Does the magnetic permeability satisfy the Kramers-Kronig relations for passive media?

Mário G. Silveirinha†

†Departamento de Engenharia Electrotécnica, Instituto de Telecomunicações,
Universidade de Coimbra, Pólo II, 3030-290 Coimbra, Portugal
Phone: +351-239796231, Fax: +351-239796293, e-mail: mario.silveirinha@co.it.pt

Abstract

One of the most debated features of the extracted effective permeability of structured media is that often, even in case of very low loss, it may exhibit an anti-resonant frequency response that is inconsistent with the Kramers-Kronig's formulas for passive media. Here we critically analyse such features and study the restrictions of the dispersive behaviour of the magnetic permeability in the spectral range where its usual physical meaning is preserved.

1. Introduction

The Kramers-Kronig's (KK) formulas establish a relation between the real and imaginary parts of the frequency response function, $\chi(\omega)=\chi'(\omega)+i\chi''(\omega)$, of a physical system. Specifically, provided the system is causal and the response ceases for large frequencies, we have [1]:

$$\chi'(\omega)=\frac{1}{\pi}P.V.\int_{-\infty}^{+\infty}\frac{\chi''(\Omega)}{\Omega-\omega}d\Omega \tag{1a}$$

$$\chi''(\omega)=-\frac{1}{\pi}P.V.\int_{-\infty}^{+\infty}\frac{\chi'(\Omega)}{\Omega-\omega}d\Omega \tag{1b}$$

In particular, the above relations must certainly be satisfied by the magnetic susceptibility, $\chi(\omega)=\mu(\omega)-1$, that describes the magnetic response of some medium. Indeed, the permeability as a physical response can certainly be measured (or defined as the result of some measurement), and thus it must be causal and satisfy the above KK relations.

On the other hand, for passive media it is necessary that $\mu''(\omega)>0$ for $\omega>0$ in the spectral range where the permeability retains its usual physical meaning (in this work, we assume a time variation of the form $e^{-i\omega t}$ and that $\mu=\mu'+i\mu''$) [1-3]. However, it has been noted that this passivity condition together with the KK relations can lead to results that are inconsistent with the actual physical response of some materials. For example, it is simple to check that the passivity condition together with Eq. (1a) implies that $\mu(0)>1$, i.e. that the material's response in the static limit is paramagnetic. In general, this cannot be correct because it would contradict the existence of diamagnetic materials in the static limit [4].

Clearly, the reason for this inconsistency is the fact the magnetic permeability may lose its usual physical meaning at relatively low frequencies [1]. As a consequence, the condition $\mu''(\omega)>0$ may not hold for all ω, and it may be violated away from the spectral range $|\omega|<\omega_{\max}$ where the permeability retains its usual physical meaning. For example, in principle in frequency bands where the material is characterized by strong spatial dispersion, it may not be possible to define the permeability self-consistently, and even if it were possible it is doubtful that passivity would still require that $\mu''(\omega)>0$ because the magnetic effects may be strongly correlated with those of other concurrent higher-order multipoles.

To avoid such problems, the most common approach – described in details in the book of Landau and Lifshitz [1] – formulates the KK relations in such a way that only the frequency range $|\omega| < \omega_{\max}$ where the permeability retains its usual physical meaning is involved in the formulas. In particular, this Kramers-Kronig-Landau-Lifshitz (KK-LL) framework yields the following relation between the real and imaginary parts of the magnetic permeability:

$$\mu'(\omega) - \mu_1 = \frac{1}{\pi} P.V. \int_{-\omega_{\max}}^{+\omega_{\max}} \frac{\mu''(\Omega)}{\Omega - \omega} d\Omega \qquad (2)$$

where $\mu_1 = \mu(\omega_{\max})$. We emphasize that it is assumed that the permeability retains its usual physical meaning for $|\omega| < \omega_{\max}$, and thus it must satisfy $\mu''(\omega) > 0$ for $0 < \omega < \omega_{\max}$. However, the analytic continuation of μ outside $|\omega| < \omega_{\max}$, may not necessarily be consistent with the usual passivity condition.

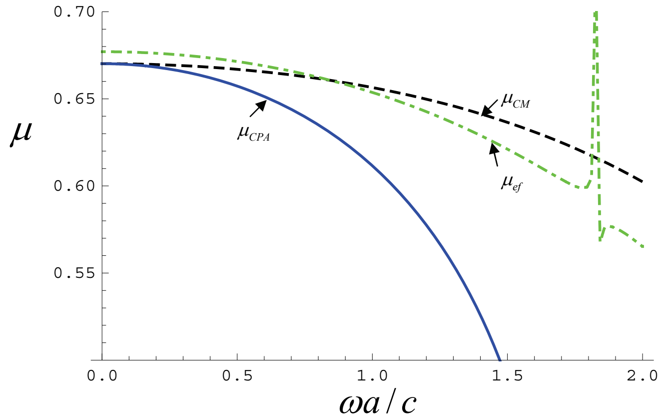

Fig 1. Magnetic permeability of a triangular array of metallic particles in the limit of vanishingly small loss. Solid blue line: coherent potential approximation [5]. Black dashed line: Clausius-Mossotti with a dispersive interaction constant. Green dot dashed line: full wave simulation [6].

