The transformation optics of chiral metamaterials

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Abstract

The geometrical interpretation of electromagnetism in transparent media (transformation optics) is extended to include media with isotropic, inhomogeneous chirality. It is found that both reciprocal and non–reciprocal chiral media may be described in terms of geometry through introducing the geometrical property of space–time torsion into the Maxwell equations. I then show how such a formalism may be applied to the design of optical devices.

1. Introduction–geometry and Maxwell's equations in continuous media

Transformation optics establishes an equivalence between Riemannian geometry and the behaviour of light within a transparent medium. Formally, light propagation through a curved space–time can be understood as propagation through an impedance matched medium [1]. This equivalence has been used as a design strategy for optical devices, where the 'design parameter' is the space time metric, $g_{\mu\nu}$, and the intuition is guided by the knowledge that light rays follow geodesics and the polarization undergoes parallel transport along each ray. The material parameters that arise from Maxwell's equations when desribed within a Riemannian geometry can be summarized within the following constitutive relations [2],

$$\begin{aligned} \boldsymbol{D} &= \epsilon_0 \boldsymbol{\eta} \cdot \boldsymbol{E} + \boldsymbol{g} \times \boldsymbol{H} \\ \boldsymbol{B} &= \mu_0 \boldsymbol{\eta} \cdot \boldsymbol{H} - \boldsymbol{g} \times \boldsymbol{E} \end{aligned} \tag{1}$$

where $\boldsymbol{\eta}$ is a dimensionless, symmetric 3×3 tensor, and $\boldsymbol{g}$ is a 3 component vector with the dimensions of inverse velocity. The tensor, $\boldsymbol{\eta}$, is determined by the measurement of *spatial* distance, while the magnetoelectric coupling, $\boldsymbol{g}$, is related to the frame–dragging [3] of the equivalent space–time geometry. The particular form of magnetoelectric coupling given in (1) may be practically realized with a moving dielectric medium (e.g. a fluid), where $\boldsymbol{g}$ is proportional to the local velocity of the medium [4].

The term transformation optics came from the initial use of co–ordinate transformations to arrive at material parameters—i.e. Euclidean geometry [5, 6]. This was the approach that led to the design and manufacture of a monochromatic cloaking device [5], and in this original sense, transformation optics works through the specification of three functions of position. The full Riemannian geometry has greater freedom, with a symmetric space–time metric containing *ten* independent functions of position, translating into nine independent material parameters [2]. For comparison, the full Riemannian geometry allows us to envisage an invisibility device that works over a broad spectrum of frequencies [7]. Here we show that the geometrical analogy is deeper than this, and we can use *non–Riemannian* differential geometry to examine materials with more general magnetoelectric coupling (in particular chiral materials) within transformation optics.

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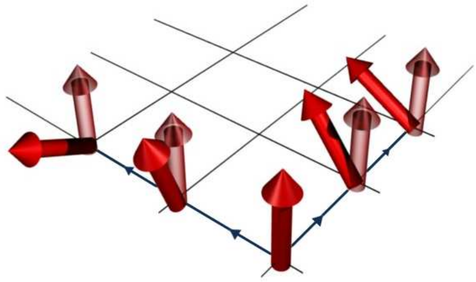

Fig. 1: Parallel transport in a space with an antisymmetric torsion tensor. Consequently, the polarization is rotated around the direction of propagation.

2. A chiral medium as a non–Riemannian geometry for light

The form of (1) represents only a particular kind of (non–reciprocal) magnetoelectric coupling, and many other possibilities exist [8]. Realistic examples of such media have a natural symmetry in the coupling of the electric polarization to the magnetic field and the magnetization to the electric field, which is suggestive of a geometrical interpretation. For instance, an isotropic chiral medium can be described with the following constitutive relations,

$$\begin{aligned} \boldsymbol{D} &= \epsilon \boldsymbol{E} + \frac{i\kappa}{c}\boldsymbol{B} \\ \boldsymbol{H} &= \mu^{-1}\boldsymbol{B} + \frac{i\kappa}{c}\boldsymbol{E} \end{aligned} \tag{2}$$

which—besides the distinction between $\boldsymbol{H}$ and $\boldsymbol{B}$—cannot be described in terms of (1).
We consider a simple example where (2) can be interpreted geometrically: the full theory can be found in [9]. Take a medium with $\epsilon(\omega, \boldsymbol{r})/\epsilon_0 = \mu(\omega, \boldsymbol{r})/\mu_0 = n(\omega, \boldsymbol{r})$, and where $\kappa(\omega, \boldsymbol{r})$ is a very slowly varying function of position. The source Maxwell equations can then be written in the frequency domain as,

$$\begin{aligned} \boldsymbol{\nabla} \cdot \boldsymbol{E} &= -\frac{1}{n}(\boldsymbol{\nabla} n) \cdot \boldsymbol{E} \\ \boldsymbol{\nabla} \times \boldsymbol{B} &= \frac{1}{n}(\boldsymbol{\nabla} n) \times \boldsymbol{B} - \frac{2\mu_0 n \omega \kappa}{c}\boldsymbol{B} - \frac{i\omega n^2}{c^2}\boldsymbol{E} \end{aligned} \tag{3}$$

