Sum rules and physical limitations for passive metamaterials

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Abstract

Bandwidth is an important parameter in many metamaterial applications. It has been shown that Herglotz functions and sum rules offer a powerful methodology to analyze the trade-off between bandwidth and design parameters. Here, this approach is described for the temporal dispersion of constitutive relations and high-impedance surfaces.

1. Introduction

Bandwidth is an important figure of merit in metamaterial applications such as high-impedance surfaces, cloaking, lenses, and antennas. In this paper, Herglotz functions and sum rules are used to derive constraints on passive metamaterials [1, 2, 3]. The results are exemplified for the temporal dispersion of the constitutive relations and the admittance of high-impedance surfaces.

Metamaterials are temporally dispersive, *e.g.*, the permittivity and permeability depend on frequency. The sum rule [1] relate weighted integrals of the constitutive parameter over all spectrum with the instantaneous and static response of the material model. Various sum rules are presented in [1] that constrain the dispersion of the constitutive relations. The bounds constrain the temporal dispersion over a frequency interval of the material parameter, *e.g.*, how close $\epsilon(\omega)$ can be to a constant ϵ_{m}.

In [2], a sum rule for high-impedance surfaces was introduced. It relates frequency intervals having impedance above an arbitrary threshold with the static properties of the structure. The sum rule is valid for periodic structures composed by arbitrary dielectric and magnetic materials above a perfect conductor. The sum rule is used to derive physical bounds that show how the bandwidth depends on thickness, angle of incidence, polarization, and material properties.

2. Temporal dispersion for constitutive relations

The linear, causal, time translational invariant, continuous, isotropic, passive and non-magnetic constitutive relations are

$$\boldsymbol{D}(t) = \epsilon_0\epsilon_\infty\boldsymbol{E}(t) + \epsilon_0\int_{-\infty}^{t}\chi_{\mathrm{ee}}(t-t')\boldsymbol{E}(t')\,\mathrm{d}t' \quad \text{such that } 0 \leq \int_{-\infty}^{T}\boldsymbol{E}(t)\cdot\frac{\partial\boldsymbol{D}(t)}{\partial t}\,\mathrm{d}t \qquad (1)$$

for all times T and smooth fields $\boldsymbol{E}$ tending to zero as $t\to-\infty$. The Fourier transform (time dependence $\mathrm{e}^{-\mathrm{i}\omega t}$) of (1) gives the frequency domain model $\boldsymbol{D}(\omega)=\epsilon_0\epsilon(\omega)\boldsymbol{E}(\omega)$ where the symbols $\boldsymbol{D}$ and $\boldsymbol{E}$ are reused to denote the electromagnetic fields as functions of the angular frequency ω. Passivity restricts the permittivity ϵ such that $h_\epsilon=\omega\epsilon(\omega)$ is a Herglotz function [4, 5], *i.e.*, $h_\epsilon(\omega)$ is holomorphic and $\operatorname{Im} h_\epsilon(\omega)\geq 0$ in the upper half plane $\operatorname{Im}\omega>0$, see [3, 4].

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In [1], it is shown that a permittivity (1) satisfies the sum rule

$$\int_0^\infty \mathrm{Im}\{h_1(\omega)\}\,\mathrm{d}\omega = \int_0^\infty \frac{1}{\pi}\arg\left(\frac{\omega\big(\epsilon(\omega)-\epsilon_\mathrm{m}\big)-\omega_0\Delta}{\omega\big(\epsilon(\omega)-\epsilon_\mathrm{m}\big)+\omega_0\Delta}\right)\mathrm{d}\omega = \frac{\omega_0\Delta}{\epsilon_\infty-\epsilon_\mathrm{m}} \quad \text{for } \epsilon_\mathrm{m}<\epsilon_\infty \qquad (2)$$

from which the following bound on the temporal dispersion around $\epsilon(\omega)\approx\epsilon_\mathrm{m}$ is derived

$$\max_{\omega\in[\omega_1,\omega_2]}|\epsilon(\omega)-\epsilon_\mathrm{m}| \geq \frac{B}{1+B/2}(\epsilon_\infty-\epsilon_\mathrm{m})\begin{cases}1/2 & \text{lossy case}\\ 1 & \text{lossless case.}\end{cases} \qquad (3)$$

where $B=(\omega_2-\omega_1)/\omega_0$ and $\omega_0=(\omega_1+\omega_2)/2$. The bound (3) shows that the deviation $|\epsilon(\omega)-\epsilon_\mathrm{m}|$ is proportional to the fractional bandwidth B and the difference $\epsilon_\infty-\epsilon_\mathrm{m}$.

