dieks, 1982; Wootters and Zurek, 1982).

However, Alice can send the qubit to Bob at the price of simultaneously erasing it at her site. This is the

essence of *teleportation*: a quantum state is transferred from one place to another: not copied to other place, but *moved* to that place. But how to perform this with a pair of maximally entangled qubits? Bennett, Brassard, Crepeau Jozsa, Peres and Wootters have found the answer to this question in 1993 (Bennett *et al.*, 1993).

To perform teleportation, Alice needs to measure her qubit and part of a maximally entangled state. Interestingly, this measurement is itself entangling: it is projection onto the basis of four Bell states (3). Let us follow the situation in which she wants to communicate a qubit in state  $|q\rangle = a|0\rangle + b|1\rangle$  on system A with use of a singlet state residing on her system A' and Bob's system B. The total initial state which is

$$|\psi_{AA'B}\rangle = |q\rangle_A \otimes \frac{1}{\sqrt{2}} [|0\rangle|0\rangle + |1\rangle|1\rangle]_{A'B}$$
(8)

can be written using the Bell basis (3) on the system AA' in the following way:

$$|\psi_{AA'B}\rangle = \frac{1}{2}[|\phi^{+}\rangle_{AA'}(a|0\rangle_{B} + b|1\rangle_{B}) +|\phi^{-}\rangle_{AA'}(a|0\rangle_{B} - b|1\rangle_{B}) +|\psi^{+}\rangle_{AA'}(a|1\rangle_{B} + b|0\rangle_{B}) +|\psi^{-}\rangle_{AA'}(a|1\rangle_{B} - b|0\rangle_{B})].$$
(9)

Now when Alice measures her systems AA' in this basis, she induces equiprobably the four corresponding states on Bob's system. The resulting states on system B are very similar to the state of qubit  $|q\rangle$  which Alice wanted to send him. They however admix to the initial state of system B. Thus Bob does not get any information instantaneously. However the output structure revealed in the above equation can be used: now Alice tells to Bob her result via telephone. Accordingly to those two bits of information (which of Bell states occurred on AA') Bob rotates his gubit by one of the four Pauli transformations (7). This is almost the end. After each rotation, Bob gets  $|q\rangle$  at his site. At the same time, Alice has just one of the Bell states: the systems A and A' becomes entangled after measurement, and no information about the state  $|q\rangle$  is left with her. That is, the no cloning principle is observed, while the state  $|q\rangle$  was transferred to Bob.

There is a much simpler way to send qubit to Bob: Alice could just send it directly. Then however, she has to use a quantum channel, just at the time she wants to transmit the qubit. With teleportation, she might have send half of EPR pair at any earlier time, so that only classical communication is needed later.

This is how quantum teleportation works in theory. This idea has been developed also for other communication scenarios (see (Dür and Cirac, 2000c)). It became immediately an essential ingredient of many quantum communication protocols. After pioneering experiments (Boschi *et al.*, 1998; Bouwmeester *et al.*, 1997; Furusawa *et al.*, 1998), there were beautiful experiments performing teleportation in different scenarios during last decade (see e.g. (Barrett *et al.*, 2004; Marcikic *et al.*, 2004; Nielsen *et al.*, 1998; Riebe *et al.*, 2004; Ursin *et al.*, 2004)). For the most recent one with mesoscopic objects, see (Sherson *et al.*, 2006).

#### D. Entanglement swapping

Usually quantum entanglement originates in certain *direct interaction* between two particles placed close together. Is it possible to get entangled (correlated in quantum way) some two particles which have never interacted in the past? The answer is positive (Bennett *et al.*, 1993; Yurke and Stoler, 1992b; Żukowski *et al.*, 1993).

Let Alice share a maximally entangled state  $|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)_{AC}$  with Clare, and Bob share the same state with David:

$$|\phi^+\rangle_{AC} \otimes |\phi^+\rangle_{BD}.\tag{10}$$

Such state can be obviously designed in such a way, that particles A and D have never seen each other. Now, Clare and Bob perform a *joint* measurement in Bell basis. It turns out that for any outcome, the particles A and D collapse to some Bell state. If Alice and Bob will get to know the outcome, they can perform local rotation, to obtain entangled state  $\Phi_{AD}^+$ . In this way particles of Alice and David are entangled although they never interacted directly with each other, as they originated from different sources.

One sees that this is equivalent to teleporting one of the EPR pair through the second one. Since the protocol is symmetric, any of the pairs can be chosen to be either the channel, or the teleported pair.

This idea has been adopted in order to perform quantum repeaters (Dür et al., 1999a), to allow for distributing entanglement — in principle — between arbitrarily distant parties. It was generalized to a multipartite scenario in (Bose et al., 1998). Swapping can be used as a tool in multipartite state distribution, which is for example useful in quantum cryptography.

The conditions which should be met in optical implementation of entanglement swapping (as well as teleportation) have been derived in (Żukowski *et al.*, 1993). Along those lines entanglement swapping was realized in lab (Pan *et al.*, 1998).

# E. Beating classical communication complexity bounds with entanglement

Yao (Yao, 1979) has asked the following question: how much communication is needed in order to solve a given problem distributed among specially separated computers? To be more concrete, one can imagine Alice having got the *n*-bit string *x* and Bob having got *n*-bit string *y*. Their task is to infer the value of some a priori given function f(x, y) taking value in  $\{0, 1\}$ , so that finally both parties know the value. The minimal amount of bits needed in order to achieve this task is called the *communication complexity* of the function f.

Again, one can ask if entanglement can help in this case. This question was first asked by Cleve and Buhrman (Cleve and Buhrman, 1997) and in a different context independently by Grover (Grover, 1997), who showed the advantage of entanglement assisted distributed computation over the classical one.

Consider the following example which is a three-party version of the same problem (Buhrman *et al.*, 2001a). Alice Bob and Clare get two bits each:  $(a_1, a_0)$   $(b_1, b_0)$  and  $(c_1, c_0)$  representing a binary two-digit numbers  $a = a_1a_0$ ,  $b = b_1b_0$  and  $c = c_1c_0$ . They are promised to have

$$a_0 \oplus b_0 \oplus c_0 = 0. \tag{11}$$

The function which they are to compute is given by

$$f(a,b,c) = a_1 \oplus b_1 \oplus c_1 \oplus (a_0 \lor b_0 \lor c_0).$$
(12)

It is easy to see, that announcing of four bits is sufficient so that all three parties could compute f. One party announces both bits (say it is Alice)  $a_1a_0$ . Now if  $a_0 = 1$ , then the other parties announces their first bits  $b_1$  and  $c_1$ . If  $a_0 = 0$ , then one of the other parties, (say Bob) announces  $b_1 \oplus b_0$  while Clare announces  $c_1$ . In both cases all parties compute the function by adding announced bits modulo 2. Thus four bits are enough. A bit more tricky is to show that four bits are *necessary*, so that the classical communication complexity of the above function equals four (Buhrman *et al.*, 2001a).

Suppose now, that at the beginning of the protocol Alice Bob and Clare share a quantum three-partite entangled state:

$$|\psi_{ABC}\rangle = \frac{1}{2}[|000\rangle - |011\rangle - |101\rangle - |110\rangle]_{ABC},$$
 (13)

such that each party holds a corresponding qubit. It is enough to consider the action of Alice as the other parties will do the same with their classical and quantum data respectively.

Alice checks her second bit. If  $a_0 = 1$  she does a Hadamard<sup>20</sup> on her qubit. Then she measures it in  $\{|0\rangle, |1\rangle\}$  basis noting the result  $r_A$ . She then announces a bit  $a_1 \oplus r_A$ .

One can check, that if all three parties do the same, the r-bits with "quantum origin" fulfills  $r_A \oplus r_B \oplus r_C = a_0 \lor b_0 \lor c_0$ . It gives in turn

$$(a_1 \oplus r_A) \oplus (b_1 \oplus r_B) \oplus (c_1 \oplus r_C) =$$

$$a_1 \oplus b_1 \oplus c_1 \oplus (r_A \oplus r_B \oplus r_C) =$$

$$a_1 \oplus b_1 \oplus c_1 \oplus (a_0 \lor b_0 \lor c_0) = f(a, b, c).$$
(14)

<sup>&</sup>lt;sup>20</sup> Hadamard transformation is a unitary transformation of basis which changes  $|0\rangle$  into  $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$  and  $|1\rangle$  into  $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$ .

Thus three bits are enough. The fourth, controlled by common quantum entanglement, got hidden in the three others, hence need not be announced.

One can bother that the quantum state (13) should enter the communication score, as it needs to be distributed among the parties. Thus in addition to 3 bits they need to send qubits. However similarly as in case of dense coding, the entangled state is independent of the function, i.e. it can be prepared prior to the knowledge of the form of the function.

It is known, that if exact evaluation of the function is required (as in the described case), sharing additionally correlated bit-strings (shared randomness) by the parties can not help them. Thus we see, that entanglement which can serve itself as a source of shared randomness, is a stronger resource. If one allows nonzero probability of error, then classical correlations can help, but it was shown in a bipartite scenario, that entanglement allows for smaller error. (Buhrman *et al.*, 2001a). Entanglement can beat classical correlations even exponentially as shown by R. Raz (Raz, 1999). Other schemes with exponential advantage are provided in (Gavinsky *et al.*, 2006b). They are closely related to the phenomenon called *quantum fingerprinting* (Buhrman *et al.*, 2001b; Gavinsky *et al.*, 2006a).

Interestingly, though the effect is of practical importance, in its roots there is purely philosophical question of type: "Is the Moon there, if nobody looks?" (Mermin, 1985). Namely, if the outcomes of the Alice, Bob and Clare measurements would exist prior to measurement, then they could not lead to reduction of complexity, because, the parties could have these results written on paper and they would offer three bit strategy, which is not possible. As a matter of fact, the discoveries of reduction of communication complexity used GHZ paradox in Mermin version, which just says that the outcomes of the four possible measurements given by values of  $a_0$ ,  $b_0$ and  $c_0$  (recall the constraint  $a_0 \oplus b_0 \oplus c_0$ ) performed on the state (13) cannot preexist.

In (Brukner *et al.*, 2004) it was shown that this is quite generic: for *any* correlation Bell inequality for *n* systems, a state violating the inequality allows to reduce communication complexity of some problem. For recent experiment see (Trojek *et al.*, 2005).

# IV. CORRELATION MANIFESTATIONS OF ENTANGLEMENT: BELL INEQUALITIES.

## A. Bell theorem: CHSH inequality.

The physical consequences of the existence of entangled (inseparable) states are continuously subject of intensive investigations in the context of both EPR paradox and quantum information theory. They manifest itself, in particular, in correlation experiments via Bell theorem which states that the probabilities for the outcomes obtained when suitably measuring some quantum states cannot be generated from classical correlations. As a matter of fact, Bell in his proof assumed perfect correlations exhibited by the singlet state. However, in real experiments such correlations are practically impossible. Inspired by Bell's paper, Clauser, Horne, Shimony and Holt (CHSH) (Clauser *et al.*, 1969) derived a correlation inequality, which provides a way of experimental testing of the LHVM as independent hypothesis separated from quantum formalism. Consider correlation experiment in which the variables  $(A_1, A_2)$  are measured on the one of the subsystems of the whole system while  $(B_1, B_2)$  on the other one, and that the subsystems are spatially separated. Then the LHVM imposes the following constraints on the statistics of the measurements on the sufficiently large ensemble of systems<sup>21</sup>

$$|E(A_1, B_1) + E(A_1, B_2) + E(A_2, B_1) - E(A_2, B_2)| \le 2,$$
(15)

where  $E(A_i, B_j)$  is the expectation value of the correlation experiment  $A_i B_j$ .

This is the CHSH inequality that gives a bound on any LHVM. It involves only two-partite correlation function for two alternative dichotomic measurements and it is complete in a sense that if full set of such inequalities is satisfied there exists a joint probability distribution for the outcomes of the four observables, which returns the measured correlation probabilities as marginals (Fine, 1982).

In quantum case the variables convert into operators and one can introduce the CHSH operator

$$\mathcal{B}_{CHSH} = \mathbf{A}_1 \otimes (\mathbf{B}_1 + \mathbf{B}_2) + \mathbf{A}_2 \otimes (\mathbf{B}_1 - \mathbf{B}_2), \quad (16)$$

where  $\mathbf{A}_1 = \mathbf{a}_1 \cdot \sigma$ ,  $\mathbf{A}_2 = \mathbf{a}_2 \cdot \sigma$ , similarly for  $\mathbf{B}_1$  and  $\mathbf{B}_2$ ,  $\sigma = (\sigma_x, \sigma_y, \sigma_z)$  is the vector of Pauli operators,  $\mathbf{a} = (a_x, a_y, a_z)$  etc., are unit vectors describing the measurements that the parties: A (Alice) and B (Bob) perform. Then, the CHSH inequality requires that the condition

$$|\mathrm{Tr}(\mathcal{B}_{CHSH}\rho)| \le 2 \tag{17}$$

is satisfied for all states  $\rho$  admitting a LHVM.

Quantum formalism predicts Tsirelson inequality (Tsirelson, 1980)

$$|\langle \mathcal{B}_{CHSH} \rangle_{QM}| = |\text{Tr}(\mathcal{B}_{CHSH}\rho)| \le 2\sqrt{2} \qquad (18)$$

for all states  $\rho$  and all observables  $\mathbf{A}_1, \mathbf{A}_2, \mathbf{B}_1, \mathbf{B}_2$ . Clearly, the CHSH inequality can be violated for some choices of observables, implying violation of LHVM. For singlet state  $\rho = |\psi^-\rangle\langle\psi^-|$ , there is maximal violation  $|\text{Tr}(\mathcal{B}_{CHSH}\rho)| = 2\sqrt{2}$  which corresponds to Tsirelson bound.

<sup>&</sup>lt;sup>21</sup> It is assumed here that the variables  $A_i$  and  $B_i$  for i = 1, 2 are dichotomic i.e. have values  $\pm 1$ .

#### **B.** The optimal CHSH inequality for $2 \times 2$ systems

At the beginning of 90's there were two basic questions: Firstly, it was hard to say whether a given state violates the CHSH inequality, because one has to construct a respective Bell observable for it. In addition, given a mixed state, there was no way to ensure whether ii satisfies the CHSH inequality for each Bell observable. This problem was solved completely for an arbitrary quantum states  $\rho$ of two qubits by using Hilbert-Schmidt space approach (Horodecki *et al.*, 1995). Namely, *n*-qubit state can be written as

$$\rho = \frac{1}{2^n} \sum_{i_1 \dots i_n = 0}^3 t_{i_1 \dots i_n} \sigma_{i_1}^1 \otimes \dots \otimes \sigma_{i_n}^n, \qquad (19)$$

where  $\sigma_0^k$  is the identity operator in the Hilbert space of qubit k, and the  $\sigma_{i_k}^k$  correspond to the Pauli operators for three orthogonal directions  $i_k = 1, 2, 3$ . The set of real coefficients  $t_{i_1...i_n} = \text{Tr}[\rho(\sigma_{i_1} \otimes \cdots \otimes \sigma_{i_n})]$  forms a correlation tensor  $T_{\rho}$ . In particular for the two-qubit system the  $3 \times 3$  dimensional tensor is given by  $t_{ij} :=$  $\text{Tr}[\rho(\sigma_i \otimes \sigma_j)]$  for i, j = 1, 2, 3.

In this case one can compute the mean value of an arbitrary  $\mathcal{B}_{CHSH}$  in an arbitrary fixed state  $\rho$  and then maximize it with respect to all  $\mathcal{B}_{CHSH}$  observables. In result we have:  $\max_{\mathcal{B}_{CHSH}} |\operatorname{Tr}(\mathcal{B}_{CHSH}\rho)| = 2\sqrt{M(\rho)} = 2\sqrt{t_{11}^2 + t_{22}^2}$ , where  $t_{11}^2$  and  $t_{22}^2$  are the two largest eigenvalues of  $T_{\rho}^T T_{\rho}$ ,  $T_{\rho}^T$  is the transposed of  $T_{\rho}$ .

It follows that for  $2 \times 2$  systems the necessary and sufficient criterion for the violation of the CHSH inequality can be written in the form

$$M(\rho) > 1. \tag{20}$$

The quantity  $M(\rho)$  depends only on the state parameters and contains all the information that is needed to decide whether a state violates a CHSH inequality. Note that using  $M(\rho)$  one can define a measure of violation of the CHSH inequality  $B(\rho) = \sqrt{\max\{0, \sqrt{M(\rho)} - 1\}}$ (Miranowicz, 2004b), which for an arbitrary two-qubit pure state equals to measures of entanglement: negativity and *concurrence* (see Sec. XV.C.2.b). Clearly, in general, both  $M(\rho)$  and  $B(\rho)$  are not measures of entanglement as they do not behave monotonously under LOCC. However, they can be treated as entanglement parameters, in a sense that they characterize a degree of entanglement. In particular the inequality (20) provides practical tool for the investigation of nonlocality of the arbitrary two qubits mixed states in different quantum-information contexts (see e.g. (Hyllus *et al.*, 2005; Scarani and Gisin, 2001a; Walther et al., 2005)).

### C. Nonlocality of quantum states and LHV model

#### 1. Pure states

The second more fundamental question was: Are there many quantum states, which do not admit LHV model? more precisely: which quantum states do not admit LHV model? Even for pure states the problem is not completely solved.

Gisin proved that for the standard (ie. nonsequential) projective measurements the only pure 2-partite states which do not violate correlation CHSH inequality (15) are product states and they are obviously local (Gisin, 1991; Gisin and Peres, 1992). Then Popescu and Rohrlich showed that any n-partite pure entangled state can always be projected onto a 2-partite pure entangled state by projecting n-2 parties onto appropriate local pure states (Popescu and Rohrlich, 1992). Clearly, it needs an additional manipulations (postselection). Still the problem whether Gisin theorem, can be generalized without postselection for an arbitrary n-partite pure entangled states, remains open. In the case of three parties there are generalized GHZ states (Scarani and Gisin, 2001b; Zukowski et al., 2002) that do not violate the MABK inequalities (Ardehali, 1992; Belinskii and Klyshko, 1993; Mermin, 1990a) (see next subsec.). More generally it has been shown (Żukowski *et al.*, 2002) that these states do not violate any Bell inequality for n-partite correlation functions for experiments involving two dichotomic observables per site. Acin *et al.* and Chen *et al.* considered a Bell inequality that shows numerical evidence that all three-partite pure entangled state violate it (Acin et al., 2004a; Chen et al., 2004). Recently a stronger Bell inequalities with more measurement settings was presented (Laskowski et al., 2004; X.-H.Wu and Zong, 2003) which can be violated by a wide class of states, including the generalized GHZ states (see also (Chen *et al.*, 2006b)).

### 2. Mixed states

In the case of noisy entangled states the problem appeared to be much more complex. A natural conjecture was that only separable mixed states of form (38) admit a LHV model. Surprisingly, Werner constructed one parameter of family of  $U \otimes U$  invariant states (see Sec. VI.B.9) in  $d \times d$  dimensions, where U is a unitary operator and showed that some of them can be simulated by such model (Werner, 1989a). In particular, two-qubit (d = 2) Werner states are mixtures of the singlet  $|\psi^-\rangle$  with white noise of the form

$$\rho = p |\psi^{-}\rangle \langle \psi^{-}| + (1-p) \frac{\mathrm{I}}{4}.$$
(21)

It has been shown, using the criterion (20) (Horodecki *et al.*, 1995), that the CHSH inequality is violated when  $2^{-\frac{1}{2}} .$ 

However Popescu noticed that Werner's model accounts only for the correlations obtained for a reduced

class of local experiments involving projective measurements and he found that some Werner mixtures if subjected to sequence of local generalized measurements can violate CHSH inequality , (Popescu, 1995). Then Gisin demonstrated that even for the case of two qubits the "hidden nonlocality" can be revealed by using local filters in the first stage of the process (a procedure of this kind can be treated as a preprocessing consisting of stochastic-LOCC) (Gisin, 1996b). In the same year Peres discovered that some states admitting a LHVM for the single copy violate Bell inequalities when jointly measuring more than one copy with postselection procedure (Peres, 1996b). The merging of these two tests leads to the stronger detection of hidden nonlocality (Masanes, 2006).

A generic structure of any LHVM for experiments with postselection was considered and it was proved that LHVM for joint probabilities of outcomes of sequences of measurements cannot exist in the case of "hidden nonlocality" (Popescu, 1995; Teufel *et al.*, 1997; Żukowski *et al.*, 1998a). Barrett constructed a LHV model which simulates arbitrary single (nonsequential) positive-operator-valued (POV) measurements of single copies of a class of entangled Werner states (see Sec.IV) (Barrett, 2002).

It is interesting that the enhancement of nonlocal correlations is possible even without postselection by collective measurements (Liang and Doherty, 2006). The first experimental demonstration of "hidden nonlocality" has been reported by Kwiat *et al.* (Kwiat *et al.*, 2001), (see in this context (Walther *et al.*, 2005)).

Remarkably, the nonlocal properties of entangled states can be related to Grothendieck's constant, a mathematical constant appearing in Banach space theory (Grothendieck, 1953). Namely Grothendieck's constant gives a bound on the strength of violation of all bipartite Bell inequalities (Acin et al., 2006b; Tsirelson, 1987) Interestingly, it has been recently shown that it is no longer the case for tripartite Bell inequalities (Perez-Garcia et al., 2007). The situation changes if according to Bell's idea one assumes maximal correlations in choice of observables (see (Pitowsky, 2007)). It also turns out that nonlocal properties of the Werner states can be expressed via critical values  $p_c$  of the parameter p, for which the states cease to be nonlocal under projective measurements and which are related to Grothendieck's constant (Acin *et al.*, 2006b). In this way the existence of the LHVM for two-qubit Werner states (21) and projective measurements has been proved up to  $p \approx 0.66$ close to the region of the CHSH violation  $2^{-\frac{1}{2}} .$ 

Banaszek and Wódkiewicz first investigated nonlocality of the original EPR state in the Wigner representation and showed that EPR state is strongly nonlocal, though its Wigner function is positive (Banaszek and Wódkiewicz, 1998). The observables producing violation of Bell-type inequalities are displaced photon-number parities rather than continuous variables. It has been shown, that all squeezed Gaussian states are nonlocal despite they have positive Wigner function (van Loock and Braunstein, 2001).

There is another algorithmic approach for the problem of nonlocality of quantum states (Terhal *et al.*, 2003a). It is based on constructing a symmetric quasi-extension of the quantum state via semidefinite program. It is interesting that the method allows to construct (analytically) LHV model for bound entangled states with two measurement settings generated by real unextendible product basis (see in this context (Kaszlikowski *et al.*, 2002)).

The above results allow to understand the deep nature of noisy entanglement not only in the context of LHV model. Clearly the CHSH inequality can be used as a tool for two nonequivalent tasks: testing quantum formalism against LHV model (nonlocality witness) and testing for entanglement within quantum formalism (entanglement witness). On the other hand, the idea of "hidden nonlocality" leads to the concept of distillation of entanglement - an important notion in the quantum information theory (see Sec. XII). Finally, it turned out that hidden nonlocality allows to reveal nonclassical features of arbitrary entangled state. Namely, Masanes et al. by use of specific entanglement witnesses characterized the states that violate CHSH inequality after local filtering operators (Masanes et al., 2007). Then they has proved that any bipartite entangled state  $\sigma$  exhibits a hidden nonlocality which can be "activated" in a sense that there exist another state  $\rho$  not violating CHSH such that the state  $\rho \otimes \sigma$  violates CHSH (see sec. XII.I in this context).

### D. Bell theorem beyond CHSH-setting

The CHSH inequality is one elementary inequality, that can be viewed as a special case of an infinite hierarchy of Bell inequalities related to the type of correlation measurements with n-partite systems, where each of the parties has the choice of m *l*-valued observables to be measured. For the CHSH inequality n = m = l = 2.

As early as 1990 several generalizations of the latter were derived for the case n, 2, 2 (MABK inequalities). The complete set of such inequalities was constructed by Werner and Wolf (Werner and Wolf, 2001b) and independently by Żukowski and (Žukowski and Brukner, 2002),Brukner see also (Weinfurter and Žukowski, 2001). The inequality WWZB is given by a linear combination of the correlation expectation values

$$\sum_{k} f(k)E(k) \le 2^n, \tag{22}$$

where coefficients are given by  $f(k) = \sum_{s} S(s)(-1)^{\langle k,s \rangle}$ , S(s) is an arbitrary function of  $s = s_1...s_n \in \{-1,1\}^n$ , such that  $S(s_1...s_n) = \pm 1$ ;  $\langle k,s \rangle = \sum_{j=1}^n k_j s_j$  and  $E(k) = \langle \prod_{j=1}^n A_j(k_j) \rangle_{av}$  is the correlation function (average over many runs of experiment) labeled by a bit string  $k = k_1...k_n$ , binary variables  $k_j \in 0, 1$  indicate the choice of the  $\pm 1$ -valued, observable  $A_j(k_j)$  at site j.

There are  $2^{2^n}$  different functions S(s), and correspondingly  $2^{2^n}$  inequalities. In particular, putting  $S(s_1...s_n) = \sqrt{2} \cos(-\frac{\pi}{4} + (s_1 + ... + s_n - n)\frac{\pi}{4})$  one recovers the Mermintype inequalities and for n = 2 the CHSH inequality (15) follows.

Fortunately the set of linear inequalities (22) is equivalent to a single non-linear inequality

$$\sum_{s} \left| \sum_{k} (-1)^{\langle k, s \rangle} E(k) \right| \le 2^n, \tag{23}$$

which characterizes the structure of the accessible classical region for correlation function for *n*-partite systems being a hyper-octahedron in  $2^n$  dimensions as the unit sphere of the Banach space  $l^1$  (Werner and Wolf, 2001b). Hence it is necessary and sufficient condition for correlation function of *n*-partite systems to admit LHVM.

In the quantum case the observables  $A_j(k_j)$  convert into Hermitian operators  $A_j(k_j) = \mathbf{a}_j(k_j) \cdot \vec{\sigma}$  with spectrum in  $\{-1, +1\}$  where  $\sigma$  is the vector of Pauli matrices and  $\mathbf{a}_j(k_j)$  is a normalized vector in  $\mathbb{R}^3$ . Consequently the n-qubit quantum correlation function has the form:

$$E_q(k) = \operatorname{Tr}[\rho \otimes_{j=1}^n \mathbf{a}_j(k_j) \cdot \vec{\sigma}] = \langle T, \mathbf{a}_1 \otimes \cdots \otimes \vec{\mathbf{a}}_n \rangle.$$
(24)

Inserting  $E_q(k)$  to the expression (23) and using (19) one obtains that the correlations between the measurement on n qubits admit an LHVM iff in any set of local coordinate system and for any set of unit vectors  $\mathbf{c}_j = (c_1^j, c_2^j) \in \mathbb{R}^2$  the following inequality holds (Żukowski and Brukner, 2002)

$$T_{c_1...c_n}^{mod} \equiv \sum_{i_1...i_n}^2 c_{i_1}^1...c_{i_n}^n |t_{i_1...i_n}| \le 1,$$
(25)

where  $T_{c_1...c_n}^{mod}$  is a component of moduli of correlation tensor T, along directions defined by the unit vectors  $\mathbf{c}_j$ . In the spirit of the Peres separability criterion one can express the above condition as follows: within LHVM  $T^{mod}$  is also a possible correlation tensor.

Note that the Bell-WWZB operator for n-partite system with two dichotomic observables per site admits spectral decomposition into n-qubit GHZ states (82) which lead to maximal violations (Scarani and Gisin, 2001b), (Werner and Wolf, 2001c). Thus the GHZ states play a similar role for the WWZB inequality as the Bell states for the CHSH one.

The WWZB inequalities are important tool for the investigation of possible connections between quantum nonlocality distillability and entanglement for n-qubit systems. In particular, it has been shown that the violation of the WWZB inequality by a multi-qubit states implies that pure entanglement can be distilled from it. But the protocol may require that some of the parties join into several groups (Acin, 2002; Acin *et al.*, 2003c).

This result was generalized to the asymptotic scenario (Masanes, 2006).

Other n-particle Bell type inequalities based on the assumption of partial separability were found which are maximally violated by n-particle GHZ-states (Seevinck and Svetlichny, 2002). There were different generalizations of Bell inequalities for bipartite systems using more than two dichotomic measurements per site ((Collins and Gisin, 2004; Pitowsky and Svozil, 2001; Śliwa, 2003; Żukowski, 2006). The only in 2004 Collins and Gisin has showed that the inequality involving three dichotomic observables per site can "detect" nonlocality of the two-qubit states which escapes the detection via CHSH inequality (Collins and Gisin, 2004). Collins et al. developed a novel approach to Bell inequalities based on logical constraints local variables models must satisfy and introduced family of Bell inequalities for bipartite systems of arbitrarily high dimensionality (Collins et al., 2002). A very appealing form of these inequalities has been recently presented in (Zohren and Gill, 2006). Surprisingly for the above inequalities in contrast to the CHSH inequality the optimal state in dimension d > 2 is not maximally entangled.

Interestingly, it has been recognized that there is a tradeoff between the local commutativity and Bell inequality non-violating (Roy, 2005). This idea was developed in (Seevinck and Uffink, 2007).

Note that there are Bell inequalities of other type such as: Bell inequalities in phase space (Auberson *et al.*, 2003); and entropic Bell inequalities, which involve classical entropy (Braunstein and Caves, 1988). However, the entropic ones are weaker than the CHSH inequality. In (Nagata *et al.*, 2004) Bell inequalities were considered in the context of rotational invariance. It should be noted, that the pioneering investigation of the Bell inequalities in the relativistic context was performed by Werner and Summers (Summers and Werner, 1985) and Czachor (Czachor, 1997).

#### E. Logical versions of Bell's theorem

The violation of Bell inequalities as a paradigmatic test of quantum formalism has some unsatisfactory feature as it only applies to a statistical measurement procedure. It is intriguing, that the quantum formalism via quantum entanglement offers even stronger departure from classical intuition based on the logical argument which discusses only perfectly correlated states. This kind of argument was discovered by Greenberger, Horne and Zeilinger (GHZ) and applies for all multipartite systems that are in the same GHZ state i.e.  $|\psi\rangle_{ABC} = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)_{ABC}$ . The authors showed that any deterministic LHVM predicts that a certain outcome always happen while quantum formalism predicts it never happens.

The original GHZ argument for qubits has been subsequently developed (Cabello, 2001a,b; Cerf *et al.*, 2002a; Chen *et al.*, 2003; Greenberger *et al.*, 2005a,b; Hardy, 1993; Mermin, 1990b) and extended to continuous variables (Chen and Zhang, 2001; Clifton, 2000; Massar and Pironio, 2001) and it is known as the "allversus-nothing" proof of Bell theorem or "Bell theorem without inequalities". The proofs are purely logical. However, in real experiments ideal measurements and perfect correlations are practically impossible. To overcome the problem of a "null experiment" Bell-type inequalities are needed.

An important step in this direction was the reduction of the GHZ proof to the two-particle nonmaximally (Hardy, 1993) and maximally entangled systems of high-dimensionality (Chen *et al.*, 2005d; Durt *et al.*, 2001; Kaszlikowski *et al.*, 2000; Pan *et al.*, 2001; Torgerson *et al.*, 1995). Recently a two-particle allversus-nothing nonlocality tests were performed by using two-photon so-called hyperentanglement (Cinelli *et al.*, 2005; Yang *et al.*, 2005b). It is instructive to see how the GHZ argument works in this special case, when a single source produces a two-photon state simultaneously maximally entangled both in the polarization and in the path degrees of freedom. It can be written in the form:

$$|\Psi\rangle = \frac{1}{2} (|H\rangle_A |V\rangle_B - |V\rangle_A |H\rangle_B) (|R\rangle_A |L\rangle_B - |L\rangle_A |R\rangle_B).$$
(26)

Here photon-A and photon-B pertain to spatially separated observers Alice and Bob,  $|H\rangle$  and  $|V\rangle$  stand for photons with horizontal and vertical polarization and R, L are the spatial (path) modes.

Now, one can define the set of the nine local observables corresponding to  $|\Psi\rangle$  to be measured by Alice and Bob as follows:  $a_i = (z'_i, x_i, x_i \cdot z'_i), b_i = (z_i, x'_i, z_i \cdot x'_i),$  $c_i = (z_i z'_i, x_i x'_i, z_i z'_i \cdot x_i x'_i)$ , where i = A, B and  $z_i = \sigma_{z_i},$  $x_i = \sigma_{x_i}, z'_i = \sigma'_{z_i}, x'_i = \sigma'_{x_i}$  are Pauli-type operators for the polarization and the path degree of freedom respectively. The observables are arranged in three groups a, b, and c, where the observables of each group are measured by one and the same apparatus.

Then quantum formalism makes the following predictions for perfect correlations:

$$\begin{aligned} z_A \cdot z_B \left| \Psi \right\rangle &= - \left| \Psi \right\rangle, z'_A \cdot z'_B \left| \Psi \right\rangle = - \left| \Psi \right\rangle, \\ x_A \cdot x_B \left| \Psi \right\rangle &= - \left| \Psi \right\rangle, x'_A \cdot x'_B \left| \Psi \right\rangle = - \left| \Psi \right\rangle, \\ z_A z'_A \cdot z_B z'_B \left| \Psi \right\rangle &= \left| \Psi \right\rangle, x_A x'_A \cdot x_B x'_B \left| \Psi \right\rangle = \left| \Psi \right\rangle, \\ z_A \cdot x'_A \cdot z_B x'_B \left| \Psi \right\rangle &= \left| \Psi \right\rangle, x_A \cdot z'_A \cdot x_B z'_B \left| \Psi \right\rangle = \left| \Psi \right\rangle, \\ z_A z'_A \cdot x_A x'_A \cdot z_B x'_B \cdot x_B z'_B \left| \Psi \right\rangle &= - \left| \Psi \right\rangle. \end{aligned}$$

According to EPR the perfect correlations allow assign the pre-existing measurement values v(X) to the above observables X. Then the consistency between quantum predictions and any LHVM gives the following relations between the "elements of reality":

$$\begin{aligned} v(z_A)v(z_B) &= -1, v(z'_A)v(z'_B) = -1, \\ v(x_A)v(z_B) &= -1, v(x'_A)v(x'_B) = -1, \\ v(z_A z'_A)v(z_B)v(z'_B) &= 1, v(x_A x'_A)v(x_B)v(x'_B) = 1, \\ v(z_A)v(x'_A)v(z_B x'_B) &= 1, v(x_A)v(z'_A)v(x_B z'_B) = 1, \end{aligned}$$

$$v(z_A z'_A)v(x_A x'_A)v(z_B x'_B)v(x_B z'_B) = -1.$$

Multiplying the above equations except the last one, one gets  $v(z_A z'_A)v(x_A x'_A)v(z_B x'_B)v(x_B z'_B) = 1$ , that contradicts the last one.

In real experiments the nonlocality tests are based on the estimation of the Bell type operator  $O = -z_A \cdot z_B - z'_A \cdot z'_B - x_A \cdot x_B - x'_A \cdot x'_B + z_A z'_A \cdot z_B \cdot z'_B + x_A x'_A \cdot x_B \cdot x'_B + z_A \cdot x'_A \cdot z_B x'_B + x_A \cdot z'_A \cdot x_B z'_B - z_A z'_A \cdot x_A x'_A \cdot z_B x'_B \cdot x_B z'_B$  and Bell-Mermin type inequality (Cabello, 2001a,b; Chen *et al.*, 2003)

$$\langle O \rangle_L \le 7.$$
 (27)

It is easy to check that the operator O satisfies the relation  $O |\Psi\rangle = 9 |\Psi\rangle$ , which is in contradiction with the above inequality implied by LHVM. In a recent experiment (Yang *et al.*, 2005b) observed value for O was 8.56904  $\pm$  0.00533. It demonstrates a contradiction of quantum predictions with LHVM (see in this context related experiment (Cinelli *et al.*, 2005)). A novel nonlocality test "stronger two observer all versus nothing" test (Cabello, 2005) have been performed by use of a 4-qubit linear cluster state via two photons entangled both in polarization and linear momentum (Vallone *et al.*, 2007).

Note that, the GHZ argument has been extended to the entangled two-particle states that are produced from two independent sources (Yurke and Stoler, 1992b) by the process of "entanglement swapping" (Żukowski *et al.*, 1993). The complete proof depends crucially on the independence of the two sources and involves the factorization of the Bell function (Greenberger *et al.*, 2005a).

#### F. Violation of Bell inequalities: general remarks

There is a huge literature concerning interpretation of the "Bell effect". The most evident conclusion from those experiments is that it is not possible to construct a LHVM simulating all correlations observed for quantum states of composite systems. But such conclusion is not surprising. What is crucial in the context of the Bell theorem is just a *gap* between quantum and classical description of the correlations, which gets out of hand.

Nature on its fundamental level offers us a new kind of statistical non-message-bearing correlations, which are encoded in the quantum description of states of compound systems via entanglement. They are "nonlocal"<sup>22</sup> in the sense, that

i) They cannot be described by a LHVM.

ii) They are *nonsignaling* as the local measurements performed on spatially separated systems cannot be used to transmit messages.

 $<sup>^{22}</sup>$  The term nonlocality is somewhat misleading. In fact there is a breaking of conjunction of locality and counterfactuality.

Generally speaking, quantum compound systems can reveal *holistic nonsignaling* effects even if their subsystems are spatially separated by macroscopic distances. In this sense quantum formalism offers holistic description of Nature (Primas, 1983), where in a non-trivial way the system is more than a combination of its subsystems.

It is intriguing that entanglement does not exhaust full potential of nonlocality under the constraints of Indeed, it does not violate Tsirelson's no-signaling. bound (18) for CHSH inequalities. On the other hand, one can design family of probability distributions (see (Gisin, 2005) and ref. therein), which would violate this bound but still do not allow for signaling. They are called Popescu-Rohrlich nonlocal boxes, and represent extremal nonlocality admissible without signaling (Popescu and Rohrlich, 1994). We see therefore that quantum entanglement is situated in an intermediate level between locality and maximal non-signaling nonlocality. Needless to say, the Bell inequalities still involve many fascinating open problems interesting for both philosophers and physicists (see (Gisin, 2007)).

#### V. ENTROPIC MANIFESTATIONS OF ENTANGLEMENT

#### A. Entropic inequalities: classical versus quantum order

It was mentioned in the introduction that Schrödinger first pointed out, that entanglement does not manifest itself only as correlations. In fact he has recognized an other aspect of entanglement, which involves profoundly nonclassical relation between the information which an entangled state gives us about the whole system and the information which it gives us about subsystems.

This new "nonintuitive" property of compound quantum systems, intimately connected with entanglement, was a long-standing puzzle from both a physical and a mathematical point of view. The main difficulty was, that in contrast to the concept of correlation, which has a clear operational meaning, the concept of information in quantum theory was obscure till 1995, when Schumacher showed that the von Neumann entropy

$$S(\rho) = -\mathrm{Tr}\rho\log\rho \tag{28}$$

has the operational interpretation of the number of qubits needed to transmit quantum states emitted by a statistical source (Schumacher, 1995b). The von Neumann entropy can be viewed as the quantum counterpart of the Shannon entropy  $H(X) = -\sum_i p_i \log p_i$ ,  $\sum_i p_i = 1$ , which is defined operationally as the minimum number of bits needed to communicate a message produced by a classical statistical source associated to a random variable X.

Fortunately in 1994 Schrödinger's observation, that an entangled state gives us more information about the total system than about subsystems was, quantified just by means of von Neumann entropy. It was shown that the entropy of a subsystem can be greater than the entropy of the total system only when the state is entangled (Horodecki and Horodecki, 1994). In other words the subsystems of entangled system may exhibit more disorder than the system as a whole. In the classical world it never happens. Indeed, the Shannon entropy H(X) of a single random variable is never larger than the entropy of two variables

$$H(X,Y) \ge H(X), \quad H(X,Y) \ge H(Y).$$
(29)

It has been proved (Horodecki and Horodecki, 1996; Horodecki *et al.*, 1996c; Terhal, 2002; Vollbrecht and Wolf, 2002b) that analogous  $\alpha$ -entropy inequalities hold for separable states

$$S_{\alpha}(\rho_{AB}) \ge S_{\alpha}(\rho_A), \quad S_{\alpha}(\rho_{AB}) \ge S_{\alpha}(\rho_B),$$
(30)

where  $S_{\alpha}(\rho) = (1-\alpha)^{-1} \log \operatorname{Tr} \rho^{\alpha}$  is the  $\alpha$ -Renyi entropy for  $\alpha \geq 0$ , here  $\rho_A = \operatorname{Tr}_B(\rho_{AB})$  and similarly for  $\rho_B$ . If  $\alpha$  tends to 1 decreasing, one obtains the von Neumann entropy  $S_1(\rho) \equiv S(\rho)$  as a limiting case.

The above inequalities represent scalar separability criteria. For pure states these inequalities are violated if and only if the state is entangled. As separable states admit LHVM one can expect, that the violation of the entropic inequalities is a signature of some nonclassical features of a compound quantum system resulting from its entanglement. The above inequalities involve a nonlinear functionals of a state  $\rho$ , so they can be interpreted as a scalar separability criteria based on a nonlinear entanglement witness (see Sec. VI).

The  $\alpha$ -entropy inequalities were considered in the context of Bell inequalities (Horodecki and Horodecki, 1996) (and quantum communication for  $\alpha = 1$  (Cerf and Adami, 1997)). For  $\alpha = 2$  one get the inequalities based on the 2-Renyi entropy, which are stronger detectors of entanglement than all Bell-CHSH inequalities (Horodecki, 1996; Horodecki *et al.*, 1996c; Santos, 2004). The above criterion was expressed in terms of the Tsallis entropy (Abe and Rajagopal, 2001; Tsallis *et al.*, 2001; Vidiella-Barranco, 1999).

Another interesting family of separability of nonlinear criteria was derived (Gühne and Lewenstein, 2004a) in terms of entropic uncertainty relations (Białynicki-Birula and Mycielski, 1975; Deutsch, 1983; Kraus, 1987; Maassen and Uffink, 1988)<sup>23</sup> using Tsallis entropy. In contrast to linear functionals, such as correlation functions, the entropies, being a nonlinear functionals of a state  $\rho$ , are not directly measurable quantities. Therefore, the problem of experimental verification of the violation of the entropic inequalities was difficult from both a conceptual and a technical point of view.

<sup>&</sup>lt;sup>23</sup> Quite recently the quantum entropic uncertainty relations has been expressed in the form of inequalities involving the Renyi entropies (Białynicki-Birula, 2006).

However, recently Bovino *et al.* demonstrated experimentally a violation of 2-entropic inequalities based on the Renyi entropy (Bovino *et al.*, 2005). This also was first direct experimental measurement of a nonlinear entanglement witness (see Sec. VIII.C).

# B. 1-entropic inequalities and negativity of information

Entropic inequalities (30) can be viewed as a scalar separability criteria. This role is analogous to Bell inequalities as entanglement witnesses. In this context a natural question arises: is this all we should expect from violation of entropic inequalities? Surprisingly enough, entropic inequalities are only the tip of the iceberg which reveals dramatic differences between classical and quantum communication due to quantum entanglement. To see it consider again the entropic inequalities based on the von Neumann entropy ( $\alpha = 1$ ) which hold for separable states. They may be equivalently expressed as

$$S(\rho_{AB}) - S(\rho_B) \ge 0, \quad S(\rho_{AB}) - S(\rho_A) \ge 0$$
 (31)

and interpreted as the constraints imposed on the correlations of the bipartite system by positivity of some function

$$S(A|B) = S(\rho_{AB}) - S(\rho_B) \tag{32}$$

and similarly for S(B|A). Clearly, the entropy of the subsystem  $S(\rho_A)$  can be greater than the total entropy  $S(\rho_{AB})$  of the system only when the state is entangled. However, there are entangled states which do not exhibit this exotic property, i.e. they satisfy the constraints. Thus the physical meaning of the function S(A|B) (S(B|A)) and its peculiar behavior were an enigma for physicists.

There was an attempt (Cerf and Adami, 1997) to overcome this difficulty by replacing Shannon entropies with von Neumann one in the formula for classical conditional entropy

$$H(X|Y) = H(X,Y) - H(X)$$
 (33)

to get quantum conditional entropy in form (32) (Wehrl, 1978). Unfortunately, such approach is inconsistent with the concept of the channel capacity defined via the function -S(A|B) called *coherent information* (see Sec. XII.A) (Schumacher and Nielsen, 1996)<sup>24</sup>. Moreover, so defined conditional entropy can be negative.

Recently, the solution of the problem was presented within the quantum counterpart of the classical Slepian-Wolf theorem called "quantum state merging" ((Horodecki *et al.*, 2005h), (Horodecki *et al.*, 2006e)). In



FIG. 1 The concept of state merging: before and after.

1971 Slepian and Wolf considered the following problem: how many bits does the sender (Alice) need to send to transmit a message from the source, provided the receiver (Bob) already has some prior information about the source. The amount of bits is called the partial information. Slepian and Wolf proved (Slepian and Wolf, 1971), that the partial information is equal to the conditional entropy (33), which is always positive:  $H(X|Y) \ge 0$ .

In the quantum state merging scenario, an unknown quantum state is distributed to spatially separated observers Alice and Bob. The question is: how much quantum communication is needed to transfer Alice's part of the state to Bob in such a way that finally Bob has total state (Fig. 1). This communication measures the partial information which Bob needs conditioned on its prior information S(B). Surprisingly, Horodecki at al. proved that necessary and sufficient number of qubits is given by the formula (32), even if this quantity is negative.

Remarkably, this phenomenon as unders into two levels: "classical" and quantum depend on sign of the partial information S(A|B):

1) partial information S(A|B) is positive (the inequalities (31) are not violated): the optimal state merging protocol requires sending  $r \cong S(A|B)$  qubits.

2) partial information S(A|B) is negative (the inequalities (31) are violated): the optimal state merging does not need sending qubits; in addition Alice and Bob gain  $r \cong -S(A|B)$  pairs of qubits in a maximally entangled state.

It is remarkable that we have clear division into two regimes: when the 1-entropy inequalities are not violated, in the process of quantum state merging consumed and we have a nice analogy to the classical Slepian-Wolf theorem where partial information (equal to conditional entropy (33)) is always positive; when the inequalities are violated, apart from state merging we get additional entanglement. Thus the quantum and classical regimes are determined by the relations between the knowledge about the system as a whole and about its subsystem, as considered by Schrödinger.

Finally, let us note that the early recognized manifestations of entanglement: (*nonlocality* (EPR, Bell) and what we can call *insubordination* (Schrödinger)) had been

<sup>&</sup>lt;sup>24</sup> The term "coherent information" was originally defined to be a function of a state and a channel, but further its use has been extended to apply to a bipartite state.

seemingly academic issues, of merely philosophical relevance. What is perhaps the most surprising twist, is that both the above features qualify entanglement as a resource to perform some concrete tasks.

Indeed, the violation of the Bell inequalities determines usefulness of quantum states for the sake of specific nonclassical tasks, such as e.g. reduction of communication complexity, or quantum cryptography (see Sec.III). Similarly, the violation of the entropic inequalities based on the von Neumann entropy (31) determines the usefulness of states as a *potential* for quantum communication. It is in agreement with the earlier results, that the negative value of the function S(A|B) is connected with the ability of the system to perform teleportation (Cerf and Adami, 1997; Horodecki and Horodecki, 1996) as well as with nonzero capacity of a quantum channel (Devetak, 2003; Lloyd, 1997; Schumacher and Nielsen, 1996).

### C. Majorization relations

In 2001 Nielsen and Kempe discovered a stronger version of the "classical versus quantum order" (Nielsen and Kempe, 2001), which connects the majorization concept and entanglement. Namely they proved, that if a state is separable than the following inequalities

$$\lambda(\rho) \prec \lambda(\rho_A) \quad \lambda(\rho) \prec \lambda(\rho_B)$$
 (34)

has to be fulfilled. Here  $\lambda(\rho)$  is a vector of eigenvalues of  $\rho$ ;  $\lambda(\rho_A)$  and  $\lambda(\rho_B)$  are defined similarly. The relation  $x \prec y$  between d-dimensional vectors x and y(x is majorized by y) means that  $\sum_{i=1}^{k} x_i^{\downarrow} \leq \sum_{i=1}^{k} y_i^{\downarrow}$ ,  $1 \leq k \leq d-1$  and the equality holds for k = d,  $x_i^{\downarrow}(1 \leq i \leq d)$  are components of vector x rearranged in decreasing order;  $y_i^{\downarrow}(1 \leq i \leq d)$  are defined similarly. Zeros are appended to the vectors  $\lambda(\rho_A)$ ,  $\lambda(\rho_B)$  in (34), in order to make their dimension equal to the one of  $\lambda(\rho)$ .

The above inequalities constitute necessary condition for separability of bipartite states in arbitrary dimensions in terms of the local and the global spectra of a state. This criterion is stronger than entropic criterion (30) and it again supports the view that "separable states are more disordered globally than locally" (Nielsen and Kempe, 2001). An alternative proof of this result have been found (Gurvits and Barnum, 2005).

## VI. BIPARTITE ENTANGLEMENT

### A. Definition and basic properties

The fundamental question in quantum entanglement theory is which states are entangled and which are not. Only in few cases this question has simple answer. The simplest is the case of pure bipartite states. In accordance with definition of multipartite entangled states (Sec. (II) any bipartite pure state  $|\Psi_{AB}\rangle \in \mathcal{H}_{AB} =$   $\mathcal{H}_A \otimes \mathcal{H}_B$  is called separable (entangled) iff it can be (can not be) written as a product of two vectors corresponding to Hilbert spaces of subsystems:

$$|\Psi_{AB}\rangle = |\phi_A\rangle|\psi_B\rangle. \tag{35}$$

In general if the vector  $\Psi_{AB}$  is written in any orthonormal product basis  $\{|e_A^i\rangle \otimes |e_B^j\rangle\}$  as follows<sup>25</sup>:

$$|\Psi_{AB}\rangle = \sum_{i=0}^{d_A-1} \sum_{j=0}^{d_B-1} A^{\Psi}_{ij} |e^i_A\rangle \otimes |e^j_B\rangle, \qquad (36)$$

then it is product *if and only if* the matrix of coefficients  $A^{\Psi} = \{A_{ij}^{\Psi}\}$  is of rank one. In general the rank  $r(\Psi) \leq k \equiv min[d_A, d_B]$  of this matrix is called *Schmidt rank of vector*  $\Psi$  and it is equal to either of ranks of the reduced density matrices  $\varrho_A^{\Psi} = \text{Tr}_B |\Psi_{AB}\rangle \langle \Psi_{AB}|, \ \varrho_B^{\Psi} = \text{Tr}_A |\Psi_{AB}\rangle \langle \Psi_{AB}|$  (which satisfy  $\varrho_A^{\Psi} = A^{\Psi}(A^{\Psi})^{\dagger}$  and  $\varrho_A^{\Psi} = [(A^{\Psi})^{\dagger}A^{\Psi}]^T$  respectively<sup>26</sup>). In particular there always exists such a product bi-orthonormal basis  $\{|\tilde{e}_A^i\rangle \otimes |\tilde{e}_B^i\rangle\}$  in which the vector takes the Schmidt decomposition:

$$|\Psi_{AB}\rangle = \sum_{i=0}^{r(\Psi)} a_i |\tilde{e}_A^i\rangle \otimes |\tilde{e}_B^i\rangle, \qquad (37)$$

where the strictly positive numbers  $a_i = \{\sqrt{p_i}\}$  correspond to the nonzero singular eigenvalues (Nielsen and Chuang, 2000) of  $A^{\Psi}$ , and  $p_i$  are the nonzero elements of the spectrum of either of the reduced density matrices.

Quantum entanglement is in general both quantitatively and qualitatively considered to be a property *invariant* under product unitary operations  $U_A \otimes U_B$ . Since in case of pure vector and the corresponding pure state (projector)  $|\Psi_{AB}\rangle\langle\Psi_{AB}|$  the coefficients  $\{a_i\}$  are the only parameters that are invariant under such operations, they completely determine entanglement of bipartite pure state.

In particular, it is very easy to see that pure state  $(\text{projector})|\Psi_{AB}\rangle\langle\Psi_{AB}|$  is separable iff the vector  $\Psi_{AB}$  is product. Equivalently the rank of either of reduced density matrices  $\varrho_A$ ,  $\varrho_B$  is equal to 1, or there is single nonzero Schmidt coefficient. Thus for bipartite pure states it is elementary to decide whether the state is separable or not by diagonalizing its reduced density matrix.

So far we have considered entanglement of pure states. Due to decoherence phenomenon, in laboratories we unavoidably deal with mixed states rather than pure ones. However mixed state still can contain some "noisy" entanglement. In accordance with general definition for *n*-partite state (II) any bipartite state  $\rho_{AB}$  defined on Hilbert space  $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$  is separable (see (Werner,

 $<sup>^{25}</sup>$  Here the orthonormal basis  $\{|e_X^i\rangle\}$  spans subspace  $\mathcal{H}_X,\;X=A,B.$ 

 $<sup>^{26}\</sup> T$  denotes transposition.

1989a)) if and only if it can neither be represented nor approximated by the states of the following form

$$\varrho_{AB} = \sum_{i=1}^{k} p_i \varrho_A^i \otimes \varrho_B^i, \qquad (38)$$

where  $\varrho_A^i$ ,  $\varrho_B^i$  are defined on local Hilbert spaces  $\mathcal{H}_A$ ,  $\mathcal{H}_B$ . In case of finite dimensional systems, i.e. when dim $\mathcal{H}_{AB} < \infty$  the states  $\varrho_A^i$ ,  $\varrho_B^i$  can be chosen to be pure. Then, from the Caratheodory theorem it follows (see (Horodecki, 1997; Vedral and Plenio, 1998)) that the number k in the convex combination can be bounded by the square of the dimension of the global Hilbert space:  $k \leq d_{AB}^2 = (d_A d_B)^2$  where  $d_{AB} = dim \mathcal{H}_{AB}$ etc. It happens that for two-qubits the number of states (called sometimes cardinality) needed in the separable decomposition is always four which corresponds to dimension of the Hilbert space itself (see (Sanpera *et al.*, 1998; Wootters, 1998)). There are however  $d \otimes d$  states that for  $d \geq 3$  have cardinality of order of  $d^4/2$  (see (DiVincenzo *et al.*, 2000b)).

We shall restrict subsequent analysis to the case of finite dimensions unless stated otherwise.

The set  $S_{AB}$  of all separable states defined in this way is convex, compact and invariant under the product unitary operations  $U_A \otimes U_B$ . Moreover the separability property is preserved under so called (stochastic) separable operations (see Sec. XI.B).

The problem is that given any state  $\rho_{AB}$  it is very hard to check whether it is separable or not. In particular, its separable decomposition may have nothing common with the eigendecomposition, i.e. there are many separable states that have their eigenvectors entangled, i.e. nonproduct.

It is important to repeat, what the term entanglement means on the level of mixed states: all states that do not belong to S, i.e. are not separable (in terms of the above definition) are called *entangled*.

In general the problem of separability of mixed states appears to be extremely complex, as we will see in the next section. The operational criteria are known only in special cases.

#### B. Main separability/entanglement criteria in bipartite case

### 1. Positive partial transpose (PPT) criterion

Let us consider the characterization of the set of mixed bipartite separable states. Some necessary separability conditions have been provided in terms of entropic inequalities, but a much stronger criterion has been provided by Asher Peres (Peres, 1996b), which is called positive partial transpose (PPT) criterion. It says that if  $\varrho_{AB}$ , is separable then the new matrix  $\varrho_{AB}^{T_B}$  with matrix elements defined in some fixed product basis as:

$$\langle m | \langle \mu | \varrho_{AB}^{T_B} | n \rangle | \nu \rangle \equiv \langle m | \langle \nu | \varrho_{AB} | n \rangle | \mu \rangle$$
(39)

is a density operator (i.e. has nonnegative spectrum), which means automatically that  $\varrho_{AB}^{T_B}$  is also a quantum state (It also guarantees positivity of  $\varrho_{AB}^{T_A}$  defined in analogous way). The operation  $T_B$ , called partial transpose<sup>27</sup>, corresponds just to transposition of indices corresponding to the second subsystem and has interpretation as a partial time reversal (Sanpera *et al.*, 1998). If the state is represented in a block form

$$\varrho_{AB} = \begin{pmatrix} \varrho_{00} & \varrho_{01} & \cdots & \varrho_{0} & d_{A-1} \\ \varrho_{10} & \varrho_{11} & \cdots & \varrho_{1} & d_{A-1} \\ \cdots & \cdots & \cdots & \cdots \\ \varrho_{d_{A}-1} & 0 & \varrho_{d_{A}-1} & 1 & \cdots & \varrho_{d_{A}-1} & d_{A-1} \end{pmatrix}$$
(40)

with block entries  $\rho_{ij} \equiv \langle i | \otimes I | \rho_{AB} | j \rangle \otimes I$ , then one has

$$\varrho_{AB}^{\Gamma} = \begin{pmatrix} \varrho_{00}^{T} & \varrho_{01}^{T} & \cdots & \varrho_{0}^{T} d_{A-1} \\ \varrho_{10}^{T} & \varrho_{11}^{T} & \cdots & \varrho_{1 \ d_{A}-1}^{T} \\ \cdots & \cdots & \cdots & \cdots \\ \varrho_{d_{A}-1 \ 0}^{T} & \varrho_{d_{A}-1 \ 1}^{T} & \cdots & \varrho_{d_{A}-1 \ d_{A}-1}^{T} \end{pmatrix}. \quad (41)$$

Thus the PPT condition corresponds to transposing block elements of matrix corresponding to second subsystem. PPT condition is known to be stronger than all entropic criteria based on Renyi  $\alpha$ -entropy (V) for  $\alpha \in [0, \infty]$  (Vollbrecht and Wolf, 2002b). A fundamental fact is (Horodecki *et al.*, 1996a; Peres, 1996b) that *PPT* condition is necessary and sufficient condition for separability of  $2 \otimes 2$  and  $2 \otimes 3$  cases. Thus it gives a complete characterization of separability in those cases (for more details or further improvements see Sec. VI.B.2).

