Non-linear transformation optics

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Abstract

The concept of transformation optics is extended to non-linear media. It is shown that transformation optics can be generalized to include arbitrary non-linear response. In a systematic expansion, transformation optics favors implicit constitutive relations in terms of energy densities $\mathbf{D} \cdot \mathbf{E}$ and $\mathbf{B} \cdot \mathbf{H}$ rather than $\mathbf{E}^2$ and $\mathbf{H}^2$. As an example, the Kerr non-linearity is studied in some details.

1. Introduction

Transformation optics (TO) [1, 2, 3] has become one of the most powerful tools in the design of artificial media (metamaterials). TO allows to derive a linear constitutive relation directly from a desired trajectory of light – in a completely algebraic way. The key ingredient of TO is to define the trajectory in the specified medium as a result of a transformation applied on the trajectories in free space. Unlike normal coordinate transformations, under which physics is invariant, the field lines are considered as attached to the coordinates during the transformation, thereby changing physics and deriving the constitutive law of the desired medium. Since transformation media are linked to normal coordinate transformations, it follows immediately that all transformation media obtained as a deformation of the free-space solution must be linear. The formalism of TO does not require a free-space solution as starting point. Instead, complicated trajectories of light in a complex medium can be derived as the result of a deformation of the trajectories in a known, simpler medium. As far as non-birefringent, linear media are concerned, this generalization only adds some additional impedance factors compared to standard TO [4]. Here, we suggest to use a similar technique to introduce TO of non-linear media, which opens the door to a completely new application of this tool. It turns out that this generalization includes some important technical issues since not only the constitutive parameters, but also the parametrization of the nonlinear terms is affected by the coordinate transformations.

2. Transformation optics for nonlinear media

In TO the component notation of the Maxwell equations in conjunction with Einstein summation convention

$$\nabla_i B^i = 0\ , \qquad \frac{d}{dt}B^i + \epsilon^{ijk}\partial_j E_k = 0\ , \qquad \nabla_i D^i = \rho\ , \qquad \epsilon^{ijk}\partial_j H_k - \frac{d}{dt}D^i = j^i\ , \quad (1)$$

ISBN 978-952-67611-0-7

is extremely convenient to keep track of the transformation properties of all quantities. Here, ϵ^{ijk} is the Levi-Civita tensor and ∇_i is the covariant derivative in three dimensions with respect to the space metric γ_{ij}: $\nabla_i A^i = \frac{1}{\sqrt{\gamma}}\partial_i(\sqrt{\gamma}A^i)$. As an arbitrary coordinate transformation applied on a linear constitutive relation results in another linear constitutive relation, nonlinear TO must start with a nonlinear medium relation. This motivates as starting point the generic relation

$$D^i = \left(\varepsilon^{ij} + \chi_1^{ijk}E_k + \chi_2^{ijkl}E_kE_l + \ldots\right)E_j\ , \quad B^i = \left(\mu^{ij} + \xi_1^{ijk}H_k + \xi_2^{ijkl}H_kH_l + \ldots\right)H_j\ . \quad (2)$$

Under an arbitrary coordinate transformation, $\mathbf{x} \Rightarrow \bar{\mathbf{x}}$, all medium parameters transform as tensors with respect to the Jacobian $T^{i'}{}_j = \frac{\partial \bar{x}^{i'}}{\partial x^j}$,

$$\bar{\varepsilon}^{i'j'} = T^{i'}{}_iT^{j'}{}_j\varepsilon^{ij}\ , \qquad \bar{\chi}_1^{i'j'k'} = T^{i'}{}_iT^{j'}{}_jT^{k'}{}_k\chi_1^{ijk}\ , \qquad \bar{\chi}_2^{i'j'k'l'} = T^{i'}{}_jT^{j'}{}_jT^{k'}{}_kT^{l'}{}_l\chi_2^{ijkl}\ , \quad (3)$$
$$\bar{\mu}^{i'j'} = T^{i'}{}_iT^{j'}{}_j\mu^{ij}\ , \qquad \bar{\xi}_1^{i'j'k'} = T^{i'}{}_iT^{j'}{}_jT^{k'}{}_k\xi_1^{ijk}\ , \qquad \bar{\xi}_2^{i'j'k'l'} = T^{i'}{}_jT^{j'}{}_jT^{k'}{}_kT^{l'}{}_l\xi_2^{ijkl}\ , \quad (4)$$

