Is the perfect electromagnetic conductor the most general truly isotropic medium?

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Abstract

The perfect electromagnetic conductor (PEMC) is a paradigmatic medium in electromagnetics. It was introduced, using differential forms, as a truly isotropic medium. In this paper we will show, using spacetime algebra, that the PEMC is just an extreme case of a more general class of Tellegen media that are truly isotropic – that of Minkowskian isotropic media (MIM). A PEMC is shown to be a MIM that corresponds to an ideal electromagnetic conductor.

1. Introduction

The perfect electromagnetic conductor (PEMC) is a very important medium in electromagnetics as it generalizes the concepts of perfect electric conductor (PEC) and perfect magnetic conductor (PMC) [1]-[3]. A PEC is a medium where \( E = 0 \) and \( B = 0 \) and a PMC a medium where \( D = 0 \) and \( H = 0 \). As a biisotropic medium is characterized by the constitutive relations

\[
\begin{pmatrix}
\vec{D} \\
\vec{B}
\end{pmatrix} = \begin{pmatrix}
\varepsilon_0 \varepsilon & \sqrt{\varepsilon_0 \mu_0 \xi} \\
\sqrt{\varepsilon_0 \mu_0 \zeta} & \mu_0 \mu
\end{pmatrix} \begin{pmatrix}
\vec{E}
\\
\vec{H}
\end{pmatrix}
\]

(1)

a PEC corresponds to \( \varepsilon = \infty \) and \( \mu = 0 \) whereas a PMC corresponds to \( \varepsilon = 0 \) and \( \mu = \infty \). In both cases – for both the PEC and the PMC – parameters \( \xi \) and \( \zeta \) may have any values. A PEMC, on the other hand, is a medium where both conditions \( \vec{H} + \vec{M} \vec{E} = 0 \) and \( \vec{D} - \vec{M} \vec{B} = 0 \) are required to be valid. Then, a PEC corresponds to \( M = \pm \infty \) and a PMC to \( M = 0 \). However, what actually characterizes a PEMC is the fact that it is a truly isotropic medium: its isotropy is a Lorentz invariant property – in the sense that the medium is actually isotropic for any inertial observer (i.e., in terms of special relativity). Accordingly, we will call such a medium a Minkowskian isotropic medium (MIM). The following question then arises: is the PEMC the most general case of a MIM? The main goal of this paper is to answer that question. Our answer, however, is negative: we will show that a PEMC is just a very special case of a MIM and that the associated condition \( \varepsilon \mu - \xi \zeta = 0 \) is actually wrong.

The main difficulty presented by the definition of a PEMC stems from (1): a PEMC corresponds to the case where [1]

\[
\left( \frac{\varepsilon_0 \varepsilon}{\sqrt{\varepsilon_0 \mu_0 \xi}} \right) = q \begin{pmatrix}
M \\
1
\end{pmatrix}, \quad q \to \infty.
\]

(2)

This leads to infinite values for the four parameters \( (\varepsilon, \mu, \xi, \zeta) \) unless \( M = 0 \) or \( M = \infty \). In this paper we will show that the most general case of a MIM corresponds actually to
\[
\left( \frac{\varepsilon_0 e}{\sqrt{\varepsilon_0 \mu_0 \zeta}}, \frac{\sqrt{\varepsilon_0 \mu_0 \zeta}}{\mu_0 \mu} \right) = q \left( \begin{pmatrix} 1 + q^{-2} c^{-2} & M & 1 \\ 1 & 1 / M \end{pmatrix}, \right) \text{ with } \left\{ c = (\varepsilon_0 \mu_0)^{1/2}, \eta_0 = (\mu_0 / \varepsilon_0)^{1/2} \right\}
\]

where \((q, M)\) are the two scalars that define the medium. Only when \(q \to \infty\) do we get the very special (and ideal) case of a PECM corresponding to (2). One should stress, however, that

\[
\varepsilon = \left( \alpha_r^2 + \beta_r^2 \right) / \alpha_r, \quad \mu = 1 / \alpha_r, \quad \zeta = \zeta = \kappa = -\beta_r / \alpha_r \quad \text{with } \left( \alpha_r, \beta_r \right) = M \left( \mu_0 / q, -\eta_0 \right).
\]

This means that a MIM is a Tellegen medium with two parameters \((\alpha_r, \beta_r)\) such that

\[
n_0^2 = \varepsilon / \mu - \zeta \zeta = \varepsilon / \mu - \kappa^2 = 1, \quad \eta^2 = \varepsilon / \mu = \alpha_r^2 + \beta_r^2 = M^2 \left( \eta_0^2 + \mu_0^2 / q^2 \right).
\]

Hence, for the particular case of a PEMC (with \(q \to \infty\)), one should also have \(n_0^2 = 1\) and \(\varepsilon / \mu = \beta_r^2 = \eta_0^2 M^2\). Accordingly, the statement that, in a PEMC, one has \(\varepsilon / \mu = \zeta \zeta\) as in equation (19) of [1], in equation (21) of [2] or in p. 26-6 of [3] is wrong. In fact, the phase velocity (and the group velocity, if there is no dispersion) and also the energy velocity, in there are no losses) inside a MIM is

\[
v_p = c / n_0 = c / \sqrt{\varepsilon / \mu - \kappa^2} = c.
\]

For a truly isotropic medium, then, one should have \(n_0 = 1\) and \(v_p = c\) for all inertial observers. A value \(n_0 = 0\), corresponding to \(\kappa^2 = \varepsilon / \mu\), would violate – for a dispersionless and loss-less medium – the foundations of special relativity. In fact, the calculation of the determinant in (2) can lead to erroneous conclusions if one does take into account that – in terms of a PEMC – it is an indeterminate of the form \(\infty \to \infty\) according to (3) and (4), reduces to

\[
\Delta = \varepsilon_0 \mu_0 \left( \varepsilon / \mu - \zeta \zeta \right) = \left( \varepsilon / \mu - \kappa^2 \right) / c^2 = n_0^2 / c^2 = \left( q^2 \alpha_r^2 \right) / \beta_r^2 \quad : \quad \Delta = 1 / c^2.
\]

For a PEMC one has \(q \to \infty\) and \(\alpha_r \to 0\), though, \(q \alpha_r = -\beta_r / c = \mu_0 M\) according to (4).