#### 2. Separability via positive, but not completely positive maps

Peres PPT condition initiated a general analysis of the problem of the characterization of separable (equivalently entangled) states in terms of linear positive maps (Horodecki *et al.*, 1996a). Namely, it can be seen that the PPT condition is equivalent to demanding the positivity <sup>28</sup> of the operator  $[I_A \otimes T_B](\varrho_{AB})$ , where  $T_B$  is the transposition map acting on the second subsystem. The transposition map is a positive map (i.e. it maps any positive operator on  $\mathcal{H}_B$  into a positive one), but it is not completely positive<sup>29</sup>. In fact,  $I_A \otimes T_B$  is not a positive map and this is the source of success of Peres criterion.

It has been recognized that any positive (P) but not completely positive (CP) map  $\Lambda : \mathcal{B}(\mathcal{H}_B) \to \mathcal{B}(\mathcal{H}_{A'})$  with

<sup>&</sup>lt;sup>27</sup> Following (Rains, 1998) instead of  $\varrho_{AB}^{T_B}$  we will write  $\varrho_{AB}^{\Gamma}$  (as the symbol  $\Gamma$  is a right "part" of the letter T).

<sup>&</sup>lt;sup>28</sup> The operator is called positive iff it is Hermitian and has nonnegative spectrum.

<sup>&</sup>lt;sup>29</sup> The map  $\hat{\Theta}$  is completely positive iff  $I \otimes \Theta$  is positive for identity map I on any finite-dimensional system.

codomain related to some new Hilbert space  $\mathcal{H}_{A'}$  provides nontrivial necessary separability criterion in the form:

$$[\mathbf{I}_A \otimes \Lambda_B](\varrho_{AB}) \ge 0. \tag{42}$$

This corresponds to nonnegativity of spectrum of the following matrix:

$$\begin{bmatrix} I_A \otimes \Lambda_B ](\varrho_{AB}) \\ = \begin{pmatrix} \Lambda(\varrho_{00}) & \dots & \Lambda(\varrho_0 \ d_{A-1}) \\ \Lambda(\varrho_{10}) & \dots & \Lambda(\varrho_1 \ d_{A-1})) \\ \dots & \dots & \dots \\ \Lambda(\varrho_{d_{A-1} \ 0}) & \dots & \Lambda(\varrho_{d_{A-1} \ d_{A-1}}) \end{pmatrix}$$
(43)

with  $\rho_{ij}$  defined again as in (40).

It happens that using the above technique one can provide a necessary and sufficient condition for separability (see (Horodecki et al., 1996a)): the state  $\rho_{AB}$  is separable if and only if the condition (42) is satisfied for all P but not CP maps  $\Lambda : \mathcal{B}(\mathcal{H}_B) \to \mathcal{B}(\mathcal{H}_A)$  where  $\mathcal{H}_A, \mathcal{H}_B$ describe the left and right subsystems of the system AB.

Note that the set of maps can be further restricted to all P but not CP maps that are identity-preserving (Horodecki, 2001b) (the set of witnesses can be then also restricted via the isomorphism). One could also restrict the maps to trace preserving ones, but then one has to enlarge the codomain (Horodecki *et al.*, 2006d).

Given characterization in terms of maps and witnesses it was natural to ask about a more practical characterization of separability/entanglement. The problem is that in general the set of P but not CP maps is not characterized and it involves a hard problem in contemporary linear algebra (for progress in this direction see (Kossakowski, 2003) and references therein).

However for very low dimensional systems there is surprisingly useful solution (Horodecki *et al.*, 1996a): *the states of*  $d_A \otimes d_B$  *with*  $d_A d_B \leq 6$  *(two-qubits or qubitqutrit systems) are separable if and only if they are PPT.* Recently even a simpler condition for two-qubit systems (and only for them) has been pointed out (Augusiak *et al.*, 2006) which is important in context of physical detections (see Secs. VIII.C.2 and VIII.C.3) : two-qubit state  $\varrho_{AB}$  is separable iff

$$det(\varrho_{AB}^{\Gamma}) \ge 0. \tag{44}$$

This is the simplest two-qubit separability condition. It is a direct consequence of two facts known earlier but never exploited in such a way: the partial transpose of any entangled two-qubit state (i) is of full rank and has only one negative eigenvalue. (Sanpera *et al.*, 1998; Verstraete *et al.*, 2001a). Note that some generalizations of (44) for other maps and dimensions are also possible (Augusiak *et al.*, 2006).

The sufficiency of the PPT condition for separability in low dimensions follows from the fact (Størmer, 1963; Woronowicz, 1976) that all positive maps  $\Lambda : \mathcal{B}(\mathcal{C}^d) \to \mathcal{B}(\mathcal{C}^{d'})$  where d = 2, d' = 2 and d = 2, d' = 3 are decomposable, i.e. are of the form:

$$\Lambda^{dec} = \Lambda^{(1)}_{CP} + \Lambda^{(2)}_{CP} \circ T, \qquad (45)$$



FIG. 2 The line represents hyperplane corresponding to the entanglement witness W. All states located to the left of the hyperplane or belonging to it (in particular all separable states) provide nonnegative mean value of the witness, i.e.  $Tr(W\varrho_{sep}) \geq 0$  while those located to the right are entangled states detected by the witness.

where  $\Lambda_{CP}^{(i)}$  stand for some CP maps and T stands for transposition. It can be easily shown (Horodecki *et al.*, 1996a) that among all decomposable maps the transposition map T is the "strongest" map i.e. there is no decomposable map that can reveal entanglement which is not detected by transposition.

#### 3. Separability via entanglement witnesses

Entanglement witnesses (Horodecki *et al.*, 1996a; Terhal, 2000a) are fundamental tool in quantum entanglement theory. They are observables that completely characterize separable states and allow to detect entanglement physically. Their origin stems from geometry: the convex sets can be described by hyperplanes. This translates into the statement (see (Horodecki *et al.*, 1996a; Terhal, 2000a)) that the state  $\rho_{AB}$  belongs to the set of separable if it has nonnegative mean value

$$\operatorname{Tr}(W\varrho_{AB}) \ge 0 \tag{46}$$

for all observables W that (i) have at least one negative eigenvalue and (ii) have nonnegative mean value on product states or — equivalently — satisfy the nonnegativity condition

$$\langle \psi_A | \langle \phi_B | W | \psi_A \rangle | \phi_B \rangle \ge 0. \tag{47}$$

for all pure product states  $|\psi_A\rangle|\phi_B\rangle$  The observables W satisfying conditions (i) and (ii) above <sup>30</sup> have been named *entanglement witnesses* and their physical importance as entanglement detectors was stressed in (Terhal, 2000a) in particular one says that *entanglement of*  $\rho$  *is detected by witness* W iff  $\text{Tr}(W\rho) < 0$ , see fig. 2. (We

<sup>&</sup>lt;sup>30</sup> The witnesses can be shown to be isomorphic to P but not CP maps, see Eq. (49).

shall discuss physical aspects of entanglement detection in more detail subsequently). An example of entanglement witness for  $d \otimes d$  case is (cf. (Werner, 1989a)) the Hermitian swap operator

$$V = \sum_{i,j=0}^{d-1} |i\rangle\langle j| \otimes |j\rangle\langle i|.$$
(48)

To see that V is an entanglement witness let us note that we have  $\langle \psi_A | \langle \phi_B | V | \psi_A \rangle | \phi_B \rangle = | \langle \psi_A | \phi_B \rangle |^2 \ge 0$  which ensures property (ii) above. At the same time  $V = P^{(+)} - P^{(-)}$  where  $P^{(+)} = \frac{1}{2}(I+V)$  and  $P^{(-)} = \frac{1}{2}(I-V)$ correspond to projectors onto symmetric and antisymmetric subspace of the Hilbert space  $\mathcal{C}^d \otimes \mathcal{C}^d$  respectively. Hence V also satisfies (i) since it has some eigenvalues equal to -1. It is interesting that V is an example of so called *decomposable entanglement witness* (see (45) and analysis below).

The P but not CP maps and entanglement witnesses are linked by so called Choi-Jamiołkowski isomorphism (Choi, 1982; Jamiołkowski, 1972a):

$$W_{\Lambda} = [\mathbf{I} \otimes \Lambda](P_d^+) \tag{49}$$

with pure projector

$$P_d^+ = |\Phi_d^+\rangle \langle \Phi_d^+|, \tag{50}$$

where state vector  $\Phi_d^+ \in \mathcal{H}_A \otimes \mathcal{H}_A$  is defined as

$$|\Phi_d^+\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle \otimes |i\rangle, \ d = \dim \mathcal{H}_A.$$
 (51)

The pure projector  $P_d^+$  is an example of maximally entangled state<sup>31</sup> on the space  $\mathcal{H}_A \otimes \mathcal{H}_A$ .

The very important observation is that while the condition (46) as a whole is equivalent to (42), a particular witness is not equivalent to a positive map associated via isomorphism: the map proves a stronger condition (see discussion further in the text).

As we already said, the special class of decomposable P but not CP maps (i.e. of the form 45) which provide no stronger criterion than the PPT one, is distinguished. Consequently, all the corresponding entanglement witnesses are called *decomposable* and are of the form (see (Lewenstein *et al.*, 2000)):

$$W^{dec} = P + Q^{\Gamma}, \tag{52}$$

where P, Q are some positive operators. It can be easily shown that decomposable witnesses (equivalently — decomposable maps) describe the new set  $S_{PPT}$  of all states that satisfy PPT criterion. Like the set S of separable states this set is also convex, compact and invariant under product unitary operations. It has been also found that stochastic separable operations preserve PPT property (Horodecki *et al.*, 1998a). In general we have  $S \subsetneq S_{PPT}$ . As described in previous section the two sets are equal for  $d_A d_B \leq 6$ . In all other cases, they differ (Horodecki, 1997) (see Sec. VI.B.7 for examples), i.e. there are entangled states that are PPT. The latter states give rise to the so-called bound entanglement phenomenon (see Sec. XII).

To describe  $S_{PPT}$  it is enough to consider only a subset of decomposable witnesses where P = 0 and Q is a pure projector corresponding to entangled vector  $|\Phi\rangle$ . This gives a minimal set of entanglement witnesses that describe set of PPT states. The required witnesses are thus of the form

$$W = |\Phi\rangle\langle\Phi|^{\Gamma} \tag{53}$$

with some entangled vector  $|\Phi\rangle$ . The swap V is proportional to a witness of this kind. Indeed, we have  $V = dP_+^{\Gamma}$  (hence — as announced before — the swap is a decomposable witness).

For  $d \otimes d$  systems there is one distinguished decomposable witness which is not of the form (53) but is very useful and looks very simple. This is the operator

$$W(P^{+}) = \frac{1}{d}\mathbf{I} - P^{+}.$$
 (54)

One can prove that the condition  $\langle W(P^+) \rangle_{\varrho_{PPT}} \geq 0$  provides immediately the restriction on the parameter called *fidelity* or *singlet fraction*<sup>32</sup>:

$$F(\varrho) = \operatorname{Tr}[P^+\varrho]. \tag{55}$$

Namely (see (Rains, 2000)):

$$F(\varrho_{PPT}) \le \frac{1}{d}.\tag{56}$$

In particular this inequality was found first for separable states and its violation was shown to be sufficient for entanglement distillation (Horodecki and Horodecki, 1999).

As we have already mentioned, the *set* of maps conditions (42) is equivalent to *set* of witnesses condition (46). Nevertheless any single witness  $W_{\Lambda}$  condition is much *weaker* than the condition given by the map  $\Lambda$ . This is because the first is of scalar type, while the second represents an operator inequality condition. To see this difference it is enough to consider the two-qubit case and compare the transposition map T (which detects all entanglement in sense of PPT test) with the entanglement witness isomorphic to it, which is the swap operation V, that does not detect entanglement of any

<sup>&</sup>lt;sup>31</sup> For simplicity we will drop the dimension denoting projector onto  $\Phi^+ \equiv \Phi_d^+$  as  $P^+ \equiv P_d^+$  provided it does not lead to ambiguity.

<sup>&</sup>lt;sup>32</sup> One has  $0 \le F(\varrho) \le 1$  and  $F(\varrho) = 1$  iff  $\varrho = P^+$ .

symmetric pure state. Indeed it is not very difficult to see (see (Horodecki and Ekert, 2002)) that the condition based on one map  $\Lambda$  is equivalent to a *continu*ous set of conditions defined by all witnesses of the form  $W_{\Lambda,A} \equiv A \otimes IW_{\Lambda}A^{\dagger} \otimes I$  where A are operators on  $C^d$  of rank more than one. Thus after a bit of analysis it is not difficult to see why PPT condition associated with a single map (transposition T) is equivalent to set of all the conditions provided by the witnesses of the form (53).

On the other hand one must stress condition based on a witness is naturally directly measurable (Terhal, 2000b) (and it has found recently many experimental implementations, see introduction section) while physical implementation of separability condition based on (unphysical) P but not CP maps is much more complicated, though still possible (see Sec. VIII.C.1). Moreover, the power of witnesses can be enhanced with help of nonlinear corrections (see Sec. VIII.B).

For higher dimensional systems there are many nondecomposable P but not CP maps but their construction is in general hard (see (Kossakowski, 2003) and reference therein). In particular checking that given map is P is very hard and equivalent to checking the positivity condition (ii) in the definition of witnesses. Note that long time ago in literature an algorithm that corresponds to checking strict positivity of witnesses was found (ie. (ii) condition with strict inequality) (Jamiołkowski, 1972b)) but its complexity is very high (it has an interesting consequences though which will be mentioned later in more details).

The important question one can ask about enregards tanglement witnesses their optimality (Lewenstein *et al.*, 2000, 2001). We say that an entanglement witness  $W_1$  is finer than  $W_2$  iff the entanglement of any  $\rho$  detected by  $W_2$  is also detected by  $W_1$ . A given witness W is called optimal iff there is no witness finer than it. The useful sufficient condition of optimality (Lewenstein et al., 2000) is expressed in terms of the Hilbert subspace  $\mathcal{P}_W = \{ |\phi\rangle |\psi\rangle : \langle \phi | \langle \psi | W | \phi \rangle |\psi\rangle = 0 \}.$ Namely if  $\mathcal{P}_W$  spans the whole Hilbert space then the witness is optimal. In a sense it is then fully "tangent" to set of separable states. The systematic method of optimization of given entanglement witness have been worked out first in (Lewenstein et al., 2000, 2001) (for alternative optimization procedure see (Eisert *et al.*, 2004), cf. optimization of witnesses for continuous variables (Hyllus and Eisert, 2006)).

In some analogy to the pure bipartite case, we can define the Schmidt rank for density matrices (Terhal and Horodecki, 2000) as  $r_S(\varrho) =$ min(max<sub>i</sub>[ $r_S(\psi_i)$ ]) where minimum is taken over all decompositions  $\varrho = \sum_i p_i |\psi_i\rangle \langle \psi_i|$  and  $r_S(\psi_i)$  are the Schmidt ranks of the corresponding pure states (see Sec. VI.A). One can easily prove that separable operations can not increase it.

Now, for any k in the range  $\{1, \ldots, r_{max}\}$  with  $r_{max} = \min[d_A, d_B]$ , we have a set  $S_k$  of states with Schmidt number not greater than k. For such sets we can build

a theory similar to that for "simple" entanglement, with Schmidt-number witnesses in place of usual witnesses. The family of sets  $S_k$  satisfies inclusion relations:  $S_1 \subset S_2 \subset \ldots \subset S_{r_{max}}$ . Note that here  $S_1$  corresponds just to the set of separable states, while  $S_{r_{max}}$ — to the set of all states. Each set is compact, convex and again closed under separable operations. Moreover, each such set is described by k-positive maps (Terhal and Horodecki, 2000) or by Schmidt-rank k witnesses (Sanpera *et al.*, 2001). A Schmidt rank k witness is an observable  $W_k$  that satisfies two conditions (i) must have at least one negative eigenvalue and (ii) must satisfy:

$$\langle \Psi_k | W_k | \Psi_k \rangle \ge 0, \tag{57}$$

for all Schmidt rank k vectors  $\Psi_k \in \mathcal{H}_{AB}$ . As in case of separability problem the k-positive maps are related via Choi-Jamiołkowski isomorphism to special maps that are called k-positive (i.e. such that  $[I_k \otimes \Lambda_k]$  is positive for  $I_k$  being identity on  $\mathcal{B}(\mathcal{C}^k)$  but not completely positive. The isomorphism is virtually the same as the one that links entanglement witnesses  $W = W_1$ with k-positive maps  $\Lambda_1$ . Important maps (respectively witnesses) are the ones which discriminate between  $\mathcal{S}_k$ and  $\mathcal{S}_{k+1}$ . They are those maps, which are k-positive but not k + 1 positive. A nice example is the following family of maps  $\Lambda_p(\varrho) = \operatorname{ITr}(\varrho) - p(\varrho)$  which is k but not k + 1 positive for  $\frac{1}{k+1} (see$ (Terhal and Horodecki, 2000)). The case of p = 1 corresponds to special so called *reduction map* which plays a role in entanglement distillation defining in particular reduction separability criterion which will be discussed subsequently. Many techniques have been generalized from separability to mixed state Schmidt rank analysis (see (Sanpera et al., 2001)). For general review of separability problem including especially entanglement witnesses see (Bruß, 2002; Bruß et al., 2002; Terhal, 2002). The Schmidt number witnesses and maps description is reviewed extensively in (Bruß et al., 2002).

#### 4. Witnesses and experimental detection of entanglement

As already mentioned, entanglement witnesses have been found very important in experimental detection of entanglement (Horodecki *et al.*, 1996a; Terhal, 2000b): for any entangled state  $\rho_{ent}$  there are witnesses which are signatures of entanglement in a sense that they are negative on this state  $\text{Tr}(W\rho_{ent}) < 0$ . Here we shall describe few further aspects of the detection showing both the importance of entanglement witnesses and their possible applications.

The first issue concerns the following question (Horodecki *et al.*, 1999c): Given experimental mean values of incomplete set of observables  $\langle A_i \rangle = a_i$  what information about entanglement should be concluded basing on those data? The idea of the paper was that *if entanglement is finally needed as a resource then the observer* 

should consider the worst case scenario, i.e. should minimize entanglement under experimental constraints. In other words, experimental entanglement should be of the form:

$$E(a_1,\ldots,a_k) = \inf_{\{\langle A_i \rangle_{\varrho} = a_i\}_{i=1}^k} E(\varrho).$$
(58)

Such minimization of entanglement of formation and relative entropy of entanglement was performed for given mean of Bell observable on unknown 2-qubit state. It is interesting that in general there are many states achieving that minimum — to get single final state the authors also have proposed final application of maximum entropy Jaynes principle (Horodecki *et al.*, 1999c).

Quite recently the idea of minimization of entanglement under experimental constraints was applied with help of entanglement witnesses (Eisert et al., 2007; Gühne et al., 2006b). In (Gühne et al., 2006b) using convex analysis the authors have performed minimization of convex entanglement measures for given mean values of entanglement witnesses basing on approximation of convex function by affine functions from below. Specific estimates have been performed for existing experimental data. Independently a similar analysis of lower bounds for many entanglement measures has been performed in (Eisert et al., 2007), where the emphasis was put on analytical formulas for specific examples. The derived formulas (Eisert et al., 2007; Gühne et al., 2006b) provide a direct quantitative role for results of entanglement witnesses measurements. Note that more refined analysis was focused on correlations obtained in the experiment, identifying which types of correlations measured in incomplete experiments may be already signature of entanglement (Audenaert and Plenio, 2006).

Another experimental issue, where entanglement witnesses have been applied, is the problem of macroscopic entanglement at finite temperature. A threshold temperature for existence of entanglement can be identified. The relation between thermal equilibrium state and entanglement was hidden already in 2-qubit analysis of Javnes principle and entanglement (Horodecki et al., 1999c). The first explicit analysis of entanglement in thermal state was provided by Nielsen (Nielsen, 1998) where first calculation of temperatures for which entanglement is present in two-qubit Gibbs state was performed. A fundamental observation is that entanglement witnesses theory can be exploited to detect entanglement in general (multipartite) thermal states including systems with large number of particles (Brukner and Vedral, 2004; Toth, 2005). In the most elegant approach, for any observable O one defines entanglement witness as follows (see (Toth, 2005)):

$$W_O = O - \inf_{|\Psi_{prod}\rangle} \langle \Psi_{prod} | O | \Psi_{prod} \rangle, \tag{59}$$

where infimum is taken over all product pure states  $\Psi_{prod}$ (note that the method can be extended to take into account partial separability<sup>33</sup> as well). Now if  $W_O$  has negative eigenvalue becomes immediately an entanglement witness by construction. In case of spin lattices one takes O = H where H is a Hamiltonian of the system and calculates  $\langle W_H\rangle_{\varrho}$  for quantum Gibbs state  $\varrho_{Gibbs} = \exp(-H/kT)/\operatorname{Tr}[\exp(-H/kT)]$ . It can be immediately seen that for H with discrete spectrum the observable  $W_H$  has a negative eigenvalue iff the lowest energy state is entangled and then the observable becomes entanglement witness by construction (see (59)). In this one can estimate the range of temperatures for which the mean value  $\langle W_H \rangle_{\varrho_{Gibbs}}$  is negative (Toth, 2005). Further improvements involve uncertainty based entanglement witnesses (Anders et al., 2005) and applications of entanglement measures like robustness of entanglement (Markham et al., 2006) to thermal entanglement.

Finally let us recall an important issue: how to decompose given witness into locally measurable observables (Gühne *et al.*, 2002), (Gühne *et al.*, 2003). This is an important issue since if we want to detect entanglement between spatially separated systems we can only measure mean values of tensor products of local observables consistent with spatial separation (cf. Sec. XI). For a given witness W in  $d_A \otimes d_B$  one can ask about the minimal number of local measurements on systems A, B that can reconstruct the mean value of the whole witness. The problem is to find optimal decomposition

$$W = \sum_{k=1}^{r} \gamma_k X_A^k \otimes Y_B^k \tag{60}$$

onto a set of product of normalized (in Hilbert-Schmidt norm) observables such that any two pairs  $X_A^k \otimes Y_B^k$ ,  $X_A^{k'} \otimes Y_B^{k'}$  in the sum differ on at most one side. Then the cardinality of the representation (60) is calculated as  $s = r - r_I$  where  $r_I$  is a number of those product terms  $\{X_A^{k'} \otimes Y_B^{k'}\}_{k'=1}^{r_I}$  in which at least one of local observables is proportional to identity I (say  $X_A^{k'} = \alpha I$ ) and the second one  $(Y_B^{k'})$  is linearly dependent on all the other local ones (i.e.  $Y_B^{k'} = \sum_{k \neq k'} \alpha_k Y_B^k$ ). The optimal cardinality  $s_{min}$  minimize over all decompositions of the type (60) gives minimal number of different measurement settings needed to measure mean of given entanglement between spatially separated systems. The problem of finding the optimal decomposition (60) with the minimal cardinality  $s_{min}$  has been investigated in papers (Gühne *et al.*, 2002, 2003) and analytical solutions have been found for both bipartite cases discussed above as well as for multipartite generalizations.

<sup>&</sup>lt;sup>33</sup> See Sec. VII.

#### 5. Entanglement witnesses and Bell inequalities

Entanglement witnesses (see Sec. VI) are Hermitian operators that are designed directly for detection of entanglement. In 2000 Terhal first considered a possible connection between entanglement witnesses and Bell inequalities (Terhal, 2000a). From a "quantum" point of view, Bell inequalities are just nonoptimal entanglement witnesses. For example one can define the CHSH-type witness which is positive on all states which admit LHVM

$$W_{CHSH} = 2 \cdot \mathbf{I} - \mathcal{B}_{CHSH},\tag{61}$$

where  $\mathcal{B}_{CHSH}$  is the CHSH operator (16). Such defined witness is of course nonoptimal one since the latter is strictly positive on *separable* states. However, basing on the concept of the optimal witness (Lewenstein *et al.*, 2000) one can estimate how much optimal witnesses have to be shifted by the identity operator to make them positive on all states admitting a LHVM. Moreover, there exists a natural decomposition of the CHSH witness into two optimal witnesses, and the identity operator (Hyllus *et al.*, 2005). In multipartite case, Bell inequalities can even detect so called bound entanglement (Augusiak *et al.*, 2006; Dür, 2001; Kaszlikowski *et al.*, 2000; Sen(De) *et al.*, 2002) (see Sec. XII.J).

Inspired by CHSH inequalities Uffink proposed inequalities (Uffink, 2002), which are no longer implied by LHVM, yet constitute a (nonlinear) entanglement witness (see Sec. VIII.B), see in this context (Uffink and Seevinck, 2006).

Svetlichny and Sevnick proposed Bell inequalities that can be used as a detector of genuinely multipartite entanglement (Seevinck and Svetlichny, 2002). Toth and Gühne proposed a new approach to entanglement detection (Toth and Gühne, 2005a,b)based on the stabilizer theory (Gottesman, 1996). In particular, they found interesting connections between entanglement witnesses and Mermin-type inequalities (Toth and Gühne, 2005b).

In general, the problem of the relation between Bell inequalities and entanglement witnesses is very complex. It follows from the very large number of degrees of freedom of the Bell inequalities. Nevertheless it is basic problem, as the Bell observable is a *double* witness. It detects not only entanglement but also nonlocality.

It is interesting that the loophole problem in the experimental tests of Bell inequalities (see (Gill, 2003)) has found its analogy (Skwara *et al.*, 2006) in an entanglement detection domain. In particular the efficiency of detectors that still allows to claim that entanglement was detected, has been analyzed and related to cryptographic application.

# 6. Distinguished maps criteria: reduction criterion and its extensions

There are two important separability criteria provided by P but not CP maps. The first one is the so called reduction criterion (Cerf et al., 1999; Horodecki and Horodecki, 1999) defined by the formula (42) with the reduction map:  $\Lambda^{red}(\varrho) = \text{ITr}(\varrho) - \varrho$ . This map is decomposable but — as we shall see subsequently — plays important role in entanglement distillation theory(Horodecki and Horodecki, 1999). Only in case of two-dimensional Hilbert space the map represents just a reflection in Bloch sphere representation (Bengtsson and Życzkowski, 2006) and can be easily shown to be just equal to transpose map T followed by  $\sigma_y$  i.e.  $\sigma_y T(\varrho) \sigma_y$ . As such it provides a separability condition completely equivalent to PPT in this special (two-qubit) case. In general the reduction separability criterion [I<sub>A</sub>  $\otimes \Lambda_B^{red}](\varrho_{AB}) \geq 0$  generated by  $\Lambda^{red}$  can be written as:

$$\varrho_A \otimes \mathbf{I} - \varrho_{AB} \ge 0 \tag{62}$$

and since  $\Lambda^{red}$  is decomposable (Horodecki and Horodecki, 1999) the corresponding separability criterion is weaker than PPT one (see Sec. VI.B.2). On the other hand it is interesting that this criterion is stronger (Hiroshima, 2003) than mixing separability criteria (Nielsen and Kempe, 2001) as well as some entropic criteria with  $\alpha \in [0, 1]$  and  $\alpha = \infty$ (for the proofs see (Vollbrecht and Wolf, 2002b) and (Horodecki and Horodecki, 1999) respectively).

Another very important criterion is the one based on the map due to Breuer and, independently, Hall (Breuer, 2006a; Hall, 2006) which is a modification of reduction map on even dimensional Hilbert space d = 2k. On this subspace there exist antisymmetric unitary operations  $U^T = -U$  (for instance the one U = antidiag[1, -1, 1, -1, ..., 1, -1] (Breuer, 2006a)). The corresponding antiunitary map  $U(\cdot)^T U$ , maps any pure state to some state that is orthogonal to it. This leads to the conclusion that the map which acts on the state  $\rho$ as follows:

$$\Lambda(\varrho) = \Lambda^{red}(\varrho) - U(\varrho)^T U^{\dagger} \tag{63}$$

is positive for any antisymmetric U. This map is not decomposable and the entanglement witness  $W_{\Lambda}$  corresponding to it has an optimality property since the corresponding space  $\mathcal{P}_{W_{\Lambda}}$  (see Sec. VI.B.3) is the full Hilbert space (Breuer, 2006b). This nondecomposability property allows the map to detect special class of very weak entanglement, namely PPT entanglement mentioned already before. We shall pass to its more detailed description now.

# 7. Range criterion and its applications; PPT entanglement

The existence of nondecomposable maps (witnesses) for the cases with  $d_A \cdot d_B > 6$  implies that there are states that are entangled but PPT in all those cases. Thus the PPT test is no longer a sufficient test of separability in those cases. This has striking consequences, for quantum communication theory, including entanglement distillation and quantum key distribution which we discuss further in this paper. Existence of PPT entangled states was known already in terms of cones in mathematical literature (see for example (Choi, 1982)) sometimes expressed in direct sum language.

On physical ground, first examples of entangled states that are PPT were provided in (Horodecki, 1997), following Woronowicz construction (Woronowicz, 1976). Their entanglement had to be found by a criterion that is independent on PPT one. As we already mentioned, this might be done with properly chosen P but not CP nondecomposable map (see Sec. VI.B.2). In (Horodecki, 1997) another criterion was formulated for this purpose, which is useful for other applications (see below). This is the range criterion: if  $\rho_{AB}$  is separable, then there exists a set of product vectors  $\{\psi_A^i \otimes \phi_B^i\}$ , such that it spans range of  $\rho_{AB}$  while  $\{\psi_A^i \otimes (\phi_B^i)^*\}$  spans range of  $\rho_{AB}^{T_B}$ , where complex conjugate is taken in the same basis in which PPT operation on  $\rho_{AB}$  has been performed. In particular an example of  $3 \otimes 3$  PPT entangled state revealed by range criterion (written in a standard basis) was provided:

$$\varrho_{a} = \frac{1}{8a+1} \begin{bmatrix}
a & 0 & 0 & a & 0 & 0 & 0 & a \\
0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & a & 0 & 0 & 0 & a \\
0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1+a}{2} & 0 & \frac{\sqrt{1-a^{2}}}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & a & 0 \\
a & 0 & 0 & 0 & a & 0 & \frac{\sqrt{1-a^{2}}}{2} & 0 & \frac{1+a}{2}
\end{bmatrix}, \quad (64)$$

where 0 < a < 1. Further examples of PPT states that are entangled can be found in (Alber *et al.*, 2001a).

Especially interesting way of application of range criterion to finding PPT states is unextendible product basis (UPB) methods (Bennett *et al.*, 1999b; DiVincenzo *et al.*, 2003a). UPB is a set  $S_{UPB}$  of orthonormal product vectors in  $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$  such that there is no product vector that is orthogonal to all of them.

*Example* .- An example in the  $3 \otimes 3$  case is (Bennett *et al.*, 1999b):  $S_{UPB} \equiv \{|0\rangle(|0\rangle + |1\rangle), (|0\rangle + |1\rangle)|2\rangle, |2\rangle(|1\rangle + |2\rangle)(|1\rangle + |2\rangle)|0\rangle, (|0\rangle - |1\rangle + |2\rangle)(|0\rangle - |1\rangle + |2\rangle)\}.$ 

Since there is no product vector orthogonal to the subspace  $\mathcal{H}_{UPB}$  spanned by elements of  $\mathcal{S}_{UPB}$ , any vector from the orthogonal subspace  $\mathcal{H}_{UPB}^{\perp}$  (spanned by vectors orthogonal to  $\mathcal{H}_{UPB}$ ) is entangled. Consequently, by the range criterion above, any mixed state with support contained in  $\mathcal{H}_{UPB}^{\perp}$  is entangled. In particular, a special class of states proportional to the projector  $P_{\mathcal{H}_{UPB}^{\perp}} = \mathbf{I} - P_{\mathcal{H}_{UPB}}$  (here  $P_{\mathcal{H}}$  stands for projection onto the subspace  $\mathcal{H}$ ) is also entangled, but it can be shown to be PPT because of the special way in which the projector  $P_{\mathcal{H}_{UPB}}$  was constructed. In this way the notion of UPB

leads to construction of PPT entangled states. This result was further exploited to provide new nondecomposable maps (Terhal, 2000b). What was the mathematical origin of the construction? This is directly related to the question: what is the other way PPT entanglement can be detected? As we already announced there is a rule linked directly to the positive-map separability condition (see (Horodecki *et al.*, 1996a)): for any PPT entangled state there is a nondecomposable P but not CP map  $\Lambda$ such that the criterion (42) is violated. Thus a way to detect PPT entanglement is to find a proper nondecomposable P but not CP map. The alternative statement, saying that PPT state is entangled if an only if it is detected by some nondecomposable witness (i.e. the one that is not of the form (52)) is immediately also true.

It is rather hard to construct nondecomposable maps and witnesses respectively. En example of P but nor CP nondecomposable map due to Choi is (Choi, 1982) (see (Kossakowski, 2003) and references therein for generalization):

$$\Lambda(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}) = \begin{bmatrix} a_{11} + a_{33} & -a_{12} & -a_{13} \\ -a_{21} & a_{22} + a_{11} & -a_{23} \\ -a_{31} & -a_{32} & a_{33} + a_{22} \end{bmatrix},$$
(65)

which allows to detect PPT entangled states that help in bound entanglement activation (see (Horodecki *et al.*, 1999a)).

The new technique for achieving the nondecomposable maps was worked out by Terhal (Terhal, 2000b). Her idea was to take a projector on UPB space  $P_{\mathcal{H}_{UPB}}$  and observe that the following quantity  $\epsilon = \min_{\Psi_{sep}} \langle \Psi_{sep} | P_{\mathcal{H}_{UPB}} | \Psi_{sep} \rangle$  is strictly positive because of unextendibility property. Then the following operator on  $d \otimes d$  space

$$W_{UPB} = P_{\mathcal{H}_{UPB}} - d\epsilon |\Psi_{max}\rangle \langle \Psi_{max}|, \qquad (66)$$

where  $\Psi_{max}$  is a maximally entangled state such that  $|\Psi_{max}\rangle \notin \mathcal{H}_{UPB}$ . Since one can always show that such a vector exists any entanglement witness of the above form detects PPT entanglement of  $\varrho_{UPB} = \frac{1}{N}(I - P_{\mathcal{H}_{UPB}})$  since the mean value  $\text{Tr}(W_{UPB}\varrho_{UPB})$  will gain only the (negative) contribution form the second term of the (66). At the same time the optimization of the  $\epsilon$  above guarantees that the  $W_{UPB}$  has nonnegative mean value on any product state. Terhal has calculated explicitly the lower bounds for the parameter  $\epsilon$  in cases of few examples of UPB-s.

This idea was further generalized to the case of edge states (Lewenstein *et al.*, 2000),(see also (Horodecki *et al.*, 2000e; Kraus et al., 2000;Lewenstein and Sanpera, 1998)). A state is called "edge" and denoted as  $\delta_{edge}$  if it satisfies two properties (i) PPT property and (ii) extremal violation of the range criterion (i. e. there should be no  $|\phi\rangle|\psi\rangle \in \mathcal{R}(\rho)$  such that  $|\phi\rangle|\psi^*\rangle \in \mathcal{R}(\rho^{\Gamma})$ . It can be seen (see below) that entanglement of any PPT entangled state is due to some

"nonvanishing" admixture of edge state. An example of edge state is just any state based on UPB construction  $\rho_{UPB}$ . Another edge state is  $2 \otimes 4$  PPT entangled state from (Horodecki, 1997).

Now generalization of Terhal construction leads to a method that in fact detects any PPT entanglement is (Lewenstein *et al.*, 2001):

$$W = P + Q^{\Gamma} - \frac{\epsilon}{c}C, \tag{67}$$

with P, Q being positive operators supported on kernels of  $\delta_{edge}$  and  $\delta_{edge}^{\Gamma}$  respectively while the parameter  $\epsilon = \min_{\Psi_{sep}} \langle \Psi_{sep} | P + Q^{\Gamma} | \Psi_{sep} \rangle$  can be shown to be strictly positive (by extremal violation of range criterion by  $\delta_{edge}$  state) while C is arbitrary positive operator with  $\operatorname{Tr}(\delta_{edge}) > 0$  and  $c = \max_{\Psi_{sep}} \langle \Psi_{sep} | C | \Psi_{sep} \rangle$ . All the above witnesses are nondecomposable and it is interesting that entanglement of all PPT entangled states can be detected even by a restricted subclass of the above, when P, Q are just projectors on the kernels of  $\delta_{edge}$  while C is the identity operator (then c = 1). Of course all maps isomorphic to the above witnesses are also nondecomposable.

We would like also to mention a nice idea of construction of nondecomposable map based on k-positive maps. Namely it happens that for any map  $\Lambda_k$  that is k-positive but not k + 1 positive (cf. application in Schmidt rank of density matrix (Terhal and Horodecki, 2000)) the positive (by definition) map  $I_k \otimes \Lambda_k$  is already nondecomposable (Piani and Mora, 2006). This is also connected to another construction of PPT entangled states (Piani and Mora, 2006) which was inspired by very useful examples due to (Ishizaka, 2004) in pure states interconvertibility.

Coming back to range criterion introduced here, there is an interesting application: the so called Lewenstein-Sanpera decomposition. Namely any bipartite state  $\rho$  can be uniquely decomposed (see (Karnas and Lewenstein, 2000)) in the following way (Lewenstein and Sanpera, 1998):

$$\varrho = (1-p)\varrho_{sep} + p\sigma, \tag{68}$$

where  $\rho_{sep}$  (called best separable approximation BSA) is a separable state,  $\sigma$  is entangled and  $p^*$  is a maximal probability  $p^* \in [0,1]$  such that the decomposition of the above form but with  $\sigma$  taken to be arbitrary state is still true. Clearly  $\rho$  is separable iff  $p^* =$ 1. For two-qubits the entangled part  $\sigma$  is always pure (Lewenstein and Sanpera, 1998). Moreover the decomposition can be then found in a fully algebraic way without optimization procedure (Wellens and Kuś, 2001). In particular if  $\rho_{sep}$  is of full rank, then  $\sigma$  is maximally entangled and  $p^*$  is just equal to so called Wootters concurrence (see (Wellens and Kuś, 2001) for the proof). In general the way of construction of the decomposition requires technique of subtracting of product states from the range of the matrix (Lewenstein and Sanpera, 1998) (cf. (Horodecki et al., 2000e; Kraus et al., 2000)). This was

the first proposal of systematic way of checking separability.

Similar technique has been used to find the decomposition of PPT entangled states. Namely it happens that any PPT entangled state can be decomposed into the form  $\rho = (1-p)\rho_{sep} + p\delta_{edge}$  where  $\delta_{edge}$  is an edge state defined above. This is a systematic method leading to necessary and sufficient separability test for states which are PPT i.e. in the region where checking separability is the hardest (the state is separable if at the end the parameter p is zero). Same difficulty like in BSA method is finding product vector in the range of the matrix. The problem becomes much more tractable in case of states that satisfy low rank condition (Horodecki et al., 2000e)  $r(\varrho) + r(\varrho^{\Gamma}) \leq 2d_A d_B - d_A - d_B + 2$ . Then typically (in so called generic case) the state has only finite number of product states satisfying the range criterion which can be found by solving polynomial equations. In such cases the separability problem can be solved in finite number of steps (Horodecki et al., 2000e).

Let us also mention that recently in analogy to the BSA construction above the decomposition onto the symmetrically extendible and nonextendible parts has been found which has a very nice cryptographic application (Moroder *et al.*, 2006b) (see Sec. XIX). The BSA construction leads also to an entanglement measure (see Sec. XV.C.4).

#### 8. Matrix realignment criterion and linear contractions criteria

There is yet another strong class of criteria based on linear contractions on product states. They stem from the new criterion discovered in (Rudolph, 2003), (Chen and Wu, 2003) called computable cross norm (CCN) criterion or matrix realignment criterion which is operational and independent on PPT test (Peres, 1996b). In terms of matrix elements it can be stated as follows: if the state  $\rho_{AB}$  is separable then the matrix  $\mathcal{R}(\rho)$  with elements

$$\langle m | \langle \mu | \mathcal{R}(\varrho_{AB}) | n \rangle | \nu \rangle \equiv \langle m | \langle n | \varrho | \nu \rangle | \mu \rangle \tag{69}$$

has trace norm not greater than one (there are many other variants see (Horodecki *et al.*, 2006d).

It can be formally generalized as follows: if  $\Lambda$  satisfies

$$\|\Lambda(|\phi_A\rangle\langle\phi_A|\otimes|\phi_B\rangle\langle\phi_B|)\|_1 \le 1 \tag{70}$$

for all pure product states  $|\phi_A\rangle\langle\phi_A| \otimes |\phi_B\rangle\langle\phi_B|$  then for any separable state  $\rho_{AB}$  one has  $||\Lambda(\rho_{AB})||_1 \leq 1^{34}$ . The matrix realignment map  $\mathcal{R}$  which permutes matrix elements just satisfies the above contraction on products criterion (70). To find another interesting contractions

<sup>&</sup>lt;sup>34</sup> Here  $||X||_1 = \text{Tr}\sqrt{XX^{\dagger}}$  denotes trace norm.

of that type that are not equivalent to realignment is an open problem.

Quite remarkably the realignment criterion has been found to detect some of PPT entanglement (Chen and Wu, 2003)(see also (Rudolph, 2003)) and to be useful for construction of some nondecomposable maps. It also provides nice lower bound on concurrence function (see (Chen *et al.*, 2005b)). On the other hand it happens that for any state that violates the realignment criterion there is a local uncertainty relation (LUR) (see Sec. VIII.A) that is violated but converse statement is not always true (Gühne *et al.*, 2006a). On the other hand finding LUR-s (like finding original entanglement witnesses) is not easy in general and there is no practical characterization of LUR-s known so far, while the realignment criterion is elementary, fast in application and still powerful enough to detect PPT entanglement.

# 9. Some classes of important quantum states: entanglement regions of parameters

In this section we shall recall classes of states or which PPT property is equivalent to separability.

We shall start from Werner states that are linked to one of the most intriguing problem of entanglement theory namely NPT bound entanglement problem (DiVincenzo *et al.*, 2000a; Dür *et al.*, 2000a) (see Sec. XII).

Werner  $d \otimes d$  states (Werner, 1989a) .- Define projectors  $P^{(+)} = (I + V)/2, P^{(-)} = (I - V)/2$  with identity I, and "flip" operation V (48):

The following  $d \otimes d$  state

$$W(p) = (1-p)\frac{2}{d^2+d}P^{(+)} + p\frac{2}{d^2-d}P^{(-)}, \ 0 \le p \le 1$$
(71)

is invariant under any  $U \otimes U$  operation for any unitary U. W(p) is separable iff it is PPT which holds for  $0 \le p \le \frac{1}{2}$ .

Isotropic states (Horodecki and Horodecki, 1999) .-They are  $U \otimes U^*$  invariant (for any unitary U)  $d \otimes d$ states. They are of the form

$$\varrho_F = \frac{1-F}{d^2-1}\mathbf{I} + \frac{Fd^2-1}{d^2-1}P_+, \quad 0 \le F \le 1$$
(72)

(with  $P^+$  defined by (50)). An isotropic state is separable iff it is PPT which holds for  $0 \le F \le \frac{1}{d}$ .

"Low global rank class" (Horodecki et al., 2000e). The general class of  $d_A \otimes d_B$  state of all states which have global rank not greater than local ones:  $r(\varrho_{AB}) \leq \max[r(\varrho_A), r(\varrho_B)]$ . Here again PPT condition is equivalent to separability. In particular for  $r(\varrho_{AB}) = r(\varrho_A) =$  $r(\varrho_B)$  the PPT property of  $\varrho_{AB}$  implies separability (Horodecki *et al.*, 2000e). If  $r(\varrho_{AB}) < \max[r(\varrho_A), r(\varrho_B)]$ (which corresponds to violation of entropic criterion for  $\alpha = \infty$ ) then PPT test is violated, because reduction criterion is stronger than  $S_{\infty}$  entropy criterion (Horodecki *et al.*, 2003f).

# 10. Characterization of bipartite separability in terms of biconcurrence

In this section we shall describe a quadratic function of the state that provides necessary and sufficient condition for separability called *biconcurrence*. This function was inspired by a generalization of two-qubit Wootters' concurrence due to (Rungta *et al.*, 2001), that exploited the *universal state inverter*, which in turn is actually the reduction map  $\Lambda^{red}$  (see Sec. VI.B.6). The generalized concurrence can be written in the form  $C(\psi_{AB}) = \sqrt{\frac{1}{2}} \langle \psi_{AB} | [\Lambda_A^{red} \otimes \Lambda_B^{red}] (|\psi_{AB}\rangle \langle \psi_{AB} |) | \psi_{AB} \rangle =$  $\sqrt{2(1 - \text{Tr}(\varrho_B^2)}$ , which directly reproduces Wootters' concurrence in case of two-qubits (see Sec. XV.C.2.b). In (Badziąg *et al.*, 2002), a simplified form was obtained:

$$C(\psi) = \sqrt{\langle \psi_{AB} | [I_A \otimes \Lambda_B^{red}](|\psi_{AB}\rangle \langle \psi_{AB} |) | \psi_{AB} \rangle}$$
  
=  $\sqrt{1 - \text{Tr}(\varrho_B^2)}.$  (73)

Now for any ensemble realizing mixed state  $\varrho = \sum_{i=1}^{k} p_i |\tilde{\psi}_{AB}^i\rangle \langle \tilde{\psi}_{AB}^i |$  with  $k \leq N := (d_A d_B)^2$  and  $|\psi_i\rangle := \sqrt{p_i} |\tilde{\psi}_{AB}^i\rangle$ , the  $N \otimes N$  biconcurrence matrix  $B_{m\mu,n\nu}$  is defined as:

$$B_{m\mu,n\nu} \equiv \langle \psi_m | [\mathbf{I} \otimes \Lambda^{red}] (|\psi_\mu\rangle \langle \psi_n|) | \psi_\nu\rangle \qquad (74)$$

(we have dropped here the subsystem indices AB and extended the ensemble to N-element one by adding extra N - k zero vectors). It can be written equivalently as<sup>35</sup>

$$B_{m\mu,n\nu} = \langle \psi_m | \psi_n \rangle \langle \psi_\mu | \psi_\nu \rangle - \operatorname{Tr}[(A^{\psi_m})^{\dagger} A^{\psi_\mu} (A^{\psi_n})^{\dagger} A^{\psi_\nu}]$$
(75)

with the matrix of coefficients  $A^{\psi}$  defined by the relation  $\psi = \sum_{i=0}^{d_A-1} \sum_{j=0}^{d_B-1} A_{ij}^{\psi} |i\rangle |j\rangle$ . Now, there is an important observation namely the state  $\varrho_{AB}$  is separable if and only if the scalar biconcurrence function

$$\mathcal{B}(\varrho) := \inf_{U} \sum_{m=1}^{N} [U \otimes UBU^{\dagger} \otimes U^{\dagger}]_{mm,mm}$$
(76)

vanishes (Badziąg *et al.*, 2002). Here infimum (equal to minimum) is taken over all unitary operations U defined on Hilbert space  $\mathcal{H}$  ( $dim\mathcal{H} = d_A^2 d_B^2$ ) while the matrix B represents operator on  $\mathcal{H} \otimes \mathcal{H}$ . As a matter of fact value of the biconcurrence function (76) is just the square of Euclidean norm of concurrence vector introduced in (Audenaert *et al.*, 2001c) (see sec. XV.C.2.b).

It is interesting that, as one can easily show,  $\mathcal{B}(\varrho) = \inf_{\{p_i,\psi_i\}} \sum_{i=1}^{N} p_i^2 C(\psi_i)^2$  where infimum is taken over all *N*-element ensembles realizing given state  $\varrho$ . Note that

<sup>&</sup>lt;sup>35</sup> A simple expression for biconcurrence was exhibited in (Mintert *et al.*, 2004).  $B_{m,\mu,n,\nu} = \langle \psi_m | \langle \psi_\mu | P_{AA'}^{(-)} \otimes P_{BB'}^{(-)} | \psi_n \rangle | \psi_\nu \rangle.$ 

putting just square root under sum provides the seminal concurrence value for mixed states

$$\mathcal{C}(\varrho) = \inf_{\{p_i, \psi_i\}} \sum_{i=1}^{\kappa} p_i C(\psi_i), \tag{77}$$

which has an advantage of being an entanglement monotone and can be bounded analytically (Mintert *et al.*, 2004)). Still, as in all concurrence cases there is no direct algorithm known that finds the minimum efficiently even for low dimensional systems.

Interestingly, the biconcurrence matrix allows to formulate separability problem in terms of a single "entanglement witness", though acting on a completely different Hilbert space (Badziag *et al.*, 2007). Namely for a give state  $\rho$  one constructs a special witness  $W_{\rho}$  such that the state is separable iff this witness vanishes on some product state.

#### 11. Enhancing separability criteria by local filters

For strictly positive density matrices one can strengthen separability criteria by use of the following result of (Leinaas et al., 2006) (see also (ande J. Dehaene and Moor, 2003)). For any such state  $\rho$  there exist invertible operators A and B such that

$$\tilde{\rho} = A \otimes B\rho A^{\dagger} \otimes B^{\dagger} = \frac{1}{d_A d_B} (\mathbf{I} + \sum_{i=1}^{d_A^2 - 1} a_i E_i \otimes F_i), \quad (78)$$

where  $E_i$  ( $F_i$ ) are traceless orthonormal hermitian operators. The operation is called filtering (see sec. XI.B)). The operators A and B can be found constructively. Due to invertibility of operators A and B, the (unnormalized) state  $\tilde{\rho}$  is entangled if and only if the original state is entangled. Thus the separability problem reduces to checking states of the above form. They have, in particular, maximally mixed subsystems. Given any separability criterion, it often proves useful to apply it to the filtered state  $\tilde{\rho}$  rather than to original state (see e.g. (Gühne *et al.*, 2007)).

### VII. MULTIPARTITE ENTANGLEMENT — SIMILARITIES AND DIFFERENCES

In multipartite case the qualitative definition of separability and entanglement is much richer then in bipartite case. There is the so-called full separability, which is the direct generalization of bipartite separability. Moreover, there are many types of *partial* separability. Below we will briefly discuss the separability criteria in this more complicated situation.

### A. Notion of full (*m*-partite) separability

The definition of full multipartite separability (or just m-separability) of m systems  $A_1...A_m$  with Hilbert space

 $\mathcal{H}_{A_1...A_n} = \mathcal{H}_{A_1} \otimes ... \otimes \mathcal{H}_{A_m}$  is analogous to that in bipartite case:  $\varrho_{AB} = \sum_{i=1}^k p_i \varrho_{A_1}^i \otimes ... \otimes \varrho_{A_m}^i$ . The Caratheodory bound is kept  $k \leq dim \mathcal{H}_{A_1...A_m}^2$ . Such defined set of *m*-separable states is again (i) convex and (ii) closed (with respect to trace norm) Moreover separability is preserved under *m*-separable operations which are immediate generalization of bipartite separable ones

$$\varrho_{A_1,\dots,A_m} \to \frac{\sum_i A_i^1 \otimes \dots \otimes A_i^n \varrho_{A_1,\dots,A_m} (A_i^1 \otimes \dots \otimes A_i^n)^{\dagger}}{\operatorname{Tr}(\sum_i A_i^1 \otimes \dots \otimes A_i^n \varrho_{A_1,\dots,A_m} (A_i^1 \otimes \dots \otimes A_i^n)^{\dagger})}$$
(79)

The separability characterization in terms of positive but not completely positive maps and witnesses generalizes in a natural way (Horodecki *et al.*, 2001a). There is a condition analogous to (42) with I acting on first subsystem  $\mathcal{H}_{A_1}$  and the map  $\Lambda_{A_2...A_m} : \mathcal{B}(\mathcal{H}_{A_2,...,A_m}) \to \mathcal{B}(\mathcal{H}_{A_1})$ . Namely, in the formula (42) we take the maps  $\Lambda_{A_2...A_m} :$  $\mathcal{B}(\mathcal{H}_{A_2,...,A_m}) \to \mathcal{B}(\mathcal{H}_{A_1})$  that are *positive on product states* ie.  $\Lambda_{A_2...A_m}(|\phi_{A_2}\rangle\langle\phi_{A_2}|\otimes\cdots\otimes|\phi_{A_m}\rangle\langle\phi_{A_m}|) \geq 0$ (with arbitrary states  $\phi_{A_i} \in \mathcal{H}_{A_i}$ ) but not completely positive. The corresponding entanglement witness must have again (i) at least one negative eigenvalue and also satisfy (ii)

$$\langle \phi_{A_1} | \dots \langle \phi_{A_m} | W | \phi_{A_1} \rangle \dots | \phi_{A_m} \rangle \ge 0. \tag{80}$$

Maps and witnesses are again related by the isomorphism (49) with maximally entangled state  $P_+$  on bipartite system  $A_1A_1$ .

The above description provides full characterization of m-separability of m-partite system. An example of maps positive on product states is simply a product of positive maps. Of course there exist maps that are positive on product states, but are not of the latter form. (those are in particular maps (Horodecki *et al.*, 2001a) detecting entanglement of some semiseparable states constructed in (Bennett *et al.*, 1999b), see one of examples below). Multipartite witnesses and related maps were investigated in (Jafarizadeh *et al.*, 2006) by means of linear programming.

*Example .-* An elementary example of fully separable 3-qubit state is:

$$\varrho = p|0\rangle\langle 0|^{\otimes 3} + (1-p)|1\rangle\langle 1|^{\otimes 3}.$$
(81)

Now let us come back to the description of general mixed states separability criteria in multipartite systems. Note that in this case there is no simple necessary and sufficient condition for separability like PPT characterization of  $2 \otimes 2$  or  $2 \otimes 3$  case. Even for three qubits no such criterion has been found so far, in particular checking PPT criteria with respect to all bipartite partitions is not enough at all. However there are many criteria that may be applied. Range criterion immediately generalizes to its many variants since now we require range of  $\rho_{A_1,\ldots A_m}$  to be spanned by  $\{|\phi_{A_1}\rangle \ldots |\phi_{A_m}\rangle\}$  while range of the state  $\rho_{A_1\ldots A_m}^{T_{A_k_1}\ldots A_{k_l}}$  partially transposed with respect to subset  $\{A_{k_1},\ldots,A_{k_l}\} \subset \{A_1,\ldots,A_m\}$  is clearly required

to be spanned by the products of these vectors where all with indices  $k_1, \ldots, k_l$  are complex conjugated. Of course if the state is separable all such partial transposes must lead to matrices with nonnegative spectrum, more precisely all matrices of the type  $\rho_{A_1...A_m}^{T_{A_{k_1}...A_{k_l}}}$  should be states themselves.

The realignment criteria are generalized to permutational criteria (Chen and Wu, 2002; Horodecki et al., 2006d) which state that if state  $\rho_{A_1A_2...A_n}$  is separable then the matrix  $[\mathcal{R}_{\pi}(\varrho)]_{i_1j_1,i_2j_2,\ldots,i_nj_n}$  $\equiv$  $\varrho_{\pi(i_1j_1,i_2j_2,\ldots,i_nj_n)}$  (obtained from the original state via permutation  $\pi$  of matrix indices in product basis) satisfies  $\|\mathcal{R}_{\pi}(\varrho)\|_{1} \leq 1$ . Only some permutations give nontrivial criteria, that are also different from partial transpose. It is possible to significantly reduce the number of permutations to the much smaller set of those that provide independent criteria (Wocjan and Horodecki, 2005). In the case of three particles one has a special case of partial realignment (Chen and Wu, 2003; Horodecki et al., 2006d)). Finally, let us recall that the contraction criterion (70) generalizes immediately.

Let us now consider the case of pure states in more detail. A pure *m*-partite state is fully separable if and only if it is a product of pure states describing *m* elementary subsystems. To check it, it is enough to compute reduced density matrices of elementary subsystems and check whether they are pure. However, if one asks about the possible ways this simple separability condition is violated then the situation becomes more complicated.