Since μ_1 can be an arbitrary real number, the theory of Landau-Lifshitz permits a diamagnetic response in the static limit, and apparently solves the contradiction. However, this may not be the end of the story. To illustrate this, we consider a metamaterial formed by a triangular array of cylindrical metallic particles with circular cross-section and radius $R = 0.4a$, where a is the lattice constant. We restrict our attention to the case where the wave propagates in the *xoy* direction and the magnetic field is parallel to the axes of the cylindrical particles, i.e. the *z*-axis (for simplicity the geometry is inherently two-dimensional but similar results are obtained in the three-dimensional case). The rods have a plasmonic-type electrical response characterized by a Drude dispersion model $\varepsilon_{inc} = 1 - \omega_p^2 / \omega(\omega + i\Gamma)$ and $\mu = 1$. In Fig. 1 we plot the effective permeability (along the *z*-direction) for $\omega_p a / c = 10.0$ and in the limit of vanishing loss $\Gamma = 0^+$, calculated with three different homogenization formalisms. As seen, the effective medium theories concur reasonably well, and demonstrate that the material's response is diamagnetic in the static limit. However, what is truly striking, is that these homogenization theories suggest that the despite the loss being vanishingly small, the permeability may be characterized by an anomalous dispersion for arbitrarily low frequencies. It is well known that such regime is forbidden by the KK-LL relations for passive media, i.e. by Eq. (2).

2. An alternative to the KK-LL formulas

The results of Fig. 1 cannot possibly be explained in the framework of KK-LL, which requires that in a transparency band $d\mu / d\omega > 0$ [1]. Let us critically analyse the derivation of Eq. (6). A detailed analysis shows that Eq. (6) is valid only if the term $\chi_C(\omega) = \frac{1}{\pi} \int_0^{\pi} \frac{\mu(\omega_{\max} e^{i\theta}) - \mu_1}{1 - e^{-i\theta} \omega / \omega_{\max}} d\theta$ is negligible as compared to $\mu(\omega)-\mu_1$. This is the case if $\mu(\omega_{\max}e^{i\theta})\approx\mu_1$ for *any* angle $0\le\theta\le\pi$. However, it is unlikely that in general the magnetic response may be nearly constant in the pertinent semi-circumference. Actually, this can only happen if $\omega_{\max}>>\omega_{res}$, where ω_{res} is the largest resonant frequency of μ. Otherwise we may estimate that for small frequencies $|\chi_C(\omega)|\sim|\mu_1-1|$ and hence the KK-LL formulation is valid only if $|1-\mu_1|<<|\mu(\omega)-\mu_1|$. We argue that in case μ_1 differs significantly from unity, as it may happen in case of strong diamagnetism (e.g. in Fig. 1), the requirement $|1-\mu_1|<<|\mu(\omega)-\mu_1|$ may be too stringent, and thus the KK-LL relations may be inapplicable.

As discussed next, it is possible to formulate a weaker version of KK-LL formulas that overcomes the identified problems. Indeed, suppose that as usual the permeability $\mu(\omega)$ is regular at the origin and consider the function $\chi=(\mu-\mu_s)/\omega^2$, with $\mu_s=\mu(0)$. Clearly, χ is an analytic function in the upper-half plane and in the real axis (loss, even if vanishingly small, is necessarily present), with the exception of the point $\omega=0$, where it has a pole of order one. It is possible to prove that such properties enable to link μ' and μ'' as [7],

$$\frac{\mu'(\omega)-\mu_s}{\omega^2}=\frac{1}{\pi}P.V.\int_{-\omega_{\max}}^{\omega_{\max}}\frac{\mu''(\Omega)}{\Omega^2}\frac{1}{\Omega-\omega}d\Omega \tag{3a}$$

$$\frac{\mu''(\omega)}{\omega^2}=\frac{1}{i\omega}\left(\left.\frac{d\mu}{d\omega}\right|_{\omega=0}\right)-\frac{1}{\pi}P.V.\int_{-\omega_{\max}}^{\omega_{\max}}\frac{\mu'(\Omega)-\mu_s}{\Omega^2}\frac{1}{\Omega-\omega}d\Omega \tag{3b}$$

provided $|(\mu_s-1)/\omega_{\max}^2|<<|(\mu-\mu_s)/\omega^2|$. Clearly, unlike the condition associated with the usual KK-LL formula, this condition can be easily satisfied by μ provided $\omega<<\omega_{\max}$, and therefore Eqs. (3) should be valid even when the magnetic permeability has a resonance close or beyond $\omega_{\max}$, which typically is the case in metamaterials. Thus, we propose such formulas as an alternative to the standard KK-LL framework. They are based on weaker assumptions, and in the same manner as the KK-LL formulas enable linking the real and imaginary parts of the magnetic permeability in the spectral range where the permeability retains its physical meaning and is consistent with the usual passivity condition.

In particular, it is simple to check that Eq. (3a) implies that in a transparency band $(\mu-\mu_s)/\omega^2$ is an increasing function of frequency. In general this forces $d\mu/d\omega>0$ only if $\mu>\mu_s$. Thus, a regime of anomalous dispersion may be possible when $\mu<\mu_s$. This finding is totally consistent, and may explain, indeed, the anomalous dispersion effects in the example of Fig. 1. In this talk, we will further develop the outlined ideas, present several examples of two- and three-dimensional metamaterials characterized by anomalous dispersion of the permeability function in both the real and imaginary frequency axes, and explain such properties in light of our theory.

References

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