which allows to rewrite the constitutive relation in the new coordinates as

$$\bar{D}^i = \left(\bar{\varepsilon}^{ij} + \bar{\chi}_1^{ijk}E_k + \bar{\chi}_2^{ijkl}E_kE_l + \ldots\right)\bar{E}_j\ , \quad \bar{B}^i = \left(\bar{\mu}^{ij} + \bar{\xi}_1^{ijk}H_k + \bar{\xi}_2^{ijkl}H_kH_l + \ldots\right)\bar{H}_j\ . \quad (5)$$

This describes the same system as Eq. (2), but formulated in different coordinates. As key ingredient of TO the field lines are "attached" to the coordinates during the transformation. This can be achieved by treating the new coordinates $\bar{\mathbf{x}}$ as if they belonged to the original formulation with metric γ_{ij} instead of $\bar{\gamma}_{ij}$, which makes the simple rescalings [3, 5]

$$\tilde{\mathbf{E}} = \bar{\mathbf{E}}\ , \qquad \tilde{\mathbf{D}} = \frac{\sqrt{\bar{\gamma}}}{\sqrt{\gamma}}\bar{\mathbf{D}}\ , \qquad \tilde{\mathbf{H}} = \bar{\mathbf{H}}\ , \qquad \tilde{\mathbf{B}} = \frac{\sqrt{\bar{\gamma}}}{\sqrt{\gamma}}\bar{\mathbf{B}}\ . \quad (6)$$

necessary. With this rule one arrives at the nonlinear transformation medium

$$\tilde{D}^i = \frac{\sqrt{\bar{\gamma}}}{\sqrt{\gamma}}\left(\bar{\varepsilon}^{ij} + \bar{\chi}_1^{ijk}\tilde{E}_k + \bar{\chi}_2^{ijkl}\tilde{E}_k\tilde{E}_l + \ldots\right)\tilde{E}_j\ , \quad \tilde{B}^i = \frac{\sqrt{\bar{\gamma}}}{\sqrt{\gamma}}\left(\bar{\mu}^{ij} + \bar{\xi}_1^{ijk}\tilde{B}_k + \bar{\xi}_2^{ijkl}\tilde{B}_k\tilde{B}_l + \ldots\right)\tilde{H}_j\ . \quad (7)$$

The transformation medium (7) is not physically equivalent to the original medium (2), but the exact solution of the former follows in an algebraic way from the solution of Eq. (2) via the transformations (3) and (4) and the rescalings (6), which makes this tool extremely powerful also for nonlinear media.

3. Transformations of the Kerr nonlinearity

Let us simplify Eq. (2) to the Kerr nonlinearity

$$D^i = \left(\varepsilon + \chi|\mathbf{E}|^2\right)\delta^{ij}E_j\ , \qquad B^i = \mu_0\delta^{ij}H_j\ , \quad (8)$$

where we assumed an orthonormal frame in this equation. In trying to apply the formalism developed in the previous section to this example an additional complication arises. Indeed, $|\mathbf{E}|^2 = E_i\delta^{ij}E_j$ implicitly depends on the spatial metric, which in the orthonormal frame is just the unit matrix. From the result (7) it follows that the Kerr nonlinearity transforms into

$$\tilde{D}^i = \sqrt{\bar{\gamma}}\left(\varepsilon + \chi\,(\tilde{E}_k\bar{\gamma}^{kl}\tilde{E}_l)\right)\bar{\gamma}^{ij}\tilde{E}_j\ , \qquad \tilde{B}^i = \mu_0\sqrt{\bar{\gamma}}\bar{\gamma}^{ij}\tilde{H}_j\ . \quad (9)$$