The first problem is that in multipartite case (in comparison to bipartite one) only very rarely pure states admit the generalized Schmidt decomposition  $|\Psi_{A_1,\ldots,A_m}\rangle = \sum_{i=1}^{\min[d_{A_1},\ldots,d_{A_m}]} a_i |\tilde{e}_{A_1}^i\rangle \otimes \cdots \otimes |\tilde{e}_{A_m}^i\rangle$  (see (Peres, 1995; Thaplyial, 1999)). An example of the state admitting Schmidt decomposition in the  $d^{\otimes m}$  case is the generalized Greenberger-Horne-Zeilinger state

$$|GHZ\rangle_{d}^{(m)} = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} (|i\rangle^{\otimes m}),$$
 (82)

which is a generalization of original GHZ state (Greenberger *et al.*, 1989) that is a three-qubit vector  $|GHZ\rangle = \frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|1\rangle|1\rangle$ ). To give an example of state which does not admit Schmidt decomposition, note that the latter implies that if we trace out any subsystem, the rest is in fully separable state. One easily finds that the following state

$$|W\rangle = \frac{1}{\sqrt{3}} (|0\rangle|0\rangle|1\rangle + |0\rangle|1\rangle|0\rangle + |1\rangle|0\rangle|0\rangle)$$
(83)

has entangled two qubit subsystem, hence does not admit Schmidt decomposition (Dür *et al.*, 2000b).

Thus, in general entanglement of pure state is described by spectra of the reduced density matrices produced by all bipartite partitions. As implied by the full separability definition it is said to be fully *m*-partite separable iff:

$$\Psi_{A_1,\dots,A_m}\rangle = |\psi_{A_1}\rangle \otimes \dots \otimes |\psi_{A_m}\rangle. \tag{84}$$

However violation of this condition clearly does not automatically guarantee what can be intuitively considered as "truly" *m*-partite entanglement (to understand it see for instance the 4-qubit state  $\Psi_{A_1A_2A_3A_4} = |\Phi_{A_1A_2}\rangle \otimes |\Phi_{A_3A_4}\rangle$  where at least one vector  $\Phi_{A_1A_2}$ ,  $\Phi_{A_3A_4}$  is entangled).

One says that the state is m-partite entangled iff all bipartite partitions produce mixed reduce density matrices (note that both reduced states produced in this way have the same nonzero eigenvalues). This means that there does not exist cut, against which the state is product. To this class belong all those pure states that satisfy generalized Schmidt decomposition (like the GHZ state above). But there are many others, e.g. the mentioned W state. In Sec. XIII we will discuss how one can introduce classification within the set of *m*-partite entangled states. One can introduce a further classification by means of stochastic LOCC (SLOCC) (see Sec. XIII), according to which for 3 qubits there are two classes of truly 3-partite entangled states, represented just by the GHZ and W states. There are furthermore three classes of pure states which are partially entangled and partially separable: this is the state  $|\Phi^+\rangle|0\rangle$  (where  $\Phi^+ = \frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|1\rangle$ ) and its twins produced by two cyclic permutations of subsystems. We see that in the case of those pure states only 2-qubit entanglement is present and explicitly "partial" separability can be seen. This leads us to the various notions of partial separability which will be described in next section. Here we will present an important family of pure entangled states.

Example: quantum graphstates.General form of graph states has been introduced in (Raussendorf et al., 2003) as a generalization of cluster states (Briegel and Raussendorf, 2001) that have been shown to be a resource for one-way quantum computer (Raussendorf and Briegel, 2001). Universality of quantum computer based on graph states is one of the fundamental application of quantum entanglement in a theory of quantum computer (see (Hein *et al.*, 2005). In general, any graph state is a pure *m*-qubit state  $|G\rangle$  corresponding to a graph G(V, E). The graph is described by the set V of vertices with cardinality |V| = m (corresponding to qubits of  $|G\rangle$ ) and the set E of edges, i.e. pairs of vertices (corresponding to pairs of qubits of  $|G\rangle$ )<sup>36</sup>. Now, the mechanism of creating  $|G\rangle$  is very simple. One takes as the initial state  $|+\rangle^{\otimes m}$  with  $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ . Then according to graph G(V, E), to any of pairs of qubits corresponding to vertices connected by an edge from E one applies a controlled phase gate:  $U_{C-phase} = |0\rangle \langle 0| \otimes I + |1\rangle \langle 1| \otimes \sigma_3$ . Note that since all such operations commute even if performed according to

<sup>&</sup>lt;sup>36</sup> Usually *E* is represented by symmetric *adjacency matrix* with elements  $A_{uv} = A_{vu} = 1$  iff  $u \neq v$  are connected by the edge and  $A_{uv} = 0$  otherwise. Note, that there are no more than  $\frac{m(m-1)}{2}$  of the edges — pairs of vertices.

the edges with a common vertex, the order of applying the operations is arbitrary. Remarkably, the set of graph states constructed is described by a *polynomial* number m(m-1)/2 of discrete parameters, while in general the set of all states in the m-qubit Hilbert space is described by an exponential  $2^m$  number of continuous parameters. Moreover, local unitary interconvertibility under  $\otimes_{i=1}^m U_i$ of two graphs states is equivalent to convertibility under stochastic local operations and classical communications (SLOCC).

Any connected<sup>37</sup> graph state  $|G\rangle$  is fully entangled mparticle state and violates some Bell inequality. The latter fact has been proven (Gühne et al., 2005) using alternative (equivalent) description of graph states in terms of stabilizer formalism (Gottesman, 1997). Moreover the idea of efficiently locally measurable entanglement witnesses that detect entanglement of graph states has been proposed (Toth and Gühne, 2005a). This idea can be developed, to show that entanglement of any connected graph violates some Bell inequality that requires only two measurements per each qubit site (Toth et al., 2006). This is a very useful proposal of local detection of the mean value of corresponding entanglement witness. For of a review of many important and very interesting applications of graph states in quantum information, including details of their fundamental role in one-way quantum computing, see (Hein *et al.*, 2005).

#### B. Partial separability

Here we shall consider two other very important notions of partial separability. The first one is just separability with respect to partitions. In this case, the state of  $A_1, \ldots, A_m$  elementary subsystems is separable with respect to a given partition  $\{I_1, \ldots, I_k\}$ , where  $I_i$ are disjoint subsets of the set of indices  $I = \{1, \ldots, m\}$  $(\bigcup_{l=1}^k I_l = I)$  iff  $\varrho = \sum_{i=1}^N p_i \varrho_1^i \otimes \cdots \otimes \varrho_k^i$  where any state  $\varrho_l^i$  is defined on tensor product of all elementary Hilbert spaces corresponding to indices belonging to set  $I_i$ . Now, one may combine *several separability conditions* with respect to several different partitions. This gives many possible choices for partial separability.

Let us show an interesting example of partial separability which requires even number of qubits in general.

*Example* .- Consider four-qubit Smolin states (Smolin, 2001):

$$\varrho_{ABCD}^{unlock} = \frac{1}{4} \sum_{i=1}^{4} |\Psi_{AB}^{i}\rangle \langle \Psi_{AB}^{i}| \otimes |\Psi_{CD}^{i}\rangle \langle \Psi_{CD}^{i}|, \qquad (85)$$

where  $|\Psi^i\rangle$  are four Bell states. It happens that it is symmetrically invariant under any permutations (to see it one can use the symmetric Hilbert-Schmidt representation  $\varrho_{ABCD}^{unlock} = \frac{1}{4}(I^{\otimes 4} + \sum_{i=1}^{3} \sigma_{i}^{\otimes 4})$ . Thus, the state is separable under any partition into two two-qubit parts. Still, it is entangled under any partition 1 versus 3 qubits since it violates PPT criterion with respect to this partition ie.  $(\varrho_{ABCD}^{unlock})^{T_A} \geq 0$ . This state has been shown to have applications in remote concentration of quantum information (Murao and Vedral, 2001). The Smolin state has been also shown to be useful in reduction of communication complexity via violation of Bell inequalities (Augusiak and Horodecki, 2006a). No bound entangled states with fewer degrees of freedom useful for communication complexity are known so far. Generalization of the state to multipartite case is possible (Augusiak and Horodecki, 2006b; Wu *et al.*, 2005).

A particularly interesting from the point of view of low partice case systems is a special class of partially separable states called *semiseparable*. They are separable under all 1-(m-1) partitions :  $\{I_1 = \{k\}, I_2 = \{1, \ldots, k 1, k + 1, \ldots, m\}$ ,  $1 \le k \le m$ . It allows to show a new type of entanglement: there are semiseparable 3-qubit states which are still entangled. To see it consider the next example (DiVincenzo *et al.*, 2003a):

the following *Examples* .- Consider product  $2 \otimes 2 \otimes 2$  state composed on 3 parts states: ABC generated by set defined as  $S_{Shift}$ = $\begin{array}{l} |0\rangle|0\rangle,|+\rangle|1\rangle|-\rangle,|1\rangle|-\rangle|+\rangle,|-\rangle|+\rangle|1\rangle\}, \quad \text{(with } \\ |\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)). \quad \text{This set can be proved to} \end{array}$ define multipartite unextendible product basis in full analogy to the bipartite case discussed in Sec. (VI.B.7): there is no product state orthogonal to subspace spanned by  $S_{\text{Shift}}$ . Thus, in analogy to bipartite construction, the state  $\rho_{\text{Shift}} = (I - P_{\text{Shift}})/4$  where  $P_{\text{Shift}}$  projects onto the subspace spanned by  $S_{\mathbf{Shift}}$  can be easily shown to be entangled as a whole (i.e. not fully separable) but PPT under all cuts (i.e. A|BC, AB|C, B|AC). However it happens that it is not only PPT but also *separable* under all cuts. This means that semiseparability is not equivalent to separability even in the most simple multipartite case like 3-qubit one.

Entanglement of semiseparable states shown here immediately follows from construction by application of multipartite version of range criterion. It is also detected by permutation criteria. Finally, one can use maps that are positive on all product states of two qubits but not positive in general, described in Sec. VII.A (see (Horodecki *et al.*, 2000e)).

Another interesting class is the set of  $U \otimes U \otimes U$  invariant  $d \otimes d \otimes d$  states which comprises semiseparable and fully 3-separable subclasses of states in one 5-parameter family of states (see (Eggeling and Werner, 2001)).

The moral of the story is that *checking bipartite separability with respect to all possible cuts in not enough to guarantee full separability.* However separability with respect to some partial splittings gives still important generalization of separability and have interesting applications (see (Dür and Cirac, 2000a,b; Dür *et al.*, 1999b; Smolin, 2001) and Sec. XII). In this context we shall

<sup>&</sup>lt;sup>37</sup> A graph state  $|G\rangle$  is called *connected* if the corresponding graph G(V, E) is connected, that is when any two vertices from V are linked by the path of subsequent edges from E.

describe below a particularly useful family of states.

Example .- Separability of the family of the states presented here is fully determined by checking PPT criterion under any possible partitions. To be more specific, PPT condition with respect to some partition guarantees separability along that partition. The states found some (Dür and Cirac, 2000b; Dür *et al.*, 1999b) important applications in activation of bound entanglement (Dür and Cirac, 2000a), nonadditivity of multipartite quantum channels (Dür *et al.*, 2004) and multipartite bound information phenomenon (Acin *et al.*, 2004b). This is the following *m*-qubit family (Dür *et al.*, 1999b):

$$\varrho^{(m)} = \sum_{a=\pm} \lambda_0^a |\Psi_0^a\rangle \langle \Psi_0^a| + \sum_{k\neq 0} \lambda_k (|\Psi_k^+\rangle \langle \Psi_k^+| + |\Psi_k^-\rangle \langle \Psi_k^-|),$$
(86)

where  $|\Psi_k^{\pm}\rangle = \frac{1}{\sqrt{2}}(|k_1\rangle|k_2\rangle...|k_{m-1}\rangle|0\rangle \pm |\overline{k_1}\rangle|\overline{k_2}\rangle...|\overline{k_{m-1}}\rangle|1\rangle$  with  $k_i = 0, 1, \ \overline{k_i} = k_i \oplus 1 \equiv (k_i + 1) \mod 2$  and k being one of  $2^{m-1}$  real numbers defined by binary sequence  $k_1, \ldots, k_{m-1}$ .

Let us put  $\Delta = \lambda_0^+ - \lambda_0^- \ge 0$  and define *bipartite* splitting into two disjoint parts  $A(k) = \{$  subset with last (m-th) qubit  $\}$ ,  $B(k) = \{$  subset without last qubit  $\}$  with help of binary sequence k such that *i*-th qubit belongs to A(k) (and not to B(k)) if and only if the sequence  $k_1, \ldots, k_{m-1}$  contains  $k_i = 0$ . Then one can prove (Dür *et al.*, 1999b) that (i)  $\varrho^{(m)}$  is separable with respect to partition  $\{A(k), B(k)\}$  iff  $\lambda_k \ge \Delta/2$  which happens to be equivalent to PPT condition with respect to that partition (ie.  $\varrho^{T_B} \ge 0$ ) (ii) if PPT condition is satisfied for all bipartite splittings then  $\varrho^{(m)}$  is fully separable. Note that the condition (ii) above does not hold in general for other mixed states which can be seen easily on 3-qubit semiseparable state  $\varrho_{\mathbf{Shift}}$  recalled in this section. That state is entangled but clearly satisfies PPT condition under all bipartite splittings.

There is yet another classification that allows for stratification of entanglement involved. For instance *m*particle system may be required to have at most s-particle entanglement which means that it is a mixture of all states such that each of them is separable with respect to some partition  $\{I_1, \ldots, I_k\}$  where all sets of indices  $I_k$  have cardinality not greater than s. All m-partite states that have at most m - 1-particle entanglement satisfy special Bell inequalities (see (Svetlichny, 1987) for bipartite and (Seevinck and Svetlichny, 2002) for general case) and nonlinear separability criteria like that of (Maassen and Uffink, 1988) which we shall pass to just in next section (see Sec. VIII.B).

# VIII. FURTHER IMPROVEMENTS OF ENTANGLEMENT TESTS: NONLINEAR SEPARABILITY CRITERIA

Then the nonlinear separability criteria into two different classes. The first ones are based on functions of results of few measurements performed in noncollective manner (on one copy of  $\varrho_{AB}$  a time). This first time comprises separability conditions in terms of uncertainty relations that have been started to be developed first for continuous variables (We must stress here that we do not consider entanglement measures which are also a nonlinear functions of the state, but belong to a very special class that has — in a sense — its own philosophy.). Such conditions will be described in subsections (VIII.A, VIII.B) below.

The second class of nonlinear separability conditions is based on *collective* measurements on several copies and has attracted more and more attention recently. We shall present them in Sec. VIII.C.2.

#### A. Uncertainty relation based separability tests

The uncertainty relations have been first developed for continuous variables and applied to Gaussian states (Duan *et al.*, 2000) (see also (Mancini *et al.*, 2002)). For the bipartite case nonlinear inequalities for approximations of finite dimensional Hilbert spaces in the limit of high dimensions have been exploited in terms of global angular momentum-like uncertainties in (Kuzmich, 2000) with further experimental application (Julsgaard *et al.*, 2001)<sup>38</sup>.

General separability criteria based on uncertainty relation and valid both for discrete and continuous variables (CV) have been introduced in (Giovannetti et al., 2003), and (Hofmann and Takeuchi, 2003) (the second one was introduced for discrete variables but its general formulation is valid also for the CV case). Soon further it has been shown (Hofmann, 2003) that PPT entanglement can be detected by means of uncertainty relation introduced in (Hofmann and Takeuchi, 2003). This approach has been further developed and simplified by Gühne (Gühne, 2004) and developed also in entropic terms (Gühne and Lewenstein, 2004a). Another separability criterion in the two-mode continuous systems based on uncertainty relations with the particle number and the destruction operators was presented (Toth *et al.*, 2003), which may be used to detect entanglement in light field or in Bose-Einstein condensates.

Let us recall briefly the key of the approach of local uncertainty relations (LUR) (Hofmann and Takeuchi, 2003) which has found a very nice application in the idea of macroscopic entanglement detection via magnetic susceptibility (Wieśniak et al., 2005). Consider the set of local observables  $\{A_i\}_{i=0}^N$ ,  $\{B_i\}_{i=1}^N$  on Hilbert spaces  $\mathcal{H}_A$ ,  $\mathcal{H}_B$  respectively. Suppose that one has bounds on sum of local variances ie.  $\sum_i \delta(A_i)^2 \geq c_a, \sum_i \delta(B_i)^2 \geq$  $c_b$  with some nonnegative values  $c_a, c_b$  and the variance definition  $\delta(M)_{\varrho}^2 \equiv \langle M^2 \rangle_{\varrho} - \langle M \rangle_{\varrho}^2$ . Then for

<sup>&</sup>lt;sup>38</sup> The first use of uncertainty relation to detect entanglement (with theoretically and experimentally) can be found in (Hald *et al.*, 1999) for spins of atomic ensembles.

any separable state  $\rho_{AB}$  the following inequality holds (Hofmann and Takeuchi, 2003):

$$\sum_{i} \delta(A_i \otimes \mathbf{I} + \mathbf{I} \otimes B_i)^2_{\varrho_{AB}} \ge c_A + c_B.$$
(87)

Note that by induction the above inequality can be extended to the multipartite case. Quite remarkably if the observables  $A_i$  and  $B_i$  are chosen in a special asymmetric way, then the above inequality can be shown (Hofmann, 2003) to detect entanglement of the family of (64) PPT states. The LUR approach has been generalized in (Gühne, 2004) to separability criteria *via* nonlocal uncertainty relations. That approach is based on the observation that for any convex set S (here we choose it to be set of separable states), any set of observables  $M_i$  and the state  $\rho = \sum_i p_i \rho_i$ ,  $\rho_i \in S$  the following inequality holds:

$$\sum_{i} \delta(M_i)_{\varrho}^2 \ge \sum_{k} p_k \sum_{i} \delta(M_i)_{\varrho_k}^2.$$
(88)

It happens that in many cases it is relatively easy to show that right hand side (RHS) is separated from zero only if  $\rho$  is separable, while at the same time LHS vanishes for some entangled states <sup>39</sup>.

Especially, as observed in (Gühne, 2004) if the observables  $M_i$  have no common product eigenvector then the RHS must be strictly greater than zero since  $\delta(M)^2_{\Psi}$  vanishes iff  $\Psi$  is an eigenvector of M. Consider now an arbitrary state that violates the range criterion in such a way that it has no product vector in its range  $R(\rho)$ (an example is just the PPT entangled state produced by UPB meted, but discrete value PPT entangled states from (Horodecki and Lewenstein, 2000). Now there is a simple observation: any subspace  $\mathcal{H}^{\perp}$  orthonormal to the subspace  $\mathcal{H}$  having no product vector can be spanned by (maybe nonorthogonal) entangled vectors  $\{|\Psi_i\rangle\}_{i=1}^r$ ,  $r = dim \mathcal{H}$ . The technique of (Gühne, 2004) is now to take  $M_i = |\Psi_i\rangle\langle\Psi_i|, i = 1, \ldots, r$  and  $M_{r+1} = P_{R(\rho)}$ where the last one is a projector onto the range of the state, which — by definition — has no product state in the range. Note that since all  $|\Psi_i\rangle$  belong to the kernel<sup>40</sup> of  $\rho$  one immediately has LHS of (88) vanishing. But as one can see there is no product eigenvector that is common to all  $M_i$ -s hence as mentioned above, RHS must be strictly positive which gives expected violation of the inequality (88).

An alternative interesting way is to consider an entropic version of uncertainty relations, as initiated in (Giovannetti, 2004) and developed in (Gühne and Lewenstein, 2004a). This technique is a next step in application of entropies in detecting entanglement. The main idea is (see (Gühne and Lewenstein, 2004a)) to prove that if sum of local Klein entropies<sup>41</sup> satisfies uncertainty relations  $S(A_1) + S(A_2) \geq C$ ,  $S(B_1) + S(B_2) \geq C$  for systems A, B respectively then for any separable state global Klein entropy must satisfy the same bound  $S(A_1 \otimes B_1) + S(A_2 \otimes B_2) \geq C$ . This approach was also extended to multipartite case (Gühne and Lewenstein, 2004a).

#### B. Nonlinear improvement of entanglement witnesses

In general any bound on nonlinear function of means of observables that is satisfied by separable states but violated by some entangled states can be in broad sense considered as "nonlinear entanglement witnesses". Here we shall consider those separability conditions that had their origin in entanglement witnesses (like for instance Bell operators) and lead to nonlinear separability test. They are still all based on functions of mean values of noncollective measurements (ie. single measurement per copy).

Let us come back to more general nonlinear conditions inspired by Bell inequalities criteria. This interesting approach has been first applied to the multipartite case (Maassen and Uffink, 1988), where on the basis of Bell inequalities nonlinear inequalities have been constructed. These can discriminate between full *m*-particle entanglement of the system  $A_1...A_m$  and the case when it contains at most m - 1-particle entanglement (cf. Sec. VII.B). It is worth to note here that linear inequalities discriminating between those two cases have been provided as Bell-like inequalities early in (Svetlichny, 1987) for 3 particles and extended in (Seevinck and Svetlichny, 2002) to m particles. Though they were not related directly to entanglement, they automatically serve as an entanglement criteria since any separable state allows for LHVM description of all local measurements, and as such satisfy all Bell inequalities.

For small-dimensional systems, a nonlinear inequality inspired by Bell inequality that is satisfied by all separable two-qubit states has been provided in (Yu *et al.*, 2003). Namely the two-qubit state  $\rho$  is separable if and only if the following inequality holds:

$$\sqrt{\langle A_1 \otimes B_1 + A_2 \otimes B_2 \rangle_{\varrho}^2 + \langle A_3 \otimes \mathbf{I} + \mathbf{I} \otimes B_3 \rangle_{\varrho}^2} - \langle A_3 \otimes B_3 \rangle_{\varrho} \le 1, \quad (89)$$

<sup>&</sup>lt;sup>39</sup> There is a more general form of this inequality in terms of covariance matrices, which gives rise to new separability criteria; a simple (strong) necessary and sufficient criterion for two-qubit states was presented which violates LUR (Gühne *et al.*, 2007). (see also (Abascal and Björk, 2007) in this context) Other criterion for symmetric for n-qubit states have been presented in the form of a hierarchy of inseparability condition on the intergroup covariance matrices of even order (Devi *et al.*, 2007).

<sup>&</sup>lt;sup>40</sup> By kernel of the state we mean the space spanned by vectors corresponding to zero eigenvalues of the state.

<sup>&</sup>lt;sup>41</sup> The Klein entropy associated with an observable and a state is the classical entropy of probability distribution coming out of measurement of this observable on the quantum state.

for all sets of dichotomic observables  $\{A_i\}_{i=1}^3 \{B_i\}_{i=1}^3$ that correspond to two local bases orthonormal vectors  $\mathbf{a}_i$ ,  $\mathbf{b}_i$  that have the same orientation in Cartesian frame (the relation  $A_i = \mathbf{a}_i \vec{\sigma}, B_i = \mathbf{b}_i \vec{\sigma}$  holds). There is also a link with PPT criterion: for any entangled state  $\rho$  with negative eigenvalue  $\lambda_{min}$  of  $\rho^{\Gamma}$ , the optimal setting of observables makes LHS equal to  $1 - 4\lambda_{max}$ .

The above construction was inspired by a Bell-type entanglement witness (see (Yu *et al.*, 2003) for details). In this context a question arises if there is any systematic way of nonlinear improvements of entanglement witnesses. Here we shall describe a very nice nonlinear improvement that can be applied to any entanglement witness (see (47)) and naturally exploits the isomorphism (49). Before showing that, let us recall that the whole entanglement witnesses formalism can be translated to the level of covariance matrix in continuous variables, and the nonlinear corrections to such a witnesses that are equivalent to some uncertainty relations can also be constructed (Hyllus and Eisert, 2006).

The above announced (quadratic) nonlinear correction worked out in (Gühne and Lütkenhaus, 2006a) involves a set of operators  $X_k = X_k^{(h)} + iX_k^{(antih)}$  and its mean  $\langle X_k \rangle_{\varrho}$ , which (in practice) must be collected from mean values of their Hermitian  $(X_k^{(h)})$  and antihermitian  $X_k^{(antih)}$  parts respectively. The general form of the nonlinear improvement of the witness W corresponds to the condition

$$\mathcal{F}_{\varrho} = \langle W \rangle_{\varrho} - \sum_{k} \alpha_{k} |\langle X_{k} \rangle_{\varrho}|^{2} \ge 0, \tag{90}$$

where real numbers  $\alpha_k$  and operators  $X_k$  are both chosen in such a way that for all possible separable states  $\varrho_{AB}$  the condition  $\mathcal{F}_{\varrho_{AB}} \geq 0$  is satisfied. One can see that the second term is a quadratic correction to the original (linear) mean value of entanglement witness. Higher order corrections are also possible though they need not automatically guarantee stronger condition (Gühne and Lütkenhaus, 2006a). Let us illustrate this by use of example of partial transpose map  $I_A \otimes T_B(\cdot) = (\cdot)^{\Gamma}$ and the corresponding decomposable witness (see Eq. (52) coming from the minimal set of witnesses describing set of states satisfying PPT conditions  $W = |\Psi\rangle\langle\Psi|^{\Gamma}$ . In general, virtually any entanglement witness condition can be improved in this way, though one has to involve Hermitian conjugate of the map in Hilbert-Schmidt space and exploit the fact (see discussion in Sec. VI.B.3) that single map condition is equivalent to continuous set of entanglement witnesses conditions. Now, for the witness of the form  $W = |\Psi\rangle\langle\Psi|^{\Gamma}$  one of the two versions are natural. In the first one chooses single  $X = |\Phi\rangle\langle\Psi|$  (where both vectors are normalized) and then single parameter  $\alpha$  is equal to the inverse of maximal Schmidt coefficient of  $|\Psi\rangle$ . The second option is to choose  $X_i = |\Phi\rangle\langle\Psi_i|$  with  $\Psi_i$ being an orthonormal basis in global Hilbert space  $\mathcal{H}_{AB}$ and then the choose  $\alpha_i = 1$  for all parameters guarantees positivity of the value (90).

# C. Detecting entanglement with collective measurements

# 1. Physical implementations of entanglement criteria with collective measurements

The idea of direct measurement of pure states entanglement was considered first in (Sancho and Huelga, 2000) and involved the first explicit application of collective measurements to entanglement detection (Acin *et al.*, 2000). In general the question here is how to detect entanglement physically by means of a little number of collective measurements that do not lead to complete tomography of the state. Here we focus on the number of estimated parameters (means of observables) and try to diminish it. The fact that the mean of an observable may be interpreted as single binary estimated parameter (equivalent just to one-qubit polarization) has been proven in (Horodecki, 2003a; Paz and Roncaglia, 2003), cf. (Brun, 2004).

The power of positive maps separability criteria and entanglement measures has motivated work on implementations of separability criteria *via* collective measurements, introduced in (Horodecki, 2003b; Horodecki and Ekert, 2002) and significantly improved in (Carteret, 2005; Horodecki *et al.*, 2006f). On the other hand the entropic separability criteria has led to the separate notion of *collective entanglement witnesses* (Horodecki, 2003d) which will be described in more detail in the next section.

The collective measurement evaluation of nonlinear state functions (Ekert et al., 2002; Filip, 2002) (see also (Fiurasek, 2002a; Horodecki, 2003d)) was implemented experimentally very recently in distance lab paradigm (Bovino et al., 2005). The method takes an especially striking form in the two-qubit case, when not only unambiguous entanglement detection (Horodecki and Ekert, 2002) but also estimation of such complicated entanglement measure as entanglement of formation and Wootters concurrence can be achieved by measuring only four collective observables (Horodecki, 2003b), much smaller than 15 required by state estimation. The key idea of the latter scheme is to measure four collective observables  $A^{(2k)}$  on 2k copies of the state that previously have been subjected to physical action of some maps<sup>42</sup>. The mean values of these descrables reproduce all four moments  $\langle A^{(2k)} \rangle = \sum_i \lambda_i^k$  of spectrum  $\{\lambda_k\}$  of the square of the Wootters concurrence matrix  $\hat{C}(\varrho) = \sqrt{\sqrt{\varrho}\sigma_2 \otimes \sigma_2 \varrho^* \sigma_2 \otimes \sigma_2 \sqrt{\varrho}}$ . Note that, due to the link (Wootters, 1998) between Wootters concurrence and entanglement of formation, the latter can be also inferred in such an experiment. Recently, a collective observable very similar to those of (Horodecki, 2003d), acting on two copies of quantum state which

 $<sup>^{42}</sup>$  They are so called *physical structural approximations*, which we describe further in this section.

detect two-qubit concurrence has been constructed and implemented (Walborn *et al.*, 2003). The observable is much simpler, however the method works under the promise that the state is pure. This approach can be also generalized to multiparty case using suitable factorisable observable corresponding to the concurrence (see (Aolita and Mintert, 2006)).

In methods involving positive maps criteria the main problem of how to physically implement unphysical maps has been overcome by the help of structural physical approximations (SPA) of unphysical maps. In fact for any Hermitian trace nonincreasing map L there is a probability p such that the new map  $\hat{L} = p\mathcal{D} + (1-p)\Lambda$ can be physically implemented if  $\mathcal{D}$  is just a fully depolarizing map that turns any state into maximally mixed one. Now one can apply the following procedure: (i) put  $L = I \otimes \Lambda$  for some P but not CP map  $\Lambda$ , (ii) apply the new (physical) map  $\tilde{L}$  to many copies of bipartite system in unknown state  $\rho$ , (iii) estimate the spectrum of the resulting state  $\tilde{L}(\varrho) = [I \otimes \tilde{\Lambda}](\varrho)$ . Infer the spectrum of  $L(\rho) = [I \otimes \Lambda](\rho)$  (which is an easy affine transformation of the measured spectrum of  $\tilde{L}(\rho)$  and check in this way whether the condition (42) for map  $\Lambda$  is violated. Generalization to multipartite systems is immediate. This test requires measurement of  $d^2$  observables instead of  $d^4 - 1$ ones needed to check the condition with prior state tomography. The corresponding quantum networks can be easily generalized to multipartite maps criteria including realignment or linear contractions criteria (Horodecki, 2003c). The implementation with help of local measurements has also been developed (see (Alves et al., 2003).

However, as pointed out by Carteret (Carteret, 2005), the disadvantage of the method is that SPA involved here requires in general significant amount of noise added to the system. The improved method of noiseless detection of PPT criterion, concurrence and tangle has been worked out (Carteret, 2003, 2005) with help of polynomial invariants technique which allows for very simple and elegant quantum network designing. The problem whether noiseless networks exist for all other positive maps (or contraction maps) have been solved quite recently where general noiseless networks have been designed (Horodecki *et al.*, 2006f).

Finally let us note that the above techniques have been also developed on the ground of continuous variables (Fiurasek and Cerf, 2004; Pregnell, 2006; Stobińska and Wódkiewicz, 2005).

### 2. Collective entanglement witnesses

There is yet another technique introduced (Horodecki, 2003d) that seems to be more and more important in context of experimental implementations. This is the notion of *collective entanglement witness*. Consider a bipartite system AB on the space  $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ . One introduces here the notion of collective observable  $A^{(n)}$  with respect to the single system Hilbert space  $\mathcal{H}$  as an

observable defined on  $\mathcal{H}^{\otimes n}$  and measured on n copies  $\varrho^{\otimes n}$  of given state  $\varrho$ . Also one defines the notion of mean of collective observable in single copy of state  $\varrho^{\otimes n}$  as  $\langle \langle A^{(n)} \rangle \rangle := \operatorname{Tr}(A^{(n)}\varrho^{\otimes n})$ . Now any observable  $W^{(n)}$  defined on  $(\mathcal{H}_{AB})^{\otimes n}$  that satisfies the following condition

$$\langle \langle W^{(n)} \rangle \rangle_{\varrho_{sep}} := \operatorname{Tr}(W^{(n)} \varrho_{sep}^{\otimes n}) \ge 0,$$
 (91)

but there exists an entangled state  $\rho_{ent}$  such that

$$\langle \langle W^{(n)} \rangle \rangle_{\varrho_{ent}} < 0,$$
 (92)

is called *collective* (n-copy) entanglement witness. In (Horodecki, 2003d) collective entanglement witnesses reproducing differences of Tsallis entropies<sup>43</sup>, and as such verifying entropic inequalities with help of single observable, have been designed.

Here we shall only recall the case of n = 2. The following joint observables Let us consider

$$X_{AA'BB'}^{(2)} = V_{AA'} \otimes (V_{BB'} - I_{BB'}),$$
  

$$Y_{AA'BB'}^{(2)} = (V_{AA'} - I_{AA'}) \otimes V_{BB'}.$$
(93)

represent collective entanglement witnesses since they reproduce differences of Tsallis entropies with q = 2:  $\langle \langle X \rangle \rangle = S_2(\varrho_{AB}) - S_2(\varrho_A), \ \langle \langle Y \rangle \rangle = S_2(\varrho_{AB}) - S_2(\varrho_B)$  which are always nonnegative for separable states (Horodecki *et al.*, 1996c).

Collective (2-copy) entanglement witnesses were further applied in continuous variables for Gaussian states (Stobińska and Wódkiewicz, 2005). Recently it has been observed that there is a *single* 4-copy collective entanglement witness that detects all (unknown) 2-qubit entanglement (Augusiak *et al.*, 2006). On this basis a corresponding universal quantum device, that can be interpreted as a quantum computing device, has been designed.

It is very interesting that the following collective entanglement witness (Mintert and Buchleitner, 2006):

$$\tilde{W}_{AA'BB'}^{(2)} = 2P_{AA'}^{(+)} \otimes P_{BB'}^{(-)} + 2P_{AA'}^{(-)} \otimes P_{BB'}^{(-)} -4P_{AA'}^{(-)} \otimes P_{BB'}^{(-)}, \quad (94)$$

with  $P^{(+)}$  and  $P^{(-)}$  being projectors onto symmetric and antisymmetric subspace respectively (see VI.B.3), has been shown to provide a lower bound on bipartite concurrence  $C(\rho_{AB})$  (Mintert *et al.*, 2004)

$$\mathcal{C}(\varrho_{AB}) \ge -\langle \langle \tilde{W}_{AA'BB'}^{(2)} \rangle \rangle_{\varrho_{AB}}.$$
(95)

Note that  $\tilde{W}^{(2)}_{AA'BB'}$  shown above is just twice the sum of the witnesses given in Eq. (93).

<sup>&</sup>lt;sup>43</sup> The family of Tsallis entropies parametrized by q > 0 is defined as follows:  $S_q(\rho) = \frac{1}{1-q}(\text{Tr}\rho^q - 1).$
Further the technique leading to the above formula was applied in the construction of (single-copy) entanglement witnesses quantifying concurrence as follows (Mintert, 2006). Suppose we have a given bipartite state  $\sigma$  for which we know the concurrence  $C(\sigma)$  exactly. Then one can construct the following entanglement witness which is parametrised by state  $\sigma$  in a nonlinear way:  $W_{AB}(\sigma_{A'B'}) = -4 \operatorname{Tr}_{A'B'}(I \otimes \sigma_{A'B'}V_{AA'} \otimes V_{BB'})/\mathcal{C}(\sigma_{A'B'})$  which has the following nice property:

$$\mathcal{C}(\varrho_{AB}) \ge -\langle W_{AB}(\sigma_{A'B'}) \rangle_{\varrho_{AB}}.$$
(96)

## 3. Detection of quantum entanglement as quantum computing with quantum data structure

It is interesting that entanglement detection schemes involve schemes of quantum computing. The networks detecting the moments of the spectrum of  $L(\rho)$  that are elements of the entanglement detection scheme (Carteret, 2005; Horodecki, 2003b,c; Horodecki and Ekert, 2002) can be considered as having fully quantum input data (copies of unknown quantum state) and give classical output — the moments of the spectrum. The most striking network is a universal quantum device detecting 2qubit entanglement (Augusiak *et al.*, 2006)<sup>44</sup>. This is a quantum computing unit with an input where (unknown) quantum data comes in. The input is represented by 4 copies of unknown 2-qubit state. Then they are coupled by a special unitary transformation (the network) to a single qubit polarization of which is finally measured. The state is separable iff it is polarization is not less than certain value.

This rises the natural question of whether in the future it will be possible to design a quantum algorithm working on fully quantum data (ie. copies of unknown multipartite states, entanglement of which has to be checked) that will solve the separability/entanglement problem much faster than any classical algorithms (cf. next section).

# IX. CLASSICAL ALGORITHMS DETECTING ENTANGLEMENT

The first systematic methods of checking entanglement of given state was worked out in terms of finding the decomposition onto separable and entangled parts of the state (see (Lewenstein and Sanpera, 1998) and generalizations to case of PPT states (Kraus *et al.*, 2000; Lewenstein *et al.*, 2001)). The methods were based on the systematic application of the range criterion involving however the difficult analytical part of finding product states in the range of a matrix. A further attempt to provide an algorithm deciding entanglement was based on checking variational problem based on concurrence vector (Audenaert et al., 2001c). The problem of existence of classical algorithm that unavoidably identifies entanglement has been analyzed in (Doherty et al., 2002, 2004) both theoretically and numerically and implemented by semidefinite programming methods. This approach is based on a theorem concerning the symmetric extensions of bipartite quantum state (Fannes et al., 1988; Raggio and Werner, 1989). It has the following interpretation. For a given bipartite state  $\rho_{AB}$  one asks about the existence of a hierarchy of symmetric extensions, i.e. whether there exists a family of states  $\rho_{AB_1...B_n}$ (with *n* arbitrary high) such that  $\rho_{AB_i} = \rho_{AB}$  for all  $i = 1, \ldots, n$ . It happens that the state  $\rho_{AB}$  is separable if and only if such a hierarchy exists for each natural n(see Sec. XVI). However, for any fixed n checking existence of such symmetric extension is equivalent to an instance of semidefinite programming. This leads to an algorithm consisting in checking the above extendability for increasing n, which always stops if the initial state  $\varrho_{AB}$  is entangled. However the algorithm never stops if the state is separable. Further another hierarchy has been provided together with the corresponding algorithm in (Eisert et al., 2004), extended to involve higher order polynomial constraints and to address multipartite entanglement question.

The idea of dual algorithm was provided in (Hulpke and Bruß, 2005), based on the observation that in checking separability of given state it is enough to consider countable set of product vectors spanning the range of the state. The constructed algorithm is dual to that described above, in the sense that its termination is guaranteed iff the state is separable, otherwise it will not stop. It has been further realized that running both the algorithms (ie. the one that always stops if the state is entangled with the one that stops if the state is separable) in parallel gives an algorithm that always stops and decides entanglement definitely (Hulpke and Bruß, 2005).

The complexity of both algorithms is exponential in the size of the problem. It happens that it must be so. The milestone result that has been proved in a mean time was that solving separability problem is NP hard (Gurvits, 2002, 2003, 2004). Namely is it known (Yudin and Nemirovskii, 1976) that if a largest ball contained in the convex set scales properly, and moreover there exists an efficient algorithm for deciding membership, then one can efficiently minimize linear functionals over the convex set. Now, Gurvits has shown that for some entanglement witness optimization problem was intractable. This, together with the results on radius of the ball contained within separable states (see Sec. X), shows that problem of separability cannot be efficiently solved.

Recently the new algorithm via analysis of weak membership problem as been developed together with analysis of NP hardness ((Ioannou, 2007; Ioannou and Travaglione, 2006a; Ioannou *et al.*, 2004)).

<sup>&</sup>lt;sup>44</sup> It exploits especially the idea of measuring the mean of observable by measurement of polarization of specially coupled qubit (Horodecki, 2003a; Paz and Roncaglia, 2003), cf. (Brun, 2004).

The goal of the algorithm is to solve what the authors call "witness" problem. This is either (i) to write separable decomposition up to given precision  $\delta$  or (ii) to find an (according to slightly modified definition) entanglement witness that separates the state from a set of separable states by more than  $\delta$  (the notion of the separation is precisely defined). The analysis shows that one can find more and more precisely a likely entanglement witness that detects the entanglement of the state (or find that it is impossible) reducing the set of "good" (ie. possibly detecting the state entanglement) witnesses by each step of the algorithm. The algorithm singles out a subroutine which in the standard picture (Horodecki et al., 1996a; Terhal, 2000b) can be, to some extent, interpreted as an oracle calculating a "distance" of the given witness to the set of separable states.

Finally, note that there are also other proposals of algorithms deciding separability like (Zapatrin, 2005) (for review see (Ioannou, 2007) and references therein).

## X. QUANTUM ENTANGLEMENT AND GEOMETRY

Geometry of entangled and separable states is a wide branch of entanglement theory (Bengtsson and Życzkowski, 2006). The most simple and elementary example of geometrical representation of separable and entangled states in three dimensions is a representation of two-qubit state with maximally mixed subsystems (Horodecki and Horodecki, 1996). Namely any two-qubit state can be represented in Hilbert-Schmidt basis  $\{\sigma_i \otimes \sigma_j\}$  where  $\sigma_0 = I$ , and in this case the correlation matrix T with elements  $t_{ij} = \text{Tr}(\rho\sigma_i \otimes \sigma_j), i, j = 1, 2, 3$  can be transformed by local unitary operations to the diagonal form. This matrix completely characterizes the state iff local density matrices are maximally mixed (which corresponds to vanishing of the parameters  $r_i = \text{Tr}(\sigma_i \otimes I\varrho)$ ,  $s_j = \text{Tr}(I \otimes \sigma_j \rho)$ , for i, j = 1, 2, 3. It happens that after diagonalizing 45, T is always a convex combination of four matrices  $T_0 = diag[1, -1, 1], T_1 = diag[-1, 1, 1],$  $T_2 = diag[1, 1, -1], T_3 = diag[-1, -1, -1]$  which corresponds to maximally mixed Bell basis. This has a simple interpretation in threedimensional real space: a tetrahedron  $\mathcal{T}$  with four vertices and coordinates corresponding to the diagonals above ((1, 1, -1) etc.). The subset of separable states is an octahedron that comes out from intersection of  $\mathcal{T}$  with its reflection around the origin of the set of coordinates. It is remarkable that all the states with maximally mixed subsystems are equivalent (up to product unitary operations  $U_A \otimes U_B$ ) to Bell diagonal states (a mixture of four Bell states (3)). Moreover, for all states (not only those with

maximally mixed subsystems) the singular values of the correlation matrix T are invariants under such product unitary transformations. The Euclidean lengths of the real three-dimensional vectors with coordinates  $r_i$ ,  $s_j$  defined above are also similarly invariant.

Note, that a nice analogon of the tetrahedron  $\mathcal{T}$  in the state space for entangled two qudits was defined and investigated in the context of geometry of separability (Baumgartner *et al.*, 2007). It turns out that the analogon of the octahedron is no longer a polytope.

One can naturally ask about reasonable set of the parameters or in general — functions of the state — that are invariants of product unitary operations. Properly chosen invariants allow for characterization of local orbits i.e. classes of states that are equivalent under local unitaries. (Note that any given orbit contains either only separable or only entangled states since entanglement property is preserved under local unitary product transformations). The problem of characterizing local orbits was analyzed in general in terms of polynomial invariants in (Grassl et al., 1998; Schlienz and Mahler, 1995). In case of two qubits this task was completed explicitly with 18 invariants in which 9 are functionally independent (Makhlin, 2002) (cf. (Grassl et al., 1998)). Further this result has been generalized up to four qubits (Briand et al., 2003; Luque and Thibon, 2003). Other way of characterizing entanglement in terms of local invariants was initiated in (Linden and Popescu, 1998; Linden et al., 1999b) by analysis of dimensionality of local orbit. Full solution of this problem for mixed two-qubit states and general bipartite pure states has been provided in (Kuś and Zyczkowski, 2001) and (Sinołęcka et al., 2002) respectively. For further development in this direction see (Grabowski et al., 2005) and references therein. There are many other results concerning geometry or multiqubit states to mention only (Heydari, 2006; Levay, 2006; Miyake, 2003).

There is another way to ask about geometrical properties of entanglement. Namely to ask about volume of set of separable states, its shape and the boundary of this set. The question about the volume of separable states was first considered in (Życzkowski *et al.*, 1998) and extended in (Życzkowski, 1999). In (Życzkowski *et al.*, 1998) it has been proven with help of entanglement witnesses theory that for any finite dimensional system (bipartite or multipartite) the volume of separable states is nonzero. In particular there exists always a ball of separable states around maximally mixed state. An explicit bound on the ratio of volumes of the set of all states S and that of separable states  $S_{sep}$ 

$$\frac{\operatorname{vol}(\mathcal{S})}{\operatorname{vol}(\mathcal{S}_{sep})} \ge \left(\frac{1}{1+d/2}\right)^{(d-1)(N-1)}$$
(97)

for N partite systems each of dimension d was provided in (Vidal and Tarrach, 1999). This has inspired further discussion which has shown that experiments in NMR quantum computing may not correspond to real quan-

<sup>&</sup>lt;sup>45</sup> Diagonalizing matrix T corresponds to applying product  $U_A \otimes U_B$  unitary operation to the state.

tum computing since they are performed on pseudopure states which are in fact separable (Braunstein *et al.*, 1999). Interestingly, one can show (Kendon *et al.*, 2002; Życzkowski *et al.*, 1998) that for any quantum system on some Hilbert space  $\mathcal{H}$  maximal ball inscribed into a set of mixed states is located around maximally mixed states and is given by the condition  $R(\varrho) = \operatorname{Tr}(\varrho^2) \geq \frac{1}{\dim \mathcal{H}^2 - 1}$ . Since this condition guarantees also the positivity of any unit trace operator, and since, for bipartite states  $\operatorname{Tr}(\varrho^2_{AB}) = \operatorname{Tr}[(\varrho^{\Gamma}_{AB})^2]$  this means that the maximal ball contains PPT states (the same argument works also for multipartite states (Kendon *et al.*, 2002)). In case of  $2 \otimes 2$ or  $2 \otimes 3$  this implies also separability giving a way to estimate volume of separable states from below.

These estimates have been generalized to multipartite states (Braunstein *et al.*, 1999) and further much improved providing very strong upper and lower bounds with help of a subtle technique exploiting among others entanglement witnesses theory (Gurvits and Barnum, 2002, 2003, 2005). In particular it was shown that for bipartite states, the largest separable ball around mixed states. One of applications of the largest separable ball results is the proof of NP-hardness of deciding weather a state is separable or not (Gurvits, 2003) (see Sec. IX).

There is yet another related question: one can define the state (bipartite or multipartite)  $\rho$  that remains separable under action of any unitary operation U. Such states are called *absolutely separable* (Kuś and Zyczkowski, 2001). In full analogy one can define what we call here absolute PPT property (ie. PPT property that is preserved under any unitary transformation). The question of which states are absolutely PPT has been fully solved for  $2 \otimes n$  systems (Hildebrand, 2005): those are all states spectrum of which satisfies the inequality:  $\lambda_1 \leq \lambda_{2n-1} + \sqrt{\lambda_{2n-2}\lambda_{2n}}$  where  $\{\lambda_i\}_{i=1}^{2n}$  are eigenvalues of  $\varrho$  in decreasing order. This immediately provides the characterization of absolutely separable states in  $d_A \otimes d_B$  systems with  $d_A d_B \leq 6$  since PPT is equivalent to separability in those cases. Note that for  $2 \otimes 2$  states this characterization has been proven much earlier by different methods (Verstraete et al., 2001b). In particular it follows that for those low dimensional cases the set of absolutely separable states is strictly larger than that of maximal ball inscribed into the set of all states. Whether it is true in higher dimensions remains an open problem.

Speaking about geometry of separable states one can not avoid a question about what is a boundary  $\partial S$ of set of states? This, in general not easy question, can be answered analytically in case of two-qubit case when it can be shown to be smooth (Djokovic, 2006) relatively to set of all 2-qubit states which is closely related to the separability characterization (Augusiak *et al.*, 2006)  $det(g_{AB}^{\Gamma}) \geq 0$ . Interestingly, it has been shown, that the set of separable state is not polytope (Ioannou and Travaglione, 2006b) it has no faces (Gühne and Lütkenhaus, 2006b).

There are many other interesting geometrical issues

that can be addressed in case of separable states. We shall recall one of them: there is an interesting issue of how probability of finding separable state (when the probability is measured up an a priori probability measure  $\mu$ ) is related to the probability (calculated by induced measure) of finding a random boundary state to be separable. The answer of course will depend on a choice of probability measure, which is by no means unique. Numerical analysis suggested (Slater, 2005a.b) that in two-qubit case the ratio of those two probabilities is equal to two if one assumes measure based the Hilbert-Schmidt distance. Recently it has been proven that for any  $d_A \otimes d_B$  system this rate is indeed 2 if we ask about set of PPT states rather than separable ones (Szarek *et al.*, 2006). For  $2 \otimes 2$  and  $2 \otimes 3$  case this reproduces the previous conjecture since PPT condition characterizes separability there (see Sec. VI.B.2). Moreover, it has been proven (see (Szarek *et al.*, 2006) for details) that bipartite PPT states can be decomposed into the so called pyramids of constant height.

## XI. THE PARADIGM OF LOCAL OPERATIONS AND CLASSICAL COMMUNICATION (LOCC)

## A. Quantum channel — the main notion

Here we shall recall that the most general quantum operation that transforms one quantum state into the other is a *probabilistic* or *stochastic* physical operation of the type

$$\rho \to \Lambda(\rho) / \operatorname{Tr}(\Lambda(\rho)),$$
 (98)

with trace nonincreasing CP map, i.e. a map satisfying  $Tr(\Lambda(\varrho)) \leq 1$  for any state  $\varrho$ , which can be expressed in the form

$$\Lambda(\varrho) = \sum_{i} V_i(\varrho) V_i^{\dagger}, \qquad (99)$$

with  $\sum_i V_i^{\dagger} V_i \leq I$  (domain and codomain of operators  $V_i$  called Kraus operators ((Kraus, 1983)) are in general different). The operation above takes place with the probability  $\text{Tr}(\Lambda(\varrho))$  which depends on the argument  $\varrho$ . The probability is equal to one if an only if the CP map  $\Lambda$  is tracepreserving (which corresponds to  $\sum_i V_i^{\dagger} V_i = I$  in (99); in such a case  $\Lambda$  is called *deterministic* or a quantum channel.

## **B. LOCC operations**

We already know that in the quantum teleportation process Alice performs a local measurement with maximally entangled projectors  $P_{AA'}^i$  on her particles AA'and then sends *classical* information to Bob (see Sec III.C). Bob performs accordingly a local operation  $U_B^i$ on his particle B. Note that the total operation acts on  $\rho$  as:  $\Lambda_{AA'B}(\rho) = \sum_{i} P^{i}_{AA'} \otimes U^{i}_{B}(\rho) P^{i}_{AA'} \otimes (U^{i}_{B})^{\dagger}$ . This operation belongs to so called one-way LOCC class which is very important in quantum communication theory. The general LOCC paradigm was formulated in (Bennett et al., 1996d).

In this paradigm all what the distant parties (Alice and Bob) are allowed is to perform arbitrary local quantum operations and sending classical information. No transfer of quantum systems between the labs is allowed. It is a natural class for considering entanglement processing because classical bits cannot convey quantum information and cannot create entanglement so that entanglement remains a resource that can be only manipulated. Moreover one can easily imagine that sending classical bits is much more cheaper than sending quantum bits, because it is easy to amplify classical information. Sometimes it is convenient to put some restrictions also onto classical information. One then distinguishes in general the following subclasses of operations described below. We assume all the operations (except of the local operations) to be trace nonincreasing and compute the state transformation as in (98) since the transformation may be either stochastic or deterministic (ie. quantum channel).

C1 - class of local operations .- In this case no communication between Alice and Bob is allowed. The mathematical structure of the map is elementary:  $\Lambda^{\emptyset}_{AB} =$  $\Lambda_A \otimes \Lambda_B$  with  $\Lambda_A$ ,  $\Lambda_B$  being both quantum channels. As we said already this operation is always deterministic.

C2a - class of "one-way" forward LOCC operations .-Here classical communication from Alice to Bob is allowed. The form of the map is:  $\Lambda_{AB}^{\rightarrow}(\varrho) = \sum_{i} V_{A}^{i} \otimes I_{B}([I_{A} \otimes \Lambda_{B}^{i}](\varrho))(V_{A}^{i})^{\dagger} \otimes I_{B}$  with deterministic maps  $\Lambda_{B}^{i}$ which reflect the fact that Bob is not allowed to perform "truly stochastic" operation since he cannot tell Alice whether it has taken place or not (which would happen only with some probability in general).

C2b - class of "one-way" backward LOCC operations .-Here one has  $\Lambda_{AB}^{\leftarrow}(\varrho) = \sum_i \mathbf{I}_A \otimes V_B^i [\Lambda_A^i \otimes \mathbf{I}_B](\varrho) \mathbf{I}_A \otimes (V_B^i)^{\dagger}$ . The situation is the same as in C2a but with the roles of Alice and Bob interchanged.

C3 - class of "two-way" classical communication .-Here both parties are allowed to send classical communication to each other. The mathematical form of the operation is guite complicated and the reader is referred to (Donald et al., 2002). Fortunately, there are other two larger in a sense of inclusion classes, that are much more easy to deal with: the classes of separable and PPT operations.

C4 - Class of separable operations.- This class was considered in (Rains, 1997; Vedral and Plenio, 1998). These are operations with product Kraus operators:

$$\Lambda_{AB}^{sep}(\varrho) = \sum_{i} A_i \otimes B_i \varrho A_i^{\dagger} \otimes B_i^{\dagger}, \qquad (100)$$

which satisfy  $\sum_{i} A_{i}^{\dagger} A_{i} \otimes B_{i}^{\dagger} B_{i} = \mathbf{I} \otimes \mathbf{I}.$  *C5. PPT operations* .- Those are operations (Rains, 1999, 2001)  $\Lambda^{PPT}$  such that  $(\Lambda^{PPT}[(\cdot)^{\Gamma}])^{\Gamma}$  is completely positive. We shall see that the simplest example of such

operation is  $\rho \to \rho \otimes \rho_{PPT}$  i. e. the process of adding some PPT state.

There is an order of inclusions  $C1 \subset C2a, C2b \subset C3 \subset$  $C4 \subset C5$ , where all inclusions are strict ie. are not equalities. The most intriguing is nonequivalence  $C3 \neq C4$ which follows from so called *nonlocality without entangle*ment (Bennett et al., 1999a): there are examples of product basis which are orthonormal (and hence perfectly distinguishable by suitable von Neumann measurement) but are not products of two orthonormal local ones which represent vectors that cannot be perfectly distinguished by parties that are far apart and can use only LOCC operations. Let us stress, that the inclusion  $C3 \subset C4$  is extensively used in context of LOCC operations. This is because they are hard to deal with, as characterized in a difficult way. If instead one deals with separable or PPT operations, thanks to the inclusion, one can still conclude about LOCC ones.

Subsequently if it is not specified, the term LOCC will be referred to the most general class of operations with help of local operations and classical communication namely the class C3 above.

Below we shall provide examples of some LOCC operations

*Example* .- The (deterministic) " $U \otimes U$ ,  $U \otimes U^*$ twirling" operations:

$$\tau(\varrho) = \int dUU \otimes U\varrho (U \otimes U)^{\dagger},$$
  
$$\tau'(\varrho) = \int dUU \otimes U^* \varrho (U \otimes U^*)^{\dagger}.$$
 (101)

Here dU is a uniform probabilistic distribution on set of unitary matrices. They are of "one-way" type and can be performed in the following manner: Alice pick up randomly the operation U rotates her subsystem with it and sends the information to Bob which U she had chosen. Bob performs on his side either U or  $U^*$  (depending on which of the two operations they wanted to perform). The integration above can be made discrete which follows from Caratheodory's theorem.

The very important issue is that any state can be depolarized with help of  $\tau$ ,  $\tau'$  to Werner (71) and isotropic (72) state respectively. This element is crucial for entanglement distillation recurrence protocol (Bennett et al., 1996c) (see Sec. XII.B).

Another example concerns local filtering (Bennett et al., 1996b; Gisin, 1996b):

*Example* .- The (stochastic) local filtering operation is

$$\mathcal{L}_{LOCCfilter}(\varrho) = \frac{A \otimes B\varrho A^{\dagger} \otimes B^{\dagger}}{\operatorname{Tr}(A \otimes B\varrho A^{\dagger} \otimes B^{\dagger})},$$
$$A^{\dagger}A \otimes B^{\dagger}B \leq I \otimes I. \tag{102}$$

This operation requires two-way communication (each party must send a single bit), however if A = I (B = I) then it becomes one-way forward (backward) filtering. This one-way filtering is a crucial element of a universal protocol of entanglement distillation (Horodecki *et al.*, 1997) (see Sec. XII.D).