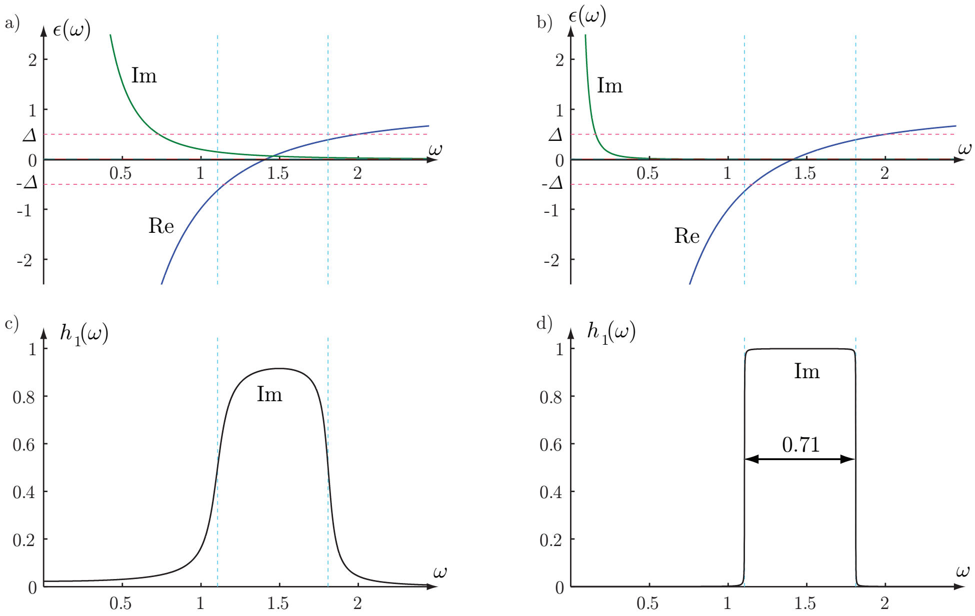

Fig. 1: Illustrations of the Drude model with $\nu=\{0.1,0.001\}$ and sum rule (2) for $\epsilon\approx 0$. a) the permittivity $\epsilon(\omega)$ for $\nu=0.1$. c) integrand $\mathrm{Im}\{h_1(\omega)\}$ in the sum rule (2) for $\nu=0.1$. bd) corresponding cases for $\nu=0.001$.

As an example consider the Drude model $\epsilon(\omega)=1+2/(-\mathrm{i}\omega(\nu-\mathrm{i}\omega))$ where ω is a dimensionless frequency variable and $\nu=\{0.1,0.001\}$ see Fig. 1ab. It is close to zero for $\omega\approx\omega_0\approx 1.4$. The sum rule (2) is evaluated for $\epsilon_\mathrm{m}=0$ and $\Delta=1/2$. The integrand in (2) is depicted in Fig. 1cd, where it is observed that it has most of its area in the region around $\omega_0\approx 1.4$, *i.e.*, in the region where $|\epsilon(\omega)|\leq\Delta$. Moreover it is seen that the integral approaches a box as the losses decrease. The area of this box is $\omega_0\Delta/(\epsilon_\infty-\epsilon_\mathrm{m})\approx 0.71$ that gives the bandwidth (width of the box) as the height is 1 in the lossless case. The additional factor of 2 for the lossy case in the bound (3) comes from potentially reducing the height of the box to $1/2$, see [1].

3. High-impedance surfaces

High-impedance surfaces are artificial surfaces synthesized from periodic structures above a ground plane [6, 7]. The properties of the high-impedance surface depend on frequency, polarization, and angle of incidence, and they have high impedance only over finite frequency bands [6, 7].