In this equation, the above-mentioned re-interpretation of the transformed coordinates is problematic, since the nonlinear term is no longer parametrized in terms of the absolute value squared of the electric field in laboratory space $|\tilde{\mathbf{E}}|^2 = \tilde{E}_i\delta^{ij}\tilde{E}_j$, but in terms of the absolute value squared in fictitious electromagnetic space. To circumvent such issues one should ensure that the original ansatz of the Kerr nonlinearity is free of hidden metrics. A simple way to achieve this is by realizing that the combination $\mathbf{D} \cdot \mathbf{E}$, which has the interpretation of an energy density, is a true scalar without any metric involved. With this formulation at hand, the above restriction to an orthonormal frame can be relaxed. Thus, the starting point of a transformed Kerr nonlinearity must be

$$D^i = (\varepsilon + \kappa \mathbf{D} \cdot \mathbf{E})\, \gamma^{ij} E_j \ , \qquad \kappa = \frac{\chi}{\varepsilon} + \text{higher contributions,} \tag{10}$$

which under a transformation $\mathbf{x} \Rightarrow \bar{\mathbf{x}}$ results in the transformation medium

$$\tilde{D}^i = \left(\frac{\sqrt{\bar{\gamma}}}{\sqrt{\gamma}} \varepsilon + \kappa \tilde{\mathbf{D}} \cdot \tilde{\mathbf{E}} \right) \bar{\gamma}^{ij} \tilde{E}_j \ , \qquad \tilde{B}^i = \mu_0 \frac{\sqrt{\bar{\gamma}}}{\sqrt{\gamma}} \bar{\gamma}^{ij} \tilde{H}_j \ . \tag{11}$$

As in TO for linear media, the effect of the transformation is encoded in the inverse metric in fictitious electromagnetic space, $\bar{\gamma}^{ij}$, that defines up to overall constants the permittivity, permeability and – newly – nonlinearity tensors, complemented by the rescaling factors $\sqrt{\bar{\gamma}}/\sqrt{\gamma}$.

Important aspects of nonlinear optics such as self-focusing or self-trapping are conveniently described in terms of an intensity dependent refractive index profile [6]. In the original Kerr medium the standard expressions

$$n = n_0 + n_2 I \ , \qquad n_0 = \sqrt{\varepsilon_R} \ , \qquad n_2 = \frac{3\chi_R}{4n_0} = \frac{3\kappa}{4\varepsilon n_0} \ , \tag{12}$$

are valid. As an example, let us consider a laser pulse entering the Kerr medium in z-direction. Coordinate transformations of the form $\bar{x} = f(z)x$, $\bar{y} = g(z)y$, $\bar{z} = \alpha z$ allow to manipulate directly the self-focusing effect in the non-linear medium in its profile (transformations of x and y) as well as in the length of the self-focusing distance (transformation of z). While the behavior of self-focusing under the transformation is dictated by the exact map of the original solution onto the solution in the transformation medium, it should be mentioned that the resulting transformation medium will include a linear magnetic response and thus the transformation of Eqs. (12) has to be considered carefully.

4. Conclusions

We have introduced the generalization of TO to nonlinear media. To make this possible a nonlinear medium instead of a free space solution was chosen as the starting point of TO. As in standard, linear TO, the solution in the – possibly extremely involved – transformation medium follows in an algebraic way from the solution in the simpler medium without transformation. The specific example of the Kerr nonlinearity was presented in some detail. Here, we encountered the additional technical issue that the nonlinear constitutive relation should be parametrized in terms of the energy densities $\mathbf{D} \cdot \mathbf{E}$ and $\mathbf{B} \cdot \mathbf{H}$. Nonetheless it was shown that transformation optics is capable to manipulate typical effects of nonlinear media such as self-focusing.

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