The local filtering operation is special case of a more general class operations called *stochastic separable operations*, which includes class C4. They are defined as follows (Rains, 1997; Vedral and Plenio, 1998):

$$\Lambda_{AB}^{sep}(\varrho) = \frac{\sum_{i} A_{i} \otimes B_{i} \varrho A_{i}^{\dagger} \otimes B_{i}^{\dagger}}{\sum_{i} \operatorname{Tr} A_{i}^{\dagger} A_{i} \otimes B_{i}^{\dagger} B_{i} \varrho}$$
(103)

where  $\sum_{i} A_{i}^{\dagger} A_{i} \otimes B_{i}^{\dagger} B_{i} \leq I \otimes I$ . These operations can be applied probabilistically via LOCC operations.

## **XII. DISTILLATION AND BOUND ENTANGLEMENT**

Many basic effects in quantum information theory based exploit pure maximally entangled state  $|\psi^+\rangle$ . However, in laboratories we usually have mixed states due to imperfection of operations and decoherence. A natural question arises, how to deal with a *noise* so that one could take advantages of the interesting features of pure entangled states. This problem has been first considered by Bennett, Brassard, Popescu, Schumacher, Smolin, and Wootters in 1996 (Bennett et al., 1996c). In their seminal paper, they have established a paradigm for *purification* (or distillation) of entanglement. When two distant parties share n copies of a bipartite mixed state  $\rho$ , which contain noisy entanglement, they can perform some LOCC and obtain in turn some (less) number of kcopies of systems in state close to a singlet state which contains *pure* entanglement. A sequence of LOCC operations achieving this task is called *entanglement pu*rification or entanglement distillation protocol. We are interested in optimal entanglement distillation protocols i.e. those which result in maximal ratio  $\frac{k}{n}$  in limit of large number n of input copies. This optimal ratio is called distillable entanglement and denoted as  $E_D$  (see Sec. XV for formal definition). Having performed entanglement distillation, the parties can use obtained nearly singlet states to perform quantum teleportation, entanglement based quantum cryptography and other useful pure entanglement based protocols. Therefore, entanglement distillation is one of the fundamental concepts in dealing with quantum information and entanglement in general. In this section we present the most important entanglement distillation protocols. We then discuss the possibility of entanglement distillation in general and report the bound entangled states which being entangled can not be distilled.

### A. One-way hashing distillation protocol

If only one party can tell the other party her/his result during the protocol of distillation, the protocol is called *one-way*, and *two-way* otherwise. One-way protocols are closely connected to error correction, as we will see below. In (Bennett *et al.*, 1996d) (see also (Bennett *et al.*, 1996c)) there was presented a protocol for two qubit states which originates from cryptographic privacy amplification protocol, called *hashing*. Following this work we consider here the so called Bell diagonal states which are mixtures of two qubit Bell basis states (3). Bell diagonal states  $\rho_{Bdiag}$  are naturally parametrized by the probability of mixing  $\{p\}$ . For these states, the one-way hashing protocol yields singlets at a rate 1 - H(p), thus proving<sup>46</sup>  $E_D(\rho_{Bdiag}) \ge 1 - H(p)$  In two-qubit case there are four Bell states (3). The n copies of the two qubit Bell diagonal state  $\rho_{Bdiag}$  can be viewed as a classical mixture of strings of n Bell states. Typically, there are only about  $2^{nH(\{p\})}$  of such strings that are likely to occur (Cover and Thomas, 1991). Since the distillation procedure yields some (shorter) string of singlets solely, there is a straightforward "classical" idea, that to distill one needs to know what string of Bells occurred. This knowledge is sufficient as one can then rotate each non-singlet Bell state into singlet state easily as in a dense coding protocol (see sec III).

Let us note, that sharing  $\phi^-$  instead of  $\phi^+$  can be viewed as sharing  $\phi^+$  with a phase error. Similarly  $\psi^+$ means bit error and  $\psi^-$  – both bit and phase error. The identification of the string of all Bell states that have occurred is then equivalent to learning which types of errors occurred in which places. Thus the above method can be viewed as error correction procedure<sup>47</sup>.

Now, as it is well known, to distinguish between  $2^{nH(\{p\})}$  strings, one needs  $\log 2^{nH(\{p\})} = nH(\{p\})$  binary questions. Such a binary question can be the following: what is the sum of the bit-values of the string at certain  $i_1, \ldots, i_k$  positions, taken modulo 2? In other words: what is the *parity* of the given subset of positions in the string. From probabilistic considerations it follows, that after r such questions about *random* subset of positions (i.e. taking each time random k with  $1 \le k \le 2n$ ) the probability of distinguishing two distinct strings is no less than  $1 - 2^{-r}$ , hence the procedure is efficient.

The trick of the "hashing" protocol is that it can be done in quantum setting. There are certain local unitary operations for the two parties, so that they are able to collect the parity of the subset of Bell states onto a *single* Bell state and then get to know it locally measuring this Bell state and comparing the results. Since each answer to binary question consumes one Bell state, and there are  $nH(\{p\})$  questions to be asked one needs at least  $H(\{p\}) < 1$  to obtain nonzero amount of not measured Bell states. If this is satisfied, after the protocol, there are  $n-nH(\{p\})$  unmeasured Bell states in a known state.

<sup>&</sup>lt;sup>46</sup> It is known, that if there are only *two* Bell states in mixture, then one-way hashing is optimal so that distillable entanglement is equal 1 - H(p) in this case.

<sup>&</sup>lt;sup>47</sup> Actually this reflects a deep relation developed in (Bennett *et al.*, 1996d) between entanglement distillation and the large domain of quantum error correction designed for quantum computation in presence of noise.

The parties can then rotate them all to a singlet form (that is correct the bit and phase errors), and hence distill singlets at an announced rate  $1 - H(\{p\})$ .

This protocol can be applied even if Alice and Bob share a non Bell diagonal state, as they can twirl the state applying at random one of the four operations:  $\sigma_x \otimes \sigma_x$ ,  $\sigma_y \otimes \sigma_y, \sigma_z \otimes \sigma_z$ ,  $I \otimes I$  (which can be done one two-way). The resulting state is a Bell diagonal state. Of course this operation often will kill entanglement. We will see how to improve this in Secs. XII.B and XII.D.

The above idea has been further generalized leading to general one-way hashing protocol which is discussed in Sec. XII.F.

### B. Two-way recurrence distillation protocol

The hashing protocol cannot distill all entangled Belldiagonal states (one easily find this, knowing that those states are entangled if and only if some eigenvalue is greater than 1/2). To cover all entangled Bell diagonal states one can first launch a *two-way* distillation protocol to enter the regime where one-way hashing protocol works. The first such protocol, called *recurrence*, was announced already in the very first paper on distillation (Bennett *et al.*, 1996c), and developed in (Bennett *et al.*, 1996d). It works for two qubit states satisfying F = $\mathrm{Tr}\rho |\phi^+\rangle \langle \phi^+| > \frac{1}{2}$  with  $|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ .

The protocol is defined by application of certain recursive procedure. The procedure is probabilistic so that it may fail however with small probability. Moreover it consumes a lot of resources. In each step the procedure uses half of initial number of states.

- 1. Divide all systems into pairs:  $\rho_{AB} \otimes \rho_{A'B'}$  with AB being source system and A'B' being a target one. For each such pair do:
  - 1.1 Apply CNOT transformation with control at A(B) system and target at A'(B') for Alice (Bob).
  - 1.2 Measure the target system in computational basis on both A' and B' and compare the results.
  - 1.3 If the results are the same, keep the AB system and discard the A'B'. Else remove both systems.
- 2. If no system survived, then stop the algorithm failed. Else twirl all the survived systems turning them into Werner states.
  - 2.1 For one of the system do: if its state  $\rho'$ satisfies  $1 - S(\rho') > 0$  then stop (hashing protocol will work) else go to step (1).

In the above procedure, one deals with Werner states, because of twirling in step 2. In two-qubit case, Werner states are equivalent to isotropic states and hence are parameterized only by the singlet fraction F (See Sec. (VI.B.9). In one step of the above recurrence procedure this parameter improves with respect to the preceding one according to the rule:

$$F'(F) = \frac{F^2 + \frac{1}{9}(1-F)^2}{F^2 + \frac{2}{3}F(1-F) + \frac{5}{9}(1-F)^2}.$$
 (104)

Now, if only  $F > \frac{1}{2}$ , then the above recursive map converges to 1 for sufficiently big initial number of copies.

The idea behind the protocol is the following. Step 1 decreases bit error (i.e. in the mixture the weight of correlated states  $|\phi^{\pm}\rangle$  increases). At the same time, the phase error increases, i.e. the bias between states of type + and those of type – gets smaller. Then there is twirling step, which equalizes bit and phase error. Provided that bit error went down more than phase error went up, the net effect is positive. Instead of twirling one can apply deterministic transformation (Deutsch *et al.*, 1996), which is much more efficient.

## C. Development of distillation protocols — bipartite and multipartite case

The idea of recurrence protocol was developed in different ways. The CNOT operation, that is made in step 1.1 of the above original protocol, can be viewed as a permutation. If one apply some other permutation acting locally on  $n \ge 2$  qubits and performs a testing measurements of steps 1.2 - 1.3 on  $m \ge 1$  one obtains a natural generalization of this scheme, developed in (Dehaene *et al.*, 2003) for the case of two-qubit states. It follows that in case of n = 4, m = 1, and a special permutation this protocol yields higher distillation rate. This paradigm has been further analyzed in context of so called code based entanglement distillation protocols (Ambainis and Gottesman, 2003; Matsumoto, 2003) in (Hostens et al., 2004). The original idea of (Bennett et al., 1996c) linking entanglement distillation protocols and error correction procedures (Gottesman, 1997) have been also developed in context of quantum key distribution. See in this context (Ambainis et al., 2002; Gottesman and Lo, 2003) and discussion in section XIX.A.

The original recurrence protocol was generalized to higher dimensional systems in two ways in (Alber *et al.*, 2001b; Horodecki and Horodecki, 1999). An interesting improvement of distillation techniques due to (Vollbrecht and Verstraete, 2005) where a protocol that interpolates between hashing and recurrence one was provided. This idea has been recently developed in (Hostens *et al.*, 2006a). Also, distillation was considered in the context of topological quantum memory (Bombin and Martin-Delgado, 2006).

The above protocols for distillation of bipartite entanglement can be used for distillation of a multipartite entanglement, when n parties are provided many copies of a multipartite state. Namely any pair of the parties can distill some EPR pairs and then, using e.g. teleportation, the whole group can redistribute a desired multipartite state. Advantage of such approach is that it is independent from the target state. In (Dür and Cirac, 2000b; Dür *et al.*, 1999b) a sufficient condition for distillability of arbitrary entangled state from *n*-qubit multipartite state has been provided, basing on this idea.

However, as it was found by Murao et. al in the first paper (Murao *et al.*, 1998) on multipartite entanglement distillation, the efficiency of protocol which uses bipartite entanglement distillation is in general less then that of *direct* distillation. The direct procedure is presented there which is a generalization of bipartite recurrence protocol of *n*-partite GHZ state from its "noisy" version (mixed with identity). In (Maneva and Smolin, 2002) the bipartite hashing protocol has been generalized for distillation of *n*-partite GHZ state.

In multipartite scenario, there is no distinguished state like a singlet state, which can be a universal target state in entanglement distillation procedures. There are however some natural classes of interesting target states, including the commonly studied GHZ state. An exemplary is the class of the graph states (see Sec. VII.A), related to one-way quantum computation model. A class of multiparticle entanglement purification protocols that allow for distillation of these states was first studied in (Dür et al., 2003) where it is shown again to outperform bipartite entanglement purification protocols. It was further developed in (Aschauer et al., 2005) for the subclass of graph states called two-colorable graph states. The recurrence and breeding protocol which distills all graph states has been also recently found (Kruszyńska et al., 2006) (for a distillation of graph states under local noise see (Kay *et al.*, 2006)).

The class of two-colorable graph states is locally equivalent to the class of so called CSS states that stems from the quantum error correction (Calderbank and Shor, 1996; Steane, 1996a,b). The distillation of CSS states has been studied in context of multipartite quantum cryptographic protocols in (Chen and Lo. 2004) (see Sec. XIX.E.3). Recently, the protocol which is direct generalization of original hashing method (see Sec. XII.A) has been found, that distills CSS states (Hostens et al., 2006b). This protocol outperforms all previous versions of hashing of CSS states (or their subclasses such as Bell diagonal states) (Aschauer et al., 2005; Chen and Lo, 2004; Dür et al., 2003; Maneva and Smolin, 2002). Distillation of the state W which is not a CSS state, has been studied recently in (Miyake and Briegel, 2005). In (Glancy et al., 2006) a protocol of distillation of all stabilizer states (a class which includes CSS states) based on stabilizer codes (Gottesman, 1997; Nielsen and Chuang, 2000) was proposed. Based on this, a breeding protocol for stabilizer states was provided in (Hostens et al., 2006c).

## D. All two-qubit entangled states are distillable

The recurrence protocol, followed by hashing can distill entanglement only from states satisfying  $F > \frac{1}{2}$ . One can then ask what is known in the general case. In this section we present the protocol which allows to overcome this bound in the case of two qubits. The idea is that with certain (perhaps small) probability, one can conclusively transform a given state into a more desired one, so that one knows if the transformation succeeded. There is an operation which can increase the parameter F, so that one can then perform recurrence and hashing protocol. Selecting successful cases in the probabilistic transformation is called *local filtering* (Gisin, 1996b), which gives the name of the protocol. The composition of filtering, recurrence and hashing proves the following result (Horodecki *et al.*, 1997):

• Any two-qubit state is distillable if and only if it is entangled.

Formally, in filtering protocol Alice and Bob are given n copies of state  $\rho$ . They then act on each copy with an operation given by Kraus operators (see note in Sec. VI)  $\{A \otimes B, \sqrt{\mathbf{I} - A^{\dagger}A} \otimes B, A \otimes \sqrt{\mathbf{I} - B^{\dagger}B}, \sqrt{\mathbf{I} - A^{\dagger}A} \otimes \sqrt{\mathbf{I} - B^{\dagger}B}\}$ . The event corresponding to Kraus operator  $A \otimes B$  is the desired one, and the complement corresponds to failure in improving properties of  $\rho$ . In the case of success the state can be transformed into

$$\sigma = \frac{A \otimes B\rho A^{\dagger} \otimes B^{\dagger}}{\operatorname{Tr}(A \otimes B\rho A^{\dagger} \otimes B^{\dagger})}, \qquad (105)$$

however only with probability  $p = \text{Tr}(A \otimes B\rho A^{\dagger} \otimes B^{\dagger})$ . After having performed the operation, Alice and Bob select all good events and keep them, while removing the copies, for which they failed.

Suppose now that Alice and Bob are given n copies of entangled state  $\rho$  having  $F \leq \frac{1}{2}$ . They would like to obtain some number k of states with  $F > \frac{1}{2}$ . We show now how one can build up a filtering operation which does this task. Namely any two-qubit entangled state becomes not positive after partial transpose (is NPT). Then, there is a pure state  $|\psi\rangle = \sum_{ij} a_{ij} |ij\rangle$  such, that

$$\langle \psi | \rho_{AB}^{\Gamma} | \psi \rangle < 0, \tag{106}$$

with  $\Gamma$  denoting partial transpose on Bob's subsystem. It can be shown, that a filter  $A \otimes B = M_{\psi} \otimes I$  with  $[M_{\psi}]_{ij} = a_{ij}$  is the needed one. That is the state  $\sigma$  given by

$$\sigma = \frac{M_{\psi} \otimes \mathrm{I}\rho_{AB} M_{\psi}^{\dagger} \otimes \mathrm{I}}{\mathrm{Tr}(M_{\psi} \otimes \mathrm{I}\rho_{AB} M_{\psi}^{\dagger} \otimes \mathrm{I})}, \qquad (107)$$

which is the result of filtering, fulfills  $F(\sigma) > \frac{1}{2}$ .

The filtering distillation protocol is just this: Alice applies to each state a POVM defined by Kraus operators

 $\{M_{\psi} \otimes \mathbf{I}, [\sqrt{\mathbf{I} - M_{\psi}^{\dagger} M_{\psi}}] \otimes \mathbf{I}\}$ . She then tells Bob when she succeeded. They select in turn on average np pairs with  $p = \operatorname{Tr}(M_{\psi}^{\dagger} M_{\psi} \otimes \mathbf{I} \rho_{AB})$ . They then launch recurrence and hashing protocols to distill entanglement, which yields nonzero distillable entanglement.

Similar argument along these lines, gives generalization of two-qubit distillability to the case of NPT states acting on  $C^2 \otimes C^N$  Hilbert space (Dür and Cirac, 2000a; Dür *et al.*, 2000a). It follows also, that for N = 3 all states are also distillable if and only if they are entangled, since any state on  $C^2 \otimes C^3$  is entangled if and only if it is NPT. The same equivalence has been shown for all rank two bipartite states (Horodecki *et al.*, 2006f)

## E. Reduction criterion and distillability

One can generalize the filtering protocol by use of reduction criterion of separability (see Sec. VI). It has been shown that any state that violates reduction criterion is distillable (Horodecki and Horodecki, 1999). Namely, for states which violate this criterion, there exists filter, such that the output state has fidelity  $F > \frac{1}{d}$ , where F is overlap with maximally entangled state  $|\Phi^+\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |ii\rangle$ . Such states is distillable, similarly, as for two-qubit states with  $F > \frac{1}{2}$ . The simplest protocol (Braunstein *et al.*, 1999) is the following: one projects such state using local rank two projectors  $P = |0\rangle\langle 0| + |1\rangle\langle 1|$ , and finds that obtained two-qubit state has  $F > \frac{1}{2}$ , hence is distillable.

The importance of this property of reduction criterion lies in the fact, that its generalization to continuous variables allowed to show that all two-mode Gaussian states which violate PPT criterion are distillable.

## F. General one-way hashing

One can ask what is the maximal yield of singlet as a function of a bipartite state  $\rho_{AB}$  that can be obtained by means of one way classical communication. In this section we will discuss protocol which achieves this task (Devetak and Winter, 2005; Horodecki *et al.*, 2005h, 2006e).

In Sec. (XII.A) we learned a protocol called one-way hashing which for Bell diagonal states with spectrum given by distribution  $\{p\}$ , gives  $1 - H(\{p\})$  of distillable entanglement. Since in case of these states, the von Neumann entropy of subsystem reads  $S(\rho_{Bdiag}^B) = 1$  and the total entropy of the state is equal to  $S(\rho_{Bdiag}^{AB}) = H(\{p\})$ , there has been a common believe that in general there should be a one-way protocol that yields

$$E_D(\rho_{AB}) \ge S(\rho_B) - S(\rho_{AB}). \tag{108}$$

This conjecture has been proved by Devetak and Winter (Devetak and Winter, 2005).

The above inequality, called *hashing inequality*, states that distillable entanglement is lower bounded by *coher*- The original proof of hashing inequality (Devetak and Winter, 2004b, 2005) was based on cryptographic techniques, where one first performs error correction (corresponding to correcting bit) and then privacy amplification (corresponding to correcting phase), both procedures by means of random codes (we will discuss it in Sec. XIX).

Another protocol that distills the amount  $I_{A\rangle B}^{coh}$  of singlets from a given state is the following (Horodecki *et al.*, 2005h, 2006e): given many copies of the state, Alice projects her system onto so-called typical subspace (Schumacher, 1995a) (the probability of failure is exponentially small with number of copies). Subsequently, she performs incomplete measurement  $\{P_i\}$  where the projectors  $P_i$  project onto blocks of size  $2^{n(S_B-S_{AB})}$ . If the measurement is chosen at random according to uniform measure (Haar measure), it turns out that for any given outcome Alice and Bob share almost maximally entangled state, hence equivalent to  $n(S_B - S_{AB})$  e-bits. Of course, for each particular outcome *i* the state is different, therefore one-way communication is needed (Bob has to know the outcome *i*).

The above bound is often too rough (e.g. because one can distill states with negative coherent information using recurrence protocol). Coherent information  $I_{A)B}^{coh}$  is not an entanglement monotone. It is then known, that in general the optimal protocol of distillation of entanglement is some two-way protocol which increases  $I^{coh}$ , followed by general hashing protocol: That is we have (Horodecki *et al.*, 2000c)

$$E_D(\rho) = \sup_{\Lambda \in LOCC} I^{coh}(\Lambda(\rho)).$$
(109)

It is however not known how to attain the highest coherent information via two-way distillation protocol.

## G. Bound entanglement — when distillability fails

Since the seminal paper on distillation (Bennett *et al.*, 1996c), there was a common expectation, that all entangled bipartite states are distillable. Surprisingly it is not the case. It was shown in (Horodecki *et al.*, 1998a) that the PPT states cannot be distilled. It is rather obvious, that one cannot distill from separable states. Interestingly, the first example of *entangled* PPT state had been already known from (Horodecki, 1997). Generally the states that are entangled yet not distillable are called *bound entangled*. It is not known if there are other bound entangled states than PPT entangled states. There are quite many interesting formal approaches allowing to obtain families of PPT entangled state. There is however no operational, intuitive "reason" for existence of this mysterious type of entanglement.

Let us first comment on the non-distillability of separable states. The intuition for this is straightforward: separable states can be created via LOCC from scratch. If one could distill singlets from them, it would be creating something out of nothing. This reasoning of course does not hold for entangled PPT states. However, one can look from a different angle to find relevant *formal* similarities between PPT states and separable states. Namely, concerning separable states one can observe, that the fidelity  $F = \text{Tr}\sigma_{sep}|\Phi^+\rangle\langle\Phi^+|$  is no greater then  $\frac{1}{d}$  for  $\sigma_{sep}$ acting on  $\mathcal{C}^d \otimes \mathcal{C}^d$ . Since LOCC operations can only transform separable states is closed under LOCC operations), one cannot distill singlet from separable states since one cannot increase the singlet fraction.

It turns out that also PPT states do not admit higher fidelity than 1/d as well as are closed under LOCC operations. Indeed we have  $\text{Tr}\rho_{AB}|\Phi^+\rangle\langle\Phi^+| = \frac{1}{d}\text{Tr}\rho_{AB}^{\Gamma}V$ which can not exceed  $\frac{1}{d}$  (here V is the swap operator (48)). Indeed,  $\rho_{AB}^{\Gamma} \geq 0$ , so that  $\text{Tr}\rho_{AB}^{\Gamma}V$  can be viewed as an average value of a random variable which can not exceed 1 since V has eigenvalues  $\pm 1$  (Rains, 1999, 2001). To see the second feature, note that any LOCC operation  $\Lambda$  acts on a state  $\rho_{AB}$  as follows

$$\rho_{out} = \Lambda(\rho_{AB}) = \sum_{i} A_i \otimes B_i(\rho_{AB}) A_i^{\dagger} \otimes B_i^{\dagger}, \quad (110)$$

which after partial transpose on subsystem B gives

$$\rho_{out}^{\Gamma} = \sum_{i} A_i \otimes (B_i^{\dagger})^T (\rho_{AB}^{\Gamma}) A_i^{\dagger} \otimes B_i^T.$$
(111)

The resulting operator is positive, if only  $\rho_{AB}^{\Gamma}$  was positive.

Since the discovery of first bound entangled states quite many further examples of such states were found, only a few of which we have discussed (see Sec. VI.B.7). The comprehensive list of achievements in this field, as well as the introduction to the subject can be found in (Clarisse, 2006b).

### H. The problem of NPT bound entanglement

Although it is already known that there exist entangled nondistillable states, still we do not have a characterization of the set of such states. The question which remains open since the discovery of bound entanglement properties of PPT states is: are all NPT states distillable? For two main attempts <sup>48</sup> to solve the problem see (DiVincenzo *et al.*, 2000a; Dür *et al.*, 2000a). In (Horodecki and Horodecki, 1999) it was shown that this holds if and only if all NPT Werner states (equivalently entangled Werner states) are distillable. It simply follows from the fact, that as in case of two qubits any entangled state can be filtered, to such a state, that after proper twirling, one obtains entangled Werner state. The question is hard to answer because of its asymptotic nature. A necessary and sufficient condition for entanglement distillation (Horodecki *et al.*, 1998a) can be stated as follows:

• Bipartite state  $\rho$  is distillable if and only if there exists *n* such, that  $\rho$  is *n*-copy distillable (i.e.  $\rho^{\otimes n}$  can be filtered to a two-qubit entangled state)<sup>49</sup>.

for some n. It is however known, that there are states which are not n-copy distillable but they are n + 1-copy distillable (Watrous, 2003) (see also (Bandyopadhyay and Roychowdhury, 2003)), for this reason, the numerical search concerning distillability basing on few copies may be misleading, and demanding in computing resources. There is an interesting characterization of n copy distillable states in terms of entanglement witnesses found in (Kraus *et al.*, 2002). Namely, a state is separable iff the following operator

$$W_n = P_{A'B'}^+ - [\rho_{AB}^{\otimes n}]^{\Gamma} \tag{112}$$

is not an entanglement witness (here A'B' is two qubit system,  $P^+$  is maximally entangled state).

One may also think, that the problem could be solved by use of simpler class of maps - namely PPT operations. Remarkably, in (Eggeling *et al.*, 2001) it was shown that any NPT state can be distilled by PPT operations. Recently the problem was attacked by means of positive maps, and associated "witnesses" in (Clarisse, 2005) (see also (Clarisse, 2006a)).

The problem of existence of NPT BE states has important consequences. If they indeed exist, then distillable entanglement is nonadditive and nonconvex. Two states of zero  $E_D$  will together give nonzero  $E_D$ . The set of bipartite BE states will not be closed under tensor product, and under mixing. For the extensive review of the problem of existence of NPT BE states see (Clarisse, 2006b).

Schematic representation of the set of all states including hypothetical set of NPT bound entangled states is shown on fig. 3.

## I. Activation of bound entanglement

Entanglement is always considered as a resource useful for certain task. It is clear, that pure entanglement can be useful for many tasks, as it was considered in Sec. III. Since discovery of bound entanglement a lot of effort was devoted to find some nontrivial tasks that this type of entanglement allows to achieve.

<sup>&</sup>lt;sup>48</sup> Two recent attempts are unfortunately incorrect (Chattopadhyay and Sarkar, 2006; Simon, 2006).

<sup>&</sup>lt;sup>49</sup> Equivalently, there exists pure state  $|\psi\rangle$  of Schmidt rank 2 such that  $\langle \psi | (\rho^{\otimes n})^{\Gamma} | \psi \rangle < 0$  (DiVincenzo *et al.*, 2000a; Dür *et al.*, 2000a).



FIG. 3 Schematic representation of the set of all states with example of entanglement witness and its optimization

The first phenomenon which proves usefulness of bound entanglement, called *activation* of bound entanglement, was discovered in (Horodecki et al., 1999a). A parameter which is improved after activation is the so called probabilistic maximal singlet fraction, that is  $F_{\max}^{(p)}(\rho) = \max_{\Lambda} \frac{1}{\operatorname{Tr}[\Lambda(\rho)]} \operatorname{Tr}[\Lambda(\rho)|\Phi^+\rangle\langle\Phi^+|]$  for  $\rho$  acting on  $\mathcal{C}^d \otimes \mathcal{C}^d$ , where  $\Lambda$  are local filtering operations <sup>50</sup>.

Consider a state  $\sigma$  with  $F_{\max}^{(p)}$  bounded away from 1, such that no LOCC protocol can go beyond this bound. Then a protocol was found, involving k pairs of some BE state  $\rho_{be}$ , which takes as an input a state  $\sigma$ , and outputs a state  $\sigma'$  with singlet fraction arbitrarily close to 1 (which depends on k). That is:

$$\rho_{be}^{\otimes k} \otimes \sigma \longrightarrow \sigma', \quad \lim_{k} F_{\max}^{(p)}(\sigma') = 1$$
(113)

The protocol is actually closely related to recurrence distillation protocol (see Sec. XII.B), generalized to higher dimension and with twirling step removed. In the above scenario the state  $\rho_{be}$  is an *activator* for a state  $\sigma$ . The probability of success in this protocol decreases as the output fidelity increases. To understand the activation recall that probabilistic maximal singlet fraction of every bound entangled state is by definition bounded by  $\frac{1}{d}$ , as discussed in Sec. XII.G. It implies, that  $\rho_{be}^{\otimes k}$  also have  $F_{max}^{(p)}$  bounded away from 1. We have then two states with probabilistic maximal fidelity bounded away from 1, which however changes if they are put together. For this reason, the effect of activation demonstrates a sort of *nonadditivity* of maximal singlet fraction.

The effect of activation was further developed in various directions. It was shown in (Vollbrecht and Wolf,

LOCC operation, that is called maximal singlet fraction.

2002a), that any NPT state, can be made one-copy distillable<sup>51</sup> by use of BE states. Moreover to this end one needs BE states which are arbitrarily close to separable states.

A remarkable result showing the power of LOCC operations supported with arbitrarily small amount of BE states is established in (Ishizaka, 2004). Namely, the interconversion of *pure* bipartite states is ruled by entanglement measures. In particular, a pure state with smaller measure called *Schmidt rank* (see Sec. XV) cannot be turned by LOCC into a state with higher Schmidt rank. However any transition between pure states is possible with some probability, if assisted by arbitrarily weakly bound entangled states. This works also for multipartite states. Interestingly, the fact that one can increase Schmidt rank by PPT operations (i.e. also with some probability by assistance of a PPT state) is implicit already in (Audenaert et al., 2003).

Although specific examples of *activators* has been found, the general question if any (bound) entangled state can be an activator has waited until the discovery of Masanes (Masanes, 2005a,b). He showed, that every entangled state (even a bound entangled one) can enhance maximal singlet fraction and in turn a fidelity of teleportation of some other entangled state, i.e. for any state  $\rho$  there exists state  $\sigma$  such that

$$F_{max}^{(p)}(\sigma) < F_{max}^{(p)}(\rho \otimes \sigma). \tag{114}$$

This result<sup>52</sup> for the first time shows that

• Every entangled state can be used to some nonclassical task

Masanes provides an existence proof via reductio ad ab*surdum.* It is then still a challenge to construct for a given state some *activator*. even though one has a promise that it can be found. This result indicates first useful task, that can be performed by all bound entangled states.

The idea of activation for bipartite case was developed in multipartite case in (Dür and Cirac, 2000a) and (Bandyopadhyay et al., 2005) where specific families of multipartite bound entangled states were found. Interestingly, those states were then used to find analogous phenomenon in *classical key agreement* scenario (see Sec. XIX). It was recently generalized by Masanes (again in existential way) (Masanes, 2005b).

The activation considered above is a result which concerns one copy of the state. An analogous result which hold in asymptotic regime, called *superactivation* was

<sup>&</sup>lt;sup>51</sup> A state  $\rho$  is called one copy distillable, if there exists projectors <sup>50</sup> The superscript (p) emphasizes a "probabilistic" nature of this P, Q of rank two such that  $P \otimes Q \rho P \otimes Q$  is NPT. In other words, maximal singlet fraction so that it would not be confused from  $\rho$  one can then obtain by LOCC a twoqubit state with F with different parameter defined analogously as  $F_{\max}(\rho)$  = greater than 1/2.  $max_{\Lambda} \frac{1}{\operatorname{Tr}[\Lambda(\rho)]} \operatorname{Tr}[\Lambda(\rho)|\Phi^+\rangle\langle\Phi^+|]$  with  $\Lambda$  being trace preserving

<sup>&</sup>lt;sup>52</sup> Actually, the result is even stronger: it holds also for  $F_{max}$  (i.e. singlet fraction achievable with probability one).

found in (Shor *et al.*, 2003). Namely there are fourpartite states, such that no two parties even with help of other parties can distill pure entanglement from them:

$$\rho_{ABCD}^{be} = \frac{1}{4} \sum_{i=1}^{4} |\Psi_{AB}^i\rangle \langle \Psi_{AB}^i| \otimes |\Psi_{CD}^i\rangle \langle \Psi_{CD}^i|.$$
(115)

However the following state, consisting of five copies of the same state, but each distributed into different parties is no longer bound entangled

$$\rho_{free} = \rho_{ABCD}^{be} \otimes \rho_{ABCE}^{be} \otimes \rho_{ABDE}^{be} \otimes \rho_{ACDE}^{be} \otimes \rho_{BCD}^{be} (116)$$

This result is stronger than activation in two ways. First it turns two totally non-useful states (bound entangled ones) into a useful state (distillable one), and second the result does not concern one copy but it has asymptotically nonvanishing rate. In other words it shows, that there are states  $\rho_1$  and  $\rho_2$  such that  $E(\rho_1 \otimes \rho_2) > E(\rho_1) + E(\rho_2)$ , despite the fact that  $E(\rho_1) = E(\rho_2) = 0$ where E is suitable measure describing effect of distillation (see Sec. XV).

#### 1. Multipartite bound entanglement

In this case one defines bound entanglement as any entangled state (*i.e. violating full separability condition*) such that one can not distill any pure entanglement between any subset of parties. Following the fact that violation of PPT condition is necessary for distillation of entanglement it is immediate that if the state satisfies PPT condition under any bipartite cut then it is impossible to distill pure entanglement at all.

The first examples of multipartite bound entanglement were semiseparable three qubit states  $\rho_{ABC}^{Shift}$  (see Sec. VII.B). Immediately it is not distillable since it is *separable* under any bipartite cut. Thus not only pure entanglement, but simply *no* bipartite entanglement can be obtained between any two subsystems, while any pure multipartite state possesses entanglement with respect to some cut.

One can relax the condition about separability or PPT property and the nice example is provided in (Dür *et al.*, 1999b). This is an example of 3-qubit state  $\varrho_{ABC}$  from the family (86) such that it is PPT with respect to two partitions but not with respect to the third one i.e.  $\varrho_{ABC}^{T_A} \geq 0$ ,  $\varrho_{ABC}^{T_B} \geq 0$  but  $\varrho_{ABC}^{T_C} \geq 0$ . Thus not all non-trivial partitions have NPT property which, in case of this family, was proven to be equivalent to distillability (see last paragraph of the previous subsection). Still the state is entangled because of violation PPT along the AB|C partition, hence it is bound entangled.

Note, however, that the bound entanglement in this state  $\rho_{ABC}$  can be activated with partial free entanglement of the state  $|\Phi_{AB}^+\rangle|0\rangle_C$  since then part A of bound entangled state can be teleported to B and vice versa, producing in this way two states  $\sigma_{AC} = \sigma_{BC}$  between Alice and Charlie and Bob and Charlie respectively which

are  $2 \otimes 4$  NPT states, but as we already know all NPT  $2 \otimes N$  states are distillable (see previous section on distillation of bipartite entanglement). In this way one can distill entanglement between AC and BC which allows to distill three particle GHZ entanglement as already previously mentioned. Thus three qubit bound entanglement of  $\rho_{ABC}$  can be activated with help of two-qubit pure entanglement.

A simple, important family of bound entangled states are four-qubit Smolin bound entangled states (85). They have the property, that they can be "unlocked": if parties AB meet, then the parties CD can obtain an EPR pair. They are bound entangled, because any cut of the form two qubits versus two other qubits is separable. Again, if instead we could distill a pure state, the Smolin state would be entangled with respect to some cut of this form.

Recently a relationship between multipartite bound entangled states and the stabilizer formalism was founded (Wang and Ying, 2007), which allows to construct a wide class of unlockable bound entangled states in arbitrary multiqudit systems. It includes a generalized Smolin states (Augusiak and Horodecki, 2006b; Bandyopadhyay *et al.*, 2005).

#### J. Bell inequalities and bound entanglement

Since bound entanglement is a weak resource, it was natural to ask, whether it can violate Bell inequalities. The first paper reporting violation of Bell inequalities was due to Dür (Dür, 2001) who showed that some multiqubit bound entangled states violate two-settings inequalities called Mermin-Klyshko inequalities, for *n*-partite states with  $n \ge 8$ . Further improvements (Kaszlikowski *et al.*, 2000)  $(n \ge 7)$  (Nagata *et al.*, 2006; Sen(De) *et al.*, 2002)  $(n \ge 6)$  have been found. In (Augusiak *et al.*, 2006) the lowest so far number of particles was established: it was shown that the four-qubit Smolin bound entangled states (85) violate Bell inequality. The latter is very simple distributed version of CHSH inequality:

$$|a_A \otimes a_B \otimes a_C \otimes (b_D + b'_D) + a'_A \otimes a'_B \otimes a'_C \otimes (b_D - b'_D)| \le 2$$
(117)

We see that this is the usual CHSH inequality where one party was split into three, and the other on (B) was untouched. What is curious, the maximal violation is obtained exactly at the same choices of observables, as for singlet state with standard CHSH:

$$a = \sigma_x, \quad a' = \sigma_y,$$
  
$$b = \frac{\sigma_x + \sigma_y}{\sqrt{2}}, \quad b' = \frac{\sigma_x - \sigma_y}{\sqrt{2}}$$
(118)

and the value is also the same as for singlet state:  $2\sqrt{2}$ .

Werner and Wolf analyzed multipartite states (Werner, 2001b; Werner and Wolf, 2000) and have shown that states which are PPT in every cut (we will call them PPT states in short) must satisfy any Bell inequality of

type  $N \times 2 \times 2$  (i.e. involving N observers, two observables per site, each observable with two outcomes).

The relation of Bell inequalities with bipartite distillability has been also analyzed in (Acin *et al.*, 2006b) where it has been shown that any violation of Bell inequalities from the Mermin-Klyshko class always leads to possibility of bipartite entanglement distillation between some subgroups of particles.

In 1999 Peres has conjectured (Peres, 1999) that PPT states do not violate Bell inequalities:

$$LHVM \Leftrightarrow PPT.$$
 (119)

We see that the above results support this conjecture. The answer to this question, if positive, will give very important insight into our understanding of classical versus quantum behavior of states of composite systems.

Let us finally mention, that all those considerations concern violation without postselection, and operations applied to many copies. Recently, Masanes showed, that if we allow collective manipulations, and postselection, no bound entangled states violate the CHSH inequality even in asymptotic regime (Masanes, 2006).

## XIII. MANIPULATIONS OF ENTANGLEMENT AND IRREVERSIBILITY

## A. LOCC manipulations on pure entangled states — exact case

The study on exact transformations between pure states by LOCC was initiated by (Lo and Popescu, 2001). A seminal result in this area is due to Nielsen (Nielsen, 1999). It turns out that the possible transitions can be classified in a beautiful way in terms of squares of Schmidt coefficients  $\lambda_i$  (i.e. eigenvalues of local density matrix). Namely, a pure state  $|\psi\rangle = \sum_{j=1}^d \sqrt{\lambda_j^{(\psi)}} |jj\rangle$  can be transformed into other pure state  $|\phi\rangle = \sum_{j=1}^d \sqrt{\lambda_j^{(\phi)}} |jj\rangle$  if and only if for each  $k \in \{1, \ldots, d\}$  there holds

$$\sum_{j=1}^{k} \lambda_j^{(\psi)\downarrow} \le \sum_{j=1}^{k} \lambda_j^{(\phi)\downarrow}, \qquad (120)$$

where  $\lambda_j^{(\psi,\phi)\downarrow}$  are eigenvalues of subsystem of  $\psi$  ( $\phi$ ) in descending order. The above condition states that  $\phi$  majorizes  $\psi$  (see also Sec. (V.C)). Thus one can transform  $\psi$  into  $\phi$  only when subsystems of  $\psi$  are more mixed than those of  $\phi$ . This is compatible with Schrödinger approach: the more mixed subsystem, the more entangled state.

Since majorization constitutes a partial order, reversible conversion  $\psi \leftrightarrow \phi$  is possible if and only if the Schmidt coefficients of both states are equal. Moreover there exist states either of which cannot be converted into

each other. Thus generically, LOCC transformations between pure states are *irreversible*. However as we will see, it can be lifted in asymptotic limit.

Further important results in this area have been provided by (Vidal, 2003) who obtained optimal probability of success for transitions between pure states (see Sec. XV.D.1), and (Jonathan and Plenio, 1999), who considered transitions state  $\rightarrow$  ensemble.

## 1. Entanglement catalysis

The most surprising consequence of the Nielsen's laws of pure state transitions have been discovered in (Jonathan and Plenio, 1999). Namely, for some states  $\psi_1$  and  $\psi_2$  for which transition  $\psi_1 \rightarrow \psi_2$  is impossible, the following process is possible:

$$\psi_1 \otimes \phi \to \psi_2 \otimes \phi, \tag{121}$$

thus we borrow state  $\phi$ , run the transition, and obtain untouched  $\phi$  back! The latter state it plays exactly a role of *catalyst*: though not used up in the reaction, its presence is necessary to run it. Interestingly, it is not hard to see that the catalyst cannot be maximally entangled. The catalysis effect was extended to the case of mixed states in (Eisert and Wilkens, 2000).

## 2. SLOCC classification

For multipartite pure states there does not exist Schmidt decomposition. Therefore Nielsen result cannot be easily generalized. Moreover analysis of LOCC manipulations does not allow to classify states into some coarse grained classes, that would give a rough, but more transparent picture. Indeed, two pure states can be transformed into each other by LOCC if and only if they can be transformed by local unitary transformations so that to parameterize classes one needs continuous labels, even in bipartite case.

To obtain a simpler, "coarse grained" classification, which would be helpful to grasp important qualitative features of entanglement, it was proposed (Dür *et al.*, 2000b) to treat states as equivalent, if with some nonzero probability they can be transformed into each other by LOCC. This is called *stochastic* LOCC, and denoted by SLOCC. It is equivalent to say that there exists reversible operators  $A_i$  such that

$$|\psi\rangle = A_1 \otimes \ldots \otimes A_N |\phi\rangle. \tag{122}$$

For bipartite pure states of  $d \otimes d$  system we obtain in this way d entangled classes of states, determined by number of nonzero Schmidt coefficients (so called Schmidt rank). Here is example of SLOCC equivalence: the state

$$|\psi\rangle = a|00\rangle + b|11\rangle \tag{123}$$

with a > b > 0 can be converted (up to irrelevant phase) into  $|\phi^+\rangle$  by filter  $A \otimes I$ , with  $A = \begin{bmatrix} \frac{b}{a} & 0\\ 0 & 1 \end{bmatrix}$  with probability  $p = 2|b|^2$ ; so it is possible to consider representative state for each class.

A surprising result is due to Ishizaka (Ishizaka, 2004) who considered SLOCC assisted by bound entangled PPT states. He then showed that every state can be converted into any other. This works for both bipartite and multipartite pure states. For multipartite states SLOCC classification was done in the case of three (Dür *et al.*, 2000b) and four qubits (Verstraete *et al.*, 2002), and also two qubits and a qudit (Miyake and Verstraete, 2004). For three qubits there are five classes plus fully product state, three of them being Bell states between two qubits (i.e. states of type  $EPR_{AB} \otimes |0\rangle_C$ ). Two others are GHZ state

$$|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) \tag{124}$$

and so called W state

$$|W\rangle = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle).$$
 (125)

They are inequivalent, in a sense, that none of them can be converted into the other one with nonzero probability (unlike in bipartite state case, where one can go from any class to any lower class i.e. having lower Schmidt rank).

In  $2 \otimes 2 \otimes d$  case (Miyake, 2004; Miyake and Verstraete, 2004), there is still discrete family of inequivalent classes, where there is maximally entangled state — two EPR state  $\phi_{AB_1}^+ \otimes \phi_{B_2C}^+$  (where the system *B* is fourdimensional). Any state can be produced from it simply via teleportation (Bob prepares the needed state, and teleports its parts to Alice and to Charlie.

In four qubit case the situation is not so simple: the inequivalent classes constitute a continuous family, which one can divide into nine qualitatively different subfamilies.

The SLOCC classification is quite elegant generalization of local unitary classification. In the latter case the basic role is played by invariants of group  $SU_{d_1} \otimes \ldots \otimes$  $SU_{d_N}$  for  $d_1 \otimes \ldots \otimes d_N$  system, while in SLOCC, the relevant group is  $SL_{d_1,C} \otimes \ldots \otimes SL_{d_N,C}$  (one restricts to filters of determinant 1, because the normalization of states does not play a role in SLOCC approach).

Finally, the SLOCC classification of pure states can be used to obtain some classification of mixed states (see (Acin *et al.*, 2001; Miyake and Verstraete, 2004)).

## B. Asymptotic entanglement manipulations and irreversibility

The classifications based on exact transformations suffer for some lack of continuity: for example in SLOCC approach  $\psi$  with squares of Schmidt coefficients (0.5, 0.49, 0.01) is in the same class as the state (1/3, 1/3, 1/3), but in different class than (0.5, 0.5, 0), while we clearly see that the first and the last have much more in common than the middle one. In order to neglect small differences, one can employ some asymptotic limit. This is in spirit of Shannon's communication theory, where one allows for some inaccuracies of information transmission, provided they vanish in asymptotic limit of many uses of channel. Interestingly, the first results on quantitative approach to entanglement (Bennett *et al.*, 1996b,c,d) were based on LOCC transformations in asymptotic limit.

In asymptotic manipulations, the main question is what is the *rate* of transition between two states  $\rho$  and  $\sigma$ . One defines the rate as follows. We assume that Alice and Bob have initially *n* copies in state  $\rho$ . They apply LOCC operations, and obtain *m* pairs<sup>53</sup> in some joint state  $\sigma_m$ . If for large *n* the latter state approaches state  $\sigma^{\otimes m}$ , i.e.

$$\|\sigma_m - \sigma^{\otimes m}\|_1 \to 0 \tag{126}$$

and the ratio m/n does not vanish, then we say that  $\rho$  can be transformed into  $\sigma$  with rate  $R = \lim m/n$ . The largest rate of transition we denote by  $R(\rho \rightarrow \sigma)$ . In particular, distillation of entanglement described in Sec. XII is the rate of transition to EPR state

$$E_D(\rho) = R(\rho \to \psi^+). \tag{127}$$

The cost of creating state out of EPR states is given by

$$E_C(\rho) = 1/R(\psi^+ \to \rho) \tag{128}$$

and it is the other basic important measure (see Sec. XV.A for description of those measures in more detail).

#### 1. Unit of bipartite entanglement

The fundamental result in asymptotic regime is that any bipartite pure state can be transformed into twoqubit singlet with rate given by entropy of entanglement  $S_A = S_B$ , i.e. entropy of subsystem (either A or B, since for pure states they are equal). And vice versa, to create any state from two-qubit singlet, one needs  $S_A$  singlets pair pair two-qubit state. Thus any pure bipartite state can be reversibly transformed into any other state. As a result, in asymptotic limit, entanglement of these states can be described by a single parameter — von Neumann entropy of subsystem. Many transitions that are not allowed in exact regime, become possible in asymptotic limit. Thus the irreversibility implied by Nielsen result is lifted in this regime, and EPR state becomes universal unit of entanglement.

 $<sup>^{53}</sup>$  Here m depends on n, which we do note write explicitly for brevity.

### 2. Bound entanglement and irreversibility

However, even in asymptotic limit one cannot get rid of irreversibility for bipartite states, due to existence of bound entangled states. Namely, to create such a state from pure states by LOCC one needs entangled states, while no pure entanglement can be drawn back from it. Thus the bound entangled state can be viewed as a sort of black holes of entanglement theory (Terhal et al., 2003b). One can also use thermodynamical analogy (Horodecki et al., 2002, 1998b). Namely, bound entangled state is like a single heat bath: to create the heat bath, one needs to dissipate energy, but no energy useful to perform mechanical work can be drawn in cyclic process (counterpart of the work is here quantum communication via teleportation). Note in this context, that the interrelations between entanglement and energy were considered also in different contexts (see e.g. (Horodecki et al., 2001c; McHugh et al., 2006; Osborne and Nielsen, 2002)).

One might hope to regain reversibility as follows: perhaps for many copies of bound entangled state  $\rho$ , though some pure entanglement is needed, a *sublinear* amount would be enough to create them (i.e. the rate  $E_C$  vanishes). In other words, it might be that for bound entangled states  $E_C$  vanishes. This would mean, that asymptotically, the irreversibility is lifted.

More general question is whether distillable entanglement is equal to entanglement cost for mixed states. Already in the original papers on entanglement distillation (Bennett *et al.*, 1996c,d) there was indication, that generically we would have a gap between those quantities<sup>54</sup>, even though for some trivial cases we can have  $E_C = E_D$  for mixed states ((Horodecki *et al.*, 1998b)). Continuing thermodynamical analogy, a generic mixed state would be like a system of two heat baths of different temperature, from which part of energy but not the whole can be transferred into mechanical work.

It was a formidable task to determine, if we have irreversibility in asymptotic setting, as it was related to fundamental, and still unsolved problem of whether entanglement cost is equal to entanglement of formation (see Sec. XV). The first example of states with asymptotic irreversibility was provided in (Vidal and Cirac, 2001). Subsequently more and more examples have been revealed. In (Vollbrecht *et al.*, 2004) mixtures of maximally entangled states were analyzed by use of uncertainty principle. It turns out that irreversibility for this class of states is generic: the reversible states happen to be those which minimize uncertainty principle, and all they turn out to be so called pseudo-pure states (see (Horodecki *et al.*, 1998b)), for which reversibility holds

<sup>54</sup> Although in their operational approach the authors meant what we now call entanglement cost, to quantify it they used nonregularized measure entanglement of formation. for trivial reasons. Example of such state is

$$\frac{1}{2}|\psi_{AB}^{+}\rangle\langle\psi_{AB}^{+}|\otimes|0\rangle_{A'}\langle0|+\frac{1}{2}|\psi_{AB}^{-}\rangle\langle\psi_{AB}^{-}|\otimes|1\rangle_{A'}\langle1|.$$
(129)

The states  $|0\rangle$ ,  $|1\rangle$  are local orthogonal "flags", which allow to return to the pure state  $\psi_+$  on system AB. This shows that, within mixtures of maximally entangled states, irreversibility is a generic phenomenon. Last but not least, it was shown that irreversibility is exhibited by *all* bound entangled states (Yang *et al.*, 2005a) (see Sec. XIII).

One could still try to regain reversibility in some form. Recall the mentioned thermodynamical system with difference of temperatures. Since only part of the energy can perform useful work, we have irreversibility. However the irreversible process occurred in the past, while creating the system by partial dissipation of energy. However, once the system is already created, the work can be reversibly drawn and put back to the system in Carnot cycle. In the case of entanglement, it would mean that entanglement can be divided into two parts: bound entanglement and free (pure entanglement) which then can be reversibly mixed with each other. One can test such hypotheses by checking first, whether having bound entanglement for free, one can regain reversibility (Horodecki et al., 1998a). In (Horodecki et al., 2002) strong evidence was provided, that even this is not the case, while conversely in (Audenaert et al., 2003) it was shown, that to some extent such reversibility can be reached (see also (Ishizaka, 2004) in this context)  $^{55}$ .

## 3. Asymptotic transition rates in multipartite states

In multipartite case there is no such a universal unit of entanglement as the singlet state. Even for three particles, we can have three different types of EPR states: the one shared by Alice and Bob, by Alice and Charlie and Bob and Charlie. By LOCC it is impossible to create any of them from the others. Clearly, one cannot create at all  $\phi^+{}_{AB}$  from  $\phi^+{}_{BC}$  because the latter has zero entanglement across A:BC cut<sup>56</sup>. From two states  $\phi^+{}_{AB}$  and  $\phi^+{}_{AC}$  one can create  $\phi^+{}_{BC}$  via entanglement swapping, i.e. by teleportation of half of one of pairs through the other pair. However this is *irreversible*: the obtained state  $\phi^+{}_{BC}$  does not have entanglement across A:BC cut, so that it is impossible to create state  $\phi^+{}_{AB}$  and for similar reason also the state  $\phi^+{}_{AC}$ .

To see why even collective operations on many copies cannot make the transformation  $\phi^+{}^{\otimes n}_{AB} + \phi^+{}^{\otimes m}_{AC} \rightarrow \phi^+{}^{\otimes k}_{AB}$ reversible it is enough to examine entropies of subsystems. For pure states they do not increase under LOCC,

 $<sup>^{55}</sup>$  It should be noted however, that in (Audenaert *et al.*, 2003) PPT operations have been used, which may be a stronger resource, than assistance of PPT states.

<sup>&</sup>lt;sup>56</sup> By  $\phi^+{}_{AB}$  we mean  $\phi^+{}_{AB} \otimes |0\rangle_C$ , similarly for  $\phi^+{}_{AC}$  and  $\phi^+{}_{BC}$ .

so they have to be constant during reversible process. Thus we have to have :  $S_X^{in} = S_X^{out}$  for X = A, B, C which gives

$$A: n+m=k; B: n=k; C: m=k.$$
 (130)

Thus we arrived at contradiction (to be rigorous, we should apply here asymptotic continuity of entropy, see Sec. XV.B.3). In the described case irreversibility clearly enters, however the reason for this is not deep.

Via such considerations, one can see that in fourparticle case not only EPR states are "independent" units, but also GHZ state cannot be created from them, hence it constitutes another independent unit. For three particles it is less immediate, but still true that GHZ cannot be created reversibly from EPR states (Linden *et al.*, 1999a). This shows that even in asymptotic setting there is truly tripartite entanglement, a distinct quality that cannot be reduced to bipartite entanglements. This shows that there is true three-particle entanglement in asymptotic limit.

Irreversibility between  $|GHZ\rangle$  and  $\phi^+$  (much less trivial than the one between EPR states themselves) can be also seen as consequence of the fact that all two-particle reduced density matrices of  $|GHZ\rangle$  are separable, while for EPR states two are separable and one is entangled. In bipartite states we cannot transform separable state into entangled one by LOCC at all. Here, the transition is possible, because the third party can help, however this requires some measurement which introduces irreversibility.

Irreversibility was further explored in (Acin *et al.*, 2003d) and irreversibility on the level of mixed bipartite states was used to show that even GHZ plus EPR's do not constitute what is called *minimal reversible entanglement generating set* (Bennett *et al.*, 2001) i.e. a minimal set of states from which any other state can be reversibly obtained by LOCC. Such set is still unknown even in the case of pure states of three qubits. Recently it was shown (Ishizaka and Plenio, 2005) that for three qubits the set cannot be also constituted by *EPR*'s and W state  $W = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$ . In (Vidal *et al.*, 2000) a nontrivial class of states was shown, which are *reversibly* transformable into *EPR*'s and *GHZ*:

$$|\psi\rangle_{ABC} = c_0|000\rangle + c_1|1\rangle \frac{1}{\sqrt{2}}(|11\rangle + |22\rangle).$$
 (131)

The irreversibility between pure multipartite states seems to be a bit different than that of mixed states in that it is less "thermodynamical". In the former case we have two "good" states, which cannot be transformed reversibly into one another (both EPR state and GHZ state are useful for some tasks), while bound entanglement is clearly "worse" than free entanglement.

## XIV. ENTANGLEMENT AND QUANTUM COMMUNICATION

In classical communication theory, the most important notion is that of correlations. To send a message, means in fact to correlate sender and receiver. Also the famous Shannon formula for channel capacity involves mutual information, a function describing correlations. Thus the ability to faithfully transmit a bit is equivalent to the ability to faithfully share maximally correlated bits. It was early recognized that in quantum communication theory it is entanglement which will play the role of correlations. In this way entanglement is the cornerstone of quantum communication theory.

In classical communication theory, a central task is to send some signals. For a fixed distribution of signals emitted by a source, there is only one ensemble of messages. In quantum case, a source is represented by a density matrix  $\rho$ , and there are *many* ensembles realizing the same density matrix. What does it then mean to send quantum information? According to (Bennett et al., 1993) it is the ability of transmitting and unknown quantum state. For a fixed source, this would mean, that all possible ensembles are properly transmitted. For example, consider a density matrix  $\rho = \frac{1}{d} \sum_{i} |i\rangle \langle i|$ . Suppose that a channel decoheres its input in basis  $\{|i\rangle\}$ . We see that the set of states  $\{|i\rangle\}$  goes through the channel without any disturbance. However complementary set consisting of states  $U|i\rangle$  where U is discrete Fourier transform, is completely destroyed by a channel, because the channel destroys superpositions. (For d = 2, example of such complementary ensemble is  $|+\rangle, |-\rangle$  where  $|\pm\rangle = \frac{1}{2}(|0\rangle \pm |1\rangle)$ ). As a matter of fact, each member of the complementary ensemble is turned into maximally mixed state.

How to recognize whether all ensembles can go through? Schumacher has noted (Schumacher, 1995a) that instead of checking all ensembles we can check whether an *entangled* state  $\psi_{AB}$  is preserved, if we send half of it (the system *B*) down the channel. The state should be chosen to be *purification* of  $\rho$ , i.e.  $\text{Tr}_A(|\psi\rangle\langle\psi|_{AB}) = \rho$ . Thus sending an unknown state is equivalent to sending faithfully entanglement.