In [2], a sum rule that relates wavelength intervals having admittance below an arbitrary threshold with the static properties of the structure was introduced, *i.e.*,

$$\int_0^\infty \mathrm{Re}\, P_\Delta(Y(\lambda))\,\mathrm{d}\lambda = \int_0^\infty \frac{1}{\pi}\arg\left(\frac{\mathrm{j}Y(\lambda)-\Delta}{\mathrm{j}Y(\lambda)+\Delta}\right)\mathrm{d}\lambda = \left(d+\frac{\gamma}{2A}\right)2\pi\Delta, \tag{4}$$

where λ denotes the wavelength, Y the admittance, d the height of the structure, γ the polarizability, and A the unit cell area. The sum rule is used to derive the bound [2]

$$\frac{\lambda_2-\lambda_1}{d} \leq 4\pi\Delta \begin{cases} 1 & \text{lossy case} \\ 1/2 & \text{lossless case.} \end{cases} \quad \text{where } \Delta = \max_{\lambda\in[\lambda_1,\lambda_2]} |Y(\lambda)| \tag{5}$$

The sum rule (4) and bound (5) are illustrated in 2 for a mushroom structure [7]. The mushroom structure modeled as PEC. This implies that the admittance is lossless below the first grating lobe and the polarizability can be negative [2].

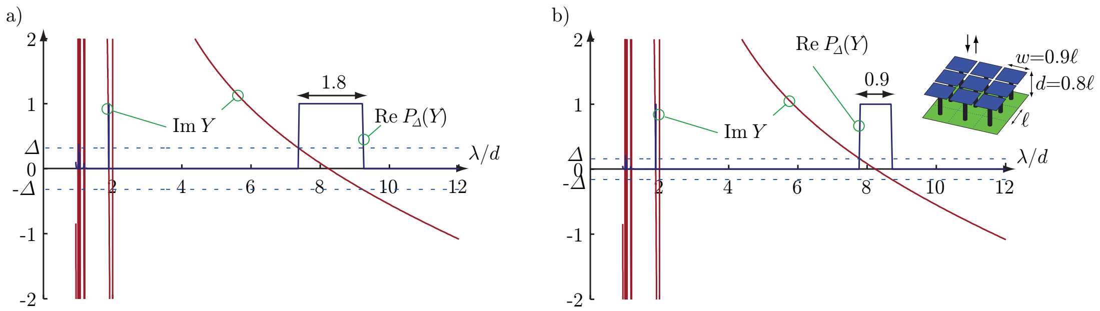

Fig. 2: Normalized admittance Y and composition $P_\Delta(Y)$ with $\Delta = \{1/\pi, 1/(2\pi)\}$ for a mushroom structure with $\ell = 10\,\mathrm{mm}$, $w = 9\,\mathrm{mm}$, and $d = 8\,\mathrm{mm}$. a) $\Delta = 1/\pi$ with the lossless bound $\lambda_2 - \lambda_1 \leq 2$. b) $\Delta = 1/(2\pi)$ with the lossless bound $\lambda_2 - \lambda_1 \leq 1$.

4. Conclusions

It is demonstrated that Herglotz functions and their associated integral identities (sum rules) provide a powerful methodology to analyze passive metamaterials. The sum rules are used to construct physical bounds that evaluate the maximal bandwidth of metamaterials.

References

[1] M. Gustafsson and D. Sjöberg. Sum rules and physical bounds on passive metamaterials. *New Journal of Physics*, 12:043046, 2010.

[2] M. Gustafsson and D. Sjöberg. Physical bounds and sum rules for high-impedance surfaces. *Antennas and Propagation, IEEE Transactions on*, 2011. in press.

[3] Anders Bernland, Annemarie Luger, and Mats Gustafsson. Sum rules and constraints on passive systems. *J. Phys. A: Math. Theor.*, 44(14):145205, 2011.

[4] H. M. Nussenzveig. *Causality and dispersion relations*. Academic Press, London, 1972.

[5] F. W. King. *Hilbert Transforms, Volume 2*. Cambridge University Press, 2009.

[6] D. Sievenpiper, Lijun Zhang, R. F. J. Broas, and N. G. Alexopolous. High impedance electromagnetic surfaces with a forbidden frequency band. *IEEE Trans. Microwave Theory Tech.*, 47(11):2059–2074, 1999.

[7] D. F. Sievenpiper. Artificial impedance surfaces for antennas. In C. A. Balanis, editor, *Modern Antenna Handbook*, pages 737–778. Wiley-Interscience, 2008.