Is the perfect electromagnetic conductor the most general truly isotropic medium?

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Abstract

The perfect electromagnetic conductor (PEMC) is a paradigmatic medium in electromagnetics. It was introduced, using differential forms, as a truly isotropic medium. In this paper we will show, using spacetime algebra, that the PEMC is just an extreme case of a more general class of Tellegen media that are truly isotropic – that of Minkowskian isotropic media (MIM). A PEMC is shown to be a MIM that corresponds to an ideal *electromagnetic conductor*.

1. Introduction

The perfect electromagnetic conductor (PEMC) is a very important medium in electromagnetics as it generalizes the concepts of perfect electric conductor (PEC) and perfect magnetic conductor (PMC) [1]-[3]. A PEC is a medium where $\vec{E}=0$ and $\vec{B}=0$ and a PMC a medium where $\vec{D}=0$ and $\vec{H}=0$. As a biisotropic medium is characterized by the constitutive relations

$$\begin{pmatrix}\vec{D}\\ \vec{B}\end{pmatrix}=\begin{pmatrix}\varepsilon_0\varepsilon & \sqrt{\varepsilon_0\mu_0}\,\xi\\ \sqrt{\varepsilon_0\mu_0}\,\zeta & \mu_0\mu\end{pmatrix}\begin{pmatrix}\vec{E}\\ \vec{H}\end{pmatrix} \tag{1}$$

a PEC corresponds to $\varepsilon=\infty$ and $\mu=0$ whereas a PMC corresponds to $\varepsilon=0$ and $\mu=\infty$. In both cases – for both the PEC and the PMC – parameters ξ and ζ may have any values. A PEMC, on the other hand, is a medium where both conditions $\vec{H}+M\vec{E}=0$ and $\vec{D}-M\vec{B}=0$ are required to be valid. Then, a PEC corresponds to $M=\pm\infty$ and a PMC to $M=0$. However, what actually characterizes a PEMC is the fact that it is a truly isotropic medium: its isotropy is a Lorentz invariant property – in the sense that the medium is actually isotropic for any inertial observer (i.e., in terms of special relativity). Accordingly, we will call such a medium a Minkowskian isotropic medium (MIM). The following question then arises: is the PEMC the most general case of a MIM? The main goal of this paper is to answer that question. Our answer, however, is negative: we will show that a PEMC is just a very special case of a MIM and that the associated condition $\varepsilon\mu-\xi\zeta=0$ is actually wrong.

The main difficulty presented by the definition of a PEMC stems from (1): a PEMC corresponds to the case where [1]

$$\begin{pmatrix}\varepsilon_0\varepsilon & \sqrt{\varepsilon_0\mu_0}\,\xi\\ \sqrt{\varepsilon_0\mu_0}\,\zeta & \mu_0\mu\end{pmatrix}=q\begin{pmatrix}M & 1\\ 1 & 1/M\end{pmatrix},\quad q\to\infty. \tag{2}$$

This leads to infinite values for the four parameters $(\varepsilon,\mu,\xi,\zeta)$ unless $M=0$ or $M=\infty$. In this paper we will show that the most general case of a MIM corresponds actually to

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$$\begin{pmatrix} \varepsilon_0 \varepsilon & \sqrt{\varepsilon_0 \mu_0}\,\xi \\ \sqrt{\varepsilon_0 \mu_0}\,\zeta & \mu_0 \mu \end{pmatrix} = q \begin{pmatrix} \left(1 + q^{-2} c^{-2}\right) M & 1 \\ 1 & 1/M \end{pmatrix}, \qquad \text{with} \qquad \begin{cases} c = \left(\varepsilon_0 \mu_0\right)^{-1/2} \\ \eta_0 = \left(\mu_0/\varepsilon_0\right)^{1/2} \end{cases} \tag{3}$$

where (q, M) are the two scalars that define the medium. Only when $q \to \infty$ do we get the very special (and ideal) case of a PEMC corresponding to (2). One should stress, however, that

$$\varepsilon = \left(\alpha_F^2 + \beta_F^2\right)/\alpha_F, \quad \mu = 1/\alpha_F, \quad \xi = \zeta = \kappa = -\beta_F/\alpha_F \quad \text{with} \quad \left(\alpha_F, \beta_F\right) = M\left(\mu_0/q, -\eta_0\right). \tag{4}$$