Indeed, in our example, the state can be chosen as  $|\Phi^+\rangle = \frac{1}{\sqrt{d}} \sum_i |i\rangle |i\rangle$ . One can see that after applying our channel to one subsystem, the state becomes classical (incoherent) mixture of states  $|i\rangle |i\rangle$ . This shows that the channel cannot convey quantum information at all. It is reflection of a mathematical fact, that if we send half of purification of a full rank density matrix down the channel, then the resulting state will encode all the parameters of the channel. This heuristic statement has its mathematical form in terms of Choi-Jamiołkowski isomorphism between states and channels. Its most standard form links the channel  $\Lambda$  with a state  $\varrho^{\Lambda}_{AB}$  having maximally mixed left subsystem  $\text{Tr}_B(\varrho^{\Lambda}_{AB}) = \frac{1}{d_A}\text{I}$  as follows:

$$\varrho_{AB}^{\Lambda} = [\mathbf{I}_A \otimes \Lambda_B] |\Phi_{AA'}^+\rangle \langle \Phi_{AA'}^+|, \qquad (132)$$

where the projector onto maximally entangled state  $P_{AA'}^+$ is defined on a product Hilbert space  $\mathcal{H}_A \otimes \mathcal{H}_{A'}$ , with  $\mathcal{H}_A \simeq \mathcal{H}_{A'}^{57}$ .

### A. Capacity of quantum channel and entanglement

The idea of quantum capacity  $Q(\Lambda)$  of quantum channel  $\Lambda$  was introduced in the seminal work (Bennett *et al.*, 1996d) which contains milestone achievements connecting quantum entanglement and quantum data transfer. The capacity Q measures the largest rate of quantum information sent asymptotically faithfully down the channel:

$$Q = \sup \frac{\# \text{ transmitted faithfully qubits}}{\# \text{ uses of channel}}, \qquad (133)$$

where the fidelity of the transmission is measured by minimal subspace fidelity  $f(\Lambda) = \min_{\psi} \langle \psi | \Lambda(|\psi\rangle \langle \psi |) | \psi \rangle$ (Bennett *et al.*, 1997). Also average fidelity transmission can be used which is direct analog of average fidelity in quantum teleportation process:  $\overline{f}(\Lambda) = \int \langle \psi | \Lambda(|\psi\rangle \langle \psi |) | \psi \rangle d\psi$  with uniform measure  $d\psi$  on unit sphere.

Other approach based on idea described above was formalized in terms of *entanglement transmission*. In particular, the quality of transmission was quantified by *entanglement fidelity* 

$$F(\Lambda, \Psi_{AB}) = \langle \Psi_{AB} | [I_A \otimes \Lambda_B] | \Psi_{AB} \rangle \langle \Psi_{AB} | ) | \Psi_{AB} \rangle,$$
(134)

with respect to a given state  $\Psi_{AB}^{58}$ . The alternative definition of quantum capacity (which has been worked out in (Barnum *et al.*, 1998; Schumacher, 1996; Schumacher and Nielsen, 1996) and shown to be equivalent to the original one of (Bennett *et al.*, 1996d, 1997) in (Barnum *et al.*, 2000)) was based on counting the optimal pure entanglement transmission under the condition of high entanglement fidelity defined above.

A variation of the entanglement fidelity (Reimpell and Werner, 2005) is when input is equal to the output  $d_A = d_B = d$  and we send half of maximally entangled state  $\Phi_d^+$  down the channel. It is measured by maximal entanglement fidelity of the channel:

$$F(\Lambda) := F(\Lambda, \Phi_d^+), \tag{135}$$

which is equal to overlap of the state  $\rho^{\Lambda}$  with maximally entangled state namely  $F(\Lambda) = F(\rho^{\Lambda}) :=$  $\operatorname{Tr}(|\Phi_d^+\rangle\langle\Phi_d^+|\rho^{\Lambda})$ . It is interesting that one has (Horodecki *et al.*, 1999b; Nielsen, 2002):

$$\overline{f}(\Lambda) = \frac{dF(\Lambda) + 1}{d+1}.$$
(136)

This formula says that possibility of sending on average faithfully quantum information happens if and only if it is possible to create maximal entanglement of  $\Phi_d^+$  with help of the channel. The above relation is an important element in the proof that, quite remarkably, definition of zero-way (or — alternatively — one-way forward) version of quantum capacity Q (see below) remains the same if we apply *any* of the fidelities recalled above (for details of the proof see the review (Kretschmann and Werner, 2004)).

Even more interesting, LHS of the above equality can be interpreted as an average teleportation fidelity of the channel that results from teleporting given state through the mixed bipartite state  $\rho_{\Lambda}$ .

An impressive connection between entanglement and quantum channels theory has been worked out already in ((Bennett *et al.*, 1996c,d)) by use of teleportation. The authors have shown how to achieve nonzero transmission rate by combining three elements (a) creating many copies of  $\rho^{\Lambda}$  by sending halves of singlets down the channel  $\Lambda$  (b) distilling maximal entanglement form many copies of the created state (c) teleporting quantum information down the (distilled) maximal entanglement. Since the last process corresponds to ideal transmission the rate of the quantum information transmission is here equal to distillation rate in step (ii). In this way one can prove the inequality linking entanglement distillation  $E_D$  with quantum channel capacity Q as (Bennett *et al.*, 1996d):

$$E_D(\varrho^{\Lambda}) \le Q(\Lambda),$$
 (137)

where  $\rho^{\Lambda}$  is given by (132). The inequality holds for one-way forward an two-way scenarios of distillation (respectively — coding). The above inequality is one of the central links between quantum channels and quantum entanglement theory (see discussion below). It is not known whether there is lower bound like  $cQ(\Lambda) \leq E_D(\rho^{\Lambda})$  for some constant c. However there is at least qualitative equivalence shown in (Horodecki, 2003e):

$$E_D(\varrho^{\Lambda}) = 0 \Rightarrow Q(\Lambda) = 0. \tag{138}$$

An alternative simple proof which works also for multipartite generalization of this problem can be found in (Dür *et al.*, 2004). It uses teleportation and Choi-Jamiołkowski isomorphism to collective channels (Cirac *et al.*, 2001).

### B. Fidelity of teleportation via mixed states

Before entanglement distillation was discovered, the fundamental idea of teleporting quantum states with help

<sup>&</sup>lt;sup>57</sup> We note that this isomorphism has operational meaning, in general, only one way: given the channel, Alice and Bob can obtain bipartite state, but usually not vice versa. However sometimes if Alice and Bob share mixed bipartite state, then by use of classical communication they can simulate the channel. Example is maximally entangled state, which allows to regain the corresponding channel via teleportation. This was first pointed out and extensively used in (Bennett *et al.*, 1996d).

<sup>&</sup>lt;sup>58</sup> For typical source all three types of fidelity — average, minimum and entanglement one are equivalent (Barnum *et al.*, 2000).

of mixed states has been put forward in the pioneering paper (Popescu, 1994). The fidelity of transmission that was used there was

$$f_{tel}(\varrho_{AB}) = \int d\phi \langle \phi | \varrho_B^{\phi} | \phi \rangle, \qquad (139)$$

with  $\varrho_B^{\phi}$  being the output state of the teleportation in Bobs hands, i.e. it is average fidelity mentioned above. Popescu used this formula to show that teleportation through some entangled 2-qubit Werner states that satisfy some local hidden variable models (Werner, 1989a) beats the classical "teleportation" fidelity threshold 2/3. Further the formula for  $f_{tel}$  was optimized in the case of pure (Gisin, 1996a) and mixed (Horodecki et al., 1996b) 2-qubit states  $\rho_{AB}$ , with Alice's measurement restricted to maximally entangled projections. For pure state  $\rho_{AB}$  general optimization was performed in  $d \otimes d$ case and it was shown to be attained just on maximally entangled Alice's projections (Banaszek, 2000)(see (Bowen and Bose, 2001) for further analysis). All that teleportation schemes that we have considered so far were deterministic. The idea of probabilistic teleportation was introduced in (Brassard et al., 2004; Mor, 1996). It was employed in effect of activation of bound entanglement (see Sec. XII.I).

Finally let us note that all possible exact teleportation and dense coding schemes were provided in (Werner, 2001a). There it was proved, that they are essentially of the same kind. In teleportation of a state of a *d*dimensional system, Alice measures in a maximally entangled basis<sup>59</sup> { $\psi_i$ }. Subsequently Bob decodes by use of unitary transformations related to the basis via  $U_i \otimes$ I | $\Psi^+$ > = | $\psi_i$ > (similarly as in dense coding). Note that for two qubits all maximally entangled bases are related by local unitary transformation (Horodecki and Horodecki, 1996). Thus there is in a sense a unique protocol of teleporting a qubit. However in higher dimension it is no longer the case (Wójcik *et al.*, 2003) which leads to inequivalent teleportation/dense coding protocols.

## C. Entanglement breaking and entanglement binding channels

The formulas (137) and (138) naturally provoke the question: which channels have capacity zero?.

Now, if the state  $\rho^{\Lambda}$  is separable, then the corresponding channel is called *entanglement breaking* (Horodecki *et al.*, 2003d) and one cannot create entanglement at all by means of such channel.<sup>60</sup> Now since it is impossible to distill entanglement from separable

state we immediately see from (138) that the capacity of entanglement breaking channel is zero i.e. no quantum faithful quantum transmission is possible with entanglement breaking channel. However the converse is not true: possibility to create entanglement with help of the channel is not equivalent to quantum communication and it is bound entanglement phenomenon which is responsible for that. To see it let us observe that if the corresponding state  $\rho^{\Lambda}$  is bound entangled, then the channel  $\Lambda$ , called in this case binding entanglement channel (introduced in (DiVincenzo et al., 2003a; Horodecki et al., 2000d)), clearly allows for creation of entanglement. However it cannot convey quantum information at all — it has capacity zero. This was first argued in (Horodecki et al., 2000d) and proof was completed in (Horodecki, 2003e) just via implication (138)).

Note that binding entanglement channels are not completely useless from general communication point of view, for they can be used to generate secure cryptographic key, see Sec. (XIX).

## D. Quantum Shannon theorem

How many qubits can be sent per use of a given channel? It turns out that the answer to the question is on one hand analogous to classical Shannon formula, and on the other hand entirely different. Capacity of classical channel is given by maximum of mutual information over all bipartite distributions, that can be obtained by use of channel.

Quantumly, even for sending qubits, as already mentioned above, we may have different scenarios: 1) quantum channel without help of classical channel (capacity denoted by  $Q^{\emptyset}$  2) with help of classical channel: oneway forward  $(Q^{\rightarrow})$ , backward  $(Q^{\leftarrow})$ , and two-way  $(Q^{\leftrightarrow})$ . There is elegant answer in the case 1), which is called Quantum Shannon Theorem. The first proof, partially heuristic, was provided in (Lloyd, 1997), subsequently it was improved in (Shor, 2002a) and finally first fully rigorous result has been obtained in (Devetak, 2003). Namely, to get asymptotic capacity formula for  $Q^{\emptyset}$  one maximizes coherent information over all bipartite states resulting from a pure state, half of it sent down the channel (see Sec.XII.F). However, one should optimize this quantity over many uses of channel, so that the formula for capacity reads

$$Q^{\emptyset}(\Lambda) = \lim \frac{1}{n} \sup_{\psi} I^{coh}_{A \rangle B}((\mathbf{I} \otimes \Lambda^{\otimes n}) |\psi\rangle \langle \psi|).$$
(140)

The fact that the formula is not "single letter", in the sense that it involves many uses of channel

<sup>&</sup>lt;sup>59</sup> That is performs a measurement with Kraus operators being the projects onto bipartite states  $|\psi_i\rangle \in \mathcal{C}^d \otimes \mathcal{C}^d$  which have maximally mixed subsystems and form a basis of  $\mathcal{C}^d \otimes \mathcal{C}^d$ 

<sup>&</sup>lt;sup>60</sup> Simply, its Kraus operators (in some decomposition) are of rank one. Impossibility of creating entanglement follows from the fact

that action of any channel of that kind can be simulated by classical channel: for any input state, one measures it via some POVM, and sends the classical outcome to the receiver. Based on this, receiver prepares some state.

was established in (Shor and Smolin, 1996), see also (DiVincenzo *et al.*, 1998b).

Coherent information can be nonzero only for entangled states. Indeed, only entangled states can have greater entropy of subsystem than that of total system (Horodecki and Horodecki, 1994). In this context, an additional interesting qualitative link between entanglement and channel capacity formula is the hashing inequality stating that  $E_{D}^{\rightarrow} \geq I_{A\rangle B}^{coh}$  (see Sec. XII). Quite remarkably sometimes  $E_{D}^{\rightarrow}(\varrho_{AB}) > 0$  for  $I_{A\rangle B}^{coh}(\varrho_{AB}) = 0$ which is related to the above mentioned Shor-Smolin result.

Finally, there is a remarkable result stating that forward classical communication does not increase quantum capacity

$$Q^{\emptyset} = Q^{\rightarrow}. \tag{141}$$

It was argued in (Bennett *et al.*, 1996d) and the proof was completed in (Barnum *et al.*, 2000). Therefore, any one-way distillation scheme, provides (through relation with  $Q^{\rightarrow}$ ) lower bound for capacity  $Q^{\emptyset}$ .

## E. Bell diagonal states and related channels

An intriguing relation between states and channels was found (Bennett et al., 1996d) — namely there is a class of channels  $\Lambda$  for which there is equality in (137) because of the following equivalence: not only given  $\Lambda$ one can produce  $\rho^{\Lambda}$  (which is easy — one sends half of maximally entangled state  $\Phi^+$  own the channel) but also given  $\rho^{\Lambda}$  and possibility of additional forward (i.e. from sender to receiver) classical communication one can simulate again the channel  $\Lambda$ . There is a fundamental observation (Bennett et al., 1996d)(which we recall here for qubit case but  $d \otimes d$  generalization is straightforward, see (Horodecki et al., 1999b)) that if the state  $\rho^{\Lambda}$  is Bell diagonal than the operation of teleporting any state  $\sigma$  through mixed state  $\rho^{\Lambda}$  produces just  $\Lambda(\sigma)$  in Bobs hands. Thus in this special case given state isomorphic to the channel plus possibility of forward classical communication one can simulate action of the chan*nel.* Thus for Bell diagonal states, we have an *operational* isomorphism, rather than just mathematical one (132). In (Bennett et al., 1996d) this was exploited to show that there is equality in (137) i.e.  $E_D^{\rightarrow}(\varrho^{\Lambda}) = Q^{\rightarrow}(\Lambda),$ and  $E_D(\varrho^{\Lambda}) = Q^{\leftrightarrow}(\Lambda)$ . The protocol attaining capacity of the channel is in this case sending half of  $|\psi^{-}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$  and perform the optimal entanglement distillation protocol.

To summarize the above description consider now the example where this analysis finds application.

*Example .-* Consider the quantum binary Pauli channel acting on single qubit as follows:

$$\Lambda_p(\rho) = p(\rho) + (1-p)\sigma_z(\rho)\sigma_z, \qquad (142)$$

where  $\sigma_z$  is a Pauli matrix. Applying the isomorphism (132) we get rank-two Bell diagonal state

$$\varrho^{\Lambda_p} = p |\phi^+\rangle \langle \phi^+| + (1-p) |\phi^-\rangle \langle \phi^-|, \qquad (143)$$

with  $|\phi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|0\rangle|0\rangle \pm |1\rangle|1\rangle)$ . The formula for distillable entanglement has been found in this case (Rains, 1999, 2001) <sup>61</sup> namely

$$E_D^{\to}(\varrho^{\Lambda_p}) = I_{B\rangle A}^{coh}(\varrho^{\Lambda_p}) = 1 - H(p), \qquad (144)$$

where  $H(p) = -p \log p - (1-p) \log(1-p)$ . (Actually, for this state  $E_D = E_D^{\rightarrow}$ ).<sup>62</sup>

Since it is Bell diagonal state we know that capacity  $Q^{\rightarrow}$  can be achieved by protocol consisting of one-way distillation of entanglement from  $\varrho^{\Lambda_p}$  followed by teleportation (see discussion above). Therefore  $Q^{\rightarrow}(\Lambda_p) = Q^{\emptyset}(\Lambda_p) = I^{coh}_{B \setminus A}(\varrho^{\Lambda_p}) = 1 - H(p)$ .

### F. Other capacities of quantum channels

One might ask weather maximizing quantum mutual information for a given channel (in analogy to maximizing coherent information, Sec. XIV.D) makes sense. Is such a maximized quantity linked to any capacity quantity? Recall that quantum mutual information is defined (in analogy to its classical version) as follows:

$$I_{A:B}(\varrho_{AB}) = S(\varrho_A) + S(\varrho_B) - S(\varrho_{AB}).$$
(145)

First of all one can prove that the corresponding (maximized) quantity would be nonzero even for entanglement breaking channels (Adami and Cerf, 1997; Belavkin and Ohya, 2001). Thus it cannot describe quantum capacity (since, as we already know, they have quantum capacity zero). Moreover, for a qubit channel it can be greater than 1, for example for noiseless channel it is just 2. This reminds dense coding, where two bits are send per sent qubit, with additional use of shared EPRpair. And these two things are indeed connected. It was established in (Bennett et al., 1999c, 2002) that the maximum of quantum mutual information taken over the states resulting from channel is exactly the classical capacity of a channel supported by additional EPR pairs (it is so called entanglement assisted capacity). Moreover, it is equal to twice quantum capacity:

$$C_{ass} = 2Q_{ass} = \sup I_{A:B} ((I_A \otimes \Lambda_B)(|\psi\rangle_{AB} \langle \psi|)) \quad (146)$$

where supremum is taken over all pure bipartite states. Note that the expression for capacity is "single letter",

<sup>&</sup>lt;sup>61</sup> That 1-H(p) is achievable was shown in (Bennett *et al.*, 1996d). <sup>62</sup> Note that this is equal to capacity of classical binary symmetric channel with error probability *p*. However, the classical capacity of the present channel is maximal, because classical information can be sent without disturbance through phase.

i.e. unlike in the unassisted quantum capacity case, it is enough to optimize over one use of channel.

A dual picture is, when one has noiseless channel, supported by noisy entangled states. Again, the formula involves coherent information (the positive coherent information says that we have better capacity, than without support of entanglement) (Bowen, 2001; Horodecki *et al.*, 2001b). There are also interesting developments concerning multipartite dense coding (Bruß *et al.*, 2004; Horodecki and Piani, 2006).

### G. Additivity questions

There is one type of capacity, where we do not send half of entangled state, but restrict just to separable states. This is classical capacity of quantum channel  $C(\Lambda)$ , without any further support. However even here entanglement comes in. Namely, it is still not resolved, whether sending signals entangled between distinct uses of channel can increase transmission rate. This problem is equivalent to the following one: can we decrease production of entropy by an operation  $\Lambda$ , by applying entangled inputs, to  $\Lambda \otimes \Lambda$ ? i.e.

$$2\inf_{\rho} S(\Lambda(\rho)) \stackrel{?}{=} \inf_{\rho} S(\Lambda \otimes \Lambda(\rho)).$$
(147)

One easily finds that if this equality does not hold it may happen only via entangled states. Quite interestingly, these problems have even further connection with entanglement: they are equivalent to additivity of one of the most important measures of entanglement — so called entanglement of formation, see (Audenaert and Braunstein, 2004; Koashi and Winter, 2004; Matsumoto, 2005; Shor, 2003) and references therein. We have the following equivalent statements

- additivity of entanglement of formation
- superadditivity of entanglement of formation
- additivity of minimum output entropy
- additivity of Holevo capacity
- additivity of Henderson-Vedral classical correlation measure  $C_{HV}$

(see Secs. XV and XIII for definitions of  $E_F$  and  $C_{HV}$ ). To prove or disprove them accounts for one of the fundamental problems of quantum information theory.

Discussing the links between entanglement and communication one has to mention about one more additivity problem inspired by entanglement behavior. This is the problem of additivity of quantum capacities Qtouched already in (Bennett *et al.*, 1996d). Due to nonadditivity phenomenon called *activation of bound entanglement* (see Sec. XII.I) it has been even conjectured (Horodecki *et al.*, 1999a) that for some channels  $Q(\Lambda_1 \otimes \Lambda_2) > 0$  even if both channels have vanishing capacities i.e.  $Q(\Lambda_1) = Q(\Lambda_2) = 0$ . Here again, as in the question of additivity of classical capacity  $C(\Lambda)$ , a possible role of entanglement in inputs of the channel  $\Lambda_1 \otimes \Lambda_2$  comes into play.

Though this problem is still open, there is its multipartite version where nonadditivity was found (Dür et al., 2004). In fact, the multiparty communication scenario can be formulated and the analog of (137) and (138) can be proved (Dür *et al.*, 2004) together with binding entanglement channels notion and construction. Remarkably in this case the application of multipartite BE channels isomorphic to multipartite bound entangled states and application of multipartite activation effect leads to nonadditivity of two-way capacity regions for quantum broadcast (equivalently multiply access) channel (Dür et al., 2004): tensor product of three binding channels  $\Lambda_i$  which automatically have zero capacity leads to the channel  $\Lambda : \otimes_{i=1}^{3} \Lambda_{i}$  with nonzero capacity. In this case bound entanglement activation (see section on activation) leads to the proof of *nonadditivity effect* in quantum information transmission. Existence of the alternative effect for one-way or zero-way capacity is an open problem. Finally let us note in this context that the multipartite generalization of the seminal result  $Q^{\emptyset} = Q^{\rightarrow}$  has been proved (Demianowicz and Horodecki, 2006) exploiting the notion of entanglement transmission.

There is an interesting application of multipartite bound entanglement in quantum communication. As we have already mentioned, bipartite bound entanglement, which is much harder to use than multipartite one, is able to help in quantum information transfer via activation effect (Horodecki *et al.*, 1999a). There is a very nice application of multipartite bound entanglement in remote quantum information concentration (Murao and Vedral, 2001). This works as follows: consider the 3-qubit state  $\psi_{ABC}(\phi)$  being an output of quantum cloning machine that is shared by three parties Alice, Bob and Charlie. The initial quantum information about cloned qubit  $\phi$ has been delocalized and they cannot concentrate it back in this distant lab scenario. If however each of them is given in addition one particle of the 4-particles in a Smolin state  $\rho_{ABCD}^{unlock}$  (85) with remaining fourth D particle handed to another party (David) then a simple LOCC action of the three parties can "concentrate" the state  $\phi$ back remotely at David's site.

## H. Miscellanea

Negative information. The fact that sending qubits means sending entanglement, can be used in even more complicated situation. Namely, suppose there is a noiseless quantum channel from Alice to Bob, and Bob already has some partial information, i.e. now the source  $\rho$  is initially distributed between Alice and Bob. The question is: how many qubits Alice has to send to Bob, in order he completes full information? First of all what does it mean to complete full information? This means that for all ensembles of  $\rho$ , after transmission, they should be in Bob's hands. Instead of checking all ensembles, one can rephrase the above "completing information" in elegant way by means of entangled state: namely, we consider purification  $\psi_{ABR}$  of  $\rho_{AB}$ . The transmission of the quantum information lacking by Bob, amounts then just to *merging* part A of the state to Bob, without disturbing it.

In (Horodecki *et al.*, 2005h, 2006e) it was shown that if classical communication is allowed between Alice and Bob then the amount of qubits needed to perform merging is equal to conditional entropy S(A|B). This holds even when S(A|B) is negative, in which case it means, that not only Alice and Bob do not need to send qubits, but they get extra EPR pairs, which can be used for some future communication. What is interesting in the context of channels capacity, this result proves the Quantum Shannon Theorem and hashing inequality. The new proof is very simple (the protocol we have recalled in Sec. XII.F).

Mother and father protocols. Several different phenomena of theory of entanglement and quantum communication have been grasped by a common formalism discovered in (Devetak et al., 2004, 2005). There are two main protocols called mother and father. In the first one, the noisy resource is state, while in the second — channel. From mother protocol one can obtain other protocols, such as dense coding via mixed states, or one-way distillation of mixed states (children). Similarly, father protocol generates protocols achieving quantum channel capacity as well as entangled assisted capacity of quantum channel. In both cases from children one can obtain parents, by using a new primitive discovered by Harrow called *cobit*.

The discovery of state merging have given a new twist. In (Abeyesinghe et al., 2006) a different version of state merging (called Fully Quantum Slepian Wolf protocol), where only quantum communication and sharing EPR pairs allowed (and counted) was provided. It turns out that the minimal number of needed qubits is equal to  $\frac{1}{2}I(A:R)$ , and the number  $\frac{1}{2}I(A:B)$  of EPR pairsis gained by Alice and Bob. The protocol is based on a beautiful observation, that a random subsystem A' of Aof size less than  $\frac{1}{2}I(A:B)$  is always maximally mixed and product with reference R. This implies that after sending the second part of A to Bob, A' is maximally entangled with Bob's system. These ideas were further developed in (Devetak and Yard, 2006), where operational meaning of conditional mutual information was discovered. It turned out that the above protocol is a powerful primitive, allowing to unify many ideas and protocols. In particular, this shows deep *operational* relations between entangled states and quantum channels.

## XV. QUANTIFYING ENTANGLEMENT

### A. Distillable entanglement and entanglement cost

The initial idea to quantify entanglement was connected with its usefulness in terms of communication (Bennett *et al.*, 1996c,d). As one knows via a two qubit maximally entangled state  $|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$  one can teleport one qubit. Teleportation is a process that involves the shared singlet, and local manipulations together with communication of classical bits. Since gubits cannot be transmitted by use of classical communication itself, it is clear that power of sending qubits should be attributed to entanglement. If a state is not maximally entangled, then it does not allow for faithful teleportation. However, in analogy to Shannon communication theory, it turns out that when having many copies in such state, one can obtain asymptotically faithful teleportation at some rate (see Sec. XII). To find how many qubits per copy we can teleport it is enough to determine how many e-bits we can obtain per copy, since every  $|\phi^+\rangle$ can then be used for teleportation. In this way we arrive at transition rates as described in Sec. XIII, and two basic measures of entanglement  $E_D$  and  $E_C$ .

Distillable entanglement. Alice and Bob start from n copies of state  $\rho$ , and apply an LOCC operation, that ends up with a state  $\sigma_n$ . We now require that for large nthe final state approaches the desired state  $(\Phi_2^+)^{\otimes m_n}$ . If it is impossible, then  $E_D = 0$ . Otherwise we say that the LOCC operations constitute a distillation protocol  $\mathcal{P}$  and the rate of distillation is given by  $R_{\mathcal{P}} = \lim_{n} \frac{m_n}{n}$ . The distillable entanglement is the supremum of such rates over all possible distillation protocols. It can be defined concisely (cf. (Plenio and Virmani, 2006)) as follows

$$E_D(\rho) = \sup\left\{r : \lim_{n \to \infty} \left(\inf_{\Lambda} \|\Lambda(\rho^{\otimes n}) - \Phi_{2^{rn}}^+\|_1\right) = 0\right\},$$
(148)

where  $\|\cdot\|_1$  is the trace norm (see (Rains, 1998) for showing that other possible definitions are equivalent). <sup>63</sup>

Entanglement cost. It is a measure dual to  $E_D$ , and it reports how many qubits we have to communicate in order to create a state. This, again can be translated to e-bits, so that  $E_C(\rho)$  is the number of e-bits one can obtain from  $\rho$  per input copy by LOCC operations. The definition is

$$E_C(\rho) = \inf\left\{r: \lim_{n \to \infty} \left(\inf_{\Lambda} \|\rho^{\otimes n} - \Lambda(\Phi_{2^{rn}}^+)\|_1\right) = 0\right\}.$$
(149)

In (Hayden *et al.*, 2001) it was shown, that  $E_C$  is equal to regularized entanglement of formation  $E_F$  — a prototype of entanglement cost, see Sec. XV.C.2. Thus, if  $E_F$ 

<sup>&</sup>lt;sup>63</sup> In place of trace norm one can use fidelity, Uhlmann fidelity  $F(\rho, \sigma) = (\text{Tr}\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}})^2$  thanks to inequality proven in (Fuchs and van de Graaf, 1997)  $1 - \sqrt{F(\rho, \sigma)} \leq \frac{1}{2} \|\rho - \sigma\|_1 \leq \sqrt{1 - F(\rho, \sigma)}$ .

is additive, then the two quantities are equal. As we have mentioned, for pure states  $E_D = E_C$ .

Distillable key. There is yet another distinguished operational measure of entanglement for bipartite states, designed in similar spirit as  $E_D$  and  $E_C$ . It is distillable private key  $K_D$ : maximum rate of bits of private key that Alice and Bob can obtain by LOCC from state  $\rho_{AB}$  where it is assumed that the rest E of the total pure state  $\psi_{ABE}$  is given to adversary Eve. The distillable key satisfies obviously  $E_D \leq K_D$ : indeed, a possible protocol of distilling key is to distill EPR states and then from each pair obtain one bit of key by measuring in standard basis. A crucial property of  $K_D$  is that it is equal to the rate of transition to a special class of states: so-called private states, which are generalization of EPR states. We elaborate more on distillable key and the structure of private states in Sec. XIX.

## B. Entanglement measures — axiomatic approach

The measures such as  $E_D$  or  $E_C$  are built to describe entanglement in terms of some tasks. Thus they arise from optimization of some protocols performed on quantum states. However one can apply axiomatic point of view, by allowing any function of state to be a measure, provided it satisfies some postulates. Let us now go through basic postulates.

### 1. Monotonicity axiom

The most important postulate for entanglement measures was proposed already in (Bennett *et al.*, 1996d) still in the context of operationally defined measures.

• Monotonicity under LOCC: Entanglement cannot increase under local operations and classical communication.

(Vedral *et al.*, 1997b) introduced idea of axiomatic definition of entanglement measures and proposed that an entanglement measure is any function that satisfies the above condition plus some other postulates. Then (Vidal, 2000) proposed that monotonicity under LOCC should be the only postulate necessarily required from entanglement measures. Other postulates would then either follow from this basic axiom, or should be treated as optional (see (Popescu and Rohrlich, 1997) in this context).

Namely, for any LOCC operation  $\Lambda$  we have

$$E(\Lambda(\rho)) \le E(\rho). \tag{150}$$

Note that the output state  $\Lambda(\rho)$  may include some registers with stored results of measurements of Alice and Bob (or more parties, in multipartite setting), performed in the course of the LOCC operation  $\Lambda$ . The mathematical form of  $\Lambda$  is in general quite ugly (see e.g. (Donald *et al.*, 2002)). A nicer mathematical expression is known as so called "separable operations" (Rains, 2000; Vedral  $et\ al.,$  1997b)

$$\Lambda(\rho) = \sum_{i} A_{i} \otimes B_{i}(\rho) A_{i}^{\dagger} \otimes B_{i}^{\dagger}$$
(151)

with obvious generalization to multipartite setting<sup>64</sup>. Every LOCC operation can be written in the above form, but not vice versa, as proved in (Bennett *et al.*, 1999a). (For more extensive treatment of various classes of operations see Sec. XI.B.)

The known entanglement measures usually satisfy a stronger condition, namely, they do not increase on average

$$\sum_{i} p_i E(\sigma_i) \le E(\rho), \tag{152}$$

where  $\{p_i, \sigma_i\}$  is ensemble obtained from the state  $\rho$  by means of LOCC operations. This condition was earlier considered as mandatory, see e.g. (Horodecki, 2001a; Plenio, 2005), but there is now common agreement that the condition (150) should be considered as the only necessary requirement <sup>65</sup>. However, it is often easier to prove the stronger condition.

Interestingly, for bipartite measures monotonicity also implies that there is maximal entanglement in bipartite systems. More precisely, if we fix Hilbert space  $C^d \otimes C^{d'}$ , then there exist states from which any other state can be created: these are states  $U_A \otimes U_B$  equivalent to singlet. Indeed, by teleportation, Alice and Bob can create from singlet any pure bipartite state. Namely, Alice prepares locally two systems in a joint state  $\psi$ , and one of systems teleports through singlet. In this way Alice and Bob share the state  $\psi$ . Then they can also prepare any mixed state. Thus E must take the greatest value to the state  $\phi^+$ .

Sometimes, one considers monotonicity under LOCC operations for which the output system has the same local dimension as the input system. For example, for n-qubit states, we are interested only in output n-qubit states. Measures that satisfy such monotonicity can be useful in many contexts, and sometimes it is easier to prove such monotonicity (Verstraete *et al.*, 2003), (see section XV.H.1).

<sup>&</sup>lt;sup>64</sup> For more extensive treatment of various classes of operations, see Sec. XI.B.

<sup>&</sup>lt;sup>65</sup> Indeed, the condition (150) is more fundamental, as it tells about entanglement of *state*, while (152) says about average entanglement of an ensemble (family  $\{p_i, \rho_i\}$ ), which is less operational notion than notion of state. Indeed it is not quite clear what does it mean to "have an ensemble". Ensemble can always be treated as a state  $\sum_i p_i |i\rangle \langle i| \otimes \rho_i$ , where  $|i\rangle$  are local orthogonal flags. However it is not clear at all why one should require a priori that  $E(\sum_i p_i |i\rangle \langle i| \otimes \rho_i) = \sum_i p_i E(\rho_i)$ .

## 2. Vanishing on separable states.

If a function E satisfies the monotonicity axiom, it turns out that it is constant on separable states. It follows from the fact that every separable state can be converted to any other separable state by LOCC (Vidal, 2000). Even more, E must be *minimal* on separable states, because any separable state can be obtained by LOCC from any other state. It is reasonable to set this constant to zero. In this way we arrive at even more basic axiom, which can be formulated already on qualitative level:

#### • Entanglement vanishes on separable states

It is quite interesting, that the LOCC monotonicity axiom almost imposes the latter axiom. Note also that those two axioms impose E to be a nonnegative function.

### 3. Other possible postulates.

The above two axioms are essentially the only ones that should be necessarily required from entanglement measures. However there are other properties, that may be useful, and are natural in some context. *Normalization*. First of all we can require that entanglement measure behaves in an "information theoretic way" on maximally entangled states, i.e. it counts e-bits:

$$E((\Phi_2^+)^{\otimes n}) = n.$$
 (153)

A slightly stronger condition would be  $E(\Phi_d^+) = \log d$ . For multipartite entanglement, there is no such a natural condition, due to nonexistence of maximally entangled state.

Asymptotic continuity. Second, one can require some type of continuity. The asymptotic manipulations paradigm suggest continuity of the form (Donald *et al.*, 2002; Horodecki *et al.*, 2000b; Vidal, 2000):

$$\|\rho_n - \sigma_n\|_1 \to 0 \quad \Rightarrow \quad \frac{|E(\rho_n) - E(\sigma_n)|}{\log d_n} \to 0, \quad (154)$$

for states  $\sigma_n$ ,  $\rho_n$  acting on Hilbert space  $\mathcal{H}_n$  of dimension  $d_n$ . This is called *asymptotic continuity*. Measures which satisfy this postulate, are very useful in estimating  $E_D$ , and other transition rates, via inequality (210) (Sec. XV.E.2). The most prominent example of importance of asymptotic continuity is that together with the above normalization and additivity is enough to obtain a *unique* measure of entanglement for pure states (see Sec. XV.E.1).

*Convexity.* Finally, entanglement measures are often convex. Convexity used to be considered as a mandatory ingredient of the mathematical formulation of monotonicity. At present we consider convexity as merely a convenient mathematical property. Most of known measures are convex, including relative entropy of entanglement, entanglement of formation, robustness of entanglement, negativity and all measures constructed by means of convex roof (see Sec. XV.C). It is open question whether distillable entanglement is convex (Shor *et al.*, 2001). In multipartite setting it is known, that a version of distillable entanglement<sup>66</sup> is not convex (Shor *et al.*, 2003).

#### 4. Monotonicity for pure states.

For many applications, it is important to know whether a given function is a good measure of entanglement just on pure states, i.e. if the initial state is pure, and the final state (after applying LOCC operation) is pure.

Since by Nielsen theorem (sec. XIII.A) for any pair of states  $\phi$  and  $\psi$ , one can transform  $\phi$  into  $\psi$  iff  $\phi$  is majorized by  $\phi$ , one finds, that a function E is monotone on pure states if and only if it is Schur-concave as a function of spectrum of subsystem. I.e. whenever x majorizes y we have  $E(x) \leq E(y)$ .

Again, it is often convenient to consider a stronger monotonicity condition (152) involving transitions from pure states to ensemble of pure states. Namely, it is then enough to require, that any *local* measurement will not increase entanglement *on average* (cf. (Linden *et al.*, 1999a)). We obtain condition

$$E(\psi) \ge \sum_{i} p_i E(\psi_i), \tag{155}$$

where  $\psi_i$  are obtained from local operation

$$\psi_i = \frac{V_i \psi}{\|V_i \psi\|}.\tag{156}$$

 $V_i = A_i \otimes I$  (or  $V_i = I \otimes B_i$ ) are Kraus operators of local measurement (satisfying  $\sum_i V_i^{\dagger} V_i = I$ ) and  $p_i$ are probabilities of outcomes. The inequality should be satisfied for both Alice and Bob measurements. Recall, that the notion of "measurement" includes also unitary operations, which are measurements with single outcomes. The condition obviously generalizes to multipartite states.

### 5. Monotonicity for convex functions

For convex entanglement measures, the strong monotonicity under LOCC of Eq. (152) has been set in a simple form by Vidal (Vidal, 2000), so that the main difficulty — lack of concise mathematical description of LOCC operations — has been overcome. Namely, a convex function f is LOCC monotone in the strong sense (152) if and only if it does not increase under

<sup>&</sup>lt;sup>66</sup> E.g. maximal amount of EPR pairs between two chosen parties, that can be distilled with help of all parties.

$$f(\rho_{AB} \otimes \sigma_X) \le f(\rho_{AB}), \quad X = A', B',$$
 (157)

b) local partial trace

$$f(\rho_{AB}) \le f(\rho_{ABX}),\tag{158}$$

- c) local unitary transformations
- d) local von Neumann measurements (not necessarily complete),

$$f(\rho_{AB}) \ge \sum_{i} p_i f(\sigma_{AB}^i), \tag{159}$$

where  $\sigma_{AB}^{i}$  is state after obtaining outcome *i*, and  $p_{i}$  is probability of such outcome

Thus it is enough to check monotonicity under uni-local operations (operations that are performed only on single site, as it was in the case of pure states). For a convex function, all those conditions are equivalent to a single one, so that we obtain a compact condition:

For convex functions, monotonicity (152) is equivalent to the following condition

$$\sum_{i} p_i E(\sigma_i) \le E(\rho), \tag{160}$$

where the inequality holds for  $\sigma_i = \frac{1}{p_i} W_A^i \otimes I_B \rho W_A^{i\dagger} \otimes I_B$ , and  $p_i = \text{Tr}(W_A^i \otimes I_B \rho W_A^{i\dagger} \otimes I_B)$ , with  $\sum_i W_A^{i\dagger} W_A^i = I_A$ , and the same for B (with obvious generalization to many parties).

Convexity allows for yet another, very simple formulation of monotonicity. Namely, for convex functions strong monotonicity can be phrased in terms of two *equalities* (Horodecki, 2005). Namely, a nonnegative convex function E is LOCC monotone (in the sense of inequality (152)), if and only if

- [LUI] E is invariant under local unitary transformations
- [FLAGS] for any chosen party X, E satisfies equality

$$E\left(\sum_{i} p_{i} \rho^{i} \otimes |i\rangle_{X} \langle i|\right) = \sum_{i} p_{i} E(\rho_{i}), \qquad (161)$$

where  $|i\rangle_X$  are local orthogonal flags.

#### 6. Invariance under local unitary transformations

Measure which satisfies monotonicity condition, is invariant under local unitary transformations

$$E(\rho) = E(U_1 \otimes \dots U_N \rho U_1^{\dagger} \otimes \dots U_N^{\dagger}).$$
(162)

Indeed these operations are particular cases of LOCC operations, and are reversible. Thus monotonicity requires that E does not change under those operations. This condition is usually first checked for candidate for entanglement measures (especially if it is difficult to prove monotonicity condition).

## C. Axiomatic measures — a survey

Here we will review bipartite entanglement measures built on axiomatic basis. Some of them immediately generalize to multipartite case. Multipartite entanglement measures we will present in Sec. XV.H

### 1. Entanglement measures based on distance

A class of entanglement measures (Vedral and Plenio, 1998; Vedral *et al.*, 1997b) are based on the natural intuition, that the closer the state is to the set of separable states, the less entangled it is. The measure is minimum distance  $\mathcal{D}^{67}$  between the given state and the states in  $\mathcal{S}$ :

$$E_{\mathcal{D},\mathcal{S}}(\varrho) = \inf_{\sigma \in S} \mathcal{D}(\varrho, \sigma).$$
(163)

The set S is chosen to be closed under LOCC operations. Originally it was just the set of separable states S. It turns out that such function is monotonous under LOCC, if distance measure is monotonous under *all* operations. It is then possible to use known, but so far unrelated, results from literature on monotonicity under completely positive maps. Moreover, it proves that it is not only a technical assumption to generate entanglement measures: monotonicity is a condition for a distance to be a measure of *distinguishability* of quantum states (Fuchs and van de Graaf, 1997; Vedral *et al.*, 1997a).

We thus require that

$$\mathcal{D}(\rho, \sigma) \ge \mathcal{D}(\Lambda(\rho), \Lambda(\sigma)) \tag{164}$$

and obviously  $\mathcal{D}(\rho, \sigma) = 0$  for  $\rho = \sigma$ . This implies nonnegativity of  $\mathcal{D}$  (similarly as it was in the case of vanishing of entanglement on separable states). More importantly, the above condition immediately implies monotonicity (150) of the measure  $E_{\mathcal{D},\mathcal{S}}$ . To obtain stronger monotonicity, one requires  $\sum_i p_i \mathcal{D}(\varrho_i, \sigma_i) \leq \mathcal{D}(\varrho, \sigma)$ . for ensembles  $\{p_i, \varrho_i\}$  and  $\{q_i, \sigma_i\}$  obtained from  $\rho$  and  $\sigma$  by applying an operation.

Once good distance was chosen, one can consider different measures by changing the sets closed under LOCC operations. In this way we obtain  $E_{\mathcal{D},PPT}$  (Rains, 2000) or  $E_{\mathcal{D},ND}$  (the distance from nondistillable states). The measure involving set PPT is much easier to evaluate. The greater the set (see fig. 3), the smaller the measure is, so that if we consider the set of separable states, those with positive partial transpose and the set of nondistillable states, we have

$$E_{\mathcal{D},ND} \le E_{\mathcal{D},PPT} \le E_{\mathcal{D},S}.$$
 (165)

In Ref. (Vedral and Plenio, 1998) two distances were shown to satisfy (164) and convexity: square of Bures metric  $B^2 = 2 - 2\sqrt{F(\varrho, \sigma)}$  where  $F(\varrho, \sigma) =$ 

<sup>&</sup>lt;sup>67</sup> We do not require the distance to be a metric.

 $[\text{Tr}(\sqrt{\rho}\sigma\sqrt{\rho})^{1/2}]^2$  is fidelity (Jozsa, 1994; Uhlmann, 1976) and relative entropy  $S(\rho|\sigma) = \text{Tr}\rho(\log \rho - \log \sigma)$ . Originally, the set of separable states was used and the resulting measure

$$E_R = \inf_{\sigma \in \text{SEP}} \operatorname{Tr} \varrho(\log \varrho - \log \sigma)$$
(166)

is called *relative entropy* of *entanglement*. It is one of the fundamental entanglement measures, as the relative entropy is one of the most important functions in quantum information theory (see (Schumacher and Westmoreland, 2000; Vedral, 2002)). Its other versions - the relative entropy distance from PPT states (Rains, 2000) and from nondistillable states (Vedral, 1998) - will be denoted as  $E_R^{PPT}$ , and  $E_R^{ND}$  respectively. Relative entropy of entanglement (all its versions) turned out to be powerful upper bound for entanglement of distillation (Rains, 2000). The distance based on fidelity received interpretation in terms of Grover algorithm (Shapira *et al.*, 2005).

## 2. Convex roof measures

Here we consider the following method of obtaining entanglement measures: one starts by imposing a measure E on pure states, and then extends it to mixed ones by convex roof (Uhlmann, 1998)

$$E(\varrho) = \inf \sum_{i} p_i E(\psi_i), \quad \sum_{i} p_i = 1, p_i \ge 0, \quad (167)$$

where the infimum is taken over all ensembles  $\{p_i, \psi_i\}$  for which  $\rho = \sum_{i} p_i |\psi_i\rangle \langle \psi_i |$ . The infimum is reached on a particular ensemble (Uhlmann, 1998). Such ensemble we call optimal. Thus E is equal to average under optimal ensemble.

The first entanglement measure built in this way was entanglement of formation  $E_F$  introduced in (Bennett *et al.*, 1996d), where  $E(\psi)$  is von Neumann entropy of the reduced density matrix of  $\psi$ . It constituted first upper bound for distillable entanglement. In Ref. (Bennett *et al.*, 1996d) monotonicity of  $E_F$  was shown. In Ref. (Vidal, 2000) general proof for monotonicity of all possible convex-roof measures was exhibited. We will recall here the latter proof, in the form (Horodecki, 2001a).

One easily checks that E is convex. Actually, convex roof measures are the largest functions that are (i) convex (ii) compatible with given values for pure states. Now, for convex functions, there is the very simple condition of Eq. (160) equivalent to strong monotonicity. Namely, it is enough to check, whether the measure does not increase on average under local measurement (without coarse graining i.e. where outcomes are given by Kraus operators).

Using this condition, we will show, that if a measure is monotone on pure states (according to Eq. (155)), then its convex roof extension is monotonous on mixed states. Thus the condition (152) is reduced to monotonicity for

pure states. To see it, consider  $\rho$  with optimal ensemble  $\{p_i, \psi_i\}$ . Consider local measurement with Kraus operators  $V_k$ . It transforms initial state  $\rho$  as follows

$$\varrho \to \{q_k, \sigma_k\}, \quad q_k = \text{Tr} V_k \varrho V_k^{\dagger}, \quad \sigma_k = \frac{1}{q_k} V_k \varrho V_k^{\dagger}.$$
(168)

The members of the ensemble  $\{p_i, \psi_i\}$  transform into ensembles of pure states (because operation is pure)

$$\psi_i \to \{q_k^i, \psi_k^i\}, \quad q_k^i = \operatorname{Tr}(V_k |\psi_i\rangle \langle \psi_i | V_k^{\dagger}), \quad \psi_k^i = \frac{1}{\sqrt{q_k^i}} V_k \psi_i$$
(169)

One finds that  $\sigma_k = \frac{1}{q_k} \sum_i p_i q_k^i |\psi_k^i\rangle \langle \psi_k^i|$ . Now we want to show that the initial entanglement  $E(\rho)$  is no less than the final average entanglement  $\overline{E} = \sum_k q_k E(\sigma_k)$ , assuming, that for pure states E is monotonous under the operation. Since  $\sigma_k$  is a mixture of  $\psi_k^i$ 's, then due to convexity of E we have

$$E(\sigma_k) \le \frac{1}{q_k} \sum_i p_i q_k^i E(\psi_k^i).$$
(170)

Thus  $\overline{E} \leq \sum_{i} p_i \sum_{k} q_k^i E(\psi_i^k)$ . Due to monotonicity on pure states  $\sum_{k} q_k^i E(\psi_i^k) \leq E(\psi_i)$ . Thus  $\overline{E} \leq \sum_{i} p_i E(\psi_i)$ . However, the ensemble  $\{p_i, \psi_i\}$  was optimal, so that the latter term is equal simply to  $E(\rho)$ . This ends the proof.

Thus, for any function (strongly) monotonous for pure states, its convex roof is monotonous for all states. As a result, the problem of monotonicity of convex roof measures reduces to pure states case. Let us emphasize that the above proof bases solely on the convex-roof construction, hence is by no means restricted to bipartite systems. In Sec. XV.D we will discuss the question of monotonicity for pure states. In particular we will see that any concave, expansible function of reduced density matrix of  $\psi$  satisfies (155). For example (Vidal, 2000) one can take  $E_{\alpha}(\psi)$  given by Renyi entropy  $\frac{1}{1-\alpha}\log_2 \operatorname{Tr}(\varrho^{\alpha})$  of the reduction for  $0 \leq \alpha \leq \infty$ . For  $\alpha = 1$  it gives  $E_F$ , while for  $\alpha = 0$  — the average logarithm of the number of nonzero Schmidt coefficients. Finally, for  $\alpha = \infty$  we obtain a measure related to the one introduced by Shimony (Shimony, 1995) when theory of entanglement did not exist:

$$E(\psi) = 1 - \sup_{\psi_{prod}} |\langle \psi | \psi_{prod} \rangle|, \qquad (171)$$

where the supremum is taken over all product pure states  $\psi_{prod}$  (see Sec. XV.H.1 for multipartite generalizations of this measure).

We will consider measures for pure bipartite states in more detail in Sec. XV.D, and multipartite in Sec. XV.H.1.

a. Schmidt number The Schmidt rank can be extended to mixed states by means of convex roof. A different extension was considered in (Sanpera *et al.*, 2001; Terhal and Horodecki, 2000) (called Schmidt number) as follows

$$r_S(\varrho) = \min(\max[r_S(\psi_i)]), \qquad (172)$$

where minimum is taken over all decompositions  $\rho = \sum_i p_i |\psi_i\rangle \langle \psi_i|$  and  $r_S(\psi_i)$  are the Schmidt ranks of the corresponding pure states. Thus instead of average Schmidt rank, supremum is taken. An interesting feature of this measure is that its logarithm is strongly nonadditive. Namely there exists a state  $\rho$  such that  $r_S(\rho) = r_S(\rho \otimes \rho)$ .

b. Concurrence For two qubits the measure called concurrence was introduced for pure states in (Hill and Wootters, 1997). Wootters (Wootters, 1998) provided a closed expression for its convex roof extension and basing on it derived computable formula for  $E_F$  in two-qubit case. For pure states concurrence is given by  $C = \sqrt{2(1 - \text{Tr}\rho^2)}$  where  $\rho$  is reduced state. For two qubits this gives  $C(\psi) = 2a_1a_2$  where  $a_1, a_2$  are Schmidt coefficients. Another way of representing C for two qubits is the following

$$C = \langle \psi | \theta | \psi \rangle, \tag{173}$$

where  $\theta$  is antiunitary transformation  $\theta \psi = \sigma_y \otimes \sigma_y \psi^*$ , with \* being complex conjugation in standard basis, and  $\sigma_y$  is Pauli matrix. It turns out that the latter expression for *C* is the most useful in the context of mixed states, for which the convex roof of *C* can be then computed as follows. Let us denote  $\tilde{\rho} = \theta \rho \theta$ , and consider operator

$$\omega = \sqrt{\rho}\sqrt{\tilde{\rho}}.$$
 (174)

Let  $\lambda_1, \ldots, \lambda_4$  be singular values of  $\omega$  in decreasing order. Then we have

$$C(\rho) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}$$
(175)

Interestingly Uhlmann has shown (Uhlmann, 2000) that for any conjugation  $\Theta$ , i.e. antiunitary operator satisfying  $\Theta = \Theta^{-1}$ , the convex roof of the function  $\Theta$ concurrence  $C_{\Theta}(\psi) = \langle \psi | \Theta | \psi \rangle$  is given by generalization of Wootters' formula:

$$C_{\Theta}(\rho) = \max\{0, \lambda_1 - \sum_{i=2}^d \lambda_i\}, \qquad (176)$$

where  $\lambda_i$  are eigenvalues of operator  $\sqrt{\rho}\sqrt{\Theta\rho\Theta}$  in decreasing order.

The importance of the measure stems from the fact that it allows to compute entanglement of formation for two qubits according to formula (Wootters, 1998)

$$E_F(\rho) = H(\frac{1 + \sqrt{1 - C^2(\rho)}}{2}), \qquad (177)$$

where H is binary entropy  $H(x) = -x \log x - (1 - x) \log(1 - x)$ . Another advantage of concurrence is that

it is very simple for pure states. Namely  $C^2$  is a polynomial function of coefficients of a state written in standard basis

$$C(\psi) = 2|a_{00}a_{11} - a_{01}a_{10}|, \qquad (178)$$

for  $\psi = a_{00}|00\rangle + a_{01}|01\rangle + a_{10}|10\rangle + a_{11}|11\rangle$ . One can extend this quantity to higher dimensions (Audenaert *et al.*, 2001c; Rungta *et al.*, 2001) (see also (Badziąg *et al.*, 2002)) by building *concurrence vector*. The elements  $C\alpha$  of the vector are all minors of rank two:

$$C_{\alpha}(\psi) = 2(A_{ij}^{\psi}A_{i'j'}^{\psi} - A_{ij'}^{\psi}A_{i'j}^{\psi})$$
(179)

where  $\psi = \sum_{ij} A_{ij}^{\psi} |ij\rangle$ ,  $\alpha = (ii', jj')$  with i < i', j < j'. So in the case of two qubits, the matrix reduces to a number, as there is only one such minor — the determinant. For  $d_1 \otimes d_2$  system, there are  $\binom{d_1}{2} \times \binom{d_2}{2}$  minors. The norm of such concurrence vector gives rise to concurrence in higher dimension

$$C(\psi) = \sqrt{\sum_{\alpha} C_{\alpha}(\psi)^2} = \sqrt{\langle \psi | \psi \rangle - \text{Tr}\rho^2}, \qquad (180)$$

where  $\rho$  is density matrix of subsystem. Strong algebraic lower bound for *C* was obtained in (Mintert *et al.*, 2004) allowing to detect states which are bound entangled (see also (Mintert *et al.*, 2005a)). Namely, consider ensemble  $\rho = \sum_{m} |\psi_m\rangle\langle\psi_m|$ 

$$C(\rho) = \inf_{\psi_m} \sum_m C(\psi_m) \tag{181}$$

One finds that

$$C(\rho) = \inf \sum_{m} \sqrt{(U \otimes UBU^{\dagger} \otimes U^{\dagger})_{mm,mm}}$$
(182)

where *B* is biconcurrence matrix 75. This matrix can be written as follows  $B_{m\mu,n\nu} = \langle \psi_{AB}^m | \langle \psi_{A'B'}^\mu | P_{AA'}^{(-)} \otimes P_{BB'}^{(-)} | \psi_{AB}^n \rangle | \psi_{A'B'}^\nu \rangle$ , from which one can show that it is positive, hence it can be written in eigendecomposition as  $B = \sum_{\alpha} |\chi_{\alpha}\rangle \langle \chi_{\alpha}|$ , with unnormalized, orthogonal  $\chi_{\alpha}$ . It was shown that

$$C(\rho) \ge \max\{\lambda_1 - \sum_{\lambda_i > 1} \lambda_i, 0\}$$
(183)

where  $\lambda_i$  are singular values of operator  $\sum_{\alpha} z_{\alpha} A^{\chi_{\alpha}}$  put in decreasing order, with  $z_{\alpha}$  being arbitrary chosen complex numbers satisfying  $\sum_{\alpha} |z_{\alpha}|^2$  and  $\chi_{\alpha} = \sum_{ij} A_{ij}^{\chi_{\alpha}} |ij\rangle$ . A very good computable bound is obtained already if the linear combination  $\sum_{\alpha} z_{\alpha} \chi_{\alpha}$  is reduced to a chosen single  $\chi_{\alpha}$ . In the case of two qubits *B* is of rank one, hence we obtain the Wootters result.

There are other interesting measures introduced in (Fan *et al.*, 2003; Sinołęcka *et al.*, 2002) and developed

in (Gour, 2005), which are built by means of polynomials of the Schmidt coefficients  $\lambda$ 's:

$$\tau_{1} = \sum_{i=1}^{d} \lambda_{i} = 1,$$
  

$$\tau_{2} = \sum_{i>j}^{d} \lambda_{i} \lambda_{j},$$
  

$$\tau_{3} = \sum_{i>j>k}^{d} \lambda_{i} \lambda_{j} \lambda_{k},$$
  

$$\vdots$$
  
(184)

The above measures  $\tau_p$  are well defined if the dimension of the Hilbert space d is no smaller than the degree p. For convenience, one can also set  $\tau_p = 0$  for p < d, so that each of the above measures is well defined for all pure states. The functions are generalizations of concurrence and can be thought as higher level concurrences. In particular  $\tau_2$  is square of concurrence. The measures are Schur-concave (i.e. they preserve majorization order), so that by Nielsen theorem (see sec. XIII.A) they satisfy monotonicity.

Due to simplicity of concurrence measure, an interesting quantitative connection has been found with complementarity between visibility and which-path information in interference experiments (Jakob and Bergou, 2003). Interesting generalizations of concurrence were found in multi-partite case (see Sec. XV.H)

## 3. Mixed convex roof measures

One can consider some variation of convex roof method by allowing decompositions of a state into arbitrary states rather than just pure ones (Horodecki *et al.*, 2006b). This allows to produce entanglement measures from other functions that are not entanglement measures themselves. Namely, consider function f which does not increase on average under local measurement (recall, that we mean here generalized measurement, so that it includes unitary transformations). I.e. f satisfies the condition (160), so that if only it were convex, it would be monotone. However, what if the function is not convex? Example of such quantity is quantum mutual information. It is obviously not a monotone, because e.g. it takes different values on separable states.