After a little consideration it is clear that (3) can indeed be re–written with the material replaced by a geometry,

$$\nabla_\mu F^{\mu\nu} = 0 \tag{4}$$

where, $g_{\mu\nu} = \mathrm{diag}(1, -n^2, -n^2, -n^2)$, the components of $F_{\mu\nu}$ are identified as, $F_{01} \to E_x$, $F_{12} \to -B_z$, etc., and ∇_μ is a covariant derivative [3]. To write (4) from the chiral medium in (3), we must assume a geometry with the connection coefficients, $\Gamma_{ijk} = [i, jk] - \mu_0 \epsilon_{ijk} \omega \kappa(\omega, \boldsymbol{r})/c$, where $[i, jk]$ is the Riemannian part, and is determined by n. Why then does (1) not include chiral media? The geometry we describe has an additional non–zero *torsion*, $T_{ijk} = \Gamma_{ijk} - \Gamma_{ikj} = -2\epsilon_{ijk}\mu_0\omega\kappa(\omega, \boldsymbol{r})/c$, and is therefore non–Riemannian. It is intuitive that this should be so, for the effect of this torsion tensor is to rotate vectors during parallel transport (see fig. 1), which seems like a natural description for the phenomenon of optical activity. In [9] it is shown that in general isotropic, inhomogenous, reciprocal, and non–reciprocal chiral media can be understood in terms of the torsion tensor, and that geometrical optics in chiral media behaves as a theory of rays in a space with torsion.

3. Applications–transformation optics with greater control over polarization

What is the use of this extended theory of transformation optics? Firstly, it provides a new unified and simple way to understand a wider range of optical phenomena. Secondly, because polarization undergoes parallel transport, ordinary transformation optics does not allow us to control propagation and polarization independently. In our more general approach, non–Riemannian geometry allows us to control the polarization independently. For example, a device equivalent to propagation on the surface of a sphere (Maxwell's fish eye) can be made optically active if we introduce torsion into the geometry, and TE light transmitted from one pole can arrive as TM light at the opposite pole (see fig. 2) [9].

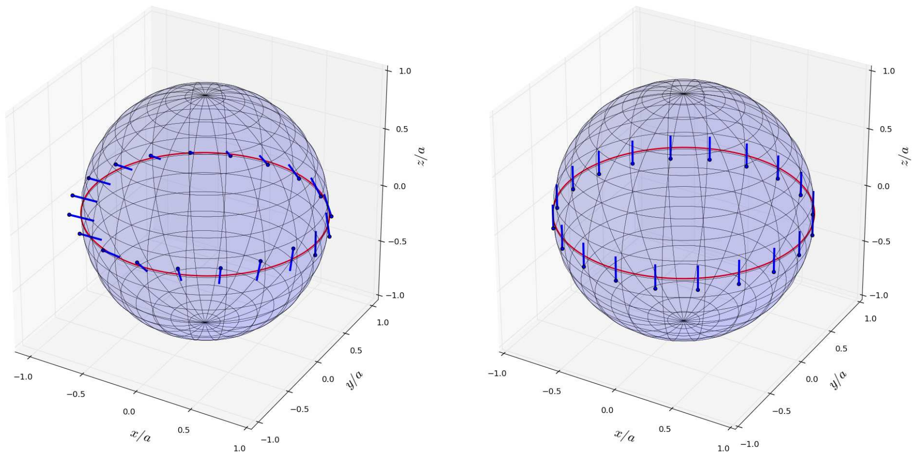

Fig. 2: Ray propagation in spherical geometry with (left) and without (right) torsion, illustrating that the introduction of torsion is equivalent to optical activity.

4. Conclusions

We have derived a more general geometrical framework for transformation optics that can describe a wider class of magnetoelectric media. In particular, we found that non–Riemannian geometry can be used to describe isotropic chiral media. This gives a natural description for optical activity in terms of the torsion tensor, and allows for polarization to be independently controlled within the formalism of transformation optics.

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