### 2. Defining a MIM through spacetime algebra

The discovery of the PEMC, by Lindell and Sihvola, is intimately linked to the formulation of electromagnetics with differential forms as developed by Lindell in [4]. In terms of differential forms and using the notation of [1]-[4] the spacetime constitutive relation of a PEMC corresponds to \(\Psi = M \Phi\).

A MIM is a more general medium: its spacetime constitutive relation corresponds, using the same notation, to \(\Psi = M \Phi + N^* \Phi\), where \(N^* \Phi\) is the Hodge dual of \(\Phi\) [5]. However, we prefer the more simple formalism of spacetime algebra (STA) [6] adopted in [7] and [8]. The Euclidean three-dimensional version of geometric algebra was also used in [9] and [10]. In terms of STA and using the same notation of [6] and [8], the spacetime constitutive relation of a MIM is the following:

\[
G = \left( \frac{1}{\eta_0} \right) (\alpha_r F + \beta_r I F) = M \left[ F / (c q) - I F \right]
\]

with \(H + M E = (M / q) B\) and \(D - M B = \left[ M / (c q^2) \right] E\). This is a manifestly covariant equation because \(\alpha_r\) and \(\beta_r\) are scalars, \(F = c^{-1} E + IB\) is the Faraday bivector, \(G = D + c^{-1} I H\) is the Maxwell bivector and \(I\) is the unit quadrivector with \(I^2 = -1\). Hence, \(I F = -B + c^{-1} I E\) is the Clifford dual of \(F\). If \(e_0 \in \mathbb{R}^4\) is a given inertial observer, with \(e_0^2 = 1\) (STA corresponds to Clifford algebra \(\mathbb{C}_{1,3}^\ast\)), then \(E = \tilde{E} e_0\) where \(\tilde{E} \in \mathbb{R}^{0,3}\) is an anti-Euclidean (relative) vector. Likewise, one has: \(B = \tilde{B} e_0\), \(D = \tilde{D} e_0\) and \(H = \tilde{H} e_0\). If \(\{ e_0, e_1, e_2, e_3 \}\) is an orthonormal basis for the quadratic space \(\mathbb{R}^{1,3}\), with \(e_0^2 = -e_1^2 = -e_2^2 = -e_3^2 = 1\), then \(I = e_1 e_2 e_3\). One should stress that \((E, B, D, H)\) are observer-dependent (i.e., relative) bivectors whereas \((F, G)\) are observer-independent (i.e., absolute) bivectors. So, in fact, from (8) we readily derive the Gibbs-Heaviside form of the constitutive relations for a
MIM, i.e., our former equation (3). One should note that, in (2)-(5), parameters $\varepsilon, \mu, \xi, \zeta, \kappa, \eta, \alpha_r, \beta_r$ are dimensionless. Furthermore, a MIM becomes an electromagnetic conductor (EMC) whenever the electromagnetic admittance

$$\gamma = \alpha_r / \eta_0 = M/(qc)$$

(9)
is such that $|\gamma| \ll 1$, thereby making $H + M E \approx 0$ and $D - M B \approx 0$ in (8). In fact, (8) can be written as $G = \gamma F - M IF$. A PEMC corresponds to an EMC when $\gamma \to 0$ or $G = -M IF$.

3. Tellegen moving media

For a general Tellegen medium one has $\xi = \zeta = \kappa$ in (1). A MIM corresponds to (3); a PEMC corresponds to (2). For a MIM, as well as for the special case of a PEMC, one has $\varepsilon \mu - \kappa^2 = 1$. For a moving Tellegen medium, as seen by any inertial observer $v = c e_0$, the manifestly covariant spacetime constitutive relation is

$$G = \frac{1}{\eta_0} \left[ \frac{1}{n_0} \exp(\Theta n_0) F - \kappa_0 IF \right], \quad \Theta = \ln(n_0), \quad \eta_0 = \frac{\mu}{n_0}, \quad \kappa_0 = \frac{\kappa}{\mu}. \quad (10)$$

In (10) we have introduced the operator $r_v(F) = -v F v$ such that $r_v^2 = 1$. This spacetime constitutive relation corresponds to a bianisotropic medium. For $n_0 = 1$ or $\Theta = 0$, (10) reduces to (8) as expected.

4. Conclusion

We have shown that a MIM is the most general class of truly isotropic media, i.e., where the isotropic characterization is completely observer-independent. To characterize a MIM two parameters $(\alpha_r, \beta_r)$ or $(\mu, M)$ are needed according to (8). Hence, for any MIM (including a PEMC), one should always have $n_0^2 = \varepsilon \mu - \xi \zeta = 1$. Actually, an EMC is a MIM that becomes a PEMC when the admittance $\gamma$ in (9) vanishes, i.e., when $q \to \infty$. Although a PEMC acts as a boundary for electromagnetic waves any actual EMC, with $\gamma \neq 0$, can support electromagnetic waves thus making relevant the value of $n_0$.

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References