From such function we could obtain entanglement measure by taking usual convex roof. However, we can also take *mixed convex roof* defined by

$$E(\rho) = \inf \sum_{i} p_i f(\rho_i), \qquad (185)$$

where infimum is taken over all decompositions  $\rho = \sum_{i} p_i \rho_i$ . The new function is already convex, and keeps

the feature of nonincreasing on average under local measurements, so that it is LOCC monotone. If we start with mutual information, we obtain measure interpolating between entanglement of formation and squashed entanglement introduced in (Tucci, 2002) and later independently in (Nagel and Raggio, 2005). Its monotonicity under LOCC was proved later (Horodecki *et al.*, 2006b).

Let us mention, that if initial function f is asymptotically continuous, then both usual convex roof, as well as mixed convex roof are asymptotically continuous too (it was shown in (Synak-Radtke and Horodecki, 2006), generalizing the result of Nielsen (Nielsen, 2000) for  $E_F$ ).

#### 4. Other entanglement measures

a. Maximal teleportation fidelity/maximal singlet fraction. For a state  $\rho$  one can consider fidelity of teleportation of a qudit averaged uniformly over inputs and maximized over trace preserving LOCC operations. Denote it by  $f_{max}$ . It is related to maximal fidelity  $F_{max}$  with  $d \otimes d$  maximally entangled state  $|\Phi^+\rangle = \sum_i \frac{1}{\sqrt{d}} |ii\rangle$  optimized over trace preserving LOCC (Rains, 1999) as follows (Horodecki *et al.*, 1999b)

$$f_{max} = \frac{dF_{max} + 1}{d+1}.$$
 (186)

Both quantities are by construction LOCC monotones. They are not equal to zero on separable states, but are constant on them:  $F_{max}(\rho_{sep}) = 1/d$ . For two qubits, the protocol to obtain  $F_{max}$  is of the following simple form (Verstraete and Verschelde, 2003): Alice applies some filter  $A_i \otimes I$  (see Sec. XI), and tells Bob whether she succeeded or not. If not, then they remove the state and produce some separable state which has overlap  $\frac{1}{d}$  with singlet state.

Another measure was constructed in (Brandao, 2005) by means of activation concept (Horodecki *et al.*, 1999a). It is connected with the maximal fidelity with  $\Psi^+$ , obtainable by means of local filtering, denoted here by  $F_{max}^{(p)}$  (i.e. probabilistic maximal singlet fraction, see sec. XII.I). Namely consider a state  $\sigma$  having some value  $F_{max}^{(p)}(\sigma)$ . With help of some other state  $\rho$  one can obtain better  $F_{max}^{(p)}$ , i.e.  $F_{max}^{(p)}(\rho \otimes \sigma)$  may be larger than  $F_{max}^{(p)}(\sigma)$ . What is rather nontrivial, this may happen even if  $\rho$  is bound entangled, i.e. for which  $F_{max}^{(p)}$  is the same as for separable states. The activation power of a given state can be quantified as follows

$$E^{(d)}(\rho) = \sup_{\sigma} \frac{F_{max}^{(p)}(\rho \otimes \sigma) - F_{max}^{(p)}(\sigma)}{F_{max}^{(p)}(\sigma)}.$$
 (187)

Since Masanes (Masanes, 2005a) showed that any entangled state can activate some other state, i.e. increase its  $F_{max}^{(p)}$ , the quantity  $E^{(d)}$  is nonzero for all entangled states, including all bound entangled states.

b. Robustness measures Robustness of entanglement was introduced in (Vidal and Tarrach, 1999). For a state  $\rho$  consider separable state  $\sigma_{sep}$ . Then  $R(\rho|\sigma_{sep})$  is defined as minimal t such that the state

$$\frac{1}{1+t}(\rho + t\sigma_{sep}) \tag{188}$$

is separable. Now robustness of entanglement is defined as

$$R(\rho) = \inf_{\sigma_{sep}} R(\rho | \sigma_{sep}).$$
(189)

It is related to the quantity  $P(\rho)$  given by minimal p such that the state

$$(1-p)\rho + p\sigma_{sep} \tag{190}$$

is separable. We have P = R/(1+R). Though P is more intuitive, it turns out that R has better mathematical properties, being e.g. convex. R satisfies monotonicity (152). In (Harrow and Nielsen, 2003; Steiner, 2003) generalized robustness  $R_g$  was considered, where the infimum is taken over all states rather than just separable ones. Interestingly it was shown that for pure states it does not make difference.  $R_g$  is monotone too. Brandao (Brandao, 2005) showed that the generalized robustness  $R_g$  has operational interpretation: it is just equal to the measure  $E^{(d)}$  quantifying the activation power. Moreover (Brandao, 2005)  $R_g$  gives rise to the following upper bound for  $E_D$ :

$$E_D(\rho) \le \log_2(1 + R_g(\rho)).$$
 (191)

c. Best separable approximation measure was introduced in (Karnas and Lewenstein, 2000) by use of best separable approximation idea (Lewenstein and Sanpera, 1998). The latter is defined as follows: one decomposes state  $\rho$ as a mixture

$$\rho = (1 - p)\delta\rho + p\sigma_{sep},\tag{192}$$

where  $\sigma_{sep}$  is separable state, and  $\delta \rho$  is arbitrary state. Denote by  $p^*$  the maximal possible p. Now, it turns out that  $E_{bsa}(\rho) = 1 - p^*$  is an entanglement measure, i.e. it is LOCC monotone and vanishes on separable states. For all pure entangled states the measure is equal to 1.

d. Witnessed entanglement In (Brandao, 2005; Brandao and Vianna, 2006) entanglement measures are constructed using entanglement witnesses as follows:

$$E = -\inf_{W} \operatorname{Tr} \rho W, \tag{193}$$

where infimum is taken over some set of entanglement witnesses. It turns out that many measures can be recast in this form. For example random robustness is equal to  $-\inf_{W:\operatorname{Tr}W=1}(\operatorname{Tr}\rho W), R_g = -\inf_{W \leq \mathrm{I}}(-\operatorname{Tr}\rho W), E_{bsa} = -\inf_{W \geq \mathrm{I}}(\operatorname{Tr}\rho W).$ 

Interestingly, the singlet fraction maximized over PPT operations (Rains, 2000) can be represented as:

$$F_{max}^{ppt} = \frac{1}{d} - \inf_{W} \text{Tr}W\rho, \qquad (194)$$

where the infimum is taken over the set  $\{W : \frac{1}{(1-d)}I \leq W \leq \frac{1}{d}I, 0 \leq W^{\Gamma} \leq 2\frac{1}{d}I\}.$ 

e. Negativity. A simple computable measure was introduced in (Życzkowski *et al.*, 1998) and then shown in (Vidal and Werner, 2002) to be LOCC monotone. It is negativity

$$\mathcal{N} = \sum_{\lambda < 0} \lambda \tag{195}$$

where  $\lambda$  are eigenvalues of  $\rho^{\Gamma}$  (where  $\Gamma$  is partial transpose). A version of the measure called logarithmic negativity given by

$$E_{\mathcal{N}}(\rho) = \log \|\rho^{\Gamma}\|_1 \tag{196}$$

isupper bound for distillable entanglement (Vidal and Werner, 2002). It can be also written as  $E_{\mathcal{N}}(\rho) = \log \frac{2\mathcal{N}(\rho)+1}{2}$ . The measure  $E_{\mathcal{N}}(\rho)$  is easily seen to satisfy monotonicity (150), because  $\mathcal{N}(\rho)$  does satisfy it, and logarithm is monotonic function. However logarithm it is not convex, and as such might be expected not to satisfy the stronger monotonicity condition (152). However it was recently shown that it does satisfy it (Plenio, 2005). The measure is moreover additive. For states with positive  $|\rho^{\Gamma}|^{\Gamma} E_{\mathcal{N}}$  has operational interpretation — it is equal to exact entanglement cost of creating state by PPT operations from singlets (Audenaert et al., 2003).

It turns out that negativity, robustness, BSA can be also obtained from one scheme originating form base norm (Plenio and Virmani, 2006; Vidal and Werner, 2002)

f. Greatest cross norm. In Ref. (Rudolph, 2001) a measure based on the so called greatest cross-norm was proposed. One decomposes  $\rho$  into sum of product operators

$$\rho = \sum_{i} A_i \otimes B_i. \tag{197}$$

Then the measure is given by

$$E(\rho) = \sup \sum_{i} \|A_i\|_1 \cdot \|B_i\|_1, \quad (198)$$

where supremum is taken over all decompositions (197). It is not known if the function satisfies monotonicity (in (Rudolph, 2001) monotonicity under local operations and

convexity was shown). However it was shown (Rudolph, 2005) that if in infimum one restricts to Hermitian operators, then it is equal to 2R + 1, where R is robustness of entanglement.

g. Rains bound Rains (Rains, 2000) has combined two different concepts (relative entropy of entanglement and negativity) to give the following quantity

$$E_{R+\mathcal{N}} = \inf_{\sigma} \left( S(\varrho|\sigma) + \|\sigma^{\Gamma}\|_1 \right), \tag{199}$$

where the infimum is taken under the set of *all* states. It is the best known upper bound on distillable entanglement. However it turns out to be an entanglement measure itself. One can show that the measure satisfies monotonicity (152). To see this, consider optimal  $\sigma$  (i.e. reaching infimum), the existence of which is assured by continuity of norm and lower semi-continuity of relative entropy ((Ohya and Petz, 1993)). For such a  $\sigma$  by monotonicity of relative entropy under any quantum operation and monotonicity of  $E_{\mathcal{N}}$  under LOCC operation we have for any LOCC map  $\Lambda$ 

$$S(\varrho|\sigma) + \|\sigma^{\Gamma}\|_{1} \ge S(\Lambda(\varrho)|\Lambda(\sigma)) + \|\Lambda(\sigma)^{\Gamma}\|_{1}.$$
 (200)

Hence the infimum can not be increased under LOCC.

h. Squashed entanglement. Squashed entanglement was introduced by (Tucci, 2002) and then independently by Christandl and Winter (Christandl and Winter, 2003), who showed that it is monotone, and proved its additivity. It is therefore the first additive measure with good asymptotic properties. In the latter paper, definition of squashed entanglement  $E_{sq}$  has been inspired by relations between cryptography and entanglement. Namely,  $E_{sq}$  was designed on the basis of a quantity called *intrin*sic information (Gisin and Wolf, 2000; Maurer and Wolf, 1997; Renner and Wolf, 2003), which was monotonic under local operations and public communication. The squashed entanglement is given by

$$E_{sq}(\rho_{AB}) = \inf_{\rho_{ABE}} \frac{1}{2} I(A:B|E)$$
 (201)

where  $I(A:B|E) = S_{AE} + S_{BE} - S_E - S_{ABE}$  and infimum is taken over all density matrices  $\rho_{ABE}$  satisfying  $\text{Tr}_E \rho_{ABE} = \rho_{AB}$ . The measure is additive on tensor product and superadditive in general, i.e.

$$E_{sq}(\rho_{AB} \otimes \rho_{A'B'}) = E_{sq}(\rho_{AB}) + E_{sq}(\rho_{A'B'});$$
  
$$E_{sq}(\rho_{AA'BB'}) \ge E_{sq}(\rho_{AB}) + E_{sq}(\rho_{A'B'}). (202)$$

It is not known whether it vanishes if and only if the state is separable. (It would be true, if infimum could be turned into minimum, this is however unknown). The measure is asymptotically continuous (Alicki and Fannes, 2004), and therefore lies between  $E_D$  and  $E_C$ . Even though it was computed for just two families of states, a clever guess for  $\rho_{ABE}$  can give very good estimates for  $E_D$  in some cases.

It is an open question if the optimization in the definition of  $E_{sq}$  can be restricted to the subsystem E being a classical register (Tucci, 2002). If it is so, then we could express  $E_{sq}$  as follows:

$$E_{sq} \stackrel{?}{=} \inf \sum_{i} p_i I_M(\rho_{AB}^i), \qquad (203)$$

where the infimum is taken over all decompositions  $\sum_i p_i \rho_{AB}^i = \rho_{AB}$ . (This measure was called c-squashed entanglement, because it can be recast as a version of  $E_{sq}$  with E being a classical register).  $E_{sq}$  would be then nothing else but a mixed convex roof measure based on mutual information mentioned above, which is an entanglement measure itself.

*i.* Conditioning entanglement It is worth to mention that there is a method to obtain a new entanglement measure from a given one as follows (Yang *et al.*, 2007):

$$CE(\rho_{AB}) = \inf_{\rho_{AA'BB'}} (E(\rho_{AA'BB'}) - E(\rho_{A'B'})), \quad (204)$$

where infimum is taken over all states  $\rho_{AA'BB'}$  such that  $\operatorname{Tr}_{A'B'}\rho_{AA'BB'} = \rho_{A'B'}$ . Provided the initial measure was convex and satisfied strong monotonicity, the same holds for the new measure. Moreover it is automatically superadditive, and is a lower bound for regularization  $E^{\infty}$  of the original measure.

## D. All measures for pure bipartite states

In Ref. (Vidal, 2000) it was shown that measures for pure states satisfying strong monotonicity (152) are in one-to-one correspondence to functions f of density matrices satisfying

- (i) f is symmetric, expansible function of eigenvalues of  $\rho$
- (ii) f is concave function of  $\rho$

(by expansibility we mean  $f(x_1, \ldots, x_k, 0) = f(x_1, \ldots, x_k)$ ). In this way all possible entanglement measures for pure states were characterized.

More precisely, let  $E_p$ , defined for pure states, satisfy  $E_p(\psi) = f(\varrho_A)$ , where  $\varrho_A$  is reduction of  $\psi$ , and f satisfies (i) and (ii). Then there exists an entanglement measure E satisfying LOCC monotonicity coinciding with  $E_p$  on pure states (E is convex-roof extension of  $E_p$ ). Also conversely, if we have arbitrary measure E satisfying (152), then  $E(\psi) = f(\varrho_A)$  for some f satisfying (i) and (ii).

We will recall the proof of the direct part. Namely, we will show that convex-roof extension E of  $E_p$  satisfies (152). As mentioned earlier, it suffices to show it for pure

states. Consider then any operation on, say, Alice side (for Bob's one, the proof is the same) which produces ensemble  $\{p_i, \psi_i\}$  out of state  $\psi$ . We want to show that the final average entanglement  $\overline{E} = \sum_i p_i E(\psi_i)$  does not exceed the initial entanglement  $E(\psi)$ . In other words, we need to show  $\sum_i p_i f(\varrho_A^{(i)}) \leq f(\varrho_A)$ , where  $\varrho_A^{(i)}$  are reductions of  $\psi_i$  on Alice's side. We note that due to Schmidt decomposition of  $\psi$ , reductions  $\varrho_A$  and  $\varrho_B$  have the same non-zero eigenvalues. Thus,  $f(\varrho_A) = f(\varrho_B)$ , due to (i). Similarly  $f(\varrho_A^{(i)}) = f(\varrho_B^{(i)})$ . Thus it remains to show that  $\sum_i p_i f(\varrho_B^{(i)}) \leq f(\varrho_B)$ . However  $\varrho_B = \sum_i p_i \varrho_B^{(i)}$  (which is algebraic fact, but can be understood as no-superluminalsignaling condition — no action on Alice side can influence statistics on Bob's side, provided no message was transmitted from Alice to Bob). Thus our question reduces to the inequality  $\sum_i p_i f(\varrho_B^{(i)}) \leq f(\sum_i p_i \varrho_B^{(i)})$ . This is however true, due to concavity of f.

As we have mentioned, examples of entanglement measures for pure states are quantum Renyi entropies of subsystem for  $0 \le \alpha \le 1$ . Interestingly, the Renyi entropies for  $\alpha > 1$  are not concave, but are Schur concave. Thus they are satisfy monotonicity (150) for pure states, but do not satisfy the strong one (152). It is not known whether the convex roof construction will work, therefore it is open question how to extend such measure to mixed states.

Historically first measure was the von Neumann entropy of subsystem (i.e.  $\alpha = 1$ ) which has operational interpretation — it is equal to distillable entanglement and entanglement cost. It is the unique measure for pure states, if we require some additional postulates, especially asymptotic continuity (see Sec. XV.E).

## 1. Entanglement measures and transition between states — exact case

Another family of entanglement measures is the following. Consider squares of Schmidt coefficients of a pure state  $\lambda_k$  in decreasing order so that  $\lambda_1 \geq \lambda_2 \geq \ldots \lambda_d$ ,  $\sum_{i=1}^d \lambda_i = 1$ . Then the sum of the d-k smallest  $\lambda$ 's

$$E_k(\psi) = \sum_{i=k}^d \lambda_i \tag{205}$$

is an entanglement monotone (Vidal, 1999). Thus for a state with n nonzero Schmidt coefficients we obtain n-1 nontrivial entanglement measures. It turns out that these measures constitute in a sense a complete set of entanglement measures for bipartite pure states.

Vidal proved the following inequality relating probability  $p(\psi \rightarrow \phi)$  of obtaining state  $\phi$  from  $\psi$  by LOCC with entanglement measures:

$$p(\psi \to \phi) \le \frac{E(\psi)}{E(\phi)}.$$
(206)

If we fix an entangled state  $\phi_0$ , then  $p(\psi \to \phi_0)$  is itself a measure of entanglement, as a function of  $\psi$ . The relation (206) gives in particular

$$p(\psi \to \phi) \le \frac{E_k(\psi)}{E_k(\phi)} \tag{207}$$

for all k. It turns out that these are the only constraints for transition probabilities. Namely (Vidal, 1999) The optimal probability of transition from state  $\psi$  to  $\phi$  is given by

$$p(\psi \to \phi) = \min_{k} \frac{E_k(\psi)}{E_k(\phi)}.$$
 (208)

This returns, in particular, Nielsen's result (Nielsen, 1999). Namely, p = 1, if for all k,  $E_k(\psi) \ge E_k(\phi)$ , which is precisely the majorization condition (120).

Thus the considered set of abstractly defined measures determines possible transformations between states, as well as optimal probabilities of such transformations. We will further see generalization of such result to asymptotic regime, where nonexact transitions are investigated.

## E. Entanglement measures and transition between states — asymptotic case

In asymptotic regime, where we tolerate small inaccuracies, which disappear in the limit of large number of copies, the landscape of entanglement looks more "smooth". Out of many measures for pure states only one becomes relevant: entropy of entanglement, i.e. any measure significant for this regime reduces to entropy of entanglement for pure states. <sup>68</sup> Moreover, only the measures with some properties, such as asymptotic continuity, can be related to operational quantities such as  $E_D$ , or more generally, to asymptotic transitions rates. We refer to such measures as good asymptotic measures.

## 1. $E_{\it D}$ and $E_{\it C}$ as extremal measures. Unique measure for pure bipartite states.

If measures satisfy some properties, it turns out that their regularizations are bounded by  $E_D$  from one side and by  $E_C$  from the other side. By regularization of any function f we mean  $f^{\infty}(\rho) = \lim_n \frac{1}{n} f(\rho^{\otimes n})$  if such a function exists. It turns out that if a function E is monotonic under LOCC, asymptotically continuous and satisfies  $E(\psi_d^+) = \log d$ , then we have

$$E_D \le E^\infty \le E_C. \tag{209}$$

<sup>&</sup>lt;sup>68</sup> One can consider half-asymptotic regime, where one takes limit of many copies, but does not allow for inaccuracies. Then other measures can still be of use, such as logarithmic negativity which is related to PPT-cost of entanglement (Audenaert *et al.*, 2003) in such regime.

In particular this implies that for pure states, there is unique entanglement measure in the sense that regularization of any possible entanglement measure is equal to entropy of subsystem<sup>69</sup>. An exemplary measure that fits this scheme is relative entropy of entanglement (related either to the set of separable states or to PPT states). Thus whenever we have reversibility, then the transition rate is determined by relative entropy of entanglement. Some versions of the theorem are useful to find upper bounds for  $E_D$  - one of central quantities in entanglement theory. We have for example that any function Esatisfying the following conditions:

- 1. E is weakly subadditive, i.e.  $E(\rho^{\otimes n}) \leq nE(\rho)$ ,
- 2. For isotropic states  $\frac{E(\rho_F^d)}{\log d} \to 1$  for  $F \to 1, d \to \infty$ ,
- E is monotonic under LOCC (i.e. it satisfies Eq. (150)),

is an upper bound for distillable entanglement. This theorem covers all known bounds for  $E_D$ .

There are not many measures which fit the asymptotic regime. apart from operational measures such as  $E_C$ ,  $E_D$  and  $K_D$  only entanglement of formation, relative entropy of entanglement (together with its PPT version) and squashed entanglement belong here. For review of properties of those measures see (Christandl, 2006).

## 2. Transition rates

One can consider transitions between any two states (Bennett *et al.*, 1996d) by means of LOCC:  $R(\rho \rightarrow \sigma)$ defined analogously to  $E_D$ , but with  $\sigma$  in place of maximally entangled state. Thus we consider *n* copies of  $\rho$ and want to obtain a state  $\sigma_n^{out}$  that will converge *m* copies of  $\sigma$  for large *n*.  $R(\rho \rightarrow \sigma)$  is then defined as maximum asymptotic rate m/n that can be achieved by LOCC operations. One then can generalize the theorem about extreme measures as follows:

$$R(\rho \to \sigma) \le \frac{E^{\infty}(\rho)}{E^{\infty}(\sigma)} \tag{210}$$

for any E satisfying:

- 1. E is nonincreasing under LOCC,
- 2. regularizations exist for states  $\rho$  and  $\sigma$  and  $E^{\infty}(\sigma) > 0$ ,
- 3. E is asymptotically continuous (see Sec. XV.B.3).

This result is an asymptotic counterpart of Vidal's relation between optimal probability of success and entanglement measures (206). Noting that, in particular, we have

$$R(\rho \to \psi_2^+) = E_D(\rho), \quad R(\psi_2^+ \to \rho) = \frac{1}{E_C(\rho)},$$
 (211)

one easily arrives at extreme measures theorem. For example to obtain bounds for transition rates between maximally correlated we can choose measures  $E_R$  and  $E_F$ , as they satisfy the conditions and are additive for those states. For more sophisticated transitions see (Horodecki *et al.*, 2003b). One notes that R gives rise to plethora entanglement measures, since the following functions

$$E^D_{\sigma}(\rho) = R(\rho \to \sigma), \quad E^C_{\sigma}(\rho) = \frac{1}{R(\sigma \to \rho)},$$
 (212)

where  $\sigma$  is an entangled state, are nonincreasing under LOCC. (Thanks to the fact that every entangled state have nonzero  $E_C$  the above measures are well defined.)

Upper bound on distillable key. This paradigm allows to provide upper bound  $K_D$  (see Sec. XIX.B.5), because distillable key is also some rate of transition under LOCC operations. Namely, any entanglement measure which is asymptotically continuous, and for so-called private state  $\gamma_d$  satisfies  $E(\gamma_d) \geq \log d$ , then it is upper bound for  $K_D$  (Christandl, 2006). Roughly speaking it follows from the fact, that distillable key can be expressed as rate of transition

$$K_D(\rho) = \sup_{\gamma_d} R(\rho \to \gamma_d) \log d, \qquad (213)$$

where supremum is taken over all p-dits  $\gamma_d$  (see Sec. XIX.B.2).

### F. Evaluating measures

It is usually not easy to evaluate measures. The only measure that is easily computable for any state is  $E_{\mathcal{N}}$ (logarithmic negativity). Entanglement of formation is efficiently computable for two-qubits (Wootters, 1998). Other measures are usually computable for states with high symmetries, such as Werner states, isotropic state, or the family of "iso-Werner" states, see (Bennett *et al.*, 1996d; Rains, 1999, 2001; Terhal and Vollbrecht, 2000; Vollbrecht and Werner, 2001).

An analytical lower bounds for concurrence for all states were provided in (Mintert *et al.*, 2004) (see also (Audenaert *et al.*, 2001c)). The bound constitutes also a new criterion of separability. A way to bound a convex roof measure, is to provide a computable convex function, that is greater than or equal to the measure on the pure states. For example we have

$$\|(|\psi\rangle\langle\psi|)^{\Gamma}\|_{1} = \|R(|\psi\rangle\langle\psi|)\|_{1} = \left(\sum_{i}\sqrt{p_{i}}\right)^{2} \qquad (214)$$

<sup>&</sup>lt;sup>69</sup> Uniqueness of entanglement measure for pure states was put forward in (Popescu and Rohrlich, 1997). The postulates that lead to uniqueness were further worked out in (Donald *et al.*, 2002; Horodecki *et al.*, 2000b; Vidal, 2000).

where  $p_i$  are squares of Schmidt coefficients of  $\psi$ , and R is realignment map VI.B.8. Comparing this with concurrence one gets a bound obtained in (Chen *et al.*, 2005a)

$$C(\rho) \ge \sqrt{\frac{2}{m(m-1)}} (\max(\|\rho^{\Gamma}\|_{1}, \|R(\rho)\|_{1}) - 1). \quad (215)$$

As far as entanglement of formation is concerned, in (Terhal and Vollbrecht, 2000) a method was introduced for which it is enough to optimize over some restricted set rather than the set of pure states. This was further successfully developed in (Chen *et al.*, 2005b; Datta *et al.*, 2006a; Fei and Li-Jost, 2006) where lower bounds for  $E_F$ were obtained based on known separability criteria such as PPT, realignment or the recent Breuer's map (see Secs. VI.B.8 and VI.B.6).

In (Vollbrecht and Werner, 2001) a surprising result was obtained, concerning possible additivity of  $E_R$ . They have shown that  $E_R$  is nonadditive for Werner asymmetric states, and moreover for large d,  $E_R$  of two copies is almost the same as for one copy. Thus, the relative entropy of entanglement can be strongly nonadditive. Therefore regularization of  $E_R$  is not equal to  $E_R$ . In (Audenaert *et al.*, 2001a) for the first time  $E_R^{\infty}$  was computed for some states. Namely for Werner states we have:

$$E_{R,S}^{\infty} = E_{R+N} = \begin{cases} 1 - H(p) & \frac{1}{2} 
(216)$$

Concerning the operational measures, we know that  $E_C = E_F^{\infty} \equiv \lim_{n\to\infty} E_F(\varrho^{\otimes n})$  (Hayden *et al.*, 2001). If  $E_F$  were additive (which is a long-standing problem) then it would be equal to  $E_C$ .  $E_D$  is bounded from above by  $E_F$  (Bennett *et al.*, 1996d). For pure states  $E_D = E_F = E_C = E_R = S(\varrho_A)$  where  $\varrho_A$  is reduced density matrix of the given pure state (Bennett *et al.*, 1996b; Vedral and Plenio, 1998). In (Vidal and Cirac, 2001) it was found that for some bound entangled state (i.e. with  $E_D = 0$ )  $E_C > 0$ . It seems that we can have  $E_D = E_C$  only for states of the form

$$\sum_{i} p_{i} |\psi_{i}\rangle \langle\psi_{i}|_{AB} \otimes |i\rangle \langle i|_{A'B'}$$
(217)

where  $|i\rangle_{A'B'}$  are product states distinguishable by Alice and Bob (Horodecki *et al.*, 1998b) (or some generalizations in similar spirit, that Alice and Bob can distinguish states, which satisfy  $E_D = E_C$  trivially).

Apart from the above trivial case of locally orthogonal mixtures the value of the measure  $E_D$  is known only for maximally correlated states  $\sum_{ij} a_{ij} |ii\rangle \langle jj|$  for which  $E_D = S_A - S_{AB}$ . It is upper bound, since it is equal to  $E_R$  (Rains, 1999). That it can be achieved follows from general result of (Devetak and Winter, 2005) stating that  $E_D \geq S_A - S_{AB}$ . Example is mixture of two maximally entangled two qubit states where we have

$$E_D(\varrho) = 1 - S(\varrho). \tag{218}$$

For higher dimension powerful tools for evaluating  $E_D$ were provided in (Rains, 2000). One knows several upper bounds for  $E_D$  (Bennett *et al.*, 1996d; Horodecki *et al.*, 2000a; Rains, 1999, 2000; Vedral and Plenio, 1998; Vidal and Werner, 2002). The best known bound is  $E_{R+\mathcal{N}}$  provided in (Rains, 2000). For Werner states it is equal to regularization of  $E_{R,\mathcal{S}}$  (this is true for more general class of symmetric states (Audenaert *et al.*, 2001b)).

Squashed entanglement has been evaluated for so called *flower state* (purification of which is given by (272)) and its generalizations in (Christandl and Winter, 2005).

Several entanglement measures have been evaluated for some multipartite pure and mixed graphs states in (Markham *et al.*, 2007). They have used results concerning distinguishing states via LOCC (276), see sec. XVIII.B.

#### 1. Hashing inequality

Any entanglement measure E that is good in asymptotic regime satisfy inequality  $E \geq S(\varrho_B) - S(\varrho_{AB}) \equiv I_{A \setminus B}^{coh}$ . For  $E_F$  the inequality follows from concavity of  $\frac{1}{d} I_{A \setminus B}^{coh}$  (Horodecki *et al.*, 1998b). For  $E_R$  the proof is much 1 more involved (Plenio *et al.*, 2000). In (Horodecki *et al.*, 2000c) the *hashing inequality* was conjectured  $E_D^{\rightarrow} \geq I_{A \setminus B}^{coh}$ , where  $E_D^{\rightarrow}$  is one-way distillable entanglement (classical communication only from Alice to Bob is allowed (Bennett *et al.*, 1996d)). In (Devetak and Winter, 2004b, 2005) the inequality was proven. However, from extreme measures result (see Sec. XV.E.1) it follows that all asymptotic measures of entanglement (in the sense that they satisfy conditions discussed in Sec. XV.E) are upper bounds for  $E_D$ , hence they are also greater than  $I_{A \setminus B}^{coh}$ .

## 2. Evaluating $E_C$ vs additivity problem

It is long open question of whether in general  $E_F$  is additive. Since  $E_D$  is lower bound for  $E_C$ , it follows that for distillable states  $E_C$  is nonzero. Though for bound entangled states (for which  $E_D = 0$ ) this does not give any hint. In (Vidal and Cirac, 2001) the first example of bound entangled state with  $E_C > 0$  was found. Later (Vidal *et al.*, 2002) examples of states with additive  $E_F$ were found.

*Example.* We will now show a simple proof (see (Horodecki *et al.*, 2005d)), that for any state of the form

$$\rho_{AB} = \sum_{ij} a_{ij} |ii\rangle\langle jj| \tag{219}$$

 $E_F$  is additive. Such states are called maximally correlated <sup>70</sup>. Namely, one considers purification  $\psi_{ABC}$  which can be taken as  $\sum_i |ii\rangle_{AB} |\tilde{\psi}_i\rangle_C$  where  $\langle \tilde{\psi}_i | \tilde{\psi}_j \rangle = a_{ij}$ . The state of subsystems *B* and *C* is of the form  $\rho_{BC} =$  $\sum_i p_i |i\rangle \langle i| \otimes |\psi_i\rangle \langle \psi_i|$ , where  $\psi_i$  are normalized vectors  $\tilde{\psi}_i$ and  $p_i = |a_{ii}|^2$ . Then using the fact that  $E_F$  can be also defined as infimum of average entanglement between Alice and Bob over all measurements performed by Charlie on system *C* one finds that

$$E_F = S_B - I_{acc}(\{p_i, \psi_i\})$$
(220)

where  $I_{acc}$  is so called accessible information<sup>71</sup>. Now, following Wootters (see (DiVincenzo *et al.*, 2002)), one can easily show by use of chain rule that  $I_{acc}$  is additive. Since  $S_B$  is additive too, we obtain that  $E_F$  is additive. More generally we have (Koashi and Winter, 2004)

$$E_F(A:B) = S(B) - C_{HV}(C \rangle B)$$
(221)

where  $C_{HV}$  is a measure of classical correlations (Henderson and Vedral, 2001). It is defined as follows:

$$C_{HV}(A\rangle B) = \sup_{\{A_i\}} \left( S(\rho_B) - \sum_i p_i S(\rho_B^i) \right)$$
(222)

where supremum is taken over all measurements  $\{A_i\}$ on system A, and  $p_i$  is probability of outcome i,  $\rho_B^i$  is the state of system B, given the outcome i occurred. In (Devetak and Winter, 2004a) it was shown that for separable states  $C_{HV}$  is additive. This is related to the result by Shor (Shor, 2002b) on additivity of classical capacity for entanglement breaking channels (actually the original results on additivity of  $E_F$  were based on Shor result).

Another result on  $E_C$  was provided in (Yang *et al.*, 2005a). Namely,  $E_C$  is lower bounded by a function G which is (mixed) convex roof of the function  $C_{HV}$ . Such a function is nonzero if and only if a state is entangled. It is worth to present a proof, as it is quite simple, and again uses duality between  $E_F$  and  $C_{HV}$ .

We first consider a pure state  $\psi_{AA'BB'}$ . For this state we have

$$E_F(\psi_{AA':BB'}) = S(AA') = E_F(\rho_{AA':B}) + C_{HV}(B' \rangle AA') \ge$$
  
$$\ge E_F(\rho_{A:B}) + C_{HV}(B' \rangle A')$$
(223)

where  $\rho_{XXX}$  denote suitable partial traces.

Now consider mixed state  $\rho_{AA':BB'}$ . We obtain

$$E_F(\rho_{AA':BB'}) \ge E_F(\rho_{A:B}) + G(B')A'), \qquad (224)$$

where G is mixed convex roof of  $C_{HV}$ . This follows from applying left hand side to optimal decomposition of  $\rho_{AA':BB'}$  into pure states (which gives mixed convex roof on the right hand side — as a result,  $E_F$  remains  $E_F$ , but  $C_{HV}$  changes into G).

Now we apply this last inequality to n copies of the same state  $\rho$  and obtain

$$E_F(\rho^{\otimes (n-1)} \otimes \rho) \ge E_F(\rho^{\otimes (n-1)}) + G(\rho).$$
(225)

Iterating this equation we obtain

$$E_F(\rho^{\otimes n}) \ge E_F(\rho) + (n-1)G(\rho). \tag{226}$$

Dividing both sides by n we obtain  $E_C \ge G(\rho)$ . It remains to prove that G is nonzero for any entangled state. This follows from standard arguments of Caratheodory's type.

#### G. Entanglement imposes different orderings

One can ask, whether different entanglement measures impose the same ordering in set of states. The question was first posed in (Virmani and Plenio, 2000). Namely, suppose that  $E(\rho) \ge E(\sigma)$ . Is it also the case that  $E'(\rho) \ge E'(\sigma)$ ? That it is not the case, we can see just on pure states. There exist incomparable states, i.e. such states  $\psi$ ,  $\phi$ , that neither  $\psi \to \phi$  nor  $\phi \to \psi$  is possible by LOCC. Since LOCC transitions is governed by entanglement measures (see Sec.XV.D) we see that there are two measures which give opposite ordering on those states.

In asymptotic regime there is unique measure for pure states. However, again it is easy to see (Virmani and Plenio, 2000), that a unique order would imply  $E_D = E_C$  for all states, while we know that it is not the case.

On can interpret this lack of single ordering as follows: there are many different types of entanglement, and in one state we have more entanglement of one type, while in other state, there is more entanglement of some other type (See (Miranowicz, 2004b; Verstraete *et al.*, 2004b) in this context).

### 223). Multipartite entanglement measures

Many of the axiomatic measures, are immediately extended to multipartite case. For example relative entropy of entanglement is generalized, by taking a suitable set in place of bipartite separable states. One can take the set of fully separable states (then the measure will not distinguish between "truly multipartite" entanglement and several instances of bipartite entanglement such as  $\phi_{AB}^+ \otimes \phi_{CD}^+$ )<sup>72</sup>. To analyze truly multipartite entanglement, one has to consider as in (Vedral *et al.*,

<sup>&</sup>lt;sup>70</sup> Additivity was shown in (Horodecki *et al.*, 2003b) along the lines of Ref. (Vidal *et al.*, 2002).

<sup>&</sup>lt;sup>71</sup> It is defined as follows:  $I_{acc}(\{p_i, \psi_i\}) = \sup_{\{A_j\}} I(i:j)$  where supremum is taken over all measurements, and I(i:j) is mutual information between symbols *i* and measurement outcomes *j*.

<sup>&</sup>lt;sup>72</sup> Some inequalities between so chosen version of  $E_R$  and bipartite entanglement were provided in (Plenio and Vedral, 2000)

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1997b) the set of all states containing no more than kparticle entanglement (see Sec. VII). Similarly one can proceed with robustness of entanglement. It is not easy however to compute such measures even for pure states (see e.g. (Plenio and Vedral, 2000)). Moreover, for multipartite states, much more parameters to describe entanglement is needed, therefore many new entanglement measures have been designed, especially for pure states. Then they can be extended to all states by convex roof (which is however also hard to compute).

Before we present several such measures, let us make a digression on behavior of k-party entanglement (see Sec. VII) under tensor product. Consider two states,  $|\phi^+\rangle_{AB}|0\rangle_C$  and  $|\phi^+\rangle_{AC}|0\rangle_B$ . Both are 2 entangled: they contain no three-partite entanglement. But if Alice, Bob and Charlie have both states, they can create e.g. *GHZ* state. It thus follows, that the "k-partyness" of entanglement is not closed under tensor product. One of implications is that distillable-*GHZ* entanglement is extremely superadditive: it jumps from zero to 1 upon tensoring the above states.

The above discussion concerns scenario where we tensor systems such a way that the number of parties does not increase: i.e.  $\psi_{ABC} \otimes \psi_{A'B'B'} = \psi_{AA':BB':CC}$ . This way of interpreting tensoring is characteristic for information theoretic applications, being analogous to many uses of channel. It seems that for analysis of physical manybody systems, it is more natural to interpret tensoring as adding new parties:  $\psi_{ABC} \otimes \psi_{DEF} = \psi_{ABCDEF}$ . In such case, "k-partyness" of entanglement is preserved under tensoring.

There is even more subtle notion of "genuinely multipartite entanglement" for pure states, which is defined recursively as follows. An entangled bipartite pure states has always genuine bipartite entanglement. A m-partite pure state has genuine m-partite entanglement if its (m-1)-particle reduced density matrices can be written as mixtures of pure states which do not have genuinely (m-1)-partite entanglement. For three qubits it is 3-tangle, which is nonzero iff the state is genuinely 3-partite entangled. It is then in GHZ class. <sup>73</sup>

#### 1. Multipartite entanglement measures for pure states

There are measures that are simple functions of sums of bipartite entanglement measures. Example is "global entanglement" of (Meyer and Wallach, 2001) which is sum of concurrences between single qubit versus all other qubits. Their monotonicity under LOCC is simply inherited from bipartite measures. The first measure that is neither easy combination of bipartite measures, nor an obvious generalization of such a measure is 3-tangle (or residual tangle) introduced in (Coffman *et al.*, 2000). It is defined as follows

$$\tau(A:B:C) = \tau(A:BC) - \tau(AB) - \tau(AC), \quad (227)$$

where 2-tangles on the right hand side are squares of concurrence (175). 3-tangle is permutationally invariant, even though the definition does not suggest it. It may be though zero for pure states that are 3-entangled (i.e. that are not product with respect to any cut). Example is so called W state. Tangle vanishes on any states that are separable under any cut, and is nonzero for example on *GHZ* state.<sup>74</sup> There are attempts to define a good generalization of tangle for multiqubit systems by means of hyper-determinant (Miyake, 2003) (see below). In (Lohmayer *et al.*, 2006) a convex roof of 3-tangle was computed for mixture of GHZ state and W state orthogonal to it.

Shortly after introducing tangle, a concept of another measure for tripartite states was introduced in the context of asymptotic rate of transitions (Linden *et al.*, 1999a):

$$E(\psi) = E_R(\rho_{AB}) + S(\rho_C) \tag{228}$$

where  $\rho_{AB}, \rho_C$  are reductions of  $\psi_{ABC}$ . The measure allowed to detect truly tripartite entanglement in GHZ state in asymptotic regime (see sec. XIII.B.3). It is easy to see that it is monotonic under Alice and Bob actions. Namely, the term  $E_R(\rho_{AB})$  represents entanglement of  $\rho_{AB}$  (hence does not increase under Alice and Bob actions) and the term  $S(\rho_C)$  represents entanglement of the total state under the cut AB : C (hence does not increase under action of any party). However the first term can be usually increased by Charlie. Still the whole measure is monotone, because this increase is always accompanied by a larger decrease of the term  $S(\rho_C)$ . Thus to increase entanglement between A and B, one has to use up entanglement between AB : C. Equivalently  $E_R(\rho_{A:B})$  goes down under mixing less than the entropy goes up. Any measure having this feature can be put to the above formula, to create a new entanglement measure. However, it is known that entanglement of formation will not work here, because it is lockable, i.e. it can decrease arbitrarily under loss of one bit, i.e. increase entropy by 1 (see section XVIII.A).

One of the first measures designed specifically for multipartite states was Schmidt measure (Eisert and Briegel, 2001). This is minimum of  $\log r$  where r is number of terms in an expansion of the state in product basis. For GHZ this measure is 1, because there are just two

<sup>&</sup>lt;sup>73</sup> Let us emphasize, that there is a different notion of genuine multiparty entanglement in asymptotic domain. There a pure state contains genuinely triparty entanglement if it cannot be reversibly transformed into EPR pairs (see sec. XIII.B.3). It is then not known, whether W state is of this sort or not.

<sup>&</sup>lt;sup>74</sup> In turns out, that if tangles in (227) are replaced by squares of negativities the obtained quantity (after symmetrizing over systems permutation) gives also rise to an entanglement monotone (Ou and Fan, 2007).

terms:  $|000\rangle$  and  $|111\rangle$ . One can show, that for W state it is impossible to write it by means of less than three terms (otherwise it would either belong to GHZ class, or to EPR class). The measure is zero if and only if the state is fully product. Therefore, it cannot distinguish truly multipartite entanglement from bipartite entanglement. However it may be useful in many contexts, see eg. (Mora and Briegel, 2005).

An interesting general class of multipartite entanglement measures was obtained in the context of classification of states via so called *normal forms* (Verstraete *et al.*, 2003). Namely, consider any homogeneous function of the state. Then if it is invariant under determinant one SLOCC i.e. it satisfies

$$f(A_1 \otimes \dots A_n \psi) = f(\psi) \tag{229}$$

for  $A_i$  being square matrices satisfying det  $A_i = 1$ , then it is entanglement monotone in strong sense (152), but under the restriction that the LOCC operation produces output states on the Hilbert space of the same *dimension*. The 3-tangle is example of such a measure. Many measures designed for pure multipartite states like those obtained in (Akhtarshenas, 2005; Miyake, 2003; Osterloh and Siewert, 2005;Wong and Christensen, 2001) are originally defined only for a fixed dimension hence it is simply not possible to check the standard monotonicity (150). However concurrence though initially defined for qubits, can be written in terms of linear entropy of subsystems, being thus well defined for all systems. Therefore there is a hope, that one can arrive at definition independent of dimension for other measures. Then to obtain full monotonicity, one will need in addition to prove, that the measure does not change, if the state is embedded into larger Hilbert spaces of subsystems (equivalently, that the measure does not change under adding local ancilla). However, for four-qubit concurrence of (Wong and Christensen, 2001)  $\langle \psi^* | \sigma_u^4 | \psi \rangle$  its natural generalization was shown to be not monotonous (Demkowicz-Dobrzański et al., 2006) (see below). Of course, even the functions that are only monotonous for fixed dimension are useful quantities in many contexts.

Measures based on hyperdeterminant. Mivake noticed that measures of entanglement such as concurrence and tangle are special cases of hyperdeterminant (Miyake, 2003). Consider for example qubits. For two qubits concurrence is simply modulus of determinant, which is hyperdeterminant of first order. Tangle is hyperdeterminant of second order — a function of tensor with three indices. Though computing hyperdeterminants of higher order than tangle is rather complex, basing just on properties of hyperdeterminant Miyake proved, that hyperdeterminants of higher degree are also entanglement monotones (Miyake, 2004). They describe truly multipartite entanglement, (in a sense, that states such as product of *EPR*'s have zero entanglement). The proof of monotonicity bases on geometric-arithmetic mean, and is closely related to the construction of entanglement measures based

on homogeneous functions described above. Explicit formula for hyperdeterminant for four qubits can be found in (Levay, 2006).

*Geometric measure.* A family of measures have been defined by Barnum and Linden (Barnum and Linden, 2001). In particular so called geometric measure is defined as

$$E_g(\psi) = 1 - \Lambda^k(\psi), \qquad (230)$$

where  $\Lambda^k(\psi) = \sup_{\phi \in S_k} |\langle \psi | \phi \rangle|^2$ , with  $S_k$  being set of k-separable states. This is generalization of Shimony measure (Shimony, 1995), which for bipartite states was related to Renyi entropy with  $\alpha = \infty$ . For relations with robustness of entanglement see (Cavalcanti, 2006). The measure was also investigated in (Wei *et al.*, 2004; Wei and Goldbart, 2003) where it was in particular computed for Smolin four qubit bound entangled states (85).

Concurrence-like measures. There were other attempts to generalize concurrence. Christensen and Wong (Wong and Christensen, 2001) obtained a measure for even number of qubits by exploiting conjugation that appeared in original definition of concurrence for two qubits. Their concurrence works for even number of qubits and is given by  $\langle \psi^* | \sigma_y^n | \psi \rangle$ . The measure is nonzero for a four partite states two pairs of EPR states. This approach was generalized in (Osterloh and Siewert, 2005, 2006) who analysed systematically possible quantities built out of antilinear operations, also of higher order in  $\psi$  than concurrence. For example, they obtained the following nice representation for 3-tangle:

$$\tau = \langle \psi | \sigma_{\mu} \otimes \sigma_{y} \otimes \sigma_{y} | \psi^{*} \rangle \langle \psi | \sigma^{\mu} \otimes \sigma_{y} \otimes \sigma_{y} | \psi^{*} \rangle \quad (231)$$

where  $\mu = 0, 1, 2, 3$  and the contraction is described by tensor  $g^{\mu,\nu} = \text{diag}[-1, 1, 0, 1]$ . They have also designed measures that distinguish between three different SLOCC classes of states (see sec. XIII.A.2):

$$\begin{aligned} |\Phi_{1}\rangle &= \frac{1}{\sqrt{2}} (|0000\rangle + |1111\rangle) \\ |\Phi_{2}\rangle &= \frac{1}{\sqrt{6}} (\sqrt{2}|1111\rangle + |1000\rangle + |0100\rangle + |0010\rangle + |0001\rangle) \\ |\Phi_{3}\rangle &= \frac{1}{2} (|1111\rangle + |1100\rangle + |0010\rangle + |0001\rangle). \end{aligned}$$
(232)

An interesting proposal is due to Akhtarshenas (Akhtarshenas, 2005), which however is not proved to be a monotone. In (Demkowicz-Dobrzański *et al.*, 2006; Mintert *et al.*, 2005b) a family of functions of the form

$$C_{\mathcal{A}}(|\psi\rangle) = 2\sqrt{\langle\psi|\otimes\langle\psi|\mathcal{A}|\psi\rangle\otimes|\psi\rangle}$$
(233)

was introduced, where  $\mathcal{A} = \sum_{s_1...s_n} p_{s_1...s_n} P_{s_1} \otimes \ldots P_{s_n}$ , with  $s_i = \pm 1$ ,  $P^{(\pm 1)}$  is projector onto symmetric (antisymmetric) subspace (see Sec. VI.B.3), and the coefficients  $p_{s_1...s_n}$  are nonnegative. They have given sufficient conditions that must be satisfied by the coefficients to ensure monotonicity of C (now without the restriction of fixed dimension). On the other hand they have shown that if  $\mathcal{A} = P^{(-)} \otimes P^{(-)} \otimes P^{(-)} \otimes P^{(-)}$  then the function returns concurrence  $\langle \psi^* | \sigma_y^4 | \psi \rangle$ , and it is not monotonic. The main tool was the following condition for monotonicity derived on basis of conditions LUI and FLAGS see sec. XV.B.5). Namely a function *C* that

- is real, nonnegative and invariant under local unitaries
- satisfies  $C(a|\psi\rangle) = |a|^2 C(|\psi\rangle)$
- is defined for mixed states as a convex roof

is an entanglement monotone if and only if  $C(a|\psi\rangle \otimes |\eta_1\rangle + b|\phi\rangle \otimes |\eta_1\rangle) \leq |a|^2 C(a|\psi\rangle \otimes |\eta_1\rangle) + bC(|\phi\rangle \otimes |\eta_1\rangle)$ with equality for a = 0 or b = 0, where  $\psi$  and  $\phi$  are arbitrary multipartite pure states, and  $\eta_1$ ,  $\eta_2$  are local orthogonal flags.

Multipartite version of squashed entanglement. We can obtain multipartite versions of both squashed and csquashed entanglement, by putting in place of mutual information of bipartite system, its generalization for multipartite systems (Horodecki *et al.*, 2006b)

$$I(A_1:\ldots:A_N) = S(A_1) + \ldots + S(A_N) - S(A_1\ldots A_N),$$
(234)

choosing normalization factor  $\frac{1}{N}$ , to get 1 on GHZ state, and applying the conditioning rule S(X|Y) = S(XY) - S(Y). So constructed multipartite entanglement measure is closely related to multipartite secrecy monotones introduced in (Cerf *et al.*, 2002b).

In the context of spin chains, another measure of entanglement was designed: localisable entanglement (Verstraete et al., 2004a). Namely, one chooses two spins, and performs LOCC operations aiming at obtaining the largest bipartite entanglement between them (measured according to a chosen entanglement measure for two bipartite states). Localisable entanglement is a generalization of entanglement of assistance, initially defined for tripartite pure states in (DiVincenzo et al., 1998a) as maximal entropy of entanglement that can be created between Alice and Bob, if Charlie helps them by measuring his system and telling the outcomes. For tripartite pure states entanglement of assistance is simply a function of bipartite state  $\rho_{AB}$  of Alice and Bob (however it reflects entanglement properties of joint state, not  $\rho_{AB}$ ). Its formula is dual to entanglement of formation: instead of infimum, there is supremum

$$E_{ass}(\rho) = \sup_{\{p_i,\psi_i\}} \sum_{i} p_i S(\rho_A^i),$$
(235)

where  $\rho_A^i$  is reduced density matrix of  $\psi_i$ , and supremum is taken over all decompositions of  $\rho$ . For example consider GHZ state. Alice and Bob themselves share classically correlated states. However, Charlie can measure in basis  $|+\rangle, |-\rangle$ , and if he tells them result, they obtain EPR pair. So  $E_{ass} = 1$  in this case.

Entanglement of assistance is not a monotone for tripartite states, because it could be increased

by cooperation between Alice, Bob and Charlie (Gour and Spekkens, 2005). It is also nonadditive (DiVincenzo *et al.*, 1998a). However in the limit of many copies, it turns out that it becomes a monotone, namely it reduces to minimum of entropies of Alice's and Bob's subsystem (Smolin *et al.*, 2005), and for larger number of parties to minimum entropy over all cuts that divide Alice from Bob. (This was generalized to N parties in (Horodecki *et al.*, 2005h, 2006e).)

#### I. Entanglement parameters

Some quantities even though are not monotonic under LOCC, seem still useful in quantitative description of entanglement. For example the parameter  $M(\rho)$  (Horodecki *et al.*, 1995) reporting maximal violation of Bell-CHSH inequality for two-qubit states, or  $N(\rho)$  reporting maximal fidelity of teleportation for a class of protocols (Horodecki et al., 1996b) (see Sec. IV). One of the most important quantities in quantum communication theory, coherent information, introduced in (Schumacher and Nielsen, 1996) (see also (Lloyd, 1997)), can be positive only if the state is entangled (Horodecki and Horodecki, 1994). One feels that the greater are such quantities, the more entangled is the state. However those quantities can increase under LOCC. For example coherent information can increase even under local partial trace.

Thus they cannot describe entanglement directly, as it would imply, that entanglement can be increased by means of LOCC. However, it is plausible that the above parameters simply underestimate some measures. Let us consider coherent information in more detail. One can consider maximum value of coherent information attainable by LOCC. This is already an entanglement measure. One can show, that this value does not exceed  $\log d$ , i.e. the value on singlet state. Thus coherent information can only underestimate the value of entanglement measure. Let us note that it is important to know the maximal value of the entanglement measure induced by the given parameter, so that we have a reference point.

## J. How much can entanglement increase under communication of one qubit?

In (Lo and Popescu, 1999) it was postulated that under sending n qubits entanglement shouldn't increase more than by n. In (Chen and Yang, 2000) it was shown for entanglement of formation. Due to teleportation, sending qubits is equivalent to bringing in a singlet. The question can then be recast as follows: which entanglement measures satisfy

$$E(\rho \otimes |\phi^+\rangle \langle \phi^+|) \le E(\rho) + n.$$
(236)

(Of course it is meaningful to ask such questions only for those entanglement measures that exhibit a sort of extensive behavior.) If a measure is subadditive, i.e.  $E(\rho \otimes \sigma) \leq E(\rho) + E(\sigma)$  then the condition is satisfied. This is the case for such measures as  $E_R$ ,  $E_F$ ,  $E_C$ ,  $E_N$ ,  $E_{sq}$ . More problematic are  $E_D$  and  $K_D$ . As far as  $E_D$  is concerned, it is easy to see that (236) is satisfied for distillable states. Simply, if by adding singlet, we can increase  $E_D$ , then we could design protocol that would produce more singlets than  $E_D$ , by using singlets obtained from distilling a first bunch of copies to distillation of the next bunch<sup>75</sup>. A more rigorous argument includes continuity of  $E_D$  on isotropic states (singlets with admixture of random noise). It is not hard to see, that for PPT states the condition also holds (it is actually enough to check it on two qubit singlet):

$$E_D(\rho \otimes |\phi^+\rangle \langle \phi^+|) \le E_N(\rho \otimes |\phi^+\rangle \langle \phi^+|)$$
  
=  $E_N(\rho) + E_N(|\phi^+\rangle \langle \phi^+|).$  (237)

This was later shown <sup>76</sup> for all states, by exploiting results of (DiVincenzo *et al.*, 2003b). The question is still open for  $K_D$ .

## XVI. MONOGAMY OF ENTANGLEMENT

One of the most fundamental properties of entanglement is monogamy (Coffman *et al.*, 2000; Terhal, 2000b). In its extremal form it can be expressed as follows: If two qubits A and B are maximally quantumly correlated they cannot be correlated at all with third qubit C. In general, there is trade-off between the amount of entanglement between qubits A and B and the same qubit A and qubit C. This property is purely quantum: in classical world if A and B bits are perfectly correlated, then there is no constraints on correlations between bits A and C. For three qubits the trade-off is described by Coffman-Kundu-Wootters monogamy inequality

$$C_{A:B}^2 + C_{A:C}^2 \le C_{A:BC}^2, \tag{238}$$

where  $C_{A:B}$  is the concurrence between A and B,  $C_{A:C}$ — between A and C, while  $C_{A:BC}$  — between system A and BC. There was a conjecture that the above inequality can be extended to n-qubits. The conjecture has been proved true only recently (Osborne and Verstraete, 2006). The monogamy is also satisfied for Gaussian states (Adesso and Illuminati, 2006; Hiroshima *et al.*, 2007) see sec. XVII.E. However, it does not hold anymore in higher dimension (Ou, 2006). Namely consider Aharonov state of three qutrits

$$\psi_{ABC} = \frac{1}{\sqrt{6}} (|012\rangle + |120\rangle + |210\rangle - |021\rangle - |210\rangle - |102\rangle).$$
(239)

One finds that  $C_{A:B}^2 = C_{AC}^2 = 1$ , while  $C_{A:BC}^2 = 3/4$ .

Interestingly the monogamy for concurrence implies monogamy of negativity (see Sec.XV.B.4) (Ou and Fan, 2007)

$$\mathcal{N}_{A:B}^2 + \mathcal{N}_{A:C}^2 \le \mathcal{N}_{A:BC}^2. \tag{240}$$

It is not known if this holds in higher dimension.

More generally in terms of entanglement measures monogamy takes the following form:

For any tripartite state of systems  $A_1$ ,  $A_2$ , B

$$E(A:B) + E(A:C) \le E(A:BC).$$
 (241)

Note that if the above inequality holds in general (i.e. not only for qubits), then it already itself implies (by induction) the inequality

$$E(A:B_1) + E(A:B_2) + \dots + E(A:B_N) \leq E(A:B_1 \dots B_N).$$
(242)

In (Koashi and Winter, 2004) it was shown that squashed entanglement satisfies this general monogamy

$$E_{sq}(A:B) + E_{sq}(A:C) \le E_{sq}(A:BC).$$
 (243)

This is the only known entanglement measure having this property.  $E_F$  and  $E_C$  are not monogamous (Coffman *et al.*, 2000; Koashi and Winter, 2004) e.g. on the above Aharonov state. This somehow support a view, that  $E_C$  does not say about entanglement contents of the state, but rather describes entanglement that must be "dissipated" while building the state by LOCC out of pure entanglement (Horodecki *et al.*, 2002; Synak *et al.*, 2005). (It is also supported by the locking effect, see Sec. XVIII.A). In the context of monogamy, one can consider other tradeoffs similar to (241). For example we have (Koashi and Winter, 2004)

$$E_F(C:A) + C_{HV}(B|A) = S_A, \qquad (244)$$

where  $C_{HV}$  is a measure of classical (Henderson and Vedral, 2001) given by 222.