This means that a MIM is a Tellegen medium with two parameters (α_F, β_F) such that

$$n_0^2 = \varepsilon\mu - \xi\zeta = \varepsilon\mu - \kappa^2 = 1, \quad \eta^{-2} = \varepsilon/\mu = \alpha_F^2 + \beta_F^2 = M^2\left(\eta_0^2 + \mu_0^2/q^2\right). \tag{5}$$

Hence, for the particular case of a PEMC (with $q \to \infty$), one should also have $n_0^2 = 1$ and $\varepsilon/\mu = \beta_F^2 = \eta_0^2 M^2$. Accordingly, the statement that, in a PEMC, one has $\varepsilon\mu = \xi\zeta$ as in equation (19) of [1], in equation (21) of [2] or in p. 26-6 of [3] is wrong. In fact, the phase velocity (and the group velocity, if there is no dispersion; and also the energy velocity, in there are no losses) inside a MIM is

$$v_p = c/n_0 = c\Big/\sqrt{\varepsilon\mu - \kappa^2} = c\,. \tag{6}$$

For a truly isotropic medium, then, one should have $n_0 = 1$ and $v_p = c$ for all inertial observers. A value $n_0 = 0$, corresponding to $\kappa^2 = \varepsilon\mu$, would violate – for a dispersionless and lossless medium – the foundations of special relativity. In fact, the calculation of the determinant in (2) can lead to erroneous conclusions if one does take into account that – in terms of a PEMC – it is an indeterminate of the form $\infty - \infty$ which, according to (3) and (4), reduces to

$$\Delta = \varepsilon_0 \mu_0 \left(\varepsilon\mu - \xi\zeta\right) = \left(\varepsilon\mu - \kappa^2\right)/c^2 = n_0^2/c^2 = \left(q^2 \alpha_F^2\right)\big/\beta_F^2 \quad \therefore \quad \boxed{\Delta = 1/c^2}\,. \tag{7}$$

For a PEMC one has $q \to \infty$ and $\alpha_F \to 0$, though, $q\alpha_F = -\beta_F/c = \mu_0 M$ according to (4).

2. Defining a MIM through spacetime algebra

The discovery of the PEMC, by Lindell and Sihvola, is intimately linked to the formulation of electromagnetics with differential forms as developed by Lindell in [4]. In terms of differential forms and using the notation of [1]-[4] the spacetime constitutive relation of a PEMC corresponds to $\mathbf{\Psi} = M\,\mathbf{\Phi}$. A MIM is a more general medium: its spacetime constitutive relation corresponds, using the same notation, to $\mathbf{\Psi} = M\mathbf{\Phi} + N\,{}^\star\mathbf{\Phi}$, where ${}^\star\mathbf{\Phi}$ is the Hodge dual of $\mathbf{\Phi}$ [5]. However, we prefer the more simple formalism of spacetime algebra (STA) [6] adopted in [7] and [8]. The Euclidean three-dimensional version of geometric algebra was also used in [9] and [10]. In terms of STA and using the same notation of [6] and [8], the spacetime constitutive relation of a MIM is the following:

$$\boxed{\mathbf{G} = \left(1/\eta_0\right)\left(\alpha_F \mathbf{F} + \beta_F \mathbf{I}\mathbf{F}\right) = M\left[\mathbf{F}/(qc) - \mathbf{I}\mathbf{F}\right]} \tag{8}$$