This says that if with one system we share too much entanglement, this must suppress classical correlations with the other system. One might think, that since classical correlations are suppressed, then also quantum should, as existence of quantum correlations imply existence of classical ones. Qualitatively it is indeed the case: no classical correlations means that a state is product, hence cannot have entanglement too. Quantitatively the issue is a bit more complicated, as  $E_F$  can be greater than  $C_{HV}$ . However, if other prominent measures of entanglement would be smaller than  $C_{HV}$ , we could treat it as an oddity of  $E_C$  as mentioned above. To our knowledge it is not known, whether  $E_R$ ,  $E_D$ ,  $K_D$  are smaller than  $C_{HV}$ . Let us note that this latter relation would imply monogamy for those measures (since they are all upper bounded by  $E_F$ ).

 $<sup>^{75}</sup>$  D. Gottesman, private communication

<sup>&</sup>lt;sup>76</sup> P. Shor, A. Harrow and D. Leung, private communication
A beautiful monogamy was found for Bell inequalities. Namely, basing on the earlier results concerning a link between the security of quantum communication protocols and violation of Bell's inequalities (Scarani and Gisin, 2001a) and theory of nonlocal games (Cleve *et al.*, 2004), Toner proved the CHSH inequality is monogamous (Toner, 2006). This means, that if three parties A,B and C share quantum state  $\rho$  and each chooses to measure one of two observables then trade-off between *AB*'s and *AC*'s violation of the CHSH inequality is given by

$$|\operatorname{Tr}(\mathcal{B}_{CHSH}^{AB}\rho)| + |\operatorname{Tr}(\mathcal{B}_{CHSH}^{AC}\rho)| \le 4.$$
(245)

It shows clearly that CHSH correlations are monogamous i.e. if AB violate the CHSH inequality, then AC cannot. This is compatible with the present ideas of drawing cryptographic key from nonlocality, where monogamy of nonlocal correlations is the main feature allowing to bound the knowledge of eavesdropper.

There is a qualitative aspect of monogamy, recognized quite early (Doherty *et al.*, 2005; Werner, 1989b). Namely, a state  $\rho_{AB}$  is separable if and only if for any *N* there exists its N + 1 partite symmetric extension, i.e. state  $\rho_{AB_1...B_N}$ , such that  $\rho_{AB_i} = \rho_{AB}$ . D. Yang in a recent paper (Yang, 2006) has provided an elegant proof of this result and has given an explicit bound on the number of shared in terms of quantity *G* (see Sec. XV.F.2) which is a mixed convex roof from Henderson-Vedral measure of classical correlations, and itself is indicator of entanglement, in a sense that it is zero if and only if a state is entangled. Namely we have

$$N \le \frac{S_A}{G(A \rangle B)}.\tag{246}$$

For pure states  $S_A = G(A \mid B)$  which shows that no symmetric extension is possible, if only state is entangled, i.e.  $G \neq 0$ .

# XVII. ENTANGLEMENT IN CONTINUOUS VARIABLES SYSTEMS

## A. Pure states

Many properties of entanglement (separability) change when passing to continuous variables since the infinite dimensional Hilbert space is not compact.

The term continuous variables comes from the fact that any infinite dimensional Hilbert space is isomorphic to any of the two spaces:

- (i)  $l^2(\mathcal{C})$  which is space of sequences  $\Psi = \{c_i\}$  with  $\sum_{\substack{i \ge 1 \\ \infty \le i = 1}}^{\infty} |c_i|^2 < \infty$  and scalar product  $\langle \Psi | \Phi \rangle = \sum_{i=1}^{\infty} a_i^* b_i$  and
- (ii) space  $L^2(R)$  of all functions  $\Psi : R \to \mathcal{C}$  with  $\int_R |\Psi(x)|^2(x)dx < \infty$  and scalar product defined as  $\int_R \Psi(x)^* \Phi(x)dx$  The variable x is just a *continuous variable* (CV) here.

An example of entangled state from such space is twomode squeezed state: which has its  $l^2 \otimes l^2$ -like representation (in so called Fock basis considered to be a standard one):

$$|\Psi_{\lambda}\rangle = \sqrt{1-\lambda^2} \sum_{n=0}^{\infty} \lambda^n |n\rangle|, n\rangle$$
 (247)

where index goes from zero for physical reasons (*n* represents the photon number). Here coefficients  $a_n := (1 - \lambda^2)\lambda^n$  are just Schmidt coefficients.

Alternatively the state has its  $L^2\otimes L^2$  representation:

$$\Psi_{\lambda}(q_1, q_2) = \sqrt{\frac{2}{\pi}} \exp[-e^{-2r}(q_1 + q_2)^2/2 - e^{2r}(q_1 - q_2)^2/2],$$
(248)

related to previous representation by

$$\lambda = \tanh(r). \tag{249}$$

In the case of infinite squeezing  $r \to \infty$  the  $\Psi(q_1, q_2)$  becomes more and more similar to the delta function  $\delta(q_1 - q_2)$  while its Fourier transform representation (changing "positions"  $q_i$  into "momenta"  $p_i$ ) becomes almost  $\delta(p_1 + p_2)$ . This limiting case was originally discussed in the famous EPR paper (Einstein *et al.*, 1935), and perfect correlations in positions as well as momenta resemble us perfect correlations of local measurements  $\sigma_x$  and  $\sigma_z$ on sides of the two-qubit state  $\Psi_+ = \frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|1\rangle$ ).

The separability property are in case of bipartite CV pure states easy — as in discrete case we have for bipartite pure states equivalence

separability 
$$\Leftrightarrow$$
 PPT criterion  
 $\Leftrightarrow$  reduced state pure  
 $\Leftrightarrow$  Schmidt rank one (250)

The entropy of entanglement of pure states remains a good measure of entanglement, exhibiting however some oddities. In the case of the state (247) it is given by (Giedke *et al.*, 2003a; Wolf *et al.*, 2004)

$$E_F(\Psi_\lambda) = \cosh^2(r) \log_2(\cosh^2(r)) - \sinh^2(r) \log_2(\sinh^2(r)).$$
(251)

However in bipartite case with both subsystems of CV type typically the entropy of entanglement is *infinite*. This is a consequence of the fact that generically quantum states on CV spaces have the entropy infinite (Wehrl, 1978) (or alternatively the set of density matrices with finite entropy is *nowhere dense* i.e. contains no ball).

As an example of such state take  $\Psi_{AB} = \sum_n \sqrt{p_n} |n\rangle_A |n\rangle_B$ , with  $p_n$  proportional (up to normalization factor) to  $1/(n+2) \log(n+2)^4$ . Then entropy of entanglement is infinite, since the series  $\sum_n p_n \log p_n$  is not convergent.

As one can expect the very important fact connected to it is that there is no maximally entangled state in such spaces. Simply the state with all Schmidt coefficients equal does not mathematically exists (it would have infinite norm).

A natural question here arises: what about usefulness of infinite entanglement which is so common phenomenon in CV? For example is it possible to distill infinite amount of two-qubit entanglement from single copy of bipartite CV quantum states with infinite entanglement? The answer to this is negative and can be proven formally (Keyl *et al.*, 2002).

Also, one can easily see that even for pure states  $E_F$  is not continuous (Eisert *et al.*, 2002b). Consider the sequence of states

$$|\Psi_{AB}^n\rangle = \sqrt{1 - \frac{1}{n\log n}} |e_0\rangle |f_0\rangle + \sqrt{\frac{1}{n\log n}} \sum_{i=1}^n |e_i\rangle |f_i\rangle$$
(252)

it is elementary to see that the sequence converges to a state with zero entanglement,  $|||\Psi_{AB}\rangle\langle\Psi_{AB}| - |e_0\rangle\langle e_0| \otimes |f_0\rangle\langle f_0||_1 \rightarrow 0$ , while its entanglement diverges  $(E_F(\Psi_{AB}^n) \rightarrow \infty).$ 

### B. Mixed states

The definition of mixed separable states has to be changed slightly if compared to discrete variables: the state is separable if it is a limit (in trace norm) of finite convex combination of, in general mixed and not pure, product states<sup>77</sup>:

$$\|\varrho_{AB}^{sep} - \sum_{i} p_{i}^{(n)} \varrho_{A}^{(n),i} \otimes \varrho_{B}^{(n),i}\|_{1} \to 0.$$
 (253)

The characterization of entanglement in terms of positive maps and entanglement witnesses is again true since the corresponding proofs are valid for general Banach spaces (see (Horodecki *et al.*, 1996a)).

There is a small difference here: since there is no maximally entangled state one has to use the original version of Jamiołkowski isomorphism between positive maps and entanglement witnesses:

$$W^{\Lambda}_{AB} = (\mathbf{I} \otimes \Lambda)(V_{AA'}), \qquad (254)$$

with  $d_A \geq d_B$  (remember that it may happen that only one of subspaces is infinite) where  $V_{AA'}$  is a swap operator on  $\mathcal{H}_{AA'} = \mathcal{H}_A \otimes \mathcal{H}_{A'}$ ,  $(\mathcal{H}_{A'}$  being a copy of  $\mathcal{H}_A$ ) and the map acting from system A' to B. This is because there is no maximally entangled state in infinite dimensional space.

The PPT criterion is well defined and serves as a separability criterion as it was in finite dimension. It has a very nice representation in terms of moments (Shchukin and Vogel, 2005a).

PPT There exist nontrivial states (Horodecki and Lewenstein, 2000) that cannot be constructed as a naive, direct sum of finite dimensional structures. It seems that such states are generic, though the definition of generic CV state in case of mixed state is not so natural as in case of pure state where infinite Schmidt rank (or rank of reduced density matrix) is a natural signature defining CV property (for discussion of the property of being generic see (Horodecki et al., 2003e)).

The first important observation concerning CV separability is (Clifton and Halvorson, 2000) that in bipartite case the set of separable states is nowhere dense or — equivalently — any state on this space is a limit (in trace norm) of sequence of entangled state. Thus set of separable states contains *no* ball of finite radius and in that sense is "of zero volume" unlike it was in finite dimensions (Życzkowski *et al.*, 1998). This result can be extended (Horodecki *et al.*, 2003e) to the set of all nondistillable states (in a sense of definition inherited from discrete variables i.e. equivalent to impossibility of producing two-qubit singlets) is also nowhere dense in set of all states. Thus CV bound entanglement like CV separability is a rare phenomenon.

Let us pass to quantitative issues involving entanglement measures. If one tries to extend definition of entanglement of formation to mixed states (Eisert *et al.*, 2002b) then again set of states with finite  $E_F$  has the same property as the set of separable states — it is again nowhere dense.<sup>78</sup> Also, it is not continuous (as we have already seen in the case of pure states).

The question was how to avoid, at least partially, the above problems with entanglement that occur when both dimensions are infinite? The authors of (Eisert *et al.*, 2002b) propose then to consider subset  $S_M(H) \subset S$  $(S \text{ set of all bipartite states defined as <math>S_M(H) := \{\varrho : \text{Tr}(\varrho H) < M\}$  for some fixed constant M and Hamiltonian H (some chosen Hermitian operator with spectrum bounded from below). The set is nowhere dense but it is defined by natural physical requirement of bounded mean energy in a physical system. Remarkably for fixed M and all states from  $S_M$ , the entanglement  $E_R$  are continuous in trace norm on pure states. Moreover those measures are asymptotically continuous on pure states of the form  $\sigma^{\otimes n}$  with finite-dimensional support of  $\sigma$ .

#### C. Gaussian entanglement

There is a class of CV states that are very well characterized with respect to separability. This is the class

<sup>&</sup>lt;sup>77</sup> This is actually the original definition of separable states (Werner, 1989a).

<sup>&</sup>lt;sup>78</sup> This problem does not occur when one of the local Hilbert spaces  $\mathcal{H}_A, \mathcal{H}_B$  is finite i.e.  $min[d_A, d_B] < \infty$  then entanglement of formation is well defined and restricted by logarithm of the dimension of the finite space (Majewski, 2002).

of Gaussian states. Formally a Gaussian state of mmodes (oscillators) is a mixed state on Hilbert space  $\mathcal{L}^2(R)^{\otimes m}$  (of functions of  $\xi = [q_1, \ldots, q_m]$  position variables) which is completely characterized by the vector of its first moments  $d_i = \operatorname{Tr}(\varrho R_i)$  (called displacement vector) and second moments covariance matrix  $\gamma_{ij} =$  $\operatorname{Tr}(\varrho \{R_i - d_i I\}, (R_j - d_j I\}_+)$ , where we use anticommutator  $\{,\}_+$  and the cannonball observables are position  $Q_k = R_{2k}$  and momentum  $P_k = R_{2k+1}$  operators of k-th oscillator which satisfy the usual Heisenberg commutation relations  $[R_k, R_{k'}] = iJ_{kk'}$  where  $J = \bigoplus_{i=1}^n J_i$ , with one mode symplectic matrices  $J_i = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ . Any general matrix S is called symplectic iff it satisfies  $SJS^T = J$ . They represent all canonical transformations  $S: \xi \to \xi' = S\xi$  where  $\xi = [q_1, p_1; q_2, p_2; \ldots; q_m p_m]^T$  is a

vector of canonical variables. The corresponding action on the Hilbert space is unitary. There is also a broader set of unitary operations called quasifree or linear Bogolubov transformations  $\xi \to S\xi + d$  where S is symplectic. The canonical operators  $Q_i, P_i$  are real and Hermitian

part of the creation  $a_k^{\dagger}$  and annihilation  $a_k$  operators that provide a natural link to  $(l^2)^{\otimes m}$  representation since they distinguish a special  $l^2$  Fock basis  $\{|n\rangle\}$  of each mode  $(a_k^{\dagger} = \sum_{n=0}^{\infty} \sqrt{n+1}|n+1\rangle_{kk}\langle n|$  and  $a_k = (a_k^{\dagger})^{\dagger})$  via number operator  $N = a_k^{\dagger}a_k = \sum_{n=0}^{\infty} n|n\rangle\langle n|$  which is diagonal in that basis.

Since displacement d can be easily removed by quasifree *local* (i.e. on each mode separately) unitary operations (Duan *et al.*, 2000)), only the properties of variance matrix are relevant for entanglement tests. Before recalling them we shall provide conditions for  $\gamma$  to be physical. Let us recall that *via* Williamson theorem  $\gamma$  can be diagonalised with some symplectic matrix  $\gamma_{diag} = S_V \gamma S_V^T = diag[\kappa_1, \kappa_1; \ldots; \kappa_m, \kappa_m]$ , with  $\kappa_i$  real. The physical character of the variance matrix  $\gamma$  is guaranteed by the condition:

$$\gamma + iJ \ge 0 \Leftrightarrow \tag{255}$$

$$\gamma \ge J^T \gamma^1 J \Leftrightarrow \tag{256}$$

$$\gamma \ge S^T S \Leftrightarrow \tag{257}$$

$$\kappa_i \ge 1, i = 1, \dots, m. \tag{258}$$

There is a remarkable fact(Simon, 2000): Gaussian state is pure if and only if its variance matrix is of the form

$$\gamma = S^T S, \tag{259}$$

for some symplectic matrix S. Generally, given m modes can be divided into k groups containing  $m_1, \ldots, m_2, m_k$ modes  $(m = \sum_i m_i)$ , belonging to different local observers  $A_1, \ldots, A_k$ . We say that the state is a k-partite *Gaussian state of*  $m_1 \times m_2 \times \cdots \times m_k$  type. For instance the bipartite state is of  $m_1 \times m_2$  type iff the first  $m_1$ modes are on Alice side, and the rest  $m_2$  on Bob one. All reduced states of the systems  $A_i$  are Gaussians. With each site we associate the symplectic matrix  $J_{A_k}$  as before. There is general necessary and sufficient separability condition that can resemble to some extent the range criterion (see (Werner and Wolf, 2001a)):

$$\underline{\varrho}$$
 (Gaussian separable  $\Leftrightarrow$  (260)

$$\gamma_{AB}(\varrho) \ge \gamma_A \oplus \gamma_B \tag{261}$$

for some variance matrices  $\gamma_A, \gamma_B$  (which, as further was shown (Simon, 2003), can be chosen to be pure, i.e. of the form (259)). Quite remarkably the above criterion can be generalized to an arbitrary number of parties (see (Eisert and Gross, 2005)). In general — if the state is not Gaussian the criterion becomes only a necessary condition of separability. The criterion is rather hard to use (see however (Werner and Wolf, 2001a) and the discussion below).

The PPT separability criterion takes a very easy form for Gaussians. On the level of canonical variables partial transpose with respect to the given subsystem (say B) corresponds to reversal of its conjugate variables  $\xi = [\xi_A, \xi_B] \rightarrow \tilde{\xi} = \Lambda$  where  $\Lambda = diag[I_A, \sigma_B^z]$ . If we introduce transformation

$$X = \Lambda X \Lambda, \tag{262}$$

then we have PPT criterion for Gaussian states: bipartite Gaussian state is PPT iff  $\tilde{\gamma}$  is physical i.e. satisfies either of the conditions (256)-(258). Note that the first one satisfied by  $\tilde{\gamma}$  can be written as:

$$\gamma + i\tilde{J} \ge 0. \tag{263}$$

There is a very important separability characterization: PPT criterion has been shown to be *both* necessary and sufficient for  $1 \times 1$  (Duan *et al.*, 2000; Simon, 2000) and subsequently generalized to  $1 \times n$  Gaussians (Werner and Wolf, 2001a). Further the same result has been proven for  $m \times n$  "bisymmetric" (Serafini, 2006; Serafini *et al.*, 2005) (i.e. symmetric under permutations of Alice and Bob modes respectively) Gaussian states. The equivalence of PPT condition to separability is *not* true, in general, if both Alice and Bob have more than one mode. In particular an example of  $2 \times 2$  Gaussian bound entanglement has been shown by proving (*via* technique similar to that from (Lewenstein and Sanpera, 1998)) that the following covariance matrix

$$\gamma = \begin{pmatrix} 2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 2 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 4 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 2 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 4 \end{pmatrix}$$
(264)

does not satisfy the condition (261).

The PPT test for  $1 \times 1$  Gaussian states can be written elementary if we represent the variance as  $\gamma = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix}$ . Then the two elementary conditions (Simon, 2000) det(A)det(B) +  $(1/8 \pm \det(C))^2$  –

Tr( $AJCJBJC^TJ$ )/8 + (det(A) + det(B))/8  $\geq$  0 represent the physical and PPT character of  $\gamma$  respectively. These conditions can be further simplified if the variance is driven by local linear unitaries (corresponding to symplectic operations) to canonical form of type I, where matrices A and B are proportional to identities and C is diagonal (Simon, 2000). An interesting and more fruitful for general (nongaussian) case approach is offered by uncertainty relations approach where the so called type II is achieved by local quasi free unitary operations and PPT condition is represented by some *uncertainty relation* (see (Duan *et al.*, 2000) and subsequent section).

In (Giedke *et al.*, 2001c) the separability problem for three mode Gaussian states was also completely solved in terms of operational criterion.

Operational necessary and sufficient condition. In (Giedke et al., 2001b) operational necessary and sufficient condition for separability for all bipartite Gaussian states have been presented. It is so far the only operational criterion of separability that detects all PPT entangled states within such broad class of states. Entanglement is detected via a finite algorithm, that transforms the initial covariance matrix into a sequence of matrices which after finite number of steps (i) either becomes not physical (not represents a covariance matrix) and then the algorithm detects entanglement (ii) or its special affine transformation becomes physical and then the initial state is recognized to be separable.

## D. General separability criteria for continuous variables

One of the natural separability criteria is local projection or — in general LOCC transformation of CV state onto product of finite dimensional Hilbert spaces and then application of one of separability criteria for discrete variables. This method was used in (Horodecki and Lewenstein, 2000) where finally discrete variables range criterion was applied.

First, it must be stressed that any Gaussian separability criterion that refers only to well defined variances, and does not use the fact that the variance matrix completely describes the state, is also a separability criterion for general CV states.

Separability criteria that do not refer to discrete quantum states usually are based on some uncertainty type relations. As an example of such relation, consider position and momenta operators  $Q_{A_1}, Q_{A_2}, P_{A_1}, P_{A_2}$  for a bipartite system  $A_1A_2$ , which satisfy the commutation relations  $[Q_{A_i}, P_{A_j}] = i\delta_{ij}$  and define  $U = |a|Q_{A_1} + \frac{1}{a}Q_{A_2}$ ,  $V = |a|P_{A_1} + \frac{1}{a}P_{A_2}$  for arbitrary nonzero real number a. Then any separable bipartite CV state  $\rho$  satisfies (Duan *et al.*, 2000)

$$\langle (\Delta U)^2 \rangle_{\varrho} + \langle (\Delta V)^2 \rangle_{\varrho} \ge \frac{1}{2} (a^2 + \frac{1}{a^2}).$$
 (265)

The above criterion can be modified to the form which refers to so called type II standard form of two mode variance and becomes then necessary and sufficient separability criterion (and hence equivalent to PPT) in this two mode case (Duan *et al.*, 2000). Further, it is interesting to note that the PPT separability criterion implies a series of uncertainty principle like relations. For any  $n \times n$ mode state the physical condition  $\gamma_{AB} + iJ \geq 0$  and observables equivalent to the statement that observables  $X(d) = \vec{d\xi}, X(d') = \vec{d'\xi}$  obey the uncertainty relation (see (Simon, 2003))

$$\langle (\Delta X(d))^2 \rangle_{\varrho} + \langle (\Delta X(d'))^2 \rangle_{\varrho} \ge |\vec{d}_A J_A \vec{d}_A + \vec{d}_B J_B \vec{d}_B |,$$
(266)

for any real vectors  $\vec{d} = (\vec{d}_A, \vec{d}_B), \ \vec{d'} = (\vec{d'}_A, \vec{d'}_B)$ . The PPT condition leads to another restriction (Simon, 2003)

$$\langle (\Delta X(d))^2 \rangle_{\varrho} + \langle (\Delta X(d'))^2 \rangle_{\varrho} \ge |\vec{d}_A J_A \vec{d}_A - \vec{d}_B J_B \vec{d}_B'|,$$
(267)

which may be further written in one combined inequality with  $|\vec{d}_A J_A \vec{d}_A| + |\vec{d}_B J_B \vec{d}_B|$  in its RHS. Special case of this inequality was considered by (Giovannetti *et al.*, 2003) together with relation to other criteria. Note that the above criterion is only necessary for the state to satisfy PPT criterion since it refers to the variance including only the first and second moments.

The practical implementation of PPT criterion in terms of all moments that goes beyond variance properties of CV states is Shchukin-Vogel criterion (Miranowicz and Piani, 2006; Shchukin and Vogel, 2005a). It turned out that their criterion covers many known separability criteria. The idea is that with any state of two modes, one can associate the following matrix of moments

$$M_{ij} = \operatorname{Tr}(\hat{a}^{\dagger q} \hat{a}^{p} \hat{a}^{\dagger n} \hat{a}^{m} \otimes \hat{b}^{\dagger l} \hat{b}^{k} \hat{b}^{\dagger r} \hat{b}^{s} \rho_{AB}), \qquad (268)$$

where i = (pqrs) and j = (nmkl). The operators a, b act on systems A, B respectively. It turns out that the above matrix is positive if and only if the state is PPT<sup>79</sup>. Positivity of matrix can be expressed in terms of non-negativity of subdeterminants. It then turns out that many known separability criteria are obtained by imposing nonnegativity of a suitably chosen subdeterminant.

An example is Simon criterion which for  $1 \times 1$ Gaussian states is equivalent to PPT. One finds that it is equivalent to nonnegativity of determinant of a  $5 \times 5$  main submatrix the matrix M. Other criteria published in (Agarwal and Biswas, 2005; Duan *et al.*, 2000; Hillery and Zubairy, 2006; Mancini *et al.*, 2002; Raymer *et al.*, 2003) can be also reduced to positivity conditions of some determinants of matrix of moments.

The approach of Shchukin and Vogel was developed in (Miranowicz *et al.*, 2006). The authors introduced a modified matrix of moments:

$$M_{kk'll'} = \operatorname{Tr}((\hat{a}^{\dagger k_1} \hat{a}^{k_2})^{\dagger} \hat{a}^{\dagger k'_1} \hat{a}^{k'_2} \otimes (\hat{b}^{\dagger l_1} \hat{b}^{l_2})^{\dagger} \hat{b}^{\dagger l'_1} \hat{b}^{l'_2} \rho_{AB}),$$
(269)

<sup>&</sup>lt;sup>79</sup> See (Verch and Werner, 2005) in this context.

so that it is labeled with four indices, and can be treated itself (up to normalization) as a state of compound system. Now, it turns out that the original state is separable, then so is the matrix of moments. In this way, one can obtain new separability criteria by applying *known* separability criteria to matrix of moments. The PPT condition applied to matrix of moments turns out to be equivalent to the same condition applied to original state. Thus applying PPT to matrix of moments reproduces Shchukin-Vogel result. What is however intriguing, that so far no criterion stronger than PPT was found.

# E. Distillability and entanglement measures of Gaussian states

The question of distillability of Gaussian states has attracted a lot of effort. In analogy to the two-qubit distillability of quantum states in finite dimensions, it has been first shown, that all two-mode entangled Gaussian states are distillable (Giedke *et al.*, 2000). Subsequently it was shown, that all NPT entangled Gaussian states are distillable (Giedke et al., 2001a). In other words, there is no NPT bound entanglement in Gaussian continuous variables: any NPT Gaussian state can be transformed into NPT two-mode one, and then distilled as described in (Giedke et al., 2000). However the protocol which achieves this task involves operations which are not easy to implement nowadays. The operations feasible for present linear-optic based technology are so called Gaussian operations. The natural question was risen then, whether entangled Gaussian states are distillable by means of this restricted class of operations. Unfortunately, it is not the case: one cannot obtain pure entanglement from Gaussian states using only Gaussian operations (Giedke and Cirac, 2002) (see in this context (Fiurasek, 2002b) and (Eisert et al., 2002a)). Although these operations are restrictive enough to effectively "bind" entanglement, they are still useful for processing entanglement: by means of them, one can distill key from entangled NPT Gaussian states (Navascues et al., 2004). Interestingly, no PPT Gaussian state from which key can be distilled is known so far (Navascues and Acin, 2005).

Apart from question of distillability and key distillability of Gaussian states, entanglement measures such as entanglement of formation and negativity have been studied. It led also to new measures of entanglement called *Gaussian entanglement measures*.

In (Giedke *et al.*, 2003b) entanglement of formation was calculated for symmetric Gaussian states. Interestingly, the optimal ensemble realizing  $E_F$  consists solely from Gaussian states. It is not known to hold in general. One can however consider the so called *Gaussian entanglement of formation*  $E_G$  where infimum is taken over decompositions into Gaussian states only. Gaussian entanglement of formation was introduced and studied in (Giedke *et al.*, 2003b). It is shown there, that  $E_G$  is monotonous under Gaussian operations. For two-mode Gaussian states its value can be found analytically. If additionally, the state is symmetric with respect to sites, this measure is additive. On a single copy it is shown to be equal to  $E_F$ .

The idea of Gaussian entanglement of formation has been extended to other convex-roof based entanglement measures in (Adesso and Illuminati, 2005). The log-negativity of Gaussian states defined already in (Vidal and Werner, 2002) has also been studied in (Adesso and Illuminati, 2005). In this case the analytic formula has been found, in terms of symplectic spectrum  $\lambda_i$  of the partially transposed covariance matrix:

$$E_N = -\sum_{i=1}^n \log_2[\min(1,\lambda_i)].$$
 (270)

The continuous variable analogue *tangle* (squared concurrence, see sec XV), called *contangle* was introduced in (Adesso and Illuminati, 2006) as the Gaussian convex roof of the squared negativity. It is shown, that for three-mode Gaussian states contangle exhibits Coffman-Kundu-Wootters monogamy. Recently the general monogamy inequality for all N-mode Gaussian states was established (Hiroshima *et al.*, 2007) (in full analogy with the qubit case (Osborne and Verstraete, 2006)). For three modes, the 3-contangle - analogue of Coffman-Kundu-Wooters 3-tangle is monotone under Gaussian operations.

Surprisingly, there is a symmetric Gaussian state which is a counterpart of both GHZ as well as W state (Adesso and Illuminati, 2006). Namely, in finite dimension, when maximizing entanglement of subsystems, one obtains W state, while maximization of tangle leads to GHZ state. For Gaussian states, such optimizations (performed for a fixed value of mixedness or of squeezing of subsystems) leads to a *single* family of pure states called GHZ/W class. Thus to maximize tripartite entanglement one has to maximize also bipartite one.

An exemplary practical use of Gaussian states apart from the quantum key distribution (e.g. (Gottesman and Preskill, 2001)) is the application for continuous quantum Byzantine agreement protocol (Neigovzen and Sanpera, 2005). There are many other theoretical and experimental issues concerning Gaussian states and their entanglement properties, that we do not touch here. For a recent review on this topic see (Adesso *et al.*, 2007; Ferraro *et al.*, 2005).

## XVIII. MISCELLANEA

# A. Entanglement under information loss: locking entanglement

Manipulating a quantum state with local operations and classical communication in non-unitary way usually decreases its entanglement content. Given a quantum bipartite system of  $2 \times \log d$  qubits in state  $\rho$  one can ask how much entanglement can decrease if one traces out a single qubit. Surprisingly, a lot of entanglement measures can decrease by arbitrary large amount, i.e. from  $O(\log d)$  to zero. Generally, if some quantity of  $\rho$  can decrease by an arbitrarily large amount (as a function of number of qubits) after LOCC operation on few qubits, then it is called *lockable*. This is because a huge amount of quantity can be controlled by a person who posses only a small dimensional system which plays a role of a "key" to this quantity.

The following related question was asked earlier in (Eisert *et al.*, 2000a): how entanglement behaves under classical information loss? It was quantified by means of entropies and for convex entanglement measures it takes the form

$$\Delta E \le \Delta S \tag{271}$$

where  $\Delta E = \sum_{i} p_i E(\rho_i) - E(\sum_{i} p_i \rho_i)$  and  $\Delta S = S(\rho) - \sum_{i} p_i s(\rho_i)$ . It holds for relative entropy of entanglement (Linden *et al.*, 1999a) (see also (Synak-Radtke and Horodecki, 2006)). It turns out however that this inequality can be drastically violated, due to above locking effect<sup>80</sup>.

The phenomenon of drastic change of the content of state after tracing out one qubit, was earlier recognized in (DiVincenzo *et al.*, 2004) in the case of classical correlations of quantum states (maximal mutual information of outcomes of local measurements) (see in this context (Ballester and Wehner, 2006; Buhrman *et al.*, 2006; Koenig *et al.*, 2005; Smolin and Oppenheim, 2006)). Another effect of this sort was found in a classical key agreement (Renner and Wolf, 2003) (a theory bearing some analogy to entanglement theory, see sec. XIX.F).

Various entanglement measures have been shown to be lockable. In (Horodecki *et al.*, 2005d) it has been shown that *entanglement cost*, and *log-negativity* are lockable measures. The family of states which reveal this property of measures (called *flower states*) can be obtained via partial trace over system E of the pure state

$$|\psi_{AaBE}^{(d)}\rangle = \frac{1}{\sqrt{2d}} \sum_{i=0}^{d-1} |i\rangle_A |0\rangle_a \{|i\rangle|0\rangle\}_B |i\rangle_E \qquad (272)$$
$$+|i\rangle_A |1\rangle_a \{|i\rangle|1\rangle\}_B U |i\rangle_E,$$

where  $U = H^{\otimes \log d}$  is a tensor product of Hadamard transformations. In this case  $E_C(\rho_{AaB}) = \frac{1}{2}\log d$  and  $E_N(\rho_{AaB}) = \log(\sqrt{d} + 1)$ . However after tracing out qubit *a* one has  $E_C(\rho_{AB}) = E_N(\rho_{AB}) = 0$ , as the state is separable. The gap between initial  $E_C$  and final one, can be made even more large, using more unitaries instead of *U* and I in the above states and the ideas of randomization (Hayden *et al.*, 2004). The examples of states for which  $E_F$  can change from nearly maximal (log d) to nearly zero after tracing out  $O(\log \log d)$  qubits was found in (Hayden *et al.*, 2006). It can indicate either drastic nonadditivity of entanglement of formation, or most probably an extreme case of irreversibility in creation-distillation LOCC processes<sup>81</sup>. The analogous effect has been found earlier in classical key agreement (Renner and Wolf, 2003). It was shown that *intrinsic information*  $I(A : B \downarrow Ee)$  (an upper bound on secret key extractable from triples of random variables) can decrease arbitrarily after erasing 1-bit random variable e (see Sec. XIX.F).

In (Christandl and Winter, 2005) it was shown that squashed entanglement is also lockable measure, which is again exhibited by flower states. The authors has shown also that regularized entanglement of purification  $E_p^{\infty}$  is lockable quantity, as it can go down from maximal to zero after erasing one-qubit system a of the state

$$\omega_{aAB} = \frac{d+1}{2d} |0\rangle \langle 0|_a \otimes P_{AB}^{(+)} + \frac{d-1}{2d} |1\rangle \langle 1|_a \otimes P_{AB}^{(-)}, \quad (273)$$

with  $P_{AB}^{(\pm)}$  being normalized projectors onto symmetric and antisymmetric subspace respectively (see Sec. VI.B.3). Although  $E_p^{\infty}$  is not an entanglement measure, it is apart from  $E_C$  another example of lockable quantity which has operational meaning (Terhal *et al.*, 2002). In case of (273)  $E_p = E_p^{\infty}$ , so the  $E_p$  itself also can be locked.

It is known that all measures based on the convex roof method (see Sec. XV.C.2) are lockable. There is also a general connection between lockability and asymptotic non-continuity. Namely it was shown in (Horodecki *et al.*, 2005d) that any measure which is (i) subextensive i.e. bounded by  $M \log d$  for constant M, (ii) convex (iii) *not* asymptotically continuous, is lockable. The only not lockable measure known up to date is the relative entropy of entanglement (Horodecki *et al.*, 2005d), which can be derived with help of (271)

Since fragility of measure to one qubit erasure is a curious property, for any lockable entanglement measure E one could design its *reduced* version, that is a non-lockable version of E. One of the possible definitions of *reduced* entanglement measure is proposed in (Horodecki et al., 2005d). Possible consequences of locking for multipartite entanglement measures can be found in (Groisman et al., 2005).

Although the phenomenon of locking shows that certain measures can highly depend on the structure of particular states, it is not known how many states have a structure which can contribute to locking of measures, and what kind of structures are specific for this purpose.

<sup>&</sup>lt;sup>80</sup> Using the fact that that a loss of one qubit can be simulated by applying one of four random Pauli matrices to the qubit one easily arrives at the connection between locking, and violation of the above inequality.

<sup>&</sup>lt;sup>81</sup> Because distillable entanglement of the states under consideration is shown to be small we have either  $E_F >> E_C$  (reporting nonadditivity of  $E_F$ ) or  $E_F = E_C >> E_D$  (reporting extreme creation-distillation irreversibility).

It is an open question whether distillable entanglement can be locked, although it is known that its one-way version is lockable, which follows from monogamy of entanglement. In case of distillable key, one can consider two versions of locking: the one after tracing out a qubit from Eve (*E*-locking), and the one after the qubit of Alice's and Bob's system is traced out (*AB*-locking). It has been shown, that both classical and quantum distillable key is not *E*-lockable (Christandl et al., 2006; Renner and Wolf, 2003). It is however not known if distillable key can be AB-lockable i.e. that if erasing a qubit from Alice and Bob systems may diminish by far their ability to obtain secure correlations. Analogous question remains open in classical key agreement (see Sec. XIX.F).

#### B. Entanglement and distinguishing states by LOCC

Early fundamental results in distinguishing by LOCC are the following: there exist sets of orthogonal product states that are not perfectly distinguishable by LOCC (Bennett et al., 1999a) (see also (Walgate and Hardy, 2002)) and every two orthogonal states (even multipartite) are distinguishable by LOCC (Walgate *et al.*, 2000). In a qualitative way, entanglement was used in the problem of distinguishability in (Terhal *et al.*, 2001). To show that a given set of states cannot be distinguished by LOCC, they considered all measurements capable to distinguish them. Then applied the measurements to ABpart of the system in state  $\psi_{AA'} \otimes \psi_{BB'}$  where the components are maximally entangled states. If the state after measurement is entangled across AA' : BB' cut, then one concludes that the measurement cannot be done by use of LOCC because the state was initially product across this cut.

An interesting twist was given in (Ghosh et al., 2001) where distillable entanglement was used. Let us see how they argued that four Bell states  $\psi_i$  (3) cannot be distinguished. Consider four partite state  $\rho_{ABA'B'} =$  $\frac{1}{4}\sum_{i} |\psi_{i}\rangle \langle \psi_{i}|_{AB} \otimes |\psi_{i}\rangle \langle \psi_{i}|_{A'B'}$ . Suppose that it is possible to distinguish Bell states by LOCC. Then Alice and Bob will distinguish Bell states of system AB (perhaps destroying them). Then they will know which of Bell states they share on system A'B', obtaining then 1 e-bit of pure entanglement (hence  $E_D \geq 1$ ). However, one can check that the initial state  $\rho_{ABA'B'}$  is separable (Smolin, 2005), so that  $E_D = 0$  and we get a contradiction. This shows that entanglement measures can be used to prove impossibility of distinguishing of some states. Initially entanglement measures have not been used for this problem. It is quite nontrivial concept, as naively it could seem that entanglement measures are here useless. Indeed, the usual argument exploiting the measures is the following: a given task cannot be achieved, because some entanglement measure would increase. In the problem of distinguishing, this argument cannot be directly applied, because while distinguishing, the state is usually

destroyed, so that final entanglement is zero.

This approach was developed in (Horodecki *et al.*, 2003c) (in particular, it was shown that any measure of entanglement can be used) and it was proved that a full basis cannot be distinguished even probabilistically, if at least one of states is entangled. Other results can be found in (Fan, 2004; Nathanson, 2005; Virmani *et al.*, 2001). In (Virmani *et al.*, 2001) it was shown that for any two pure states the famous Helstrom formula for maximal probability of distinguishing by global measurements, holds also for LOCC distinguishing.

In (Badziąg *et al.*, 2003) a general quantitative bound for LOCC distinguishability was given in terms of the maximal mutual information achievable by LOCC measurements <sup>82</sup>  $I_{acc}^{LOCC}$ . The bound can be viewed is a generalization of Holevo bound:

$$I_{acc}^{LOCC} \le \log N - \overline{E}, \qquad (274)$$

where N is number of states, and  $\overline{E}$  is average entanglement (it holds for any E which is convex and is equal to entropy of subsystems for pure states). The usual Holevo bound would read just  $I_{acc} \leq \log N$ , so we have correction coming from entanglement. The following generalization of the above inequality obtained in (Horodecki *et al.*, 2004)

$$I_{acc}^{LOCC} \le \log N - \overline{E}_{in} - \overline{E}_{out}$$
(275)

shows that if some entanglement, denoted here by  $\overline{E}_{out}$ , is to survive the process of distinguishing (which is required e.g in process of distillation) then it reduces distinguishability even more<sup>83</sup>.

Finally in (Hayashi *et al.*, 2006) the number M of orthogonal multipartite states distinguishable by LOCC was bounded in terms of several measures of entanglement, in particular we have

$$M \le D2^{-\frac{1}{M}\sum_{i} \left( S(\rho_i) + E_R(\rho_i) \right)}, \tag{276}$$

where D denotes dimension of the total system, S denotes the von Neumann entropy and  $E_R$  is the relative entropy of entanglement (see Sec. XV.C.1). A striking phenomenon is possibility of "hiding" discovered in (DiVincenzo *et al.*, 2002; Terhal *et al.*, 2001). Namely, one can encode one bit into two states  $\rho_0$  and  $\rho_1$ , for which probability of error in LOCC-distinguishing can be arbitrary close to 1. In (Eggeling and Werner, 2002) it was shown that this can be achieved even with  $\rho_i$  being separable states. Thus unlike for pure states, for two mixed states, Helstrom formula does not hold any longer.

<sup>&</sup>lt;sup>82</sup> It is defined as follows:  $I_{acc}^{LOCC}(\{p_i, \psi_i\}) = \sup_{\{A_j\}} I(i : j)$  where supremum is taken over measurements that can be implemented by LOCC, and I(i : j) is mutual information between symbols *i* and measurement outcomes *j*.

<sup>&</sup>lt;sup>83</sup> The inequality was independently conjectured and proven for one way LOCC in (Ghosh *et al.*, 2004)

Finally, a lower bound on  $I_{acc}^{LOCC}$  was provided in (Sen(De) *et al.*, 2006) and relation with entanglement distillation was discussed.

#### C. Entanglement and thermodynamical work

In (Oppenheim *et al.*, 2002a) an approach to composite systems was proposed based on thermodynamics. Namely, it is known, that given a pure qubit and a heat bath, one can draw amount  $kT \ln 2$  of work<sup>84</sup>. The second law is not violated, because afterward the qubit becomes mixed serving as an entropy sink (in place of reservoir of lower temperature) (see (Alicki *et al.*, 2004; Scully, 2001; Vedral, 1999)). In this way the noisy energy from the heat bath is divided into noise, that goes to the qubit, and pure energy that is extracted. More generally, a *d* level system of entropy *S* allows to draw amount of work equal to  $kT(\log d - S)$ . If we treat  $I = \log d - S$  as information about the state (it is also called purity), then the work is proportional to information/purity (Devetak, 2004).

Suppose now that Alice and Bob have local heat baths. and want to draw work from them by means of shared bipartite state. If they do not communicate, then they can only draw the work proportional to *local* purity. If they share two maximally classically correlated qubits, local states are maximally mixed, and no work can be drawn. However if Bob will measure his qubit, and tell Alice the value, she can conditionally rotate her qubit, so that she obtains pure qubit, and can draw work. For classically correlated systems, it turns out that the full information content  $\log d - S$  can be changed into work. How much work can Alice and Bob draw from a given quantum state, provided that they can communicate via a classical channel? It turns out that unlike in the classical case, part of information cannot be changed into work, because it cannot be concentrated via classical channel. For example, for singlet state the information content is 2, but only 1 bit of work (in units kT) can be drawn.

The amount of work that can be drawn in the above paradigm is proportional to the amount of pure local qubits that can be obtained (localisable information/purity). The difference between total information I and the localisable information represents purely quantum information, and is called *quantum deficit* (Oppenheim *et al.*, 2002a). It can be viewed as a measure of quantumness of correlations, which is zero for a classically correlated states<sup>85</sup>. (A closely related quantity called *quantum discord* was obtained earlier in (Zurek, 2003b) on the basis of purely information-theoretical considerations. It was then also related to thermodynamics (Zurek, 2000).) Recently other measures of quantumness correlations based on distance from classically correlated states were introduced (Groisman *et al.*, 2007; SaiToh *et al.*, 2007).

Localisable information is a notion dual to distillable entanglement: instead of singlets we want to distill product states. Thus local states constitute valuable resource (the class of involved operations exclude bringing fresh local ancillas). It turns out that techniques from entanglement distillation can be applied to this paradigm (Horodecki et al., 2003a; Synak et al., 2005). One of the main connections between distillation of local purity and entanglement theory is the following: For pure bipartite states the quantum deficit is equal to entropy of entanglement (Horodecki et al., 2003a). Thus one arrives at entanglement by an opposite approach to the usual one: instead of looking at nonlocal part, one considers local information, and subtracts it from total information of the state. It is an open question, if for pure multipartite states, the quantum deficit is also equal to some entanglement measure (if so, the measure is most likely the relative entropy of entanglement).

Another link is that the above approach allows to define cost of erasure of entanglement in thermodynamical spirit. Namely the cost is equal to entropy that has to be produced while resetting entangled state to separable one, by *closed* local operations and classical communication (Horodecki *et al.*, 2005g). (closed means that we do not trace out systems, to count how much entropy was produced). It was proved that the cost is lower bounded by relative entropy of entanglement. (some related ideas have been heuristically claimed earlier in (Vedral, 1999)).

Note that another way of counting cost of entanglement erasure is given by robustness of entanglement. In that case it is a mathematical function, while here the cost is more operational, as it is connected with entropy production.

In a recent development the idea of drawing work from local heat baths have been connected with a separability criterion. Namely, in (Maruyama et al., 2005) the following non-optimal scheme of drawing work was considered for two qubits: Alice and Bob measure the state in the same randomly chosen basis. If they share singlet state  $\psi^{-}$ , irrespectively of choice of basis, they obtain perfect anti-correlations, allowing to draw on average 1 bit (i.e.  $kT \ln 2$ ) of work. Consider instead classically correlated state  $\frac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|)$ . We see that any measurement apart from the one in bases  $|0\rangle, |1\rangle$  will decrease correlations. Thus if we average over choice of bases, the drawn work will be significantly smaller. The authors have calculated the optimal work that can be drawn from separable states, so that any state which allows to draw more work, must be entangled.

<sup>&</sup>lt;sup>84</sup> It is reverse of Landauer principle, which says that resetting bit/qubit to pure state costs the same amount of work.

<sup>&</sup>lt;sup>85</sup> A bipartite state is properly classically correlated or shortly classically correlated, if it can be written in the form:  $\rho_{AB} = \sum_{ij} p_{ij} |i\rangle \langle i| \otimes |j\rangle \langle j|$  with coefficients  $0 \leq p_{ij} \leq 1$ ,  $\sum_{ij} p_{ij} = 1$ ;  $\{|i\rangle\}$  and  $\{|j\rangle\}$  are local bases (Oppenheim *et al.*, 2002a).

#### D. Asymmetry of entanglement

To characterize entanglement one should recognize its different qualities. Exemplary qualities are just transition rates (exact or asymptotic) between different states under classes of operations such as LOCC (see Secs. XIII.A and XIII.B).

Recently, in (Horodecki *et al.*, 2005a) a different type of transition was studied. Namely, transition of a bipartite state into its own swapped version:  $\rho \longrightarrow V \rho V$  with V being a swap operation which exchanges subsystems of  $\rho$  (see Sec. (48)). In other words this is transition from  $\rho_{AB}$  into  $\rho_{BA}$  under LOCC. If such a transition is possible for a given state, one can say, that entanglement contents of this state is symmetric. Otherwise it is asymmetric, and the *asymmetry of entanglement* can be quantified as follows

$$A(\rho) = \sup_{\Lambda} \|\Lambda(\rho) - V\rho V\|_1, \qquad (277)$$

where supremum is taken over all LOCC operations  $\Lambda$ . Note that if a state is symmetric under swap, then also its entanglement contents is symmetric, however the converse does not hold (a trivial example is a pure nonsymmetric state: one can produce its swapped version by local unitaries).

It was shown that there exist states entanglement of which is asymmetric in the above sense. Namely, one can prove if the entanglement measure called G-concurrence (Fan *et al.*, 2003; Gour, 2005; Sinołęcka *et al.*, 2002) is nonzero, then the only LOCC operation that could swap the state is just local unitary operation. Thus any state which has nonzero G-concurrence, and has different spectra of subsystems cannot be swapped by LOCC and hence any it contains *asymmetric entanglement*. It is an open question if there exist states which maintain this asymmetry even in asymptotic limit of many copies.

One should note here, that the asymmetry in context of quantum states has been already noted in literature. In particular the asymmetric correlation functions f of a quantum state are known, such as e.g. oneway distillable entanglement  $E_D^{\rightarrow}$  (Bennett *et al.*, 1996a) or quantum discord (Zurek, 2000). Considering properly symmetrised difference between  $f(\rho_{AB})$  and  $f(\rho_{BA})$  one could in principle obtain some other measure of asymmetry of quantum correlations. However it seems that such a measure would not capture the asymmetry of entanglement content exactly. The functions defined by means of asymmetric (one-way) class of operations like  $E_D^{\rightarrow}$  seems to bring in a kind of its "own asymmetry" (the same holds e.g. for one-way distillable key). Concerning the correlation functions f such as quantum discord, they seem to capture the asymmetry of quantum correlations in general rather than that of entanglement, being nonzero for separable states (the same holds e.g. for Henderson-Vedral  $C_{HV}$  quantity (see 222) and one-way distillable common randomness (Devetak and Winter, 2004a)).

The phenomenon of asymmetry has been studied also in other contexts, such as asymmetry of quantum gate capacities (Harrow and Shor, 2005; Linden *et al.*, 2005), and the possibility of *exchange* of the subsystems of a bipartite state in such a way that entanglement with a purifying reference system is preserved (Oppenheim and Winter, 2003). Although these are very different phenomena, the possible relation is not excluded.

## XIX. ENTANGLEMENT AND SECURE CORRELATIONS

A fundamental difference between classical and quantum, is that quantum formalism allows for states of composite systems to be both *pure* and *correlated*. While in classical world those two features never meet in one state, *entangled* states can exhibit them at the same time.

For this reason, entanglement in an astonishing way incorporates basic ingredients of theory of secure communication. Indeed, to achieve the latter, the interested persons (Alice and Bob) need a private key: a string of bits which is i) perfectly correlated (*correlations*) and ii) unknown to any other person (security or privacy) (then they can use it to preform private conversation by use of so-called Vernam cipher (Vernam, 1926)). Now, it is purity which enforces the second condition, because an eavesdropper who wants to gain knowledge about a quantum system, will unavoidably disturb it, randomizing phase via quantum back reaction. In modern terminology we would say, that if Eve applies CNOT gate, to gain knowledge about a *bit*, at the same time she introduces a *phase* error into the system, which destroys purity, see (Zurek, 1981).

Let us note that all what we have said can be phrased in terms of monogamy of entanglement in its strong version: If two quantum systems are maximally quantumly correlated, then they are not correlated with any other system at all (neither quantumly nor classically) (see (Koashi and Winter, 2004) in this context). In this section we shall explore the mutual interaction between entanglement theory and the concept of private correlations.

# A. Quantum key distribution schemes and security proofs based on distillation of pure entanglement

Interestingly, the first protocol to obtain a private key <sup>86</sup> - the famous BB84 (Bennett and Brassard, 1984), did not use the concept of entanglement at all. Neither uses entanglement another protocol B92 proposed by Ch. Bennett in 1992 (Bennett, 1992), and many variations of BB84 like six-state protocol (Bruß, 1998). Indeed, these QKD protocols are based on sending randomly chosen nonorthogonal quantum states. Alice prepares a random

<sup>&</sup>lt;sup>86</sup> We call such protocol Quantum Key Distribution (QKD).

signal state, measures it and send it to Bob who also measures it immediately after reception. Such protocols are called a *prepare and measure protocols* (P& M).

The first entanglement based protocol was discovered by Ekert (see Sec. III). Interestingly, even Ekert's protocol though using explicitly entanglement was still not based solely on the "purity & correlations" concept outlined above. He did exploit correlations, but along with purity argument, used violation of Bell inequalities. It seems that it was the paper of Bennett, Brassard and Mermin (BBM) (Bennett et al., 1992) which tilted the later history of entanglement-based QKD from the "Bell inequalities" direction to "disturbance of entanglement": upon attack of Eve, the initially pure entangled state becomes mixed, and this can be detected by Alice and Bob. Namely they have proposed the following protocol which is also entanglement based, but is not based on Bell inequality. Simply Alice and Bob, when given some (untrusted) EPR pairs check their quality by measuring correlations in  $\{|0\rangle, |1\rangle\}$  basis, and the conjugated basis  $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$ . So simplified Ekert's protocol is formally equivalent to the BB84 protocol. Namely the total final state between Alice, Bob and Eve is the same in both cases<sup>87</sup>. Thus entanglement looks here quite superfluous, and moreover Bell inequalities appear rather accidentally: just as indicators of possible disturbance by Eve.

Paradoxically, it turned out recently that the Bell inequalities are *themselves* a good resource for key distribution, and allow to prove security of private key *without* assuming quantum formalism but basing solely on nosignaling assumption. (Acin *et al.*, 2006a; Barrett *et al.*, 2005; Masanes and Winter, 2006). There is still an analogy with entanglement: nonlocal correlations are monogamous (Barrett *et al.*, 2006).

Concerning entanglement, Ekert noticed that the equivalence of entanglement based protocol and BB84 is not complete:<sup>88</sup> the former has an advantage that Alice and Bob can postpone measuring EPR pairs until they need the key, so that a burglar breaking into their labs trying to get some information, would disturb the pairs risking detection, while in BB84, there is no possibility of storing the key in quantum form. Thus entanglement provides *potential* key, in a similar way as it provides potential communication in dense coding (see Sec. III). However this is not the only advantage of entanglement: in fact its role turned out to be indispensable in further development of theory of secure correlations. Actually, the interaction is bilateral: also the development of entanglement theory was influenced in essential way by the ideas of secure correlations, to mention only, that first

protocols of entanglement distillation (fundamental for the whole quantum communication theory) have been designed by use of methods of generation of secure key (Bennett *et al.*, 1996c,d).

Another (at least theoretical) advantage of entanglement-based QKD protocol is that with quantum memory at disposal, one can apply dense coding scheme and obtain a protocol which has higher capacity than usual QKD as it was applied by Long and Liu (Long and Liu, 2002) (see also (Cabello, 2000)).

Moreover, entanglement may help to carry out quantum cryptography over long distances by use quantum repeaters which exploit entanglement swapping and quantum memory (Dür *et al.*, 1999a). We should note that all the above potential advantages of entanglement, would need quantum memory.

# 1. Entanglement distillation based quantum key distribution protocols.

Both Ekert's protocol, and its BBM version worked in situation, where the disturbance comes only from eavesdropper, so if only Alice and Bob detect his presence, they can abort the protocol. Since in reality one usually deals with imperfect sources, hence also with imperfect (noisy) entanglement, it is important to ask if the secure key can be drawn from noisy EPR pairs. The *purification of EPR pairs* appeared to be crucial idea in this case. The first scheme of purification (or *distillation*) of entanglement has been discovered and developed in (Bennett *et al.*, 1996c,d) (see Sec. XII). In this scheme Alice and Bob share *n* copies of some mixed state, and by means of *local quantum operations and classical communication* (LOCC) they obtain a smaller amount k < n of states which are very close to the EPR state.

$$\rho^{\otimes n} \xrightarrow{\text{Distillation}} |\phi^+\rangle^{\otimes k}. \tag{278}$$

The highest asymptotic ratio  $\frac{k}{n}$  on the above diagram is called *distillable entanglement* and denoted as  $E_D(\rho)$  (see Sec. XV.A). This concept was adopted in (Deutsch *et al.*, 1996), where distillation process for cryptographical purpose was named *Quantum Privacy Amplification* (QPA). From *n* systems in a joint state  $\rho_n$  (which may be in principle supplied by Eve), Alice and Bob distill singlets, and finally generate a key via measurement in computational basis:

$$\rho_n \xrightarrow{\text{QPA}} |\phi^+\rangle^{\otimes k}.$$
(279)

The protocol of QPA assumes that devices used for distillation are perfect. Moreover, the distillation scheme of Bennett *et al.* works if the initial state is in tensor product of many copies. The question of verification, by Alice and Bob whether they indeed share such state (or whether the final state is indeed the desired  $\phi^+ = \frac{1}{\sqrt{2}}|00\rangle + |11\rangle$  state) was not solved in (Deutsch *et al.*, 1996).

<sup>&</sup>lt;sup>87</sup> In BB84, the state consists of preparations of Alice, outcomes of Bob's measurement and quantum states of Eve. In BBM protocol it is outcomes of Alice and Bob's measurement and quantum states of Eve.

<sup>&</sup>lt;sup>88</sup> See note added in (Bennett *et al.*, 1992).

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This problem has been tackled by Lo and Chau (Lo and Chau, 1999) who have provided the first both unconditionally secure and fully entanglement-based scheme<sup>89</sup>. To cope with imperfections Alice and Bob use *fault tolerant quantum computing*. In order to obtain secure key, they perform entanglement distillation protocol of (Bennett *et al.*, 1996d) to distill singlets, and check their quality<sup>90</sup>.

The Lo-Chau proposal has a drawback: one needs a quantum computer to implement it. On the other hand, the first quantum cryptographic protocol (BB84) does not need a quantum computer. And the BB84 was already proved to be secure by Mayers (Mayers, 2001). Yet, the proof was quite complicated, and therefore not easy to generalize to other protocols.

## 2. Entanglement based security proofs

A remarkable step was done by Shor and Preskill (Shor and Preskill, 2000), who showed, that one can prove security of BB84 scheme, which is a P&M protocol, by considering *mentally* a protocol based on entanglement (a modified Lo-Chau protocol). This was something like the Bennett, Brassard, Mermin consideration, but in a noisy scenario. Namely, while using BB84 in presence of noise, Alice and Bob first obtain a so-called raw key - a string of bits which is not perfectly correlated (there are some errors), and also not perfectly secure (Eve have some knowledge about the key). By looking at the part of the raw key, they can estimate the level of error and the knowledge of Eve. They then classically process it, applying procedures of error correction and privacy amplification (the latter aims at diminishing knowledge of Eve).

In related entanglement based scheme, we have *coherent* analogues of those procedures. Without going into details, we can imagine that in entanglement based scheme, Alice and Bob share pairs in one of four Bell states (3), which may be seen as the state  $\phi^+$  with two kinds of errors: bit error, and phase error. The error from the previous scheme translates here into bit error, while knowledge of Eve's into phase error. Now the task is simply to correct both errors. Two procedures of a different kind (error correction and privacy amplification) are now both of the same type - they correct errors.

After correcting bit error, Alice and Bob are left with

Bell states which are all correlated

$$|\phi^{-}\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle),$$
  
$$|\phi^{+}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle).$$
 (280)

Then they apply phase error correcting procedure. I.e. they get to know which systems are in  $\phi^-$  and which in  $\phi^+$  so that they can rotate each  $\phi^-$  into  $\phi^+$  and finally obtain a sequence of  $\phi^+$  solely. What Shor and Preskill noticed is that these two quantum procedures are *coherent*<sup>91</sup> versions of classical error-correction and privacy amplification respectively. Thus Shor-Preskill proof can be phrased as follows: "The BB84 protocol is secure because its suitable coherent version distills EPR states".

It should be emphasized here, that the equivalence between noisy BB84 protocol and its coherent version does not continue to the very end. Namely, in distillation, Alice and Bob after finding which pair is in  $|\phi^-\rangle$  and which in  $|\phi^+\rangle$ , rotate  $|\phi^-\rangle$ . The classical procedures performed coherently cannot perform this very last step (rotation) as no classical action can act as phase gate after embedding into quantum. However, the key is secure, because Alice and Bob could have performed the rotation, but they do not have to. Indeed, note that if Alice and Bob measure the pairs in basis  $\{|0\rangle, |1\rangle\}$  they obtain the same results, independently of whether they have rotated  $|\phi^-\rangle$ or not. The very possibility of rotation, means that the key will be secure. Thus the coherent version of BB84 does not actually give  $|\phi^+\rangle$  itself, but it does if supplemented with rotations.

The concept of proving security of P&M protocols by showing that at the *logical level* they are equivalent to distillation of entanglement, has become very fruitful. In 2003 K. Tamaki, M. Koashi and N. Imoto (Tamaki *et al.*, 2003) showed that B92 is unconditionally secure, using Shor-Preskill method (see also (Tamaki and Lütkenhaus, 2004)). They showed that B92 is equivalent to special entanglement distillation protocol known as *filtering* (Gisin, 1996b; Horodecki *et al.*, 1997) (see section XII.D). In (Ardehali *et al.*, 1998) the efficient version of BB84 was proposed, which is still unconditionally secure though the

<sup>&</sup>lt;sup>89</sup> The first proof of unconditional security of quantum key distribution was provided by Mayers who proved security of BB84 (Mayers, 2001).

<sup>&</sup>lt;sup>90</sup> Using the concept of another entanglement-based communication scheme — quantum repeaters (Briegel et al., 1998; Dür et al., 1999a) Lo and Chau established quantum key distribution (QKD) over arbitrary long distances.

<sup>&</sup>lt;sup>91</sup> It is worth to observe that formally any QKD scheme can be made "entanglement based", as according to axioms of quantum mechanics any operation is unitary and the only source of randomness is the subsystem of an entangled state. From this point of view, even when Alice sends to Bob a randomly chosen signal state, as it is in P&M schemes, according to axioms she sends a part of entangled system. Moreover, any operation that Alice and Bob would perform on signal states in P&M scheme, can be done reversibly, so that the whole system shared by Alice Bob and Eve is in a pure state at each step of the protocol. This principle of maintaining purity is usually referred to as *coherent* processing. Let us note however, that not always the coherent application of a protocol that provides a key, must result in the distillation of  $\phi^+$ . In Sec. XIX.B.2 we will see that there is a more general class of states that gives private key.

number of systems that Alice and Bob use to estimate the error rate is much smaller than in BB84. Again security is proved in Shor-Preskill style. In (Gottesman and Lo, 2003) a P&M scheme with a *two-way* classical error correction and privacy amplification protocol was found. It is shown that a protocol with two-way classical communication can have substantially higher key rate than the one with the use of one-way classical communication only. Also security of key distribution using dense coding (Long and Liu, 2002) was proved by use of Shor-Preskill techniques (Zhang *et al.*, 2005) (see in this context (Degiovanni *et al.*, 2003, 2004; Wójcik, 2005)).

Thanks to simplicity of Shor-Preskill approach pure entanglement remained the best *tool for proving unconditional security* of QKD protocols (Gisin and Brunner, 2003). As we will see further this approach can be generalized by considering *mixed* entangled states containing ideal key.

## 3. Constraints for security from entanglement

So far we have discussed the role of entanglement in particular protocols of quantum key distribution. Α connection between entanglement and any QKD protocol has been established by Curty, Lewenstein and Lütkenhaus (Curty et al., 2004). They have proved that entanglement is necessary precondition of unconditional security. Namely, in case of any QKD protocol Alice and Bob perform some measurements and are left with some classical data from which they want to obtain key. Basing on these data and measurements settings, they must be able to construct a so called *entanglement wit*ness to ensure that the data could not be generated via measurement on some *separable* state (see Sec. VI.B.3). Let us emphasized, that this holds not only for entanglement based but also for prepare and measure protocols. In the latter case, a kind of "effective" entanglement is witnessed (i.e. the one which is actually never shared by Alice and Bob).

It is worth noting, that this approach can be seen as a generalization of the very first Ekert's approach (Ekert, 1991). This is because Bell inequalities can be seen as a special case of entanglement witnesses (see Sec. VI.B.5). Formally, in any OKD protocol, Alice and Bob perform repeatedly some POVM's  $\{A_i\}$  and  $\{B_i\}$  respectively, and obtain a probability distribution of the outcomes P(a,b). Now, a necessary condition for security of the protocol is that it is possible to build out some entanglement witness  $W = \sum_{i,j} c_{ij} A_i \otimes B_j$  with real  $c_{ij}$ , such that  $\sum_{ij} c_{ij} p_{ij} < 0$  and  $\text{Tr}W\sigma \ge 0$  for all separable states  $\sigma$ . Indeed, otherwise they could make key from separable states which is impossible (Gisin and Wolf, 2000) (see also discussion in sec XIX.B). This idea has been studied in case of high dimensional systems (Nikolopoulos and Alber, 2005; Nikolopoulos et al., 2006) and general upper bounds on key rates for prepare and measure schemes has been found (Moroder et al.,

2006a,b). It is also connected with optimization of entanglement measures from incomplete experimental data (see Sec. VI.B.4).

#### 4. Secure key beyond distillability - prelude

The fact that up to date techniques to prove unconditional security were based on entanglement purification i.e. distilling *pure* entangled states, has supported the belief that possibility of distilling pure entanglement (singlet) is the only reason for unconditional security. For this reason the states which are entangled but not distillable (i.e. bound entangled) found in (Horodecki et al., 1998a) have been considered as unlikely to be useful for cryptographical tasks. Moreover Acin, Masanes and Gisin (Acin *et al.*, 2003b), showed that in the case of two qubit states (and under individual Eve's attack) one can distill key by single measurement and classical postprocessing if and only if the state contains distillable entanglement. Surprisingly, as we will see further, it has been recently proved that one can obtain unconditionally secure key even from bound entangled states.

The first interesting step towards this direction was due to Aschauer and Briegel, who showed Lo and Chau's protocol provide key even without the fault tolerant computing i.e. with realistic noisy apparatuses (Aschauer and Briegel, 2002).

The crucial property of their approach was to consider all noise, (which they assumed to act *locally* in Alice and Bob's laboratories) in a *coherent* way. Keeping track of any noise introduced by imperfect devices, they can ensure that the total state of distilled *noisy singlets* together with the state of their laboratories is in fact a *pure* state. This holds despite the fact that the noisy singlets may have *fidelity*, that is overlap with a state (6), bounded away from maximal value 1. This kind of noisy entanglement, which is corrupted because of local noise, but not due to the eavesdropping, they have called *private entanglement*.

# B. Drawing private key from distillable and bound entangled states of the form $\rho^{\otimes n}$

Strong interrelation between theory of secure key and entanglement can already be seen in the scenario, where Alice and Bob share *n* bipartite systems in *the same state*  $\rho_{AB}$  and Eve holds their purification, so that the joint state of Alice, Bob and Eve systems is a pure state  $|\psi_{ABE}\rangle$ . The task of the honest parties is to obtain by means of *Local Operations and Classical Communication* (LOCC) the highest possible amount of correlated bits, that are unknown to Eve (i.e. a secure key). The difficulty of this task is due to the fact that Eve makes a copy of any classical message exchanged by Alice and Bob.

The above paradigm, allows to consider a new measure of entanglement: distillable key  $K_D$ , which is similar in spirit to distillable entanglement as it was discussed in Secs. XV.D.1 and XV.A. It is given by the number of secure bits of key that can be obtained (per input pair) from a given state.

Let us discuss in short two extreme cases:

• All distillable states are key distillable:

$$K_D(\rho_{AB}) \ge E_D(\rho_{AB}). \tag{281}$$

• All separable states are key non-distillable:

$$K_D(\sigma_{sep}) = 0. \tag{282}$$

To see the first statement, one applies the idea of quantum privacy amplification described above. Simply, Alice and Bob distill singlets and measure them locally. Due to "purity & correlation" principle, this gives a secure key.

To see that key cannot be drawn from separable states (Curty *et al.*, 2004; Gisin and Wolf, 2000), note that by definition of separability, there is a measurement on Eve's subsystem such that conditionally upon result (say *i*) Alice and Bob share a *product* state  $\rho_A^{(i)} \otimes \rho_B^{(i)}$ . This means that Alice and Bob conditionally on Eve have initially no correlations. Of course, any further communication between Alice and Bob cannot help, because it is monitored by Eve.

In Sec. XIX.B.6 it will be shown, that there holds:

• There are non-distillable states which are key distillable:

$$E_D(\rho_{be}) = 0 \quad \& \quad K_D(\rho_{be}) > 0.$$
 (283)

#### 1. Drawing key from distillable states: Devetak-Winter protocol

Here we will present a protocol due to Devetak and Winter, which shows that from any state, one can draw at least amount of key equal to coherent information. This is compatible with the idea, that one can draw key only from entangled states (states with positive coherent information are entangled, as shown in (Horodecki and Horodecki, 1994)). The coherent version of the protocol will in turn distill this amount of singlets from the state. In this way Devetak and Winter have for the first time proved the hashing inequality (XV.F.1) for *distillable entanglement*. Thus, again cryptographic techniques allowed to develop entanglement theory.

Namely, consider a state  $\rho_{AB}$ , so that the total state including Eve's system is  $\psi_{ABE}$ . As said, we assume that they have *n* copies of such state. Now Alice performs complete measurement, which turns the total state into  $\rho_{cqq} = \sum_i p_i |i\rangle \langle i|_A \otimes \rho_{BE}^i$  where subscript *cqq* reminds that Alice's system is classically correlated with Bob and Eve subsystems. The authors considered drawing key from a general cqq state as a starting point, and showed that one can draw at least the amount of key equal to

$$I(A:B) - I(A:E),$$
 (284)

where I(X : Y) = S(X) + S(Y) - S(XY) is quantum mutual information.

Without going into details, let us note that the quantity I(A:B) is the common information between Alice and Bob, hence it says how many correlated bits Alice and Bob will obtain via error correction. Since I(A:E)is common information between Eve and Alice, its subtraction means, that in the procedure of privacy amplification this amount of bits has to be removed from key, to obtain a smaller key, about which Eve does not know anything.

Now, let us note that in the present case we have

$$I(A:B) = S(\rho_B) - \sum_{i} p_i S(\rho_B^i),$$
  
$$I(A:E) = S(\rho_E) - \sum_{i} p_i S(\rho_E^i),$$
 (285)

where  $\rho_B^i = \text{Tr}_E \rho_{BE}^i$ ,  $\rho_E^i = \text{Tr}_B \rho_{BE}^i$ ,  $\rho_B = \sum_i p_i \rho_i^B = \text{Tr}_A \rho_{AB}$ , and  $\rho_E = \sum_i p_i \rho_i^E = \text{Tr}_{AB} \rho_{AE}$ .

Since the measurement of Alice was complete, the states  $\rho_{BE}^i$  are pure, hence  $S(\rho_E^i) = S(\rho_B^i)$ . Also, since the total initial state was pure, we have  $S(\rho_E) = S(\rho_{AB})$  where  $\rho_{AB}$  is initial state shared by Alice and Bob. Thus we obtain that the amount of key gained in the protocol, is actually equal to coherent information.

Now, Devetak and Winter have applied the protocol coherently, and obtained protocol of distillation of singlets proving first general lower bound for distillable entanglement

$$E_D(\rho_{AB}) \ge S(\rho_B) - S(\rho_{AB}). \tag{286}$$

This was also used by Devetak to provide the first fully rigorous proof of quantum Shannon theorem, saying that capacity of quantum channel is given by coherent information (Devetak, 2003). Thus cryptography was used also to develop entanglement and quantum communication theory.

#### 2. Private states

In the previous section we have invoked a protocol, which produces a private key, and if applied coherently, distills singlets. A fundamental question which naturally arises is how far this correspondence goes. In particular, one can ask whether it is possible to distill key only from states from which singlets can be distilled. Surprisingly, it has been shown that one can obtain key even from certain bound entangled states (Horodecki *et al.*, 2005b,e). To this end they have characterized class of states which have subsystem that measured in certain product basis gives  $\log d$  bits of perfect key. The present section is mostly devoted to this class of states which apart from being fundamental for analysis of key distillation constitute a new quality in mixed state entanglement, hence it is also of its own interest. These states, called *private states* or gamma states, have been proved to be of the following form<sup>92</sup>

$$\gamma^{(d)} = \frac{1}{d} \sum_{i,j=0}^{d-1} |ii\rangle \langle jj|_{AB} \otimes U_i \rho_{A'B'} U_j^{\dagger}, \qquad (287)$$

where  $U_i$  are arbitrary unitary transformations acting on the system A'B'. The whole state resides on two systems with distinguished subsystems AA' and BB' respectively. The AB subsystem, after measuring in computational basis, gives  $\log d$  bits of key, hence it is called the *key part* of the private state. A private state with  $d \times d$ dimensional key part is called shortly a *pdit* and in special case of d = 2 a *pbit*. The subsystem A'B' aims to *protect* the key part, hence it is referred to as a *shield* <sup>93</sup>. The class of private state has been generalized to the multipartite case (Horodecki and Augusiak, 2006).

#### 3. Private states versus singlets

A private state can be obtained via unitary transformation from a simple private state called *basic pdit* 

$$\gamma_{basic}^{(d)} = |\Phi^+\rangle \langle \Phi^+|_{AB} \otimes \rho_{A'B'}.$$
 (288)

with<sup>94</sup>.  $|\Phi^+\rangle = \sum_{i=0}^{d-1} \frac{1}{\sqrt{d}} |ii\rangle$ . Namely the following controlled transformation, called *twisting*,

$$U_{\tau} = \sum_{kl} |kl\rangle_{AB} \langle kl|_{AB} \otimes U_{kl}^{A'B'}$$
(289)

transforms  $\gamma_{basic}$  into a private state  $\gamma$  given by (287) <sup>95</sup>. In general the unitary  $U_{\tau}$  can be a highly nonlocal operation, in the sense that it cannot be inverted by means of local operations and classical communication. Thus the  $|\Phi^+\rangle$  is contained somehow in the private state, but in a "twisted" way: Therefore, unlike in the  $|\Phi^+\rangle$  state,  $E_D$ can be much smaller than  $E_C$ , as shown by the following example (Horodecki *et al.*, 2005e):

$$\rho_{\gamma} = p |\phi^{+}\rangle \langle \phi^{+}| \otimes \rho_{s} + (1-p) |\phi^{-}\rangle \langle \phi^{-}| \otimes \rho_{a}$$
 (290)

where  $|\phi^{\pm}\rangle$  are the maximally entangled states,  $\rho_{s,a}$  are normalized projectors onto symmetric and antisymmetric subspaces and  $p = \frac{1}{2}(1 - \frac{1}{d})$ . One computes logarithmic negativity, which bounds  $E_D$ , to obtain

$$E_D(\rho_\gamma) \le \log(1 + \frac{1}{d}). \tag{291}$$

One can show that in general  $K_D \leq E_C$ , so that  $E_C(\rho_{\gamma}) \geq 1$ .

Although twisting can diminish distillable entanglement, it can not set it to zero. Indeed it was shown that all private states are distillable (Horodecki and Augusiak, 2006).

Interestingly, private state can be written as  $|\Phi^+\rangle$ , where amplitudes have been replaced by q-numbers:

$$\gamma = \Psi \Psi^{\dagger}, \tag{292}$$

where  $\Psi$  is the matrix (*not necessary* a state) similar to a pure state,

$$\Psi = \sum_{i=0}^{d-1} Y_i^{A'B'} \otimes |ii\rangle_{AB}, \qquad (293)$$

with  $Y_i = U_i \sqrt{\frac{D}{d}}$  for some unitary transformation  $U_i$ and certain state  $\rho$  on A'B'. Looking at the state  $|\psi\rangle = \sum_{j=0}^{d-1} \frac{1}{\sqrt{d}} e^{i\phi_j} |jj\rangle$  we see that  $Y_i$  is a noncommutative version of  $\frac{1}{\sqrt{d}} e^{i\phi_j}$ . This special form has not yet been studied thoughtfully. It seems that focusing on non-commutativity of operators  $Y_i$  may lead to better understanding of the properties of this class of states.

#### 4. Purity and correlations: how they are present in p-bit

In the introductory paragraph to Sec. XIX we have pointed out that the origin of private correlations in quantum mechanics is the existence of pure state which still exhibits correlations (unlike in classical world). However, we have argued then that there are states which exhibit private key, but are quite mixed. They can have arbitrarily small distillable entanglement, still offering bits of perfect key. One might wonder if it implies that we need a more general principle, which explains privacy obtained in quantum mechanics. There is no simple solution of this problem. Let us first argue, that although in different context, the principle of "purity & correlations" still works here.

To this end, let us note that private state is a unitarly rotated basic p-bit. The latter is of the form  $\gamma_{basic} = |\Phi^+\rangle \langle \Phi^+|_{AB} \otimes \rho_{A'B'}$  (see Eqs. (288) and (289)). The unitary  $U_{\tau}$  might melt completely the division of  $\gamma_{basic}$  into subsystems A, B, A' and B'. However what is for sure preserved, is that the private state is a product of some pure state  $\psi$  and  $\rho$  across some tensor product. In other words, there is a virtual two-qubit subsystem, which is in singlet state  $\psi = \phi^+_{virt}$ . (This virtual singlet is what we call "twisted" version of the original singlet state.) Because of purity, the outcomes of any measurement performed on this virtual subsystem are not known to Eve. Thus we already have one ingredient of the principle — the role of purity. Let us now examine the second ingredient — correlations. The pure state is no longer a state of two qubits, one of them on Alice side, and the

<sup>&</sup>lt;sup>92</sup> More precisely, the class of private states is locally equivalent to the one given by equation (287), that is contains any state  $\gamma^{(d)}$ rotated by local unitary transformation  $U_A \otimes U_B \otimes I_{A'B'}$ .

<sup>&</sup>lt;sup>93</sup> Dimensions  $d_{A'}$  and  $d_{B'}$  are in principle arbitrary (Without losing generality, we can put them equal).

<sup>&</sup>lt;sup>94</sup> In this section we will refer to the state  $|\Phi^+\rangle$  as well as  $|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$  as to singlet state.

<sup>&</sup>lt;sup>95</sup> The unitaries  $U_i$  and  $U_j^{\dagger}$  in definition of private state corresponds to  $U_{kk}$  and  $U_{ll}^{\dagger}$  in definition of twisting.

second on Bob's. The measurements that access individually the virtual qubits are the original local measurements of Alice and Bob, rotated however by the inverse of unitary  $U_{\tau}$ . Now, it happens that local measurements on key part (AB subsystem) of the private state in standard basis commute with  $U_{\tau}$  (289), for the latter is a unitary controlled just by this basis. Thus at least one correlation measurement on the virtual qubits is still accessible to Alice and Bob, and can be performed on the qubits A and B.

It is nevertheless natural quite natural to maintain that the "purity & correlations" is too strong principle for ensuring security. Simply, the purity of a virtual subsystem is not the purity of the total state that Alice and Bob have. In the case of basic p-bit, it is easy to obtain the whole system in a pure state, simple by removing the mixed subsystem A'B'. However, for a generic private state, the purity is inaccessible: the state of the whole system is irrevocably mixed. This is because Alice and Bob have only local access to the system, hence from some p-bits Alice and Bob may be able to distill only an amount of purity much smaller than the amount of privacy they can get. Even more: if we allow arbitrary small inaccuracy, then they may not be able to obtain a system in a pure correlated state at all, while still obtaining secure key.

Thus to get privacy, one does not need to share pure states. There is a simple explanation how it can be so: A pure state  $\Phi^+$  offers secure key in *any* correlated basis, while what we need is just a single secure basis, because we need only classical secure key - pair of bits.

## 5. Distillable key as an operational entanglement measure

The concept of private states allows to represent  $K_D$  as a quantity analogous to entanglement distillation:

$$\rho_{AB}^{\otimes n} \xrightarrow{\text{LOCC key}} \gamma_{ABA'B'}^{(d)}, \qquad (294)$$

where the highest achievable ratio  $\frac{\log d}{n}$  in the asymptotic limit equals the distillable key denoted as  $K_D(\rho_{AB})$ . Instead of singlets we distill private states. Since the class of private states is broader than the class of maximally entangled states, one can expect that distillable key can be greater than distillable entanglement. Indeed, this is the case, and an example is just the private state of Eq. (290).

Thus distillable key is a new operational measure of entanglement, which is distinct from  $E_D$ . It is also distinct from  $E_C$  and it satisfies

$$E_D \le K_D \le E_C. \tag{295}$$

Moreover it is upper bounded by relative entropy of entanglement (Horodecki *et al.*, 2005e) and squashed entanglement (Christandl, 2006). There is also a bound involving the best separable approximation (Moroder *et al.*, 2006b) which exploits the fact that admixing separable state can only decrease  $K_D$ . In (Moroder *et al.*, 2006a) there is also a bound for one-way distillable key, based on the fact that for a state which has a symmetric extension, its one-way distillable key must vanish. Indeed, then Bob and Eve share with Alice the same state, so that any final key which Alice shares with Bob, she also share with Eve.

## 6. Drawing secure key from bound entanglement.

Bound entanglement is a weak resource, especially in the bipartite case. For a long time the only useful task that bipartite BE states were known to perform was activation, where they acted together with some distillable state. Obtaining private key from bound entanglement, a process which we will present now, is the first useful task which bipartite BE states can do *themselves*.

Since distillable entanglement of some private states can be low it was tempting to admix with small probability some noise in order to obtain a state which is non-distillable while being still entangled:

$$\rho_{total} = (1 - p)\gamma + p\rho_{noise} \tag{296}$$

It happens that for certain private states  $\gamma$  the state  $\rho_{noise}$  can be adjusted in such a way that the state  $\rho_{total}$  is PPT (hence  $E_D = 0$ ), and despite this, from many copies of  $\rho_{total}$  one can distill key of arbitrarily good quality. That is one can distill private state  $\gamma'$  with arbitrary small admixture of noise.

The first examples of states with positive distillable key and zero distillable entanglement were found in (Horodecki *et al.*, 2005b,e). We present here a simple one which has been recently given in (Horodecki *et al.*, 2005f). It is actually a mixture of two private bits (correlated and anticorrelated). The total state has a matrix form

$$\rho_{ABA'B'} = \frac{1}{2} \begin{bmatrix} p_1 |X_1| & 0 & 0 & p_1 X_1 \\ 0 & p_2 |X_2| & p_2 X_2 & 0 \\ 0 & p_2 X_2^{\dagger} & p_2 |X_2| & 0 \\ p_1 X_1^{\dagger} & 0 & 0 & p_1 |X_1| \end{bmatrix}, (297)$$

with  $X_1 = \sum_{i,j=0}^{1} u_{ij} |ij\rangle \langle ji|_{A'B'}$  and  $X_2 = \sum_{i,j=0}^{1} u_{ij} |ii\rangle \langle jj|_{A'B'}$  where  $u_{ij}$  are the elements of 1 qubit Hadamard transform and  $p_1 = \frac{\sqrt{2}}{1+\sqrt{2}} (p_2 = 1-p_1)$ . This state is invariant under partial transposition over Bob's subsystem. If we however project its key part (*AB* subsystem) onto a computational basis it turns out that the joint state of Alice, Bob and Eve system is fully classical and of very simple form: with probability  $p_1$  Eve knows that Alice and Bob are correlated, while with probability  $p_2$  that they are anticorrelated. Thus the mutual information I(A : E) = 0, and

 $I(A:B) = 1 - H(p_1)$ . Thus applying Devetak-Winter protocol<sup>96</sup> (see formula (284)) we obtain a key rate

$$K_D(\rho_{be}) \ge 1 - h(p_1) = 0.0213399 > E_D(\rho_{be}) = 0.$$
 (298)

Basing on this example it is argued (Horodecki *et al.*, 2005f), that the volume of bound entangled key distillable states is non-zero in the set of states occupying more then 4 qubits. It is however a nontrivial task to provide new examples. Interestingly, no previously known bound entangled state has been shown to be key-distillable.

# C. Private states — new insight into entanglement theory of mixed states

Investigations concerning distillable key were fruitful to entanglement theory itself. A new operational measure of entanglement was obtained, and also a new source of examples of irreversibility in entanglement distillation was provided. The private states, or PPT states with nonzero key, constitute a new zoo of states which are easy to deal with and have nontrivial properties, in addition to such canonical classes as Werner, isotropic, Bell diagonal states or maximally correlated states. While the simplicity of the latter classes comes from symmetries (e.g. invariance under twirling), simplicity of the class of private states is based just on special asymmetry between the key and the shield part.

The private bits called *flower states* (given by trace over subsystem E of (272)) are the ones for which the squashed entanglement has been computed. Interestingly, in this case  $E_{sq} = E_C$ . Moreover basing on this family one can find states with  $E_C$  arbitrarily greater than the squashed entanglement and the latter arbitrarily greater than  $E_D$  (Christandl and Winter, 2003). The flower states exhibit locking (see Sec. XVIII.A). There is actually a general link between locking effect and the problem of drawing key from bound entanglement. Last but not least, the description of this class of states yields a natural generalization of pure maximally entangled states to the case of mixed states with coefficients becoming operators.

# D. Quantum key distribution schemes and security proofs based on distillation of private states - private key beyond purity

Key distillation described in Sec. XIX.B relies upon important assumption. The initial state shared by Alice and Bob should be tensor product of the same state  $\rho_{AB}$ . This assumption is unreal in almost all security applications, since the eavesdropper can interrupt the communication and entangle copies of the state. It was then unclear whether one can obtain gap between distillable key and distillable entanglement (as reported in Sec. XIX.B.5) in the general scenario, where Alice and Bob do not know a priori anything about their states. It hasn't been noticed, that a positive answer to this question follows from the results on finite Quantum de Finetti theorem by R. Renner, N. Gisin and B. Kraus (Kraus et al., 2005; Renner et al., 2005) (see especially (Renner, 2005) and (Koenig and Renner, 2005)) and the results of (Horodecki et al., 2005e) on bound entangled key distillable states. In the meantime, a more entanglement-based approach has been developed (Horodecki et al., 2006a,c), which can be seen as a generalization of Lo and Chau entanglement purification based approach to the private state distillation one. It has been shown there, that an unconditionally secure key can be arbitrarily greater than the amount of entanglement which can be distilled, and even when the latter is zero. This important result can be rephrased as follows:

• There are situations in which one can not send *faithfully* any *qubit*, but one can send arbitrarily many *unconditionally secure bits*.

## 1. "Twisting" the standard protocol

The situation is as follows. Alice and Bob have quantum channel  $\Lambda$ , which can only produce bound entangled states (this is so called binding entanglement channel). Clearly, quantum information cannot be sent faithfully via such channel: otherwise it would be possible to transmit faithfully singlets(see Sec. XIV). Suppose further that if Alice sends half of singlet, then, provided Eve has not changed the channel, they obtain a state  $\rho$  of type (297). (i.e. the state  $\rho$  and the channel  $\Lambda$  are connected with each other via Choi-Jamiołkowski isomorphism  $(I \otimes \Lambda) |\Phi^+\rangle \langle \Phi^+| = \rho$ ). The state has the property that after measuring the key part in standard basis and processing the outcomes classically, Alice and Bob obtain secure key.

Since Eve might change the channel, standard approach would be to check bit error rate and phase error rate. Unfortunately, this would not work: the channel itself (without any action of Eve) produces too high errors. Indeed, low error rates would mean that the state is close to singlet, which is not the case: the state  $\rho$  is bound entangled hence its fidelity with singlet is no better than that for separable (i.e. key undistillable) state (see Sec. VI.B.3).

To overcome this problem, we have to use the fact that (as discussed in Secs. XIX.B.2 and XIX.B.3) private states contain perfect singlet twisted by some global unitary transformation. Therefore the present state, which is a noisy version of private state, contains a twisted *noisy* singlet. Should Alice and Bob have access to the twisted noisy singlet, they would estimate its quality by measur-

<sup>&</sup>lt;sup>96</sup> Since in this particular case the state is fully classical, it would be enough to use the classical predecessor of Devetak-Winter protocol, called Csiszar-Körner-Maurer protocol.

ing on selected number of systems the observables  $\sigma_z \otimes \sigma_z$ (bit error) and  $\sigma_x \otimes \sigma_x$  (phase error). In the case of twisted singlets, they can still estimate directly bit error by measuring the observable  $\sigma_z \otimes \sigma_z$  on the key part of the shared states. However the observable  $\sigma_x \otimes \sigma_x$  does not commute with twisting, so that to estimate phase error of a twisted noisy singlet means to estimate a nonlocal observable. Fortunately, this can be done, by decomposing the observable into local ones, and measuring them separately <sup>97</sup>.

The rest of the protocol, is that Alice and Bob measure the key part of the remaining systems (i.e. not used for error testing), and performs standard error correction/privacy amplification procedures based on estimated error level. How to see that such protocol is secure? One way is to notice, that it does the same to twisted state, as the standard BB84-like protocol (such as e.g. (Ardehali *et al.*, 1998; Lo *et al.*, 2005)) does to the usual state. How this looks like in terms of entanglement purification? The standard protocol is secure, because it would produce singlets. The present protocol is secure, because, if performed coherently it produces *private state*, i.e. it is twisted purification. Of course, one can easily convert it into P&M protocol along Shor-Preskill lines.

We have arrived at general principle, which is actually "if and only if": A protocol produces secure key if and only if its coherent version produces private states.

This statement connects entanglement and key distribution in an ultimate way. It was recently applied by Renes and Smith (Renes and Smith, 2007) who have found an entanglement based proof of the P&M protocol with noisy preprocessing of (Kraus *et al.*, 2005; Renner *et al.*, 2005). They have demonstrated its *coherent* version which distills private states, and hence must be secure.

We have quite a paradoxical situation. When it turned out that one can draw secure key in the situation where no singlets can be distilled, it seemed natural that to prove unconditional security, one cannot use the techniques based on entanglement purification. Surprisingly, it is not the case: everything goes through, in a twisted form.

#### E. Entanglement in other cryptographic scenarios

There are many other quantum cryptographic scenarios than quantum key agreement, where entanglement enters. Here we comment briefly on some of them. Interestingly, this time entanglement is important not only because it makes some protocols possible (like QKD, quantum secret sharing, third man quantum cryptography), but because it disallows certain schemes (like quantum bit commitment, quantum one-out-of-two oblivious transfer).

## 1. Impossibility of quantum bit commitment — when entanglement says no

Historically, it was claimed that QIT can ensure not only unconditionally secure key distribution but also a very important ingredient of classical cryptographic protocols — a bit commitment protocol (Brassard et al., 1993). If so, Alice could *commit* some decision (a bit value) to Bob, so that after committing she could not change her mind (change the bit value) but Bob also could not infer her decision before she lets to open it. Such a protocol would be one of the most important ingredients in secure transaction protocols. Unfortunately, it is not the case: Mayers (Mayers, 1996, 1997) and independently Lo and Chau (Lo and Chau, 1997, 1998) have proved under assumptions plausible in cryptographic context, that quantum bit commitment is not possible. Paradoxically, it is exactly entanglement, which assures security of QKD, that is the main reason for which the quantum bit commitment is not possible. It shows the following important fact: When the two parties do not trust each other, entanglement between them may sometimes become the most unwanted property.

There were many attempts to perform quantum bit commitment; some of them invalid as covered by the proof given by Lo and Chau and some of them being approximated versions of impossible quantum bit commitment.

While the proof of Lo and Chau is valid, as it was pointed out by H. P. Yuen (Yuen, 2005) one could weaken assumptions, so that the Lo-Chau theorem does not apply. Namely, the initial state of Bob in this two-party protocol may not be fixed at the beginning. It leads to considerations similar to a game theoretic situation. This case is almost covered by the latest result of D'Ariano *et al.* (D'Ariano *et al.*, 2006). The latter paper also provides the most recent and wide review of this topic.

It is important to note, that the same impossibility reasons share other desired protocols such as *ideal quantum coin tossing, one-out-of-two oblivious transfer* and *one-sided two party secure computations* (Lo, 1997; Lo and Chau, 1998).

#### 2. Multipartite entanglement in quantum secret sharing

There are situations in which one does not trust a single man. A secret should be shared by a few people so that no single person could know the truth without permission of the others. Classically, splitting information into pieces among parties, so that each piece is informationless unless they cooperate, is known as *secret sharing* (Blakley, 1979; Shamir, 1979).

<sup>&</sup>lt;sup>97</sup> One needs quantum de Finetti theorem to prove that such separate estimation will report correctly the value of global observable.

Quantum secret sharing protocol, as proposed in (Hillary *et al.*, 1999), solves the following related problem, which can be viewed as secure secrete sharing. Charlie wants to perform secret sharing, but is far from Alice and Bob, thus he has to face the additional problem eavesdropping. He could resolve it if he had two separate secret channels with them, sending to one person a key, and to the other the encoded message (secret). However there is a more direct way discovered in (Hillary *et al.*, 1999), using  $|GHZ\rangle$  state

$$|GHZ\rangle_{ABC} = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) \tag{299}$$

In their protocol all three parties are supplied many copies of the above multipartite state. Then each of them measures randomly either in  $\{|+\rangle, |-\rangle\}$  basis, or in  $\{\frac{1}{\sqrt{2}}(|0\rangle+i|1\rangle), \frac{1}{\sqrt{2}}(|0\rangle-i|1\rangle)\}$ . After this step each party announces the basis that was chosen, but not the result. In half of the cases, Alice and Bob can infer the result of Charlie's measurement when they meet and compare the results. Thus the results of Charlie's measurement, which are random, can serve as a key, and Alice and Bob can receive its *split* copy. Moreover it can be shown, that this protocol is secure against any eavesdropping and even a cheating strategy of either Alice or Bob.

We have discussed sharing of classical information by means of quantum entanglement. One can also share quantum information, and entanglement again helps (Hillary et al., 1999). This time Charlie wants to split securely a qubit, so that Alice and Bob would need to cooperate to recover it. Interestingly, sharing a quantum secret, is essentially teleportation of a qubit through a GHZ state. After Charlie performs teleportation, the qubit is split between Alice and Bob. To reconstruct the qubit one of the parties (i.e. Alice) measures his qubit in  $\{|+\rangle, |-\rangle\}$  basis and sends the result of his measurement to the other party (i.e. Bob). Finally the other party applies on of two unitary operations.

This scheme has been further developed within the framework of quantum error correcting codes in (Cleve *et al.*, 1999; Gottesman, 2000). Interestingly, the protocol of Hillery at al. can be changed so that it uses bipartite entanglement only (Karlsson *et al.*, 1999). This simplification made it possible to implement the scheme in 2001 (Tittel *et al.*, 2001). For the recent experimental realization see (Chen *et al.*, 2005c), and references therein.

#### 3. Other multipartite scenarios

One of the obvious generalization of quantum key distribution is the so called *conference key agreement*. When some n parties who trust each other want to talk securely, so that each of them could receive the information, they could do the following: one party makes n - 1 QKD "bipartite" protocols with all the other parties, and then using this distributes a single key, which all the parties will share finally. This however can be achieved in a much simpler way using a multipartite entangled state. A state which is mostly used for this task is the n partite GHZ state (Chen and Lo, 2004). One can also use for this task multipartite version of private states (Horodecki and Augusiak, 2006).

Another interesting application of multipartite entangled states is the so called *third man quantum cryptog*raphy (Zukowski et al., 1998b). This cryptographic scenario involves traditionally Alice and Bob, and a third person Charlie, who controls them. Only when Charlie wants, and tells them some information, they can share a quantum private channel and they verify, that indeed nobody including Charlie have access to this channel. This can be easily achieved by employing the  $|GHZ\rangle$  state and the idea of "entanglement with assistance" (see Sec. XV). Any two-qubit subsystem of  $|GHZ\rangle$  state is in separable state, hence useless for cryptography. However if Charlie measures a third qubit in  $\{|+\rangle, |-\rangle\}$  basis, and tells Alice and Bob holding other two qubits the result of his measurement then they obtain one of the maximally entangled states. Then Alice and Bob can verify that they indeed share the entangled states and use them to launch a QKD protocol.

# F. Interrelations between entanglement and classical key agreement

So far we have discussed the role of entanglement in quantum cryptography. It is interesting, that entanglement, which is originally quantum concept, corresponds to privacy in general - not only in the context of quantum protocols. Here the interaction between entanglement theory and the domain of classical cryptography called *classical key agreement* (CKA) is presented.

The problem of distilling secret key from correlations shared by Alice and Bob with presence of an eavesdropper Eve was first studied by Wyner (Wyner, 1975) and Csiszar and Korner (Csiszar and Körner, 1978). It was introduced as a classical key agreement scenario and studied in full generality by Maurer (Maurer, 1993). According to this scenario, Alice and Bob have access to n independent realizations of variables A and B respectively, while the malicious E holds n independent realizations of a variable Z. The variables under consideration have joint probability distribution P(A, B, E). The task of Alice and Bob is to obtain via local (classical) operations and public communication (LOPC) the longest bit-string which is almost perfectly correlated and about which Eve (who can listen to the public discussion) knows a negligible amount of information<sup>98</sup>.

<sup>&</sup>lt;sup>98</sup> Let us emphasize, that unlike in quantum cryptography, it is in principle not possible to bound Eve's information on classical

Here the probability distribution P(A, B, E) is a priori given. I.e. it is assumed, that Alice and Bob somehow know how Eve is correlated with their data.

# 1. Classical key agreement — analogy to distillable entanglement scenario

Classical key agreement scenario is an elder sibling of the entanglement distillation-like scenario. This relation was first found by N. Gisin and S. Wolf (Gisin and Wolf, 1999, 2000), and subsequently developed in more generality by Collins and Popescu (Collins and Popescu, 2002). The analogy has been explored in the last years and proved to be fruitful for establishing new phenomena in classical cryptography, and new links between privacy and entanglement theory. The connections are quite beautiful, however they still remain not fully understood by now.

The classical key agreement task is described by the following diagram:

$$[P(A, B, E)]^{\otimes n} \xrightarrow{\text{key}} [P(K, K, E')]^{\otimes k}, \quad (300)$$

where P(K, K, E') is perfectly secure distribution satisfying:

$$P(K,K) \equiv \{P(i,j) = \frac{1}{2}\delta_{ij}\}$$
$$P(K,K,E') = P(K,K)P(E')$$
(301)

where Alice and Bob hold variable K and E' is some Eve's variable, i.e. Alice and Bob are perfectly correlated and product with Eve. The optimal ratio  $\frac{k}{n}$  in asymptotic limit is a (classical) distillable key rate denoted here as K(A; B||E) (Maurer, 1993).

Entanglement between two parties (see Sec. XIX) reports that nobody else *is correlated* with the parties. In similar way the privacy of the distribution P(A, B, E) means that nobody knows about (i.e. is classically correlated with) the variables A and B. In other words, any tripartite joint distribution with marginal P(A, B) has a *product* form P(A, B)P(E).

Following along these lines one can see the nice correspondence between maximally entangled state  $|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$  and the private distribution (301), and also the correspondence between the problem of transformation of the state  $\rho_{AB}^{\otimes n}$  into maximally entangled states which is the entanglement distillation task and the above described task of classical key agreement. Actually, the first entanglement distillation schemes (Bennett *et al.*,

entanglement theory	key agreement
quantum entanglement	secret classical correlations
$\operatorname{quantum}$ communication	secret classical communication
classical communication	public classical communication
local actions	local actions

TABLE I Here we present relations between basic notions of key agreement and entanglement theory following (Collins and Popescu, 2002)

1996c,d) have been designed on basis of protocols of classical key agreement. The feedback from entanglement theory to classical key agreement was initiated by Gisin and Wolf (Gisin and Wolf, 2000) who asked the question, whether there is an analogue of bound entanglement, which we discuss in next section. Subsequently, in analogy to entanglement cost which measures how expensive in terms of singlet state is the creation of a given quantum state  $\rho_{AB}$  by means of LOCC operations Renner and Wolf (Renner and Wolf, 2003) have defined information of formation denoted as  $I_{form}(A; B|E)$  (sometimes called "key cost"). This function quantifies how many secure key bits (301) the parties have to share so that they could create given distribution P(A, B, E) by means of LOPC operations. The axiomatic approach to privacy, resulting in deriving secrecy monotones (also in multipartite case). has been studied in (Cerf et al., 2002b; Horodecki et al., 2005c).

An interesting formal connection between CKA and entanglement is the following (Gisin and Wolf, 2000). Any classical distribution can be obtained via POVM measurements on Alice, Bob and Eve's subsystems of a pure quantum state  $|\psi\rangle_{ABE}$ :

$$p_{ijk}^{ABE} := \text{Tr}M_A^{(i)} \otimes M_B^{(j)} \otimes M_E^{(k)} |\psi\rangle\langle\psi|_{ABE} \qquad (302)$$

where  $M_{A,B,E}^{(.)}$  are elements of POVM's of the parties satisfying  $\sum_{l} M_{A,B,E}^{(l)} = \mathbf{I}_{A,B,E}$  with I the identity operator.

Conversely, with a given tripartite distribution P(A, B, E) one can associate a quantum state in the following way:

$$P(ABE) = \{p_{ijk}^{ABE}\} \longmapsto |\psi_{ABE}\rangle = \sum_{ijk} \sqrt{p_{ijk}^{ABE}} |ijk\rangle_{ABE}.$$
(303)

According to this approach, criteria analogous to those for pure bipartite states transitions and catalytical transitions known as majorization criteria (see Secs. XIII.A and XIII.A.1) can be found (Collins and Popescu, 2002; Gisin and Wolf, 2000). Also other quantum communication phenomena such as *merging* (Horodecki *et al.*, 2005h) and *information exchange* 

ground. However, in particular situations, there may be good practical reasons for assuming this.

(Oppenheim and Winter, 2003) as well as the no-cloning principle are found to have counterparts in CKA (Oppenheim *et al.*, 2002b).

A simple and important connection between tripartite distributions containing privacy and entangled quantum states was established in (Acin and Gisin, 2005). Consider a quantum state  $\rho_{AB}$  and its purification  $|\psi_{ABE}\rangle$  to the subsystem E.

- If a bipartite state  $\rho_{AB}$  is *entangled* then there exists a measurement on subsystems A and B such, that for all measurements on subsystem E of purification  $|\psi_{ABE}\rangle$  the resulting probability distribution P(A, B, E) has *nonzero* key cost.
- If a bipartite state  $\rho_{AB}$  is *separable*, then for all measurements on subsystems A and B there exists a measurement subsystem E of purification  $|\psi_{ABE}\rangle$  such that the resulting probability distribution P(A, B, E) has zero key cost.

#### 2. Is there a bound information?

In the entanglement distillation scenario there are bound entangled states which exhibit the highest irreversibility in creation-distillation process, as the distillable entanglement is zero although the entanglement cost is a nonzero quantity (see Sec. XII). One can ask then if the analogous phenomenon holds in classical key agreement called *bound information* (Gisin and Wolf, 2000; Renner and Wolf, 2003). This question can be stated as follows:

• Does there exist a distribution  $P(A, B, E)_{bound}$  for which a secure key is needed to create it by LOPC  $(I_{form}(A; B|E) > 0)$ , but one cannot distill any key back from it (K(A; B||E) = 0)?

In (Gisin and Wolf, 2000) the tripartite distributions obtained via measurement from bound entangled states were considered as a possible way of search for the hypothetical ones with bound information. To get Eve's variable, one has first to purify a bound entangled state, and then find a clever measurement to get tripartite distribution. In this way, there were obtained tripartite distributions with non-zero key cost. However the no-key distillability still needs to be proved.

Yet there are serious reasons supporting conjecture that such distributions exist (Gisin *et al.*, 2002; Renner and Wolf, 2002). To give an example, in (Renner and Wolf, 2002) an analogue of the necessary and sufficient condition for entanglement distillation was found. As in the quantum case the state is distillable iff there exists a projection (acting on n copies of a state for some n) onto 2-qubit subspace which is entangled (Horodecki *et al.*, 1998a), in the classical case, the key is distillable iff there exists a binary channel (acting on ncopies of a distribution for some n) which outputs Alice's and Bob's variables, such that the resulting distribution has nonzero key (see in this context (Acin *et al.*, 2003a,b; Bruß *et al.*, 2003)).

A strong confirmation supporting hypothesis of bound *information* is the result presented in (Renner and Wolf, 2003), where examples of distributions which asymptotically have bound information were found. Namely there is a family of distributions  $P(A_n, B_n, E_n)$  such that  $\lim_{n\to\infty} K(A_n; B_n || E_n) = 0$  while  $I_{form}(A_n; B_n || E_n) > \frac{1}{2}$ for all n. Another argument in favor of the existence of bound information in this *bipartite* scenario, is the fact that the *multipartite bound information* has been already proved to exist, and explicit examples have been constructed (Acin et al., 2004b). More specifically this time three honest cooperating parties (Alice, Bob and Clare) and the eavesdropper (Eve) share n realizations of a joint distribution  $P(A, B, C, E)_{bound}$  with the following properties: it has nonzero secret key cost and no pair of honest parties (even with help of the third one) can distill secret key from these realizations  $P(A, B, C, E)_{bound}$ .

Its construction has been done according to a Gisin-Wolf procedure (302) via measurement of a (multipartite) purification of a bound entangled state in computational basis.

As an example consider a four partite state

$$|\psi_{be}\rangle = \left[\frac{1}{\sqrt{3}}|GHZ\rangle + \frac{1}{\sqrt{6}}(|001\rangle|1\rangle + |010\rangle|2\rangle + |101\rangle|3\rangle + |110\rangle|4\rangle)\right]_{ABCE}$$
(304)

where  $|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$ . It's partial trace over subsystem E is a multipartite bound entangled state<sup>99</sup> (Dür and Cirac, 2000b; Dür *et al.*, 1999b). If one measures  $|\psi_{be}\rangle$  in computational basis on all four subsystems, the resulting probability distribution on the labels of outcomes is the one which contains (tripartite) bound information (Acin *et al.*, 2004b).

#### XX. ENTANGLEMENT AND QUANTUM COMPUTING

#### A. Entanglement in quantum algorithms

Fast quantum computation is one of the most desired properties of quantum information theory. There are few quantum algorithms which outperform their classical counterparts. These are the celebrated Deutsch-Jozsa, Grover and Shor's algorithm, and their variations. Since entanglement is one of the cornerstones of quantum information theory it is natural to expect, that it should be the main ingredient of quantum algorithms which are better than classical. This was first pointed out by Jozsa in (Jozsa, 1997). His seminal paper opened a debate on

<sup>&</sup>lt;sup>99</sup> This is one of the states from the family given in Eq. (86), with parameters: m = 3,  $\lambda_0^+ = \frac{1}{3}$ ,  $\lambda_0^- = \lambda_{10} = 0$ ,  $\lambda_{01} = \lambda_{11} = \frac{1}{6}$ .

the role of entanglement in quantum computing. Actually, after more than decade from the discovery of the first quantum algorithm, there is no common agreement on the role of entanglement in quantum computation. We discuss major contributions to this debate. It seems that entanglement "assists" quantum speed up, but is not sufficient for this phenomenon.

Certainly *pure* quantum computation needs some level of entanglement if it is not to be simulated classically. It was shown by Jozsa and Linden, that if a quantum computer's state contains only constant (independent of number of input qubits n) amount of entanglement, then it can be simulated efficiently (Jozsa and Linden, 2002).

Next, Vidal showed that even a polynomial in n amount of entanglement present in quantum algorithm can also be simulated classically (Vidal, 2003). The result is phrased in terms of the number  $\chi$  which he defined as the maximal rank of the subsystem of the qubits that form quantum register of the computer (over all choices of the subsystem). Any quantum algorithm that maintains  $\chi$  of order O(poly(n)) can be efficiently classically simulated. In other words to give an exponential speedup the quantum algorithm needs to achieve  $\chi$  of exponential order in n, during computation.

This general result was studied by Orus and Latorre (Orus and Latorre, 2004) for different algorithms in terms of entropy of entanglement (von Neumann entropy of subsystem). It is shown among others that computation of Shor's algorithm generates highly entangled states (with linear amount of entropy of entanglement which corresponds to exponential  $\chi$ ). Although it is not known if the Shor's algorithm provides an exponential speedup over classical factoring, this analysis suggests that Shor's algorithm cannot be simulated classically.

Entanglement in Shor's algorithm has been studied in different contexts (Ekert and Jozsa, 1998; Jozsa and Linden, 2002; Parker and Plenio, 2002; Shimoni *et al.*, 2005). Interestingly, as presence of entanglement in quantum algorithm is widely confirmed (see also (Datta *et al.*, 2005; Datta and Vidal, 2006)), its role is still not clear, since it seems that amount of it depends on the type of the input numbers (Kendon and Munro, 2006b).

Note, that the above Jozsa-Linden-Vidal "no entanglement implies no quantum advantage on pure states" result shows the need of entanglement presence for exponential speed up. Without falling into contradiction, one can then ask if entanglement must be present for polynomial speed up when only pure states are involved during computation (see (Kenigsberg *et al.*, 2006) and references therein).

Moreover it was considered to be possible, that a quantum computer using only *mixed*, separable states during computation may still outperform classical ones (Jozsa and Linden, 2002). It is shown, that such phenomenon can hold (Biham *et al.*, 2004), however with a tiny speed up. It is argued that isotropic separable state cannot be entangled by an algorithm, yet it can prove

useful in quantum computing. Answering to the general question of how big the enhancement based on separable states may be, needs more algorithm-dependent approach.

That the presence of entanglement is only necessary but not sufficient for exponential quantum speed up follows from the famous Knill-Gottesman theorem (Gottesman and Chuang, 1999; Jozsa and Linden, 2002). It states that operations from the so called *Clif*ford group composed with Pauli measurement in computational basis can be efficiently simulated on classical computer. This class of operations can however produce highly entangled states. For this reason, and as indicated by other results cited above, the role of entanglement is still not clear. As it is pointed out in (Jozsa and Linden, 2002), it may be that what is essential for quantum computation is not entanglement but the fact that the set of states which can occur during computation can not be described with small a number of parameters (see also discussion in (Knill, 2001) and references therein).

## B. Entanglement in quantum architecture

Although the role of entanglement in algorithms is unclear, its role in architecture of quantum computers is crucial. First of all the multipartite cluster states provide a resource for one-way quantum computation (Raussendorf and Briegel, 2001). One prepares such multipartite state, and the computation bases on subsequent measurements of qubits, which uses up the state.

One can ask what other states can be used to perform universal one-way quantum computation. In (den Nest et al., 2006) it was assumed that universality means possibility of creating any final state on the part of lattice that was not subjected to measurements. It was pointed out that by use of entanglement measures one can rule out some states. Namely, they introduced an entanglement measure, entanglement width, which is defined as the minimization of bipartite entanglement entropy over some specific cuts. It turns out that this measure is unbounded for cluster states (if we increase size of the system). Thus any class of states, for which this measure is bounded, cannot be resource for universal computation, as it cannot create arbitrary large cluster states. For example the GHZ state is not universal, since under any cut the entropy of entanglement is just 1. One should note here, that a more natural is weaker notion of universality where one requires possibility of compute arbitrary classical function. In (Gross and Eisert, 2006) it is shown that entanglement width is not a necessary condition for this type of universality.

An intermediate quantum computing model, between circuit based on quantum gates, and oneway computing, is teleportation based computing (Gottesman and Chuang, 1999). There two and three qubit gates are performed by use of teleportation as a basic primitive. The resource for this model of computation are thus EPR states and GHZ states. Teleportation based computing is of great importance, as it allows for efficient computation by use of linear optics (Knill *et al.*, 2001), where it is impossible to perform two-qubit gates deterministically. Moreover, using it, Knill has significantly lowered the threshold for fault tolerant computation (Knill, 2004).

The need of entanglement in quantum architecture has been also studied from more "dynamical" point of view, where instead of entangled states, one asks about operations which generate entanglement. This question is important, as due to decoherence, the noise usually limits entanglement generated during the computation. This effect can make a quantum device efficiently simulatable and hence useless. To avoid this, one has to assure that operations which constitute a quantum device generate sufficient amount of (or a robust kind of) entanglement.

An interesting connection between entanglement and fault tolerant quantum computation was obtained by Aharonov (Aharonov, 1999). She has shown that a properly defined *long range entanglement*  $^{100}$  vanishes in thermodynamical limit if the noise is too large. Combining it with the fact that fault tolerant scheme allows to achieve such entanglement, one obtains a sort of phase transition (the result is obtained within phenomenological model of noise).

The level of noise under which quantum computer becomes efficiently simulatable was first studied in (Aharonov and Ben-Or, 1996). It is shown that quantum computer which operates (globally) on  $O(\log n)$  number of qubits at a time (i.e. with limited entangling capabilities) can be efficiently simulated for any nonzero level of noise. For the circuit model (with local gates), the phase transition depending on the noise level is observed (see also (Aharonov *et al.*, 1996)).

The same problem was studied further in(Harrow and Nielsen, 2003), however basing directly on the entangling capabilities of the gates used for computation. It is shown, that so called *separable* gates (the gates which can not entangle any product input) are classically simulatable. The bound on minimal noise level which turns a quantum computer to deal only with separable gates is provided there. This idea has been recently developed in (Virmani et al., 2005), where certain gates which create only bipartite entanglement are studied. This class of gates is shown to be classically simulatable. In consequence, a stronger bound on the tolerable noise level is found.

# C. Byzantine agreement — useful entanglement for quantum and classical distributed computation

As we have already learned the role of entanglement in communication networks is uncompromised. We have already described its role in cryptography (see Sec. XIX) and communication complexity. Here we comment another application - quantum solution to one of the famous problem in classical fault-tolerant distributed computing called *Byzantine agreement*. This problem is known to have no solution in classical computer science. Yet its slightly modified version can be solved using quantum entangled multipartite state (Fitzi *et al.*, 2001).

One of the goals in distributed computing is to achieve broadcast in situation when one of the stations can send faulty signals. The station achieve broadcast if they fulfill conditions which can be viewed as a natural extension of broadcast to a fault-tolerant scenario:

- 1. if the sending station is not faulty, all stations announces the sent value.
- 2. if the sending station is faulty, then all stations announce the same value.

It is proved classically, that if there are  $t \ge n/3$  stations which are out of work and can send unpredictable data, then broadcast can not be achieved. In quantum scenario one can achieve the following modification called *detectable* broadcast which can be stated as follows:

- 1. if there is no faulty station, then all stations announces the received value.
- 2. if there is faulty station, then all other stations either *abort* or announces the same value.

This problem was solved in (Fitzi *et al.*, 2001) for n = 3. We do not describe here the protocol, but comment on the role of entanglement in this scheme. In the quantum Byzantine agreement there are two stages: quantum and classical one. The goal of quantum part is to distribute among the stations some correlations which they cannot deny (even the one which is faulty). To this end, the parties perform a protocol which distributes among them the Aharonov state:

$$\psi = \frac{1}{\sqrt{6}} (|012\rangle + |201\rangle + |120\rangle - |021\rangle - |102\rangle - |210\rangle) \quad (305)$$

The trick is that it can be achieved fault-tolerantly, i.e. even when one of the stations sends fake signals.

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<sup>&</sup>lt;sup>100</sup> Another type of long range entanglement was recently defined (Kitaev and Preskill, 2006; Levin and Wen, 2006) in the context of topological order.

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