with $\mathbf{H} + M\,\mathbf{E} = (M/q)\,\mathbf{B}$ and $\mathbf{D} - M\,\mathbf{B} = \left[M/(qc^2)\right]\mathbf{E}$. This is a manifestly covariant equation because α_F and β_F are scalars, $\mathbf{F} = c^{-1}\mathbf{E} + \mathbf{I}\mathbf{B}$ is the Faraday bivector, $\mathbf{G} = \mathbf{D} + c^{-1}\mathbf{I}\mathbf{H}$ is the Maxwell bivector and $\mathbf{I}$ is the unit quadrivector with $\mathbf{I}^2 = -1$. Hence, $\mathbf{I}\mathbf{F} = -\mathbf{B} + c^{-1}\mathbf{I}\mathbf{E}$ is the Clifford dual of $\mathbf{F}$. If $\mathbf{e}_0 \in \mathbb{R}^4$ is a given inertial observer, with $\mathbf{e}_0^2 = 1$ (STA corresponds to Clifford algebra $\mathcal{C}\ell_{1,3}$), then $\mathbf{E} = \vec{E}\mathbf{e}_0$ where $\vec{E} \in \mathbb{R}^{0,3}$ is an anti-Euclidean (relative) vector. Likewise, one has: $\mathbf{B} = \vec{B}\mathbf{e}_0$, $\mathbf{D} = \vec{D}\mathbf{e}_0$ and $\mathbf{H} = \vec{H}\mathbf{e}_0$. If $\{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is an orthonormal basis for the quadratic space $\mathbb{R}^{1,3}$, with $\mathbf{e}_0^2 = -\mathbf{e}_1^2 = -\mathbf{e}_2^2 = -\mathbf{e}_3^2 = 1$, then $\mathbf{I} = \mathbf{e}_0\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$. One should stress that $(\mathbf{E}, \mathbf{B}, \mathbf{D}, \mathbf{H})$ are observer--dependent (i.e., relative) bivectors whereas $(\mathbf{F}, \mathbf{G})$ are observer-independent (i.e., absolute) bivectors. So, in fact, from (8) we readily derive the Gibbs-Heaviside form of the constitutive relations for a

MIM, i.e., our former equation (3). One should note that, in (2)-(5), parameters $\varepsilon, \mu, \xi, \zeta, \kappa, \eta, \alpha_F, \beta_F$ are dimensionless. Furthermore, a MIM becomes an *electromagnetic conductor* (EMC) whenever the *electromagnetic admittance*

$$\boxed{\gamma = \alpha_F / \eta_0 = M/(qc)} \tag{9}$$

is such that $|\gamma| \ll 1$, thereby making $\mathbf{H} + M\,\mathbf{E} \approx 0$ and $\mathbf{D} - M\,\mathbf{B} \approx 0$ in (8). In fact, (8) can be written as $\mathbf{G} = \gamma \mathbf{F} - M\,\mathbf{I}\mathbf{F}$. A PEMC corresponds to an EMC when $\gamma \to 0$ or $\mathbf{G} = -M\,\mathbf{I}\mathbf{F}$.

3. Tellegen moving media

For a general Tellegen medium one has $\xi = \zeta = \kappa$ in (1). A MIM corresponds to (3); a PEMC corresponds to (2). For a MIM, as well as for the special case of a PEMC, one has $\varepsilon\mu - \kappa^2 = 1$. For a moving Tellegen medium, as seen by any inertial observer $\mathbf{v} = c\,\mathbf{e}_0$, the *manifestly covariant* spacetime constitutive relation is

$$\mathbf{G} = \frac{1}{\eta_0}\left[\frac{1}{\eta_G}\exp\left(\Theta \mathsf{r}_{\mathbf{v}}\right)\mathbf{F} - \kappa_G \mathbf{I}\mathbf{F}\right], \qquad \Theta = \ln\left(n_0\right), \qquad \eta_G = \frac{\mu}{n_0}, \qquad \kappa_G = \frac{\kappa}{\mu}. \tag{10}$$

In (10) we have introduced the operator $\mathsf{r}_{\mathbf{v}}\left(\mathbf{F}\right) = -\mathbf{v}\mathbf{F}\mathbf{v}$ such that $\mathsf{r}_{\mathbf{v}}^2 = 1$. This spacetime constitutive relation corresponds to a bianisotropic medium. For $n_0 = 1$ or $\Theta = 0$, (10) reduces to (8) as expected.

4. Conclusion

We have shown that a MIM is the most general class of truly isotropic media, i.e., where the isotropic characterization is completely observer-independent. To characterize a MIM two parameters (α_F, β_F) or (q, M) are needed according to (8). Hence, for any MIM (including a PEMC), one should always have $n_0^2 = \varepsilon\mu - \xi\zeta = 1$. Actually, an EMC is a MIM that becomes a PEMC when the admittance γ in (9) vanishes, i.e., when $q \to \infty$. Although a PEMC acts as a boundary for electromagnetic waves any actual EMC, with $\gamma \neq 0$, can support electromagnetic waves thus making relevant the value of n